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# *Note on Equivariant Maps from Spheres to Stiefel Manifolds*

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## § 1. Introduction

Let $X=(T, X)$ be a Hausdorff space with a fixed point free involution $T$. By [2, Def. (3.1)], the *index* of $(T, X)$ is the largest integer $n$ for which there is an equivariant map of the $n$-sphere $S^n$ into $X$. The *co-index* of $(T, X)$ is the least integer $n$ for which there is an equivariant map of $X$ into $S^n$. Here the fixed point free involution of $S^n$ is the antipodal involution $A$. We abbreviate *index* and *co-index* by $\operatorname{ind}(T, X)$ and $\operatorname{co-ind}(T, X)$, respectively. It may happen for a particular $X$ that there is no upper bound on the dimension of the sphere which can be equivariantly mapped into $X$; then we write $\operatorname{ind}(T, X)=\infty$. Also if $X$ cannot be equivariantly mapped into $S^n$ no matter how large $n$, write $\operatorname{co-ind}(T, X)=\infty$.

As there is no equivariant map of $S^{n+1}$ into $S^n$, we have

$$\operatorname{ind}(A, S^n) = \operatorname{co-ind}(A, S^n) = n.$$

Let $V_{n,m}$ be the Stiefel manifold of orthonormal $m$-frames in real $n$-space $R^n$. There is a fixed point free involution $T_2$ on $V_{n,m}$ defined by sending an $m$-frame $(v_1,\dots, v_m)$ to $(-v_1,\dots, -v_m)$.

Let $\xi_k$ be the canonical line bundle over $k$-dimensional real projective space $RP^k$, and $n\xi_k$ the Whitney sum of $n$-copies of $\xi_k$. Let Span $\alpha$ denote the maximum number of the linearly independent cross-sections of a vector bundle $\alpha$.

PROPOSITION 1. $\operatorname{ind}(T_2, V_{n,m})\geqq k$ *if and only if* $\operatorname{Span} n\xi_k\geqq m$.

For example, $\operatorname{Span} n\xi_k$ is studied in [6] and [9].

COROLLARY 2. $\operatorname{ind}(T_2, V_{n,2})=\operatorname{co-ind}(T_2, V_{n,2})=n-1$, *for even* $n$.

REMARK. By [2, p. 426],

$$n-2 = \operatorname{ind}(T_2, V_{n,2}) < \operatorname{co-ind}(T_2, V_{n,2}) = n-1, \quad \text{for odd} \quad n.$$

Let $Z_q=\{e^{i\theta}|\theta=2\pi h/q, h=0,\dots, q-1\}$ be the cyclic group of order $q$. Then an action of $Z_q$ on the complex $n$-space $C^n$ is defined by $e^{i\theta}(z_1,\dots, z_n)=(e^{i\theta}z_1,\dots, e^{i\theta}z_n)$.

We define an action $T_q$ of $Z_q$ on $V_{2n,m}$ such that $e^{i\theta}$ acts on each vector of $m$-frames as before.

Similarly, we define $\operatorname{ind}(T_q, V_{2n,m})$ to be the largest integer $2k+1$ for which there is a $Z_q$-equivariant map of $S^{2k+1}=V_{2k+2,1}$ into $V_{2n,m}$. We notice that $\operatorname{ind}(T_q, V_{2n,m})\geqq 1$.

Let $\eta_{k,q}$ be the canonical real 2-plane bundle over the mod $q$ standard lens space $L^k(q)=V_{2k+2,1}/Z_q=S^{2k+1}/Z_q$.

PROPOSITION 3. $\operatorname{ind}(T_q, V_{2n,m})\geqq 2k+1$ *if and only if* $\operatorname{Span} n\eta_{k,q}\geqq m$.

Let $p$ be an odd prime and $r$ be a positive integer.

PROPOSITION 4. *Suppose* $p\leqq k-[k/2]$. *If* $\binom{k+1}{[k/2]}\not\equiv(lp)^2 \pmod{p^r}$ *for any integer* $l$ *such that* $0\leqq l<p^{r-1}$, *then*

$$\operatorname{Span}(k+1)\eta_{k,p^r} = \operatorname{Span}(\tau(L^k(p^r)) \oplus 1) = 2k+1-2[k/2],$$

*where* $\tau(L^k(p^r))\oplus 1$ *is the Whitney sum of the tangent bundle* $\tau(L^k(p^r))$ *of* $L^k(p^r)$ *and the trivial line bundle* $1$ *over* $L^k(p^r)$.

REMARK. By [7, Th. (1.7)] and [9, Lemma 2.2],

$$\operatorname{Span}(k+1)\eta_{k,p^r} \geqq 2k+1-2[k/2].$$

COROLLARY 5. *Suppose* $p\leqq k-[k/2]$. *If* $\binom{k+1}{[k/2]}\not\equiv(lp)^2 \pmod{p^r}$ *for any integer* $l$ *such that* $0\leqq l<p^{r-1}$, *then*

$$\operatorname{ind}(T_{p^r}, V_{2k+2,2k+2-2[k/2]})\leqq 2k-1.$$

REMARK. By [7, Th. (1.7)] and [9, Lemma 2.2],

$$\operatorname{Span}(k+1)\eta_{2[k/2]-1,p^r}\geqq 2k+3-2[k/2] > 2k+2-2[k/2],$$

and so, by Proposition 3,

$$\operatorname{ind}(T_{p^r}, V_{2k+2,2k+2-2[k/2]}) \geqq 4[k/2]-1.$$

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## § 2. Proofs

PROOF OF PROPOSITION 1. Let $\tilde{T}_2$ be the involution on $S^k\times V_{n,m}$ defined by $\tilde{T}_2(x, v)=(A(x), T_2(v))$ $(x\in S^k, v\in V_{n,m})$, and $p$ be the map from $(S^k\times V_{n,m})/\tilde{T}_2$ onto $RP^k$ induced by the projection from $S^k\times V_{n,m}$ onto $S^k$. Then $p$ is the projection of the $m$-frame bundle associated with $n\xi_k$. It is easily seen that the existence of a cross-section of this bundle is equivalent to the existence of an equivariant map from $S^k$ to $V_{n,m}$. Thus the proof is complete.

PROOF OF COROLLARY 2. $n\xi_{n-1}$ is isomorphic to the Whitney sum of the tangent bundle of $RP^{n-1}$ and the trivial line bundle over $RP^{n-1}$, and $RP^{n-1}$ has a tangent 1-field for even $n$. So, by Proposition 1,

$$\operatorname{ind}(T_2, V_{n,2}) \geqq n-1 .$$

As there is an equivariant map from $V_{n,2}$ to $S^{n-1}$ by sending each 2-frame in $V_{n,2}$ to the first vector in $S^{n-1}$, we have

$$\text{co-ind}(T_2, V_{n,2}) \leqq n-1 .$$

By [2, (3.3)],

$$\operatorname{ind}(T_2, V_{n,2}) \leqq \text{co-ind}(T_2, V_{n,2}) .$$

Thus the proof is complete.

PROOF OF PROPOSITION 3. $Z_q$ acts freely on $S^{2k+1} \times V_{2n,m}$ by the action on each factor. Let $\pi$ be the map from $(S^{2k+1} \times V_{2n,m})/Z_q$ onto $L^k(q)$ induced by the projection from $S^{2k+1} \times V_{2n,m}$ onto $S^{2k+1}$. Then $((S^{2k+1} \times V_{2n,m})/Z_q, \pi, L^k(q))$ is the $m$-frame bundle associated with $n\eta_{k,q}$, and the existence of a cross-section of this bundle is equivalent to the existence of a $Z_q$-equivariant map from $S^{2k+1}$ to $V_{2n,m}$. Thus the proof is complete.

PROOF OF PROPOSITION 4. The method of the proof is the same as in [4].

Put $q=p^r$, $t=2k+2-2[k/2]$. Suppose $\operatorname{Span}(\tau(L^k(q))\oplus 1) \geqq t$. Then there is a $2[k/2]$-plane bundle $\xi$ over $L^k(q)$ such that $\tau(L^k(q))\oplus 1$ is isomorphic to the Whitney sum of the trivial $t$-plane bundle and $\xi$. The square $X(\xi)^2$ of the Euler class of $\xi$ is equal to the $[k/2]$-th Pontrjagin class of $\xi$, and this is equal to $\binom{k+1}{[k/2]} x^{2[k/2]}$ for the generator $x$ of $H^2(L^k(q); Z)=Z_q$. So, by the assumption, we have

$$X(\xi)_q = m x_q^{[k/2]} \qquad (m \not\equiv 0 \ (\operatorname{mod} p))$$

where $z_q$ is the image of $z$ by the mod $q$ reduction.

The following diagram is commutative [8]:

$$\begin{array}{ccc} H^s(E(\xi), E_0(\xi); Z_q) & \xrightarrow{j^*} & H^s(E(\xi); Z_q) \\ \phi \big\uparrow \approx & & \approx \big\uparrow \pi^* \\ H^{s-2[k/2]}(L^k(q); Z_q) & \xrightarrow{\mu} & H^s(L^k(q); Z_q) \end{array}$$

where $E(\xi)$ is the total space of $\xi$, $E_0(\xi)$ is the subspace of $E(\xi)$ which consists of non-zero vectors, $j^*$ is the homomorphism induced by the injection $j: E(\xi) \to (E(\xi), E_0(\xi))$, $\pi^*$ is the isomorphism induced by the projection of $\xi$, $\phi$ is the Thom isomorphism, and $\mu$ is defined by

$$\mu(y) = yX(\xi)_q \qquad (y \in H^{s-2[k/2]}(L^k(q); Z_q)).$$

As $X(\xi)_q = mx_q^{[k/2]}$ ($m \not\equiv 0 \pmod p$), $\mu$ is an isomorphism for $2[k/2] \leqq s \leqq 2k+1$. So, for the inclusion map $\lambda$ from $L^k(q)$ into the Thom complex $L^k(q)^\xi$ of $\xi$, we have

$$\lambda^*: H^s(L^k(q)^\xi; Z_q) \approx H^s(L^k(q); Z_q) \qquad (2[k/2] \leqq s \leqq 2k+1).$$

Since $L^k(q)^\xi$ is $(2[k/2]-1)$-connected, there is a map $f$ such that the following diagram is homotopy commutative:

$$\begin{tikzcd} L^k(q) \arrow[rr, "\lambda"] \arrow[dr, "p"'] & & L^k(q)^\xi \\ & L^k(q)/L^{[k/2]-1}(q) \arrow[ur, "f"'] & \end{tikzcd}$$

where $p$ is the projection.

It is easily verified that $f$ induces isomorphisms:

$$f^*: H^s(L^k(q)^\xi; Z_q) \approx H^s(L^k(q)/L^{[k/2]-1}(q); Z_q) \qquad (0 \leqq s \leqq 2k+1).$$

By [1, Lemma (2.4)], the $t$-fold suspension $S^t L^k(q)^\xi$ of $L^k(q)^\xi$ is homeomorphic to $L^k(q)^{\xi\oplus t} = L^k(q)^{\tau(L^k(q))\oplus 1} = L^k(q)^{(k+1)\eta}$, where $\eta = \eta_{k,q}$. By [3, Th. 1] and [5, Th. 4.7], $L^k(q)^{(k+1)\eta}$ is homeomorphic to $L^{2k+1}(q)/L^k(q)$. The complex $S^t(L^k(q)/L^{[k/2]-1}(q))$ has dimension $t+2k+1$. So, by the cellular approximation theorem, there exists a map $g$ such that the following diagram is homotopy commutative:

$$\begin{tikzcd} S^t(L^k(q)/L^{[k/2]-1}(q)) \arrow[r, "S^t f"] \arrow[d, "g"'] & S^t L^k(q)^\xi \arrow[d, "\approx"] \\ L^{2k+1-[k/2]}(q)/L^k(q) \arrow[r, "i"] & L^{2k+1}(q)/L^k(q) \end{tikzcd}$$

where $S^t f$ is the $t$-fold suspension of $f$ and $i$ is the inclusion.

Then we can see that $g$ induces isomorphisms of all cohomology groups with $Z_q$ coefficients. Also $g$ defines a map

$$g_0: S^t(L_0^k(q)/L_0^{[k/2]}(q)) \longrightarrow L_0^{2k+1-[k/2]}(q)/L_0^{k+1}(q)$$

where $L_0^k(q)$ is the $2k$-skeleton of $L^k(q)$, and $g_0$ induces isomorphisms of all cohomology groups with $Z_q$ coefficients. By the universal coefficient theorem, we see that $g_0$ induces isomorphisms of all homology groups. As the spaces are simply connected, $g_0$ is a homotopy equivalence. So, $L_0^k(q)/L_0^{[k/2]}(q)$ and $L_0^{2k+1-[k/2]}(q)/L_0^{k+1}(q)$ are stably homotopy equivalent. Therefore $k+1-[k/2]\equiv 0 \pmod{p^{[(k-[k/2]-1)/(p-1)]}}$, by [5, Th. 1.1].

But this is impossible by the easy calculations using the assumption $p\leqq k-[k/2]$. So,

$$\mathrm{Span}(\tau(L^k(q))\oplus 1)\leqq t-1\,.$$

By [7, Th. (1.7)],

$$\mathrm{Span}(\tau(L^k(q))\oplus 1)\geqq t-1\,.$$

Thus the proof is complete.

## References

[1] M. F. Atiyah: *Thom complexes*, Proc. London Math. Soc., **11** (1961), 291–310.
[2] P. E. Conner and E. E. Floyd: *Fixed point free involutions and equivariant maps*, Bull. Amer. Math. Soc., **66** (1960), 416–441.
[3] T. Kambe, H. Matsunaga and H. Toda: *A note on stunted lens space*, J. Math. Kyoto Univ., **5** (1966), 143–149.
[4] T. Kobayashi: *Non-immersion theorems for lens spaces*, J. Math. Kyoto Univ., **6** (1966), 91–108.
[5] T. Kobayashi and M. Sugawara: *On stable homotopy types of stunted lens spaces*, Hiroshima Math. J., **1** (1971), 287–304.
[6] K. Y. Lam: *Sectioning vector bundles over real projective spaces*, Quart. J. Math., **23** (1972), 97–106.
[7] D. Sjerve: *Vector bundles over orbit manifolds*, Trans. Amer. Math. Soc., **138** (1969), 97–106.
[8] R. Thom: *Espaces fibrés en sphères et carrés de Steenrod*, Ann. Sci. Ecole Norm. Sup., **69** (1952), 109–182.
[9] T. Yoshida: *On the vector bundles $m\xi_n$ over real projective spaces*, J. Sci. Hiroshima Univ. Ser. A-I, **32** (1968), 5–16.

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