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3-Primary β-Family in Stable Homotopy

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§1. Introduction

Let p be an odd prime. L. Smith [9] discovered, for each $p \geqq 5$, an infinite family $\{\beta_t\}$ in the stable homotopy groups G_* of spheres. The construction of this family is assured by the existence of the stable complex $V(2)$ for p considered in [9], [15].

The case $p=3$ is quite different from the case $p \geqq 5$ [16, §6], e.g., $V(2)$ does not exist [15, Th. 1.2] and so the construction of β_t for general t is not known; it is, however, known from the results on G_* ([6], [7, Th. B], [11]) that β_t, $t \leqq 6$ except for $t=4$, exist and that β_4 can not be defined.

Let B be a stable mapping cone $S^0 \cup_{\beta_1} e^{11}$ of $\beta_1 \in G_{10}$ of order 3, and j: $S^0 \to B$ be an inclusion. The purpose of this paper is to construct non-trivial elements $\bar{\beta}_t \in \pi_{16t-6}(B)$ of order 3 for all $t \geqq 2$ such that $j\beta_t = \bar{\beta}_t$ if $\beta_t \in G_*$ exists. We shall also construct non-trivial elements $\bar{\rho}_t \in \pi_{48t-10}(B)$, $t \geqq 1$, corresponding to the elements $\rho_{t,1} \in G_*$ of [8, Th. A].

For the simplicity, we shall denote by M and V the spectra $V(0)$ and $V(1)$ for $p=3$ in [15]. In stable notations, $M = S^0 \cup_3 e^1$ and $V = M \cup_\alpha C\Sigma^4 M$, and we have the cofiberings $S^0 \xrightarrow{i} M \xrightarrow{\pi} S^1$ and $M \xrightarrow{i_1} V \xrightarrow{\pi_1} \Sigma^5 M$. Put $VB = V \wedge B$. Its Brown-Peterson homology is given by the direct sum:

$$BP_*(VB) = BP_*(V) + \Sigma^{11} BP_*(V) = BP_*/(3, v_1) + \Sigma^{11} BP_*/(3, v_1),$$

where $BP_* = \pi_*(BP) = Z_{(3)}[v_1, v_2, \dots]$, $\deg v_i = 2(3^i - 1)$ [2] [3]. Let $[\beta i_1]: \Sigma^{16} M \to V$ and $[\pi_1 \beta]: \Sigma^{11} V \to M$ be the elements having $V\left(1\frac{1}{2}\right)$ and $\Sigma^{-5}\left(V(2)/V\left(\frac{1}{2}\right)\right)$ as their mapping cones [16, §6].

THEOREM 1.1. *There exists a stable map*

$$\bar{\beta}: \Sigma^{16} VB \longrightarrow VB$$

such that

(a) *$\bar{\beta}$ induces the multiplication by v_2 on each factor of $BP_*(VB)$,*

and hence $BP_*/(3, v_1, v_2) + \Sigma^{11} BP_*/(3, v_1, v_2)$ is realizable as the BP homology

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of the mapping cone of $\bar{\beta}$. Moreover, such a $\bar{\beta}$ is unique by the equalities

(b) $\bar{\beta}(i_1 \wedge 1_B)=[\beta i_1] \wedge 1_B$, $(\pi_1 \wedge 1_B)\bar{\beta}=[\pi_1\beta] \wedge 1_B$.

The theorem, together with some additional properties, will be proved in § 3. It is known that $BP_*/(3, v_1, v_2)$ can not be realizable. We also notice that there are distinct spaces realizing $BP_*/(3, v_1, v_2)+\Sigma^{11}BP_*/(3, v_1, v_2)$. Roughly speaking, the element $\bar{\beta}$ corresponds to $\beta \wedge 1_B$ for $p \geqq 5$, and (a) asserts that $V(2) \wedge B$ exists (not uniquely) even if $V(2)$ does not.

DEFINITION 1.2. *We define $\bar{\beta}_t \in \pi_{16t-6}(B)$, $t \geqq 1$, by the following composition ($\bar{\beta}_1=0$):*

$$S^{16t} \xrightarrow{j} \Sigma^{16t}B \xrightarrow{i_1 i \wedge 1_B} \Sigma^{16t}VB \xrightarrow{\bar{\beta}^t} VB \xrightarrow{\pi\pi_1 \wedge 1_B} \Sigma^6 B .$$

D. C. Johnson and R. Zahler ([4], [18]) obtained, for any prime $p \geqq 3$, an infinite family in $\mathrm{Ext}_A^{2,*}(BP^*, BP^*)$, the E_2-term of the Adams-Novikov spectral sequence, corresponding to the β-family when $p \geqq 5$. Our family $\{\bar{\beta}_t\}$ (except $t=1$) corresponds to their family in Ext for $p=3$, and we shall prove in § 4 the non-triviality of $\bar{\beta}_t$ by Zahler's method [18].

THEOREM 1.3. *For $t \geqq 2$, $\bar{\beta}_t$ is non-zero element of order* 3.

For $t \leqq 6$, we shall see in § 5 that $\bar{\beta}_t=j\beta_t$, $t \neq 4$, and $k\bar{\beta}_4 \neq 0$, where $k: B \to S^{11}$ is the collapsing map. This suggests a definition of β's in G_* for $p=3$: *for $t \geqq 2$ such that $k\bar{\beta}_t=0$, $\beta_t \in G_{16t-6}$ is given by $j\beta_t=\bar{\beta}_t$.*

We shall also consider a similar construction corresponding to the elements ρ's of [8]. Put $W=M \cup_{\alpha^2} C\Sigma^8 M$ and $WB=W \wedge B$, whose BP homology is $BP_*/(3, v_1^2)+\Sigma^{11}BP_*/(3, v_1^2)$.

THEOREM 1.4. *There exists a stable map*

$$\bar{\rho}: \Sigma^{48}WB \longrightarrow WB$$

inducing the multiplication by v_2^3, i.e., the mapping cone of $\bar{\rho}$ realizes $BP_*/(3, v_1^2, v_2^3)+\Sigma^{11}BP_*(3, v_1^2, v_2^3)$.

Let us denote the cofibering for W by $M \xrightarrow{i_2} W \xrightarrow{\pi_2} \Sigma^9 M$.

DEFINITION 1.5. *Define $\bar{\rho}_t \in \pi_{48t-10}(B)$ by the following composition ($t \geqq 1$):*

$$S^{48t} \xrightarrow{j} \Sigma^{48t}B \xrightarrow{i_2 i \wedge 1_B} \Sigma^{48t}WB \xrightarrow{\bar{\rho}^t} WB \xrightarrow{\pi\pi_2 \wedge 1_B} \Sigma^{10}B .$$

THEOREM 1.6. *$\bar{\rho}_t \neq 0$ and $\bar{\beta}_{3t} \in \{\bar{\rho}_t, 3, \alpha_1\}$ for $t \geqq 1$.*

(1.4) and (1.6) will be proved in § § 3–4.

In contrast with (1.4), we obtain the following non-realizing result.

THEOREM 1.7. *$BP_*/(3, v_1^2, v_2^3)$ can not be realized.*

In §5 we shall proved (1.7) and the non-realizability of $BP_*/(3, v_1, v_2^t)$ for small t. In Appendix, we shall discuss a similar consideration for the *5-primary γ-family*, and show that $BP_*/(5, v_1, v_2, v_3)+\Sigma^{39}BP_*/(5, v_1, v_2, v_3)$ can be realizable.

§2. Some additional results on the algebra $\mathscr{A}_*(V)$

For any (finite) stable complexes (CW-spectra) X and Y, we shall denote by $\pi_k(X; Y)$ the additive group consisting of all homotopy classes of stable maps $\Sigma^k X \to Y$, and set $\pi_k(X)=\pi_k(S^0; X)$, $\mathscr{A}_k(X)=\pi_k(X; X)$ and $\mathscr{A}_*(X)=\sum_k \mathscr{A}_k(X)$. The composition of maps induces a product on $\mathscr{A}_*(X)$, and $\mathscr{A}_*(X)$ forms a graded ring; $1_X \in \mathscr{A}_0(X)$ being the unit.

A space (spectrum) X is called *a Z_3-space (-spectrum)* if 1_X is of order 3, or $\mathscr{A}_*(X)$ is an algebra over Z_3 [16, Lemma 1.2]. We introduced in [16, §2] the operations θ: $\pi_k(X; Y) \to \pi_{k+1}(X; Y)$ and λ_X: $\mathscr{A}_k(M) \to \mathscr{A}_{k+1}(X)$ and discussed their properties. In particular, M and V are (non-associative) Z_3-spaces [16, §6], and we shall use the same notations as in [16] for the generators of $\mathscr{A}_*(M)$ and $\mathscr{A}_*(V)$:

$\delta = i\pi \in \mathscr{A}_{-1}(M)$, $\alpha \in \mathscr{A}_4(M)$ *the attaching class of V*,

$\beta_{(1)} = \pi_1[\beta i_1] = [\pi_1 \beta] i_1 \in \mathscr{A}_{11}(M)$, $\beta_{(2)} = [\pi_1 \beta][\beta i_1] \in \mathscr{A}_{27}(M)$;

$\delta_1 = i_1 \pi_1 \in \mathscr{A}_{-5}(V)$, $\delta_0 = i_1 \delta \pi_1 \in \mathscr{A}_{-6}(V)$,

$\alpha'' \in \mathscr{A}_2(V)$ *the associator of V*,

$\beta' = \lambda_V(\delta \beta_{(1)} \delta) = \beta_1 \wedge 1_V$, $[\beta \delta_0] = [\beta i_1] \delta \pi_1 \in \mathscr{A}_{10}(V)$,

$[\beta \delta_1] = [\beta i_1] \pi_1$, $[\delta_1 \beta] = i_1 [\pi_1 \beta] \in \mathscr{A}_{11}(V)$.

The following relation is the mod 3 version of the last equality in [16, Th. 4.2].

LEMMA 2.1. $\lambda_V(\beta_{(1)}\delta)=[\beta\delta_1]-[\delta_1\beta]$.

PROOF. By [16, Cor. 2.5, (3.7), (2.8) and Th. 6.4], $\lambda_V(\beta_{(1)}\delta)i_1 = i_1\lambda_M(\beta_{(1)}\delta) = -i_1\beta_{(1)} = -[\delta_1\beta]i_1$ and $\pi_1\lambda_V(\beta_{(1)}\delta)=\pi_1[\beta\delta_1]$. Since $\lambda_V(\beta_{(1)}\delta) \in \mathscr{A}_{11}(V)=\{[\beta\delta_1], [\delta_1\beta]\}$ [16, Th. 6.11], we have the desired result. *q.e.d.*

Since θ is derivative [16, Th. 2.2], it follows immediately from [16, Th. 6.4] that

(2.2) $$\theta[\beta\delta_1] = \alpha''[\beta\delta_0].$$

By [16, (6.1) and Lemma 6.5], we have $\theta[\delta_1\beta]=\theta[\beta\delta_1]-\theta\lambda_V(\beta_{(1)}\delta)=\theta[\beta\delta_1]+\alpha''\lambda_V(\delta\beta_{(1)}\delta)$, and hence

(2.3) $$\theta[\delta_1\beta] = \alpha''[\beta\delta_0]+\beta'\alpha''.$$

THEOREM 2.4. *In* $\mathscr{A}_{22}(V)=\{[\delta_1\beta][\beta\delta_1],\ \beta'\alpha''[\beta\delta_0],\ \beta'\beta'\alpha''\}$, *the following relations hold*:

(i) $[\beta\delta_1]^2 = -[\delta_1\beta][\beta\delta_1]-\beta'\alpha''[\beta\delta_0]$,

(ii) $[\delta_1\beta]^2 = -[\delta_1\beta][\beta\delta_1]-\beta'\alpha''[\beta\delta_0]-\beta'\beta'\alpha''$.

PROOF. By [16, Th. 2.4 (iii)] with $\xi=\beta_{(1)}\delta$, we have

(*) $$([\beta\delta_1]-[\delta_1\beta])\gamma = (-1)^{\deg\gamma}\gamma([\beta\delta_1]-[\delta_1\beta])+\beta'\theta(\gamma)$$

for any $\gamma\in\mathscr{A}_*(V)$. By using (2.2)-(2.3), the desired relations follow from (*) for $\gamma=[\beta\delta_1],\ [\delta_1\beta]$. *q.e.d.*

In the same way as above, we also obtain the following relations.

(2.4)′ (i) $[\beta\delta_1][\beta i_1] \equiv -[\delta_1\beta][\beta i_1] \quad \bmod\ \operatorname{Im}\beta'_*$,

(ii) $[\pi_1\beta][\delta_1\beta] \equiv -[\pi_1\beta][\beta\delta_1] \quad \bmod\ \operatorname{Im}\beta'^*$.

An additive basis of $\mathscr{A}_*(V)$ for deg<27 is given by [16, Th. 6.11]. We shall compute $\mathscr{A}_{27}(V)$.

THEOREM 2.5. *The homomorphisms* $i_1^*\colon \mathscr{A}_{27}(V)\to\pi_{27}(M;V)$ *and* $\pi_{1*}\colon \mathscr{A}_{27}(V)\to\pi_{22}(V;M)$ *are isomorphic. Define* $[\delta_1\beta^2]$ *and* $[\beta^2\delta_1]$ *by* $i_1^*[\delta_1\beta^2]=[\delta_1\beta][\beta i_1]$ *and* $\pi_{1*}[\beta^2\delta_1]=[\pi_1\beta][\beta\delta_1]$, *and put* $[\beta\delta_1\beta]=[\beta i_1][\pi_1\beta]$. *Then,* $\mathscr{A}_{27}(V)$ *has a basis* $\{[\beta^2\delta_1],[\beta\delta_1\beta]\}$ *and there hold the relations* $[\delta_1\beta^2]=-[\beta^2\delta_1]$ *and* $\lambda_V(\beta_{(2)}\delta)=[\beta^2\delta_1]$.

PROOF. N. Yamamoto [17] computed the algebra $\mathscr{A}_*(M)$ for deg<32, cf. [16, (6.4)], and the obstruction to compute $\mathscr{A}_{32}(M)$ was the composition $\alpha_1\beta_1^3$ in G_{33}. The triviality of this composition [13] leads to the result $\mathscr{A}_{32}(M)=\{\alpha^8\}$.

From the results on $\mathscr{A}_k(M)$, $k=27, 28, 31, 32$, we obtain $\pi_{32}(M;V)=0$ and $\pi_{27}(V;M)=0$. We have proved in [16, Prop. 6.9] that $\pi_{31}(M;V)=0$, and dually we can prove $\pi_{26}(V;M)=0$. Therefore i_1^* and π_{1*} in the theorem are isomorphic by the exact sequences:

$$\pi_{32}(M;V) \longrightarrow \mathscr{A}_{27}(V) \xrightarrow{i_1^*} \pi_{27}(M;V) \longrightarrow \pi_{31}(M;V),$$

$$\pi_{27}(V;M) \longrightarrow \mathscr{A}_{27}(V) \xrightarrow{\pi_{1*}} \pi_{22}(V;M) \longrightarrow \pi_{26}(V;M).$$

From the results on $\mathscr{A}_*(M)$, in particular the relation $\delta\alpha\delta(\beta_{(1)}\delta)^2=\beta_{(\frac{2}{1})}=\pi_1[\beta\delta_1\beta]i_1$ [16, Th. 6.4.(i)], we see that $\pi_{27}(M;V)=\{i_1\beta_{(2)}=[\delta_1\beta][\beta i_1], [\beta\delta_1\beta]i_1\}$ and $\pi_{22}(V;M)=\{\beta_{(2)}\pi_1=[\pi_1\beta][\beta\delta_1], \pi_1[\beta\delta_1\beta]\}$. Hence,

$$\mathscr{A}_{27}(V) = \{[\beta^2\delta_1], [\beta\delta_1\beta]\}=\{[\delta_1\beta^2], [\beta\delta_1\beta]\}.$$

We put $\lambda_V(\beta_{(2)}\delta)=x[\beta^2\delta_1]+y[\beta\delta_1\beta]$. Then, $[\delta_1\beta][\beta\delta_1]=i_1\beta_{(2)}\pi_1=-i_1\lambda_M(\beta_{(2)}\delta)\pi_1=\delta_1\lambda_V(\beta_{(2)}\delta)=x[\delta_1\beta][\beta\delta_1]+y[\delta_1\beta]^2$ and $x=1$, $y=0$, since $[\delta_1\beta][\beta\delta_1]$ and $[\delta_1\beta]^2$ are linearly independent by (2.4). Next put $\lambda_V(\beta_{(2)}\delta)=x'[\delta_1\beta^2]+y'[\beta\delta_1\beta]$. Then, $[\delta_1\beta][\beta\delta_1]=-\lambda_V(\beta_{(2)}\delta)\delta_1=-x'[\delta_1\beta][\beta\delta_1]-y'[\beta\delta_1]^2$ and $x'=-1$, $y'=0$ by (2.4). Thus, we obtain $[\beta^2\delta_1]=\lambda_V(\beta_{(2)}\delta)=-[\delta_1\beta^2]$ as desired. *q.e.d.*

§3. Constructing elements

Let us denote the cofibering for B by

$$S^{10} \xrightarrow{\beta_1} S^0 \xrightarrow{j} B \xrightarrow{k} S^{11}. \tag{3.1}$$

We write XB, β_X, j_X and k_X for the smash products $X\wedge B$, $1_X\wedge\beta_1$, $1_X\wedge j$ and $1_X\wedge k$, respectively, and we have the cofibering

$$\Sigma^{10}X \xrightarrow{\beta_X} X \xrightarrow{j_X} XB \xrightarrow{k_X} \Sigma^{11}X. \tag{3.1$_X$}$$

It is clear that $\xi\beta_X=\beta_Y\xi$ for any $\xi\in\pi_k(X;Y)$, i.e.,

$$\beta_X^* = \beta_{Y*}\colon \pi_k(X;Y) \longrightarrow \pi_{k+10}(X;Y). \tag{3.2}$$

Consider the element $\beta_1\wedge 1_B=\beta_B\in\mathscr{A}_{10}(B)$. By [12, Lemma 3.5], $\beta_1\wedge 1_B=k^*j_*(\alpha^*)$ for some $\alpha^*\in G_{21}$. Since $G_{21}*Z_3=0$ [11], we obtain

$$\beta_1\wedge 1_B = 0 \quad \text{in} \quad \mathscr{A}_{10}(B). \tag{3.3}$$

From (3.2)–(3.3), it follows that $\beta_X^*\colon \pi_k(X;YB)\to\pi_{k+10}(X;YB)$ and $\beta_{Y*}\colon \pi_k(XB;Y)\to\pi_{k+10}(XB;Y)$ are trivial for any X and Y. Hence the following short exact sequences are obtained:

$$0 \longrightarrow \pi_{k+11}(X;YB) \xrightarrow{k_X^*} \pi_k(XB;YB) \xrightarrow{j_X^*} \pi_k(X;YB) \longrightarrow 0; \tag{3.4}$$

$$0 \longrightarrow \pi_k(XB;Y) \xrightarrow{j_{Y*}} \pi_k(XB;YB) \xrightarrow{k_{Y*}} \pi_{k-11}(XB;Y) \longrightarrow 0. \tag{3.4*}$$

We shall treat the case X, $Y=M$ or V. Then, $\beta_X=\lambda_X(\delta\beta_{(1)}\delta)$ [16, Th. 2.4. (iv)], and so

$$\beta_M = \beta_{(1)}\delta+\delta\beta_{(1)}, \quad \beta_V = \beta'. \tag{3.5}$$

LEMMA 3.6. (i) *$\pi_{16}(MB; VB)$ has a Z_3-basis*

$$\{[\beta i_1] \wedge 1_B,\ \ j_V[\delta_1\beta][\beta i_1]k_M = -j_V[\beta\delta_1][\beta i_1]k_M\}.$$

(ii) *$\pi_{11}(VB; MB)$ has a Z_3-basis*

$$\{[\pi_1\beta] \wedge 1_B,\ \ j_M[\pi_1\beta][\beta\delta_1]k_V = -j_M[\pi_1\beta][\delta_1\beta]k_V\}.$$

PROOF. From $\pi_k(M; V)=0$, $k=5, 6$, and $\pi_{16}(M; V)=\{[\beta i_1]\}$ [16, Prop. 6.9], it follows that $\pi_{16}(M; VB)=\{j_V[\beta i_1]\}$. Also $\pi_{27}(M; VB)=\{j_V[\delta_1\beta][\beta i_1] = -j_V[\beta\delta_1][\beta i_1]\}$ by using (2.4)′ (i). Then, from (3.4) for $X=M$, $Y=V$, (i) follows.

(ii) follows from similar calculations using the following results on $\pi_k = \pi_k(V; M)$: $\pi_0=\pi_1=0$, $\pi_{11}=\{[\pi_1\beta]\}$, $\pi_{12}=\{\delta[\pi_1\beta]\alpha''\}$, $\pi_{21}=\{[\pi_1\beta]\beta'\}$ and $\pi_{22} =\{[\pi_1\beta][\beta\delta_1], [\pi_1\beta][\delta_1\beta]\}$. *q.e.d.*

The Brown-Peterson homology for M and V is given by ([9], cf. [4], [18])

$$BP_*(M) = BP_*/(3), \quad BP_*(V) = BP_*/(3, v_1),$$

where $BP_*=\pi_*(BP)=Z_{(3)}[v_1, v_2,\dots]$, the polynomial ring over the integers localized at 3, $v_i \in BP_{2(3^i-1)}$ [2] and $(x_1,\dots, x_n)$ denotes the ideal generated by $x_1,\dots, x_n$. Applying $BP_*(\ \ )$ to (3.1), $(3.1)_M$ and $(3.1)_V$, we get

(3.7) (i) $BP_*(B) = BP_* + \Sigma^{11}BP_*$,

(ii) $BP_*(MB) = BP_*/(3)+\Sigma^{11}BP_*/(3)$,

(iii) $BP_*(VB) = BP_*/(3, v_1)+\Sigma^{11}BP_*/(3, v_1)$,

where *an n-fold suspension Σ^nM* of a graded module $M=(M_i)$ is given by $(\Sigma^nM)_i =M_{i-n}$, in particular $BP_*(\Sigma^nX)=\Sigma^nBP_*(X)$.

Now we shall prove Theorem 1.1.

PROOF OF (1.1). The construction of $\bar\beta$ starts from the stable map $[\beta i_1]$: $\Sigma^{16}M \to V$ having $V\left(1\frac{1}{2}\right)$ as its mapping cone [16, p. 239]. This coincides with $\tilde\psi$ of L. Smith [9, 2nd line on p. 824] up to sign, and induces the multiplication by v_2. There is a relation [16, Th. 6.7]

$$[\beta i_1]\alpha = \beta'(\beta' i_1+\delta_1[\beta\delta_1]\delta).$$

Since $V=C_\alpha$ and $VB=C_{\beta'}$ by $(3.1)_V$ and (3.5), this relation gives an element β_0: $\Sigma^{16}V \to VB$ such that $\beta_0 i_1=j_V[\beta i_1]$ and $k_V\beta_0=\beta'\delta_1+\delta_1[\beta\delta_0]$. Since $\mathscr{A}_{16}(V) =0$ and $\mathscr{A}_5(V) \cap \operatorname{Ker}\beta'_*=\{\beta'\delta_1+\delta_1[\beta\delta_0]\}$, β_0 is unique and generates $\pi_{16}(V; VB)$. By (3.4) for $X=Y=V$, there is $\bar\beta$ such that $\bar\beta j_V=\beta_0$, and so by (2.5)

(3.8) $$\mathscr{A}_{16}(VB) = \{\bar{\beta},\ j_V[\beta^2\delta_1]k_V,\ j_V[\beta\delta_1\beta]k_V\}.$$

By (3.6), (3.8) and easy calculations, we see that

(3.9) *there is* $\bar{\beta} \in \mathscr{A}_{16}(VB)$ *such that* $\bar{\beta}(i_1 \wedge 1_B) \equiv [\beta i_1] \wedge 1_B \bmod j_V[\delta_1\beta] \cdot [\beta i_1]k_M$, $(\pi_1 \wedge 1_B)\bar{\beta} \equiv [\pi_1\beta] \wedge 1_B \bmod j_M[\pi_1\beta][\beta\delta_1]k_V$ *and* $k_V\bar{\beta}j_V = \beta'\delta_1 + \delta_1[\beta\delta_0]$, *and such* $\bar{\beta}$*'s form a coset of the subgroup* $I = \{j_V[\beta^2\delta_1]k_V, j_V[\beta\delta_1\beta]k_V\}$ *of* $\mathscr{A}_{16}(VB)$.

For any $\bar{\beta}$ in (3.9), $\bar{\beta}(i_1 \wedge 1_B)$ and $[\beta i_1] \wedge 1_B$ induce the same homomorphism on $BP_*(\ )$. Since $(i_1 \wedge 1_B)_*$ is the natural epimorphism to the quotient (3.7) (iii) of (3.7) (ii), we see that *any* $\bar{\beta}$ *in* (3.9) *satisfies* (a).

Put $\bar{\beta}(i_1 \wedge 1_B) - [\beta i_1] \wedge 1_B = xj_V[\delta_1\beta][\beta i_1]k_M$ and $(\pi_1 \wedge 1_B)\bar{\beta} - [\pi_1\beta] \wedge 1_B = yj_M[\pi_1\beta][\beta\delta_1]k_V$. Then,

$$\bar{\beta}' = \bar{\beta} - (x-y)j_V[\beta^2\delta_1]k_V - (x+y)j_V[\beta\delta_1\beta]k_V$$

satisfies (b) by (2.5) and (3.6). The uniqueness of $\bar{\beta}$ satisfying (b) follows from (3.8) and

$$I \cap \operatorname{Ker}(i_1 \wedge 1_B)^* \cap \operatorname{Ker}(\pi_1 \wedge 1_B)_* = 0.$$

q.e.d.

Remark 3.10. Let $\mathscr{A}$ be the Steenrod algebra mod 3. Denote by E_n the exterior algebra generated by Milnor's primitive elements $Q_0, \dots, Q_n$. Identifying E_n with a quotient of $\mathscr{A}$, we may regard E_n as an $\mathscr{A}$-module. Then, E_0 *and* E_1 *are realized by the cohomology of* M *and* V [15, Th. 1.1]. Let M_n be an extension (as an A-module) of E_n by $\Sigma^{11}E_n$ such that $\mathscr{P}^3 a = Q_0 b$ in M_n, where a and b are the generators corresponding to E_n and $\Sigma^{11}E_n$ ($\deg a = 0$, $\deg b = 11$). *If* E_n *is realized, then so is* M_n. In fact, $H^*(V(n) \wedge B; Z_3) = M_n$ if $V(n)$ exists. In particular, M_0 *and* M_1 *are realized by* MB *and* VB. We see also that *the mapping cone* $VB(2)$ *of* $\bar{\beta}$ *realizes* M_2, *i.e.*,

$$H^*(VB(2); Z_3) = M_2,$$

though E_2 *can not be realized* [15, Th. 1.2].

Theorem 3.11. *Let* $\bar{\delta}_1 = \delta_1 \wedge 1_B \in \mathscr{A}_{-5}(VB)$. *Then the element* $\bar{\beta}\bar{\delta}_1 - \bar{\delta}_1\bar{\beta}$ *belongs to the center of* $\mathscr{A}_*(VB)$. *In particular, there is a relation*

(3.12) $$\bar{\beta}^2\bar{\delta}_1 + \bar{\beta}\bar{\delta}_1\bar{\beta} + \bar{\delta}_1\bar{\beta}^2 = 0.$$

Proof. By the definition of λ_X, $\lambda_{VB}(\beta_{(1)}\delta) = \lambda_V(\beta_{(1)}\delta) \wedge 1_B$ [16, Th. 2.4. (ii)], and so $\lambda_{VB}(\beta_{(1)}\delta) = [\beta\delta_1] \wedge 1_B - [\delta_1\beta] \wedge 1_B = \bar{\beta}\bar{\delta}_1 - \bar{\delta}_1\bar{\beta}$ by (2.1) and (1.1) (b). By (3.5), $\lambda_{VB}(\delta\beta_{(1)}\delta) = \beta_1 \wedge 1_{VB} = 0$, and hence $\lambda_{VB}(\beta_{(1)}\delta)\xi = (-1)^{\deg\xi}\xi\lambda_{VB}$

$(\beta_{(1)}\delta)$ for any $\xi \in \mathscr{A}_*(VB)$ by [16, Th. 2.4. (iii)]. Letting $\xi=\bar{\beta}$, we obtain (3.12). *q.e.d.*

From (3.12) we have immediately

COROLLARY 3.13. $\bar{\beta}^3\bar{\delta}_1=\bar{\delta}_1\bar{\beta}^3$.

Now, we denote the cofibering for $W=M\cup_{\alpha^2}C\Sigma^8M$ by $M\xrightarrow{i_2}W\xrightarrow{\pi_2}\Sigma^9M$. There is a sequence of cofiberings [8, Lemma 1.5]

$$\Sigma^4V\xrightarrow{a}W\xrightarrow{b}V\xrightarrow{\delta_1}\Sigma^5V, \tag{3.14}$$

where a and b are given by

$$ai_1=i_2\alpha,\quad \pi_2a=\pi_1;\quad bi_2=i_1,\quad \pi_1b=\alpha\pi_2. \tag{3.15}$$

PROOF OF (1.4). By (3.14), WB is the mapping cone of $\bar{\delta}_1$. Hence, by (3.13), there is $\bar{\rho}\colon \Sigma^{48}WB\to WB$ such that $\bar{\rho}\bar{a}=\bar{a}\bar{\beta}^3$ and $\bar{b}\bar{\rho}=\bar{\beta}^3\bar{b}$, $\bar{a}=a\wedge 1_B$, $\bar{b}=b\wedge 1_B$. By (3.15) and (1.1) (a), $\bar{a}$ and $\bar{\beta}^3$ induce the multiplications by v_1 and v_2^3, respectively. Hence $\bar{\rho}$ induces the multiplication by v_2^3. *q.e.d.*

In the above we have obtained

$$\bar{\rho}\bar{a}=\bar{a}\bar{\beta}^3,\quad \bar{b}\bar{\rho}=\bar{\beta}^3\bar{b}\qquad (\bar{a}=a\wedge 1_B,\ \bar{b}=b\wedge 1_B). \tag{3.16}$$

As a consequence of (3.16), we have

PROPOSITION 3.17. *For the elements $\bar{\beta}_{3t}$ in (1.2) and $\bar{\rho}_t$ in (1.6), there holds the relation $\bar{\beta}_{3t}\in\{\bar{\rho}_t, 3, \alpha_1\}$.*

PROOF.
$$\begin{aligned}\bar{\beta}_{3t}&=(\pi\pi_1\wedge 1_B)\bar{\beta}^{3t}j_Vi_1i\\ &=(\pi\pi_2\wedge 1_B)\bar{a}\bar{\beta}^{3t}j_Vi_1i &&\text{by (3.15)}\\ &=(\pi\pi_2\wedge 1_B)\bar{\rho}^tj_Wai_1i &&\text{by (3.16)}\\ &=(\pi\pi_2\wedge 1_B)\bar{\rho}^tj_Wi_2\alpha i &&\text{by (3.15)}.\end{aligned}$$

Since $(\pi\pi_2\wedge 1_B)\bar{\rho}^tj_Wi_2$ and αi are an extension of $\bar{\rho}_t$ and a coextension of α_1, $\bar{\beta}_{3t}$ lies in the bracket $\{\bar{\rho}_t, 3, \alpha_1\}$. *q.e.d.*

§4. Proof of Theorems 1.3 and 1.6

R. Zahler [18] [4] defined an invariant taking values in $\mathrm{Ext}_A^{2,*}(BP^*, BP^*)$, $A=BP^*(BP)$ the Steenrod ring of the Brown-Peterson cohomology theory, whose coefficient ring is $BP^*(=BP_{-*})=Z_{(3)}[v_1, v_2,\ldots]$, $\deg v_i=-2(3^i-1)$ [2, §6] cf. [3] (this v_i is the dual of $v_i\in BP_*$ in the previous sections). This invariant detects

β's of [9] and ρ's of [8] for $p \geqq 5$ (cf. [4, Remark at the end of § 2]). We shall follow his line with minor alteration.

Denote by W_r the mapping cone $M \cup_{\alpha^r} C\Sigma^{4r}M (W_1 = V, W_2 = W)$ and $i_r: M \to W_r$ the inclusion. Let $H_k(r)$ be the image of $(i_r i)^*: \pi_k(W_r; B) \to \pi_k(B)$. Take $\xi = \eta i_r i \in H_k(r)$. Since $i_r^* = 0: BP^*(W_r) \to BP^*$, $(\eta i_r)^* = 0$ and there is a short exact sequence of A-modules:

$$E_\eta: \qquad 0 \longrightarrow \Sigma^{k+2}BP^*/(3) \longrightarrow BP^*(C_{\eta i_r}) \longrightarrow BP^*(B) \longrightarrow 0,$$

and we obtain the class $\{E_\eta\} \in \mathrm{Ext}_A^{1,k+2}(BP^*(B), BP^*/(3))$. Denote by $\Delta: \mathrm{Ext}_A^{j,i}(-, BP^*/(3)) \to \mathrm{Ext}_A^{i+1,j}(-, BP^*)$ the connecting homomorphism associated with the short exact sequence of A-modules:

$$0 \longrightarrow BP^* \xrightarrow{\times 3} BP^* \xrightarrow{\bar{\pi}} BP/(3) \longrightarrow 0,$$

and by $\iota: BP^* \to BP^*(B) = BP^* + \Sigma^{11}BP^*$ the right inverse of $j^*: BP^*(B) \to BP^*$. Let η' also satisfy $\eta' i_r i = \xi$. Then $\eta i_r \equiv \eta' i_r \mod \pi^* \pi_{k+1}(B)$. If $k \not\equiv -1 \mod 4$ and $k \neq 10$, any element of $\pi_{k+1}(B)$ induces the trivial homomorphism, and hence $\{E_\eta\} \equiv \{E_{\eta'}\} \mod \mathrm{Im}\, \bar{\pi}_* = \mathrm{Ker}\, \Delta$. Therefore $\Delta\{E_\eta\}$ depends only on ξ. Thus, letting $e_r(\xi) = \iota^* \Delta\{E_\eta\}$, $\eta \in (i_r i)^{*-1}\xi$, we obtain a well-defined homomorphism

$$(4.1) \qquad e_r: H_k(r) \longrightarrow \mathrm{Ext}_A^{2,k+2}(BP^*, BP^*), \qquad k \not\equiv -1 \mod 4, \quad k \neq 10.$$

Let $t = 3^f a$, where $a \not\equiv 0 \mod 3$, $a \geqq 1$ and $f \geqq 0$. If $1 \leqq r \leqq 3^f$, the multiplication $v_2^t: \Sigma^{-16t}BP^* \to BP^*/(3, v_1^r)$ is an A-homomorphism [18, Lemma 2]. Hence

$$v_2^t \in \mathrm{Ext}_A^{0,16t}(BP^*, BP^*/(3, v_1^r)).$$

Denote by $\Delta_r: \mathrm{Ext}_A^{i,j}(-, BP^*/(3, v_1^r)) \to \mathrm{Ext}_A^{i+1,j-4r}(-, BP^*/(3))$ the connecting homomorphism associated with

$$E_r: \qquad 0 \longrightarrow \Sigma^{-4r}BP^*/(3) \xrightarrow{\cdot v_1^r} BP^*/(3) \longrightarrow BP^*/(3, v_1^r) \longrightarrow 0,$$

and put

$$e(r, t) = \Delta(\Delta_r(v_2^t)) \in \mathrm{Ext}_A^{2,16t-4r}(BP^*, BP^*)$$

for $1 \leqq r \leqq 3^f$, $t = 3^f a, f \geqq 0$, $a \geqq 1$, $a \not\equiv 0 \mod 3$. Then, D. C. Johnson and R. Zahler ([4, § 2], [18, Th. 1. a]) proved

THEOREM 4.2. $e(r, t) \neq 0$.

Now we shall prove Theorems 1.4 and 1.6.

PROOF OF (1.4). We shall show $e_1(\bar{\beta}_t) = e(1, t)$. Then $\bar{\beta}_t \neq 0$ follows from (4.2). Put $\eta = (\pi\pi_1 \wedge 1_B)\bar{\beta}^t j_V$, $k = 16t - 6$. Then $\bar{\beta}_t = \eta i_1 i \in H_k(1)$ and $e_1(\bar{\beta}_t)$ is defined for $t \geqq 2$.

Since $[\pi_1 \beta]$ is the Spanier-Whitehead dual of $[\beta i_1]$, it follows from (3.9) that the coset $\bar{\beta}+I$ in (3.9) is self-dual. Hence, any $\bar{\beta}$ in (3.9) induces the multiplication by v_2 on the BP-cohomology. So, $\phi=\eta^* \in \mathrm{Ext}_A^{0,16t}(BP^*(B), BP^*/(3, v_1))$ is given by $\phi\iota=v_2^t$ and $\phi k^*=0$.

Applying $BP^*(\ )$ to the cofiber sequences for i_1 and ηi_1, we obtain the commutative diagram of short exact sequences:

$$\begin{tikzcd}
E_1: & 0 \arrow[r] & \Sigma^{k+2}BP^*/(3) \arrow[r] & \Sigma^{k+6}BP^*/(3) \arrow[r] & \Sigma^{k+6}BP^*/(3, v_1) \arrow[r] & 0 \\
E_\eta: & 0 \arrow[r] & \Sigma^{k+2}BP^*/(3) \arrow[r] \arrow[u, equal] & BP^*(C_{\eta i_1}) \arrow[r] \arrow[u] & BP^*(B) \arrow[r] \arrow[u, "\eta^*=\phi"'] & 0.
\end{tikzcd}$$

Then $\{E_\eta\}=\phi^*\{E_1\}$ in $\mathrm{Ext}^{1,*}$, and we have

$$\iota^*\{E_\eta\} = (\phi\iota)^*\{E_1\} = \Delta_1(\phi\iota) = \Delta_1(v_2^t).$$

Thus, $e_1(\bar{\beta}_t)=\iota^*\Delta\{E_\eta\}=\Delta\Delta_1(v_2^t)=e(1, t)$. *q.e.d.*

PROOF OF (1.6). In the same way as above, we see that $e_2(\bar{\rho}_t)=e(2, 3t)$ and $\bar{\rho}_t \neq 0$. The relation $\bar{\beta}_{3t} \in \{\bar{\rho}_t, 3, \alpha_1\}$ is proved in (3.17). *q.e.d.*

§5. Remarks for small t and non-realizability

We shall compare our elements $\bar{\beta}_t$ and $\bar{\rho}_t$ with the results on G_*. The non-realizability of some cyclic BP_*-modules will be proved. As we only treat the 3-primary elements, we denote simply by G_* the 3-component of G_*.

It is easy to see from (1.1) (b) that

$$\bar{\beta}_1 = i\beta_1 = 0 \quad \textit{and} \quad \bar{\beta}_2 = j\beta_2$$

for $\beta_2=\pi[\pi_1\beta][\beta i_1]i \in G_{26}$.

The elements $k\bar{\beta}_t$ and $k\bar{\rho}_t$ lie in G_{16t-17} and G_{48t-21}, which contain the image of the J-homomorphism [1]. But $k\bar{\beta}_t$ and $k\bar{\rho}_t$ can not be contained in $\mathrm{Im}\,J$, because these elements factor through V or W. Since $G_{16t-17}/\mathrm{Im}\,J$ ($t=3$, $5, 6$) and $G_{27}/\mathrm{Im}\,J$ vanish ([11], [7, Th. B], [6]), we have $k\bar{\beta}_t=0$ for $t=3, 5, 6$ and $k\bar{\rho}_1=0$. Therefore,

$$\bar{\rho}_1 = \pm j\varepsilon_1, \quad \bar{\beta}_3 = \pm j\varepsilon_2,$$

where $\varepsilon_1=\{\alpha_1, \beta_1^3, 3, \alpha_1\}$ and $\varepsilon_2=\{\varepsilon_1, 3, \alpha_1\}$, and

$$\bar{\beta}_5 = j\beta_5, \quad \bar{\beta}_6 = j\beta_6.$$

These two equalities give generators β_5 of G_{74} and β_6 of G_{90}.

We proved [7] that the element β_4 does not exist. In fact, the following relation is easily seen from [7, Th. B]

$$k\bar{\beta}_4 = \pm\beta_1\varepsilon' \quad (\neq 0),$$

and $\bar{\beta}_4$ can not lie in the image of j_*.

Since $(\alpha_1\beta_2)_*: G_{46}\to G_{75}$ is monomorphic [7], we have

$$\{\varepsilon_1, 3, \alpha_2\} = \{\varepsilon_2, 3, \alpha_1\} = 0.$$

The non-existence of β_4 is equivalent to the relation

$$(5.1) \qquad \{\varepsilon_1, 3, \alpha_2, 3\} = \{\varepsilon_2, 3, \alpha_1, 3\} \equiv \pm\beta_1\varepsilon'.$$

This means that $(\pi\pi_2\wedge 1_B)\bar{\rho}j_W$ (and $(\pi\pi_1\wedge 1_B)\beta^3 j_V$ also) can not be compressed to the bottom sphere of B. Furthermore the element $k_W\bar{\rho}j_W\in\mathscr{A}_{37}(W)$ satisfies

$$(5.2) \qquad k_W\bar{\rho}j_W i_2 i = i_2 i\varepsilon', \quad \pi\pi_2 k_W\bar{\rho}j_W = -\varepsilon'\pi\pi_2 \quad \textit{for suitable sign of } \varepsilon'.$$

There are elements $\tilde{\beta}_t: S^{16t+4}\to W$, $t=1, 2$, such that $\pi\pi_2\tilde{\beta}_t=\beta_t$. Then, since $\beta_2\varepsilon'=0$, the element $(\pi\pi_2\wedge 1_B)\bar{\rho}j_W\tilde{\beta}_2$ can be compressed to the bottom sphere of B, and the compression is β_5. But, since $\beta_1\varepsilon'\neq 0$, such a compression does not exist for $t=1$.

From (5.2), we can see $k\bar{\rho}_2=(\pi\pi_2 k_W\bar{\rho})(\bar{\rho}j_W i_2 i)=(\pm\varepsilon_1)\varepsilon'-\varepsilon'(\pm\varepsilon_1)=0$. Hence we obtain an element ρ_2 such that

$$\bar{\rho}_2 = j\rho_2, \quad \beta_6 = \{\rho_2, 3, \alpha_1\}.$$

This generates G_{86} and coincides with Nakamura's ρ_1[6] up to sign.

In the following, we shall discuss the non-realizability of BP_*-modules. We first prove Theorem 1.7.

Proof of (1.7). Let assume that there is an X such that $BP_*(X)=BP_*/(3, v_1^2, v_2^3)$ as a BP_*-module. Then, in the same way as L. Smith [10, Lemmas 2.1–2.2], the homology group of X localized at 3 is calculated and we see that X is 3-equivalent to a complex

$$X'=S^0\cup{}_3e^1\cup e^9\cup{}_3e^{10}\cup e^{49}\cup{}_3e^{50}\cup e^{58}\cup{}_3e^{59}.$$

Let Y be the 10-skeleton of X' and Y' be $\Sigma^{-1}(X'/Y)$. Then there is a cofibering $Y'\to Y\to X'$ and we have a short exact sequence

$$(*) \qquad 0 \longrightarrow BP_*(Y') \longrightarrow BP_*(Y) \longrightarrow BP_*(X') \longrightarrow 0.$$

The complexes Y and Y' are mapping cones of some elements of $\mathscr{A}_8(M)=Z_3$, generated by α^2. The BP homology of the mapping cone of $x\alpha^2$ is $BP_*/(3, v_1^2)$ or $BP_*/(3)+\Sigma^9 BP_*/(3)$ according as $x\neq 0$ or $x=0$. Hence, it follows from $(*)$ that the attaching classes for Y and Y' are non-zero. Thus, we obtain a map

$f: \Sigma^{48}W \to W$ realizing the multiplication by v_2^3.

Put $\gamma = \pi\pi_2 f i_2 i \in G_{38}$. Then, exactly the same discussion as in [8], [9] shows $\gamma \neq 0$. Hence γ is a non-zero multiple of ε_1 and satisfies $\{\gamma, 3, \alpha_2, 3\} \equiv 0$. This contradicts to (5.1). *q.e.d.*

The above proof can easily be generalized, and in the same way the following results are obtained.

(5.3) *If $BP_*/(3, v_1, v_2^t)$ is realized, there is a non-zero element $\gamma \in G_{16t-6}$ such that $3\gamma = 0$, $\{\gamma, 3, \alpha_1\} \equiv 0$ and $\{\gamma, 3, \alpha_1, 3\} \equiv 0$.*

(5.4) *If $BP_*/(3, v_1^2, v_2^{3t})$ is realized, there is a non-zero element $\gamma \in G_{48t-10}$ such that $3\gamma = 0$, $\{\gamma, 3, \alpha_2\} \equiv 0$ and $\{\gamma, 3, \alpha_2, 3\} \equiv 0$.*

Since $\{\beta_2, 3, \alpha_1\} \neq 0$ [14, Prop. 15.6], $\{\varepsilon_2, 3, \alpha_1, 3\} \not\equiv 0$ and $G_{58} = 0$ [7], it follows from (5.3) that

(5.5) *for $t = 2, 3, 4$, $BP_*/(3, v_1, v_2^t)$ can not be realized.*

Appendix. 5-Primary γ-family

For $p=5$, the existence of $V(3)$ (and the construction of the γ-family) is not known. We can, however, construct γ's in $\pi_*(B)$ for $p=5$ in a similar manner.

Set $B = S^0 \cup_{\beta_1} e^{39}$ and $VB(2) = V(2) \wedge B$. A map μ: $V(2) \wedge V(2) \to VB(2)$ is called *a multiplication* if the restrictions of μ on $V(2) \wedge S^0 = V(2)$ and on $S^0 \wedge V(2) = V(2)$ are the inclusions.

By Theorem 5.2 of [15], $\pi_*(VB(2))$ is isomorphic, for deg < 197, to the graded vector space A in the theorem, and hence

$$\pi_i(VB(2)) = \begin{cases} Z_5 & \text{for } \ i = 0, 7, 39, 54, 86, 93, \\ 0 & \text{otherwise for } \ i < 197. \end{cases}$$

We can therefore extend any map $(V(2) \wedge S^0) \cup (S^0 \wedge V(2)) \to VB(2)$ over the whole of $V(2) \wedge V(2)$. Thus,

(A.1) *there exists a multiplication μ: $V(2) \wedge V(2) \to VB(2)$.*

The relation $\beta_1 \wedge 1_B = 0$ in (3.3) holds for any $p \geqq 3$, and we have

(A.2) *there exists a multiplication μ_B: $B \wedge B \to B$.*

Now, we denote by

$$\text{(A.3)} \qquad \gamma_0: S^{248} \longrightarrow V(2)$$

an element having $V\left(2\frac{1}{8}\right)$ as its mapping cone. Then,

(A.4) *γ_0 induces the multiplication by v_3 on the BP homology.*

Using the elements of (A.1)–(A.3), we define

(A.5) $$\bar{\gamma}\colon \Sigma^{248} VB(2) \longrightarrow VB(2)$$

by the following composition

$$\Sigma^{248} VB(2) = S^{248} \wedge VB(2) \xrightarrow{\gamma_0 \wedge 1} V(2) \wedge V(2) \wedge B$$
$$\xrightarrow{\mu \wedge 1} V(2) \wedge B \wedge B \xrightarrow{1 \wedge \mu_B} VB(2).$$

Let $i_0\colon S^0 \to V(2)$ be the inclusion. Then, we have easily

(A.6) $$\bar{\gamma}(i_0 \wedge 1_B) = \gamma_0 \wedge 1_B.$$

From (A.4) and (A.6), it follows that

(A.7) *$\bar{\gamma}$ induces the multiplication by v_3 on each factor of $BP_*(VB(2)) = BP_*/(5, v_1, v_2) + \Sigma^{39} BP_*/(5, v_1, v_2)$, hence $BP_*/(5, v_1, v_2, v_3) + \Sigma^{39} BP_*/(5, v_1, v_2, v_3)$ is realized by the mapping cone of $\bar{\gamma}$.*

Recently, H. R. Miller, D. C. Ravenel and W. S. Wilson [5] have announced the non-triviality of $\gamma_t \in G_{(tp^2+(t-1)p+t-2)q-3}$, $q = 2(p-1)$, for all $t \geqq 1$ and primes $p \geqq 7$. So, we expect the non-triviality of the elements $\bar{\gamma}_t \in \pi_{248t-59}(B)$ defined by the compositions

$$S^{248t} \xrightarrow{j} \Sigma^{248t} B \xrightarrow{i_0 \wedge 1_B} \Sigma^{248t} VB(2) \xrightarrow{\bar{\gamma}^t} VB(2) \xrightarrow{\pi_0 \wedge 1_B} \Sigma^{59} B,$$

where j and i_0 are the inclusions to the bottom spheres and $\pi_0\colon V(2) \to S^{59}$ is the collapsing map.

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