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# Central limit theorem for a simple diffusion model of interacting particles

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## §1. Introduction

Given a smooth function $b(x, y)$, $x, y \in \boldsymbol{R}$, we set

$$b[x, u] = \int_{\boldsymbol{R}} b(x, y)u(y)dy \qquad \left(\text{or} = \int_{\boldsymbol{R}} b(x, y)u(dy)\right)$$

for a function $u(y)$ (or a measure $u(dy)$), and consider

$$\text{(1.1a)} \qquad \frac{\partial u}{\partial t} = \frac{1}{2}\cdot\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}(b[x, u]u),$$

$$\text{(1.1b)} \qquad u(0, x) = u(x),$$

where $u(x)$ is a probability density function. In connection with Kac's work [4] on the propagation of chaos for Boltzmann's equation, McKean [7] described the diffusion process $\{X(t)\}$ associated with (1.1) as the limit process, as $n \to \infty$, of any single component process of the diffusion $X^{(n)}(t) = (X_1^{(n)}(t), \dots, X_n^{(n)}(t))$ with generator

$$\text{(1.2)} \qquad K^{(n)}\varphi = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 \varphi}{\partial x_i^2} + \sum_{i=1}^n \left(\frac{1}{n-1}\sum_{k \neq i} b(x_i, x_k)\right)\frac{\partial \varphi}{\partial x_i}$$

and with initial density $u(x_1)u(x_2)\cdots u(x_n)$; in fact, it was shown that for each fixed $m$ the process $\{(X_1^{(n)}(t), \dots, X_m^{(n)}(t))\}$ converges in law to $\{(X_1(t), \dots, X_m(t))\}$ where $\{X_k(t)\}$, $k = 1, 2, \dots$, are independent copies of $\{X(t)\}$. Thus we have the following law of large numbers:

$$\text{(1.3)} \qquad U^{(n)}(t) = n^{-1}\sum_{k=1}^n \delta_{X_k^{(n)}(t)} \longrightarrow u(t);$$

here $\delta_x$ denotes the $\delta$-distribution at $x$ and $u(t) = u(t, x)dx$ where $u(t, x)$ is the solution of (1.1). The next stage is the central limit theorem which investigates the limit of

$$\text{(1.4)} \qquad S^{(n)}(t) = n^{1/2}(U^{(n)}(t) - u(t)),$$

as $n \to \infty$. This kind of problem was considered by Kac [5] and McKean [8] for Boltzmann's equation, and by Martin-Löf [6] and Itô [3] for non-interacting

Markovian particles. The present diffusion model differs from that of [6], [3] in the sense that there are intermolecular interactions due to the dependence of $b(x, y)$ upon $y$.

The purpose of this paper is to find the limit process $\{S(t)\}$ of $\{S^{(n)}(t)\}$, as $n\to\infty$, in a very simple case in which $b(x, y)=-\lambda(x-y)$ for a positive constant $\lambda$. In this case, the diffusion process $\{X(t)\}$ associated with (1.1) can be obtained as the solution of the stochastic differential equation (SDE)

$$dX(t) = dB(t) - \lambda(X(t) - \mu)dt, \tag{1.5}$$

where $\{B(t)\}$ is a 1-dimensional Brownian motion and $\mu=E\{X(0)\}=E\{X(t)\}$. Without loss of generality we may assume that $\mu=0$ in which case the generator of $\{X(t)\}$ yields

$$K\varphi = 2^{-1}\varphi'' - \lambda x\varphi'. \tag{1.6}$$

Although the process $\{S^{(n)}(t)\}$ is signed-measure-valued, the limit process $\{S(t)\}$ turns out to be distribution-valued. Our result is that $\{S(t)\}$ is a diffusion process on an appropriate space $\Phi'_{3/2}$ of distributions and satisfies the SDE

$$d\langle S(t), \varphi\rangle = d\langle B(t), \varphi\rangle + \langle S(t), (K+L_t)\varphi\rangle dt, \qquad \varphi\in\Phi_{3/2}, \tag{1.7}$$

where $\{B(t)\}$ is certain $\Phi'_{3/2}$-Brownian motion (continuous Lévy process on $\Phi'_{3/2}$), determined by (3.7) of § 3, and $(L_t\varphi)(x)=\lambda\mu(t, \varphi')x$ with $\mu(t, \varphi')=E\{\varphi'(X(t))\}$. Making use of Hermite polynomials, the SDE (1.7) will be solved in an explicit form.

## § 2. The limiting Gaussian random field

The diffusion process $X^{(n)}(t)=(X_1^{(n)}(t),\dots, X_n^{(n)}(t))$ with generator $K^{(n)}$ can be obtained as the solution of the SDE

$$X_k^{(n)}(t) = X_k + B_k(t) + \frac{n}{n-1}\int_0^t b[X_k^{(n)}(s), U^{(n)}(s)]ds, \quad 1\leqq k\leqq n, \tag{2.1}$$

for $b(x, y)=-\lambda(x-y)$, $\lambda>0$, where the initial values $X_k$'s are mutually independent random variables with common distribution $u$ and $\{B_k(t)\}$'s are independent copies of a 1-dimensional Brownian motion; it is also assumed that $\{X_k;\ k\geqq 1\}$ and $\{B_k(t);\ k\geqq 1\}$ are independent. If $u$ has a finite expectation, that is, if $E\{|X_k|\}<\infty$, then by the result of McKean [7] (propagation of chaos) we have $E\{|X_k^{(n)}(t)-X_k(t)|\}\to 0$ $(n\to\infty)$ for each fixed $k$, where $X_k(t)$ is the solution of

$$\begin{cases} X_k(t) = X_k + B_k(t) + \displaystyle\int_0^t b[X_k(s), u(s)]ds \\ u(s) = \text{the probability distribution of } X_k(s). \end{cases} \tag{2.2}$$

If in addition $u$ has a smooth density, then $u(t)$ has also a smooth density which is the solution of (1.1).

Let $W$ be the space of continuous paths $w$: $[0, \infty) \to \boldsymbol{R}$, and denote by $\mathfrak{U}$ the probability measure on $W$ induced by the process $\{X_k(t)\}$. Any continuous process in $\boldsymbol{R}$ may be regarded as a $W$-valued random variable which is denoted by the corresponding bold face letter. The law of large numbers (1.3) is now elaborated as follows:

$$U^{(n)} = n^{-1} \sum_{k=1}^{n} \delta_{\mathbf{X}_k^{(n)}} \longrightarrow \mathfrak{U} \qquad \text{(in probability)}. \tag{2.3}$$

If we set

$$S^{(n)} = n^{1/2}(U^{(n)} - \mathfrak{U}), \tag{2.4}$$

then it is expected that $S^{(n)}$ converges in law to some Gaussian random field $S$ as $n \to \infty$. In this section we calculate the characteristic functional of this limiting random field $S$.

We notice that $E\{X_k(t)\} = E\{X_k\} = \mu$. Therefore, by making use of a translation in the phase space if necessary, we may consider only the case $\mu = 0$. In what follows we assume that $\mu = 0$, and estimate the speed of convergence of $\{X_k^{(n)}(t)\}$ to $\{X_k(t)\}$.

LEMMA 2.1. *We have*

$$X_k^{(n)}(t) = X_k(t) + n^{-1/2} Y_k^{(n)}(t), \tag{2.5}$$

*where*

$$\begin{aligned} Y_k^{(n)}(t) &= \lambda \int_0^t Z^{(n)}(s)\,ds + \frac{\lambda}{n-1} \int_0^t \exp\left\{-\frac{n\lambda(t-s)}{n-1}\right\} Z^{(n)}(s)\,ds \\ &\quad - \frac{n^{1/2}}{n-1} \int_0^t \exp\left\{-\frac{n\lambda(t-s)}{n-1}\right\} X_k(s)\,ds, \\ Z^{(n)}(t) &= n^{-1/2} \textstyle\sum_{j=1}^{n} X_j(t). \end{aligned} \tag{2.6}$$

PROOF. Solving the differential equation

$$\begin{aligned} \dot{Y}_k^{(n)}(t) &= n^{1/2} \left\{ \frac{n}{n-1} b[X_k^{(n)}(t), U^{(n)}(t)] - b[X_k(t), u(t)] \right\} \\ &= -\frac{n\lambda}{n-1} Y_k^{(n)}(t) + \frac{\lambda}{n-1} \textstyle\sum_{j=1}^{n} Y_j^{(n)}(t) + \frac{n\lambda}{n-1} Z^{(n)}(t) - \frac{n^{1/2}\lambda}{n-1} X_k(t), \end{aligned}$$

we obtain (2.6).

Denote by $\Xi_0$ the set of functions $\xi(w)$ which are expressed as

$$\xi(w) = \varphi(w(t_1), \dots, w(t_m)) \tag{2.7}$$

with some $t_1, t_2, \dots, t_m \geqq 0$ and polynomials $\varphi(x_1, \dots, x_m)$.

LEMMA 2.2. *In addition to $\mu=0$, we assume that the initial distribution $u$ has finite absolute p-th moments for all $p>0$. Then $\Xi_0 \subset L^2(W, \mathfrak{U})$.*

PROOF. If $X$ is a $u$-distributed random variable independent of a 1-dimensional Brownian motion $\{B(t)\}$, then the probability law in the path space $W$ of the process

$$X(t) = e^{-\lambda t} X + \int_0^t e^{-\lambda(t-s)} dB(s) \tag{2.8}$$

is $\mathfrak{U}$. Since $E\{|X(t)|^p\} \leqq C_p$ for any $p \geqq 0$, we have

$$\int \xi^2 \mathfrak{U}(dw) = E\{\varphi(X(t_1), \dots, X(t_m))^2\} < \infty$$

and hence $\Xi_0 \subset L^2(W, \mathfrak{U})$.

We define $S_0^{(n)}$ and $U_0^{(n)}$ by

$$S_0^{(n)} = n^{1/2}(U_0^{(n)} - \mathfrak{U}) = n^{-1/2} \textstyle\sum_{k=1}^n (\delta_{X_k} - \mathfrak{U}). \tag{2.9}$$

LEMMA 2.3. *Under the same assumption for $u$ as in the preceeding lemma, we have*

$$\langle S^{(n)}, \xi \rangle = \langle S_0^{(n)}, \xi \rangle + \textstyle\sum_{j=1}^m \overline{\partial_j \varphi} Y^{(n)}(t_j) + R_n,$$

*for $\xi(w) = \varphi(w(t_1), \dots, w(t_m)) \in \Xi_0$, where*

$$Y^{(n)}(t) = \lambda \int_0^t Z^{(n)}(s) ds,$$

$$\overline{\partial_j \varphi} = E\{\partial_j \varphi(X_k(t_1), \dots, X_k(t_m))\},$$

*and $R_n \to 0$ in probability as $n \to \infty$.*

PROOF. We can write

$$\begin{aligned}
&\xi(X_k^{(n)}) - \xi(X_k) \\
&= \varphi(X_k(t_1) + n^{-1/2} Y_k^{(n)}(t_1), \dots, X_k(t_m) + n^{-1/2} Y^{(n)}(t_m)) - \varphi(X_k(t_1), \dots, X_k(t_m)) \\
&= \textstyle\sum_{j=1}^m \partial_j \varphi(X_k(t_1), \dots, X_k(t_m)) n^{-1/2} Y_k^{(n)}(t_j) \\
&\quad + \textstyle\sum_{i,j=1}^m \int_0^1 \int_0^t \partial_{ij}^2 \varphi(X_k(t_1) + n^{-1/2} Y_k^{(n)}(t_1) s, \dots, X_k(t_m) + n^{-1/2} Y_k^{(n)}(t_m) s) ds dt \\
&\qquad \times Y_k^{(n)}(t_i) Y_k^{(n)}(t_j)/n \\
&= \textstyle\sum_{j=1}^m \partial_j \varphi(X_k(t_1), \dots, X_k(t_m)) n^{-1/2} Y_k^{(n)}(t_j) + R_{nk},
\end{aligned}$$

and hence

$$\begin{aligned}\langle S^{(n)}, \xi\rangle &= \langle S_0^{(n)}, \xi\rangle + n^{1/2}\langle U^{(n)} - U_0^{(n)}, \xi\rangle \\ &= \langle S_0^{(n)}, \xi\rangle + \textstyle\sum_{j=1}^m \{n^{-1}\sum_{k=1}^n \partial_j\varphi(X_k(t_1),\dots, X_k(t_m))\} Y_k^{(n)}(t_j) + n^{-1/2}\sum_{k=1}^n R_{nk}\end{aligned}$$

Since $E\{|X_k(t)|^p\}\leqq C_p$ for any $p>0$, it follows from the expression (2.6) that $n^{-1/2}\sum_{k=1}^n R_{nk}\to 0$ in probability as $n\to\infty$. Moreover, by the law of large numbers we have $n^{-1}\sum_{k=1}^n \partial_j\varphi(X_k(t_1),\dots, X_k(t_m))\to\overline{\partial_j\varphi}$ in probability, proving the lemma.

Let $\Xi$ be the set of functions $\xi$ on $W$ which are expressed as

$$\xi = \xi_0 + \int_a^b \theta_t\xi_1\alpha(t)dt\,, \tag{2.10}$$

with some $\xi_0$, $\xi_1\in\Xi_0$, $0\leqq a<b<\infty$ and a continuous function $\alpha(t)$ on $[0, \infty)$, where $\theta_t$ denotes the shift: $(\theta_t w)(s)=w(t+s)$, $s\geqq 0$ and $(\theta_t\xi_1)(w)=\xi_1(\theta_t w)$. We introduce a linear operator $\Lambda$: $\Xi\to L^2(W, \mathfrak{U})$ by

$$\begin{aligned}\Lambda\xi &= \lambda\textstyle\sum_{j=1}^m \overline{\partial_j\varphi}\displaystyle\int_0^{t_j} w(s)ds, && \text{if } \xi \text{ is of the form (2.7)}\\ &= \Lambda\xi_0 + \int_a^b \Lambda(\theta_s\xi_1)\alpha(s)ds, && \text{if } \xi \text{ is of the form (2.10)}.\end{aligned}$$

Then we have the following theorem.

THEOREM 2.1. *We assume that the initial distribution $u$ has finite absolute p-th moments for all $p\geqq 0$ and $\mu=0$. Then for any $\xi\in\Xi$*

$$\lim_{n\to\infty} E\{e^{i\langle S^{(n)},\xi\rangle}\} = e^{-Q(\xi)/2},$$

*where $Q(\xi)=|\xi+\Lambda\xi|^2-(\xi+\Lambda\xi, 1)^2$ and $|\cdot|$ and $(\cdot, \cdot)$ denote the norm and the inner product of $L^2(W, \mathfrak{U})$, respectively.*

PROOF. It may be enough to consider only when $\xi$ is of the form (2.7). Introducing the evaluation map $e_t$: $W\to\boldsymbol{R}$ defined by $e_t(w)=w(t)$, we have $\langle S_0^{(n)}, e_t\rangle = Z^{(n)}(t)$, and hence

$$Y^{(n)}(t) = \lambda\int_0^t Z^{(n)}(s)ds = \lambda\int_0^t \langle S_0^{(n)}, e_s\rangle ds = \langle S_0^{(n)}, \lambda\int_0^t e_s ds\rangle\,.$$

Therefore, we have

$$\begin{aligned}\lim_{n\to\infty} E\, e^{i\langle S^{(n)},\xi\rangle} &= \lim_{n\to\infty} E\exp\{i\langle S_0^{(n)}, \xi\rangle + i\textstyle\sum_{j=1}^m \overline{\partial_j\varphi}Y^{(n)}(t_j)\}\\ &= \lim_{n\to\infty} E\exp\{i\langle S_0^{(n)}, \xi + \lambda\textstyle\sum_{j=1}^m \overline{\partial_j\varphi}\displaystyle\int_0^t e_s ds\rangle\}\\ &= e^{-Q(\xi)/2},\end{aligned}$$

in which we have used the classical central limit theorem.

## §3. The SDE for $\{S(t)\}$

Let $\{S(\xi), \xi \in \Xi\}$ be a Gaussian random field such that

$$E\{e^{i\langle S,\xi\rangle}\} = e^{-Q(\xi)/2}, \qquad \xi \in \Xi ,$$

where $Q(\xi)$ is defined in Theorem 2.1, and define $S(t)$ by $\langle S(t), \varphi\rangle = \langle S, \varphi(w(t))\rangle$ for any polynomial $\varphi$ of $x \in \boldsymbol{R}$. Then we have

$$\lim_{n\to\infty} E\{e^{i\langle S^{(n)}(t),\varphi\rangle}\} = E\{e^{i\langle S(t),\varphi\rangle}\} .$$

In this section we derive the SDE for $\{S(t)\}$ by making use of a method of Itô [3].

In what follows we assume that a probability distribution $u$, which is to be the initial distribution of the diffusion process $\{X(t)\}$ with generator $K$, has a density $u(x)$ satisfying $\mu = \int xu(x)dx = 0$ and

$$u(x) < \text{const.}\, g(x), \tag{3.1}$$

where $g(x) = (\lambda/\pi)^{1/2}e^{-\lambda x^2}$. Denote by $u(t)$ the probability distribution of $X(t)$. Then it has a density $u(t, x)$ which also satisfies $u(t, x) < \text{const.}\, g(x)$, the const. being the same as the one in (3.1).

We introduce the following notations:

$$H_n(x) = H_n(x; \lambda) = (-1)^n\{n!(2\lambda)^n\}^{-1/2}e^{\lambda x^2}\frac{d^n}{dx^n}e^{-\lambda x^2}, \quad n = 0, 1,\dots,$$

$$\mathfrak{P} = \{\varphi = \textstyle\sum a_k H_k \text{ (finite sum)}, a_k = \text{real}\},$$

$$\|\varphi\|_\alpha^2 = (2\lambda)^{2\alpha} \textstyle\sum_k a_k^2(k+1/2)^{2\alpha}, \qquad \alpha \in \boldsymbol{R},$$

$$\Phi_\alpha = \|\cdot\|_\alpha\text{-completion of } \mathfrak{P},$$

$$\Phi'_\alpha = \text{the dual space of } \Phi_\alpha \ (\cong \Phi_{-\alpha}),$$

$$\Phi = \textstyle\bigcap_\alpha \Phi_\alpha, \quad \Phi' = \bigcup_\alpha \Phi'_\alpha.$$

For $\varphi \in \mathfrak{P}$ we set

$$M_t(\varphi, w) = \varphi(w(t)) - \varphi(w(0)) - \int_0^t (K\varphi)(w(s))ds .$$

Then $\{M_t(\varphi, w)\}$ is a $\mathfrak{U}$-martingale, and it is not hard to prove

$$|M_t(\varphi, w)|^2 = \int_0^t \|\varphi'\|_{u(\tau)}^2 d\tau , \tag{3.2}$$

$$(M_t(\varphi, w), M_s(\psi, w)) = \int_0^{t\wedge s} (\varphi', \psi')_{u(\tau)} d\tau , \tag{3.3}$$

where $\|\cdot\|_{u(\tau)}$ and $(\cdot,\ \cdot)_{u(\tau)}$ denote the usual norm and inner product, respectively, in $L^2(\boldsymbol{R}, u(\tau))$.

LEMMA 3.1. *If we set*

$$\xi_t(\varphi, w) = M_t(\varphi, w) - \lambda\int_0^t \mu(s, \varphi')w(s)ds$$

*for* $\varphi \in \mathfrak{P}$, *where* $\mu(s, \varphi')=E\{\varphi'(X(s))\}$, *then*

$$\xi_t(\varphi,w) + \Lambda\xi_t(\varphi, w) = M_t(\varphi, w), \tag{3.4}$$

$$\lim_{n\to\infty} E\{e^{i\langle S^{(n)}, \xi_t(\varphi,w)\rangle}\} = \exp\{-\int_0^t \|\varphi'\|^2_{u(s)}ds/2\}. \tag{3.5}$$

PROOF. It is enough to prove that $\Lambda\xi_t(\varphi, w)=\lambda\int_0^t \mu(s, \varphi')w(s)ds$. By the definition of $\Lambda$, we have

$$\Lambda\xi_t(\varphi, w) = \lambda\mu(t, \varphi')\int_0^t w(s)ds - \lambda\int_0^t \mu(s, (K\varphi)')\,ds\int_0^s w(\tau)d\tau$$
$$- \lambda^2\int_0^t \mu(s, \varphi')ds\int_0^s w(\tau)d\tau.$$

Since $\dot\mu(s, \varphi')=\mu(s, K\varphi')=\mu(s, (K\varphi)'+\lambda\varphi')$, we have

$$\Lambda\xi_t(\varphi, w) = \lambda\mu(t, \varphi')\int_0^t w(s)ds - \lambda\int_0^t \dot\mu(s, \varphi')ds\int_0^s w(\tau)d\tau$$
$$= \lambda\int_0^t \mu(s, \varphi')w(s)ds \qquad \text{(integration by part)},$$

as was to be proved.

For $\varphi \in \mathfrak{P}$ we can write

$$\begin{aligned}\langle S(t), \varphi\rangle - \langle S(0), \varphi\rangle &= \langle S, \varphi(w(t)) - \varphi(w(0))\rangle \\ &= \langle S, \xi_t(\varphi, w)\rangle + \langle S, \int_0^t (K\varphi)(w(s))ds + \lambda\int_0^t \mu(s, \varphi')w(s)ds\rangle \\ &= \langle S, \xi_t(\varphi, w)\rangle + \int_0^t \langle S(s), K\varphi\rangle ds + \lambda\int_0^t \mu(s, \varphi')\langle S(s), \varphi_1\rangle ds\end{aligned} \tag{3.6}$$

where $\varphi_1(x)\equiv x$. A method of Itô [3] is to derive the SDE for $\{S(t)\}$ from (3.6) by noticing that

$$B_t(\varphi) = \langle S, \xi_t(\varphi, w)\rangle$$

defines a $\Phi'$-Brownian motion. In fact, by Lemma 3.1 we have

$$E\{e^{i(B_t(\varphi)-B_s(\varphi))}\} = \exp\{-\int_s^t \|\varphi'\|^2_{u(\tau)}d\tau/2\}, \qquad 0 \leqq s \leqq t,$$

and because of the bound

$$\|\varphi'\|_{u(\tau)}^2 \leqq \text{const. } \|\varphi'\|_0^2 \leqq \text{const. } \|\varphi\|_{1/2}^2$$

there exists a Brownian motion (=continuous Lévy process) $\{\boldsymbol{B}(t)\}$ in $\Phi'_{3/2}$ such that

$$E\{e^{i\langle \boldsymbol{B}(t), \varphi\rangle}\} = \exp\{-\int_0^t \|\varphi'\|_{u(\tau)}^2 d\tau/2\}, \qquad \varphi \in \Phi, \tag{3.7}$$

$$\langle \boldsymbol{B}(t), \varphi\rangle = B_t(\varphi), \quad \text{a.s.} \qquad \text{for} \quad \varphi \in \mathfrak{P}. \tag{3.8}$$

Now (3.6) yields

$$d\langle S(t), \varphi\rangle = d\langle \boldsymbol{B}(t), \varphi\rangle + \langle S(t), K\varphi + L_t\varphi\rangle dt, \tag{3.9}$$

where $L_t$: $\varphi \to \lambda\mu(t, \varphi')\varphi_1$.

Our next problem is to solve the SDE (3.9). Here, the test function $\varphi$ is taken from $\Phi$. Setting

$$B_k(t) = \langle \boldsymbol{B}(t), H_k\rangle, \qquad k = 0, 1, \ldots$$

$$S_k(t) = \langle S(t), H_k\rangle, \qquad k = 0, 1, \ldots,$$

and noting that $H_k$'s are eigenfunctions of $K$ ($KH_k = -2\lambda k H_k$), we have from (3.9)

$$dS_k(t) = dB_k(t) - 2\lambda k S_k(t)dt + (\lambda/2)^{1/2}\mu(t, H_k')S_1(t)dt.$$

This can be solved easily; the result is

$$S_0(t) \equiv 0,$$

$$S_k(t) = e^{-2\lambda kt} S_k(0) + \int_0^t e^{-2\lambda k(t-s)} dB_k(s)$$

$$+ R_k(t)\int_0^t \mu(s, H_{k-1})ds S_1(0) + \int_0^t R_k(t-s)\int_s^t \mu(\tau, H_{k-1})d\tau dB_1(s),$$

where

$$R_k(t) = k^{1/2}(2k-1)^{-1}(e^{-\lambda t} - e^{-2\lambda kt})t^{-1},$$

and $\{S_k(0), k \geqq 1\}$ is a Gaussian system with

$$E\{S_j(0)S_k(0)\} = (H_j - \overline{H}_j, H_k - \overline{H}_k)_u,$$

$$E\{S_j(0)\} = 0, \quad \overline{H}_j = \int H_j(x)u(x)dx.$$

Thus, $\{S(t)\}$ is a diffusion process on $\Phi'_{3/2}$ and we obtain the following theorem.

THEOREM 3.1. *For any polynomials* $\varphi_1, \ldots, \varphi_m$ *of* $x (\in \boldsymbol{R})$ *and* $t_1, \ldots, t_m \geqq 0$, *the joint distribution of* $\langle S^{(n)}(t_1), \varphi_1\rangle, \dots, \langle S^{(n)}(t_m), \varphi_m\rangle$ *converges to that of* $\langle S(t_1), \varphi_1\rangle, \dots, \langle S(t_m), \varphi_m\rangle$ *as* $n \to \infty$, *where* $\{S(t)\}$ *is a diffusion process on* $\Phi'_{3/2}$ *satisfying the SDE* (3.9).

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