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Numerical solutions to problems of the least squares type for ordinary differential equations

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1. Introduction

We consider a real n-dimensional system of differential equations

$$\frac{dx}{dt} = X(x, t), \quad t \in [a, b] \tag{1.1}$$

and observe the problem of finding a solution of (1.1) which minimizes a real functional

$$v[x] = (g[x])^* g[x] \tag{1.2}$$

locally, where $g[x]$ is a real m-dimensional functional and A^* is the transpose of a matrix A.

In [1] H. T. Banks and G. M. Groome, Jr. proposed an iterative procedure of finding a solution to the above problem for a linear $g[x]$ by the use of the quasilinearization of the differential system (1.1) and obtained a condition for $v[x]$ to have a local minimum at a point of attraction of the iterative procedure. In [12] M. Urabe proposed the Newton iterative procedure and the generalized Newton one. The latter is simpler than that proposed by Banks and Groome. In [9] H. Shintani and Y. Hayashi studied the same problem for several types of $g[x]$ and conditions for a local minimum of (1.2). It is worthwhile to note the work [8], though it is dissimilar to the above works.

In all the above-mentioned works except [8], the original problem is reduced to the following boundary value problem:

$$\frac{dx}{dt} = X(x, t), \quad t \in [a, b] \tag{1.3}$$

with the boundary condition

$$(g'(x)[\Phi_{(x)}])^* g(x) = 0,$$

where $\Phi_{(x)}(t)$ is the fundamental matrix of the differential system

$$\frac{dy}{dt} = X_x(x(t), t)y$$

with $\Phi_{(x)}(t_0)=I$ (I: the unit matrix), t_0 is some point in $[a, b]$ and $g'(x)$ is the first Fréchet derivative of $g[x]$. For obtaining approximate solutions this reduction is very powerful.

In [4] M. Fujii observed a posteriori error estimation of an approximate solution by using finite Chebyshev series for the original problem and gave a numerical example. In the above-mentioned boundary value problem, the boundary condition includes $\Phi_{(x)}(t)$. But in general only its approximation can be obtained. He found the following fact: In order to estimate an error bound of the approximate solution directly by making use of the method proposed in [3], the knowledge of the exact fundamental matrix $\Phi_{(x)}(t)$ is required. Thus some manipulations are necessary. However, in his case the error bound obtained was somewhat of an overestimate.

In a posteriori error estimation a fundamental matrix plays an important role. In many practical applications exact fundamental matrices and their inverses are not available, so that the estimates are not applicable if the approximate fundamental matrices and their approximate inverses are not so accurate. In [6] Y. Hayashi gave a posteriori error estimates of the approximate solutions in terms of the approximate fundamental matrices and their approximate inverses.

In this paper we still treat a posteriori error estimation of the approximate solution and the local minimality of the exact solution corresponding to the approximate one. A numerical example is given.

In Section 2 we state the original problem of the least squares type for ordinary differential equations and give preparatory descriptions. In Section 3 it is shown that the problem given in Section 2 is reduced to a special boundary value problem under the assumption that a certain matrix is positive definite. We also propose a condition for a local minimum in terms of error bounds of the approximate solution and the approximate fundamental matrix. In Section 4 we obtain a theorem which is an improvement of the results in [6, Theorem 8] for saving time. In Section 5 we give a numerical example in which the same problem as in [4] is treated by using finite Chebyshev series.

Computations in this paper have been carried out by the use of FACOM M-200 at Kyushu University and OKITAC 50/10 at Toyama University.

2. Preliminaries

2.1. The problem of the least squares type

Let R^n be a real n-space with any norm $\|\cdot\|$ and let $\|\cdot\|_*$ denote the dual norm of $\|\cdot\|$. For any $k\times n$ real matrix B ($k\leqq n$), let $\|B\|$ be the natural norm induced by the norm $\|\cdot\|$. Then by [2, pp. 42–43] it holds that

$$\|e^*\| = \|e\|_* \qquad \text{for} \quad e \in R^n.$$

Let $C[J]$ be the Banach space of all real n-vector functions $x(t)$ continuous on the interval $J=[a, b]$ with the norm $\|x\|_c=\sup_{t\in J}\|x(t)\|$ and let $M[J]$ be the Banach space of all real $n\times n$ matrix functions $A(t)$ continuous on J with the norm $\|A\|_c=\sup_{t\in J}\|A(t)\|$. The identity operator and the unit matrix are denoted by the same symbol I. The sum $F+G$ and the product FG of two operators F and G are defined in the usual manner.

For two Banach spaces X and Y, we denote by $L(X, Y)$ the set of all bounded linear operators from X into Y and we abbreviate $L(X, X)$ by $L(X)$. For $F: D\subset X\to L(X, Y)$ let $F(x)$ be an element of $L(X, Y)$ associated with $x\in D$. When $F: D\subset X\to Y$ is Fréchet differentiable at $x\in D$, we denote by $F'(x)$ the Fréchet derivative of F at x.

Let $A=(a_1, a_2,\dots, a_n)\in M[J]$ and $h\in C[J]$. Then for any $T\in L(C[J])$, we define $TA\in M[J]$ by

$$TA = (Ta_1, Ta_2,\dots, Ta_n),$$

and for a bilinear operator N from $C[J]$ into $C[J]$, we define $N[h, A]$ by

$$N[h, A] = (N[h, a_1], N[h, a_2],\dots, N[h, a_n]).$$

For $Y_i\in L(C[J])\,(i=1, 2,\dots, n)$, let $Y\in L(C[J], M[J])$ be the operator defined by

$$Yh = (Y_1h, Y_2h,\dots, Y_nh).$$

Let Ω' be a domain in the tx-space intercepted by two hyperplanes $t=a$ and $t=b$ such that the cross sections R_a and R_b at $t=a$ and $t=b$ make an open set in each hyperplane. Put $\Omega=R_a\cup\Omega'\cup R_b$ and let D_0 be the domain of $C[J]$ defined by

$$D_0 = \{x\in C[J]\,|\,(t, x(t))\in\Omega \qquad \text{for all} \quad t\in J\}.$$

Let us consider the system of differential equations

$$\frac{dx}{dt} = X(x, t) \qquad \text{for} \quad t\in J \tag{2.1}$$

and the problem of finding a solution which minimizes the functional

$$v[x] = (g[x])^*g[x] \tag{2.2}$$

locally, where x and $X(x, t)$ are real n-vectors, $X(x, t)$ is continuous in Ω and twice continuously differentiable with respect to x in Ω, and $g: D_0\to R^m$ is twice continuously Fréchet differentiable in D_0. We assume that (2.1) has at least one solution in D_0.

For any fixed $t_0\in J$ let $x(t, c)$ be a solution of (2.1) on J such that $x(t_0, c)=c$, and let

$$\Delta_0 = \{c \in R^n \mid x(t, c) \in D_0\}.$$

Let $q: \Delta_0 \to R^m$ and $s: \Delta_0 \to R^1$ be defined by

(2.3) $$q[c] = g[x(t, c)],$$

(2.4) $$s[c] = (q[c])^* q[c]/2$$

respectively, and let $\Delta \subset \Delta_0$ be a convex domain. Then for any $c, c+e \in \Delta$, by [9, Lemma 3] it holds that

(2.5) $$s[c+e] = s[c] + s'(c)e + s''(c)ee + o(\|e\|^2),$$

where

(2.6) $$s'(c)e = (q[c])^* q'(c)e,$$

(2.7) $$s''(c)ee = (q'(c)e)^* q'(c)e + (q[c])^* q''(c)ee,$$

(2.8) $$q'(c)e = g'(x(t, c))[x_c(t, c)e],$$

(2.9) $$q''(c)ee = g''(x(t, c))[x_c(t, c)e, x_c(t, c)e] + g'(x(t, c))[x_{cc}(t, c)ee],$$

x_c and x_{cc} are the first and the second Fréchet derivatives of $x(t, c)$ with respect to c respectively.

From the assumption on $X(x, t)$ it follows that $x_c(t, c)$ is the fundamental matrix of the system

(2.10) $$\frac{dy}{dt} = X_x(x(t, c), t)y$$

satisfying $x_c(t_0, c) = I$, and that $x_{cc}(t, c)$ is the solution of the system

(2.11) $$\frac{dz}{dt} ee = X_x(x(t, c), t)zee + X_{xx}(x(t, c), t)[x_c(t, c)e, x_c(t, c)e]$$

satisfying $x_{cc}(t_0, c) = 0$, where X_x and X_{xx} are the first and the second Fréchet derivatives of $X(x, t)$ with respect to x respectively.

If $s[c]$ attains a local minimum at $\hat{c} \in \Delta$, as is well known, it holds that $s'(\hat{c}) = 0$. Conversely a sufficient condition under which the solution $\hat{c}$ of $s'(c) = 0$ minimizes $s[c]$ locally is given by the following [7, Theorem 5].

THEOREM 1. *Let $\hat{c} \in \Delta$ be a solution of $s'(c) = 0$ and suppose there exists a positive constant α such that*

(2.12) $$s''(c)ee \geqq \alpha\|e\|^2 \quad \textit{for all} \quad e \in R^n.$$

Then $s[c]$ attains a local minimum at $c = \hat{c}$.

In many practical applications the solution $\hat{c}$ of $s'(c) = 0$ and the solution $x(t, \hat{c})$ of (2.1) can not be obtained exactly. We can obtain only its approximation.

Therefore in the next section, using $x^{(0)} \in C[J]$ and $Z^{(0)} \in M[J]$ which are approximations of $x(t, \hat{c})$ and $x_c(t, \hat{c})$ respectively, we establish a theorem assuring that $x(t, \hat{c})$ is a solution of (2.1) minimizing (2.2) locally.

2.2. Positive definite matrices

A matrix $A \in L(R^n)$ is called positive definite if

$$e^*Ae > 0 \quad \text{for all} \quad e \in R^n \ (e \neq 0). \tag{2.13}$$

We have

LEMMA 1. *Let $A, B \in L(R^n)$ and suppose there exists a positive number α such that*

$$e^*Be \geqq \alpha \|e\|_* \|e\| \quad \textit{for} \quad e \in R^n, \tag{2.14}$$

$$\alpha > \|A-B\|. \tag{2.15}$$

Then A is positive definite.

PROOF. For any $e \in R^n$ $(e \neq 0)$, by (2.14) and (2.15) it follows that

$$\begin{aligned} e^*Ae &= e^*Be + e^*(A-B)e \\ &\geqq (\alpha - \|A-B\|)\|e\|_* \|e\| > 0. \end{aligned}$$

Hence by (2.13) A is positive definite.

Since

$$e^*Be = e^*(B+B^*)e/2 \quad \text{for all} \quad e \in R^n, \tag{2.16}$$

we have the following

COROLLARY 1. *Let μ be the least eigenvalue of $(B+B^*)/2$ and suppose*

$$\mu > \|A-B\|_2, \tag{2.17}$$

where $\|\cdot\|_2$ is the spectral norm. Then A is positive definite.

For any symmetric $C \in L(R^n)$, since $\|C\| \geqq \|C\|_2$, we have the following

COROLLARY 2. *Let A and B be symmetric and suppose that for the least eigenvalue ν of B*

$$\nu > \|A-B\|. \tag{2.18}$$

Then A is positive definite.

3. The local minimality

3.1. The boundary value problem of the least squares type

Let

$$D = \{x \in D_0 \mid x(t_0) \in \Delta\}.$$

For $x \in D$, let $\Phi_{(x)}(t)$ be the fundamental matrix of the system

$$\frac{dy}{dt} = X_x(x(t), t)y$$

with $\Phi_{(x)}(t_0) = I$ and let U be a domain in $M[J]$ including

$$U_0 = \{\Phi_{(x)} \in M[J] \mid x \in D, \Phi_{(x)}(t_0) = I\}.$$

Let f: $D \times U \to R^n$ be defined by

$$f[u] = (g'(x)Z)^* g[x] \qquad \text{for} \quad u = (x, Z) \in D \times U. \tag{3.1}$$

Substituting (2.3) and (2.8) into (2.6), for $u = (x(t, c), x_c(t, c))$ we have

$$\begin{aligned} s'(c)e &= (g[x(t, c)])^* g'(x(t, c))[x_c(t, c)e] \\ &= e^* f[u]. \end{aligned} \tag{3.2}$$

The solution $x = x(t, \hat{c})$ of (2.1) with $\hat{c}$ such that $s'(\hat{c}) = 0$ and the fundamental matrix $Z = x_c(t, \hat{c})$ of (2.10) are a solution of the following boundary value problem:

$$\begin{cases} \dfrac{dx}{dt} = X(x, t), \\ \dfrac{dZ}{dt} = X_x(x(t), t)Z, \quad Z(t_0) = I, \\ f[u] = 0 \qquad \text{for} \quad u = (x, Z) \in D \times U. \end{cases} \tag{3.3}$$

Conversely let $\hat{u} = (\hat{x}, \hat{Z})$ be a solution of (3.3). If we put $\hat{c} = \hat{x}(t_0)$, then $s'(\hat{c}) = 0$.

Let X_2: $D \times U \to L(C[J], M[J])$ and E: $M[J] \times M[J] \to L(C[J])$ be defined by

$$X_2(u)h = X_{xx}(x(t), t)[h(t), Z(t)] \qquad \text{for} \quad u = (x, Z) \in D \times U, h \in C[J], \tag{3.4}$$

$$E(P, Q)h = \int_{t_0}^{t} P(t)Q(s)h(s)ds \qquad \text{for} \quad P, Q \in M[J], h \in C[J] \tag{3.5}$$

respectively. Let $x_{cc} = x_{cc}(t, c)$ be the solution of (2.11) satisfying $x_{cc}(t_0, c) = 0$. Then we have

$$(3.6) \qquad x_{cc}ee = \int_{t_0}^{t} x_c(t)x_c(s)^{-1}X_{xx}(x(s), s)[x_c(s)e, x_c(s)e]ds$$
$$= E(x_c, x_c^{-1})X_2(u)[x_ce]e \qquad \text{for all} \quad e \in R^n,$$

where $x = x(t, c)$, $x_c = x_c(t, c)$ and $u = (x, x_c)$.

For any $u = (x, Z) \in D \times U$, $h \in C[J]$ and $P, Q, V \in M[J]$, let us define f_x: $D \times U \to L(C[J], R^n)$ and f_z: $D \times U \to L(M[J], R^n)$ by

$$(3.7) \qquad f_x(u)h = (g'(x)Z)^*g'(x)h + (g''(x)[h, Z])^*g[x],$$

$$(3.8) \qquad f_z(u)V = (g'(x)V)^*g[x]$$

and f_2: $D \times U \times M[J] \times M[J] \to L(C[J], R^n)$ by

$$(3.9) \qquad f_2(u, P, Q) = f_x(u) + f_z(u)E(P, Q)X_2(u).$$

Then substituting (2.3), (2.8) and (2.9) into (2.2), by (3.6)–(3.9), we see that

$$(3.10) \qquad s''(c)ee = e^*(g'(x)x_c)^*g'(x)x_ce + (g[x])^*\{g''(x)[x_ce, x_c]$$
$$+ g'(x)E(x_c, x_c^{-1})X_2(u)x_ce\}e$$
$$= e^*\{f_x(u)x_c + f_z(u)E(x_c, x_c^{-1})X_2(u)x_c\}e$$
$$= e^*f_2(u, x_c, x_c^{-1})x_ce \qquad \text{for all} \quad e \in R^n,$$

where $x = x(t, c)$, $x_c = x_c(t, c)$ and $u = (x, x_c)$.

For the solution $\hat{u} = (\hat{x}, \hat{Z})$ of (3.3), if the matrix $f_2(\hat{u}, \hat{Z}, \hat{Z}^{-1})\hat{Z}$ is positive definite, then from (3.2), (3.10) and Theorem 1, $\hat{x}$ is a solution of (2.1) minimizing (2.2) locally.

3.2. A condition for a local minimum

Let $\hat{u} = (\hat{x}, \hat{Z})$ be the solution of the problem (3.3) and let $u^{(0)} = (x^{(0)}, Z^{(0)}) \in D \times U$ be an approximation of $\hat{u}$. Furthermore suppose the error bounds

$$(3.11) \qquad \|\hat{x} - x^{(0)}\|_c \leqq \nu, \quad \|\hat{Z} - Z^{(0)}\|_c \leqq \sigma$$

are given. Put

$$(3.12) \qquad D_\nu = \{x \in C[J] \mid \|x - x^{(0)}\|_c \leqq \nu\} \subset D,$$

$$(3.13) \qquad U_\sigma = \{Z \in M[J] \mid \|Z - Z^{(0)}\|_c \leqq \sigma\} \subset U.$$

Let $A \in M[J]$, $Y \in L(C[J], M[J])$, $l_0 \in L(C[J], R^n)$ and $l_1 \in L(M[J], R^n)$ be operators which are independent of $u = (x, Z) \in D \times U$ and which approximate $X_x(x(t), t)$, $X_2(u)$, $f_x(u)$ and $f_z(u)$ in $D \times U$ respectively. Let $\Phi(t)$ be the fundamental matrix of the system

$$(3.14) \qquad \frac{dy}{dt} = A(t)y$$

satisfying $\Phi(t_0)=I$. We denote by $\Phi_I(t)$ the inverse matrix of $\Phi(t)$.

Let $\tilde{\Phi}(t)\in M[J]$ and $\tilde{\Phi}_I\in M[J]$ be matrices that approximate $\Phi(t)$ and $\Phi_I(t)$ respectively. We define $\tilde{l}_2\in L(C[J])$ by

$$\tilde{l}_2 = l_0 + l_1 E(\tilde{\Phi}, \tilde{\Phi}_I)Y. \tag{3.15}$$

From now on, we write $E(\tilde{\Phi}, \tilde{\Phi}_I)$ as $\tilde{E}$ for simplicity. Put

$$r_1(t) = \tilde{\Phi}_I(t) - I + \int_{t_0}^{t} \tilde{\Phi}_I(s)A(s)ds, \tag{3.16}$$

$$\rho = \max(b-t_0,\ t_0-a). \tag{3.17}$$

Then we have the following

LEMMA 2. *Suppose* (3.11) *holds. Then it follows that*

$$\|E(\hat{Z}, \hat{Z}^{-1}) - \tilde{E}\|_c \leqq \beta_1, \tag{3.18}$$

where β_1 *is a positive number such that*

$$\rho\{\beta_2 \exp(\rho\mu_3) + \beta_3(\exp(\rho\mu_3)-1)/\mu_3\} \leqq \beta_1 \tag{3.19}$$

and β_2, β_3, μ_3 are non-negative numbers such that

$$\|\hat{Z}-\tilde{\Phi}\|_c + \|\tilde{\Phi}\|_c\|r_1\|_c \leqq \beta_2, \tag{3.20}$$

$$\|\tilde{\Phi}\|_c\|\tilde{\Phi}_I\|_c\|X_x(\hat{x}(t), t) - A(t)\|_c \leqq \beta_3, \tag{3.21}$$

$$\|X_x(\hat{x}(t), t)\|_c \leqq \mu_3. \tag{3.22}$$

Proof. Put

$$\varphi(t, s) = \hat{Z}(t)\hat{Z}(s)^{-1} - \tilde{\Phi}(t)\tilde{\Phi}_I(s), \tag{3.23}$$

where $\hat{Z}(s)^{-1}$ satisfies

$$\hat{Z}(s)^{-1} - I + \int_{t_0}^{s} \hat{Z}(\tau)^{-1}X_x(\hat{x}(\tau), \tau)d\tau = 0. \tag{3.24}$$

Since by (3.16)

$$\begin{aligned}\tilde{\Phi}_I(s) - I &+ \int_{t_0}^{s} \tilde{\Phi}_I(\tau)X_x(\hat{x}(\tau), \tau)d\tau \\ &= r_1(s) + \int_{t_0}^{s} \tilde{\Phi}_I(\tau)(X_x(\hat{x}(\tau), \tau) - A(\tau))d\tau,\end{aligned} \tag{3.25}$$

it follows that

$$\begin{aligned}\varphi(t, s) &= \hat{Z}(t) - \tilde{\Phi}(t) - \tilde{\Phi}(t)r_1(s) \\ &- \int_{t_0}^{s} \{\varphi(t, \tau)X_x(\hat{x}(\tau), \tau) + \tilde{\Phi}(t)\tilde{\Phi}_I(\tau)(X_x(\hat{x}(\tau), \tau) - A(\tau))\}d\tau.\end{aligned} \tag{3.26}$$

From (3.20)–(3.22), it follows that

$$(3.27)\quad \|\varphi(t, s)\| \leqq \|\hat{Z}-\tilde{\Phi}\|_c + \|\tilde{\Phi}\|_c\|r_1\|_c + \left|\int_{t_0}^{s} \{\|\varphi(t, \tau)\|\, \|X_x(\hat{x}(\tau), \tau)\|_c + \|\Phi\|_c\|\Phi_I\|_c\|X_x(\hat{x}(\tau), \tau)-A(\tau)\|_c\}d\tau\right|$$

$$\leqq \beta_2 + \left|\int_{t_0}^{t} (\|\varphi(t, \tau)\|\mu_3 + \beta_3)d\tau\right| .$$

By Gronwall's inequality, we have

$$(3.28)\qquad \|\varphi(t, s)\| \leqq \beta_2 \exp(\rho\mu_3) + \beta_3(\exp(\rho\mu_3)-1)/\mu_3.$$

Since

$$E(\hat{Z}, \hat{Z}^{-1})h - \tilde{E}h = \int_{t_0}^{t} \varphi(t, s)h(s)ds \qquad \text{for} \quad h \in C[J],$$

by (3.28) and (3.19) we have (3.18).

REMARK. Let μ_5 and μ_1 be non-negative constants respectively such that

$$\|X_{xx}(x(t), t)-X_{xx}(x^{(0)}(t), t)\|_c \leqq \mu_5 \qquad \text{for all} \quad x \in D_v,$$
$$(\|X_{xx}(x^{(0)}(t), t)\|_c + \mu_5)v + \|X_x(x^{(0)}(t), t) - A(t)\|_c \leqq \mu_1.$$

Then $\|X_x(\hat{x}(t), t)-A(t)\|_c$ and $\|X_x(\hat{x}(t), t)\|_c$ can be evaluated by

$$\|X_x(\hat{x}(t), t)-A(t)\|_c \leqq \mu_1, \quad \|X_x(\hat{x}(t), t)\|_c \leqq \|A\|_c + \mu_1.$$

Furthermore by (3.11), we have

$$\|\hat{Z}-\tilde{\Phi}\|_c \leqq \sigma + \|Z^{(0)}-\tilde{\Phi}\|_c.$$

Let β be a positive constant such that

$$(3.29)\qquad \mu_{20}+\|\tilde{E}Y\|_c\mu_{21} + (\|l_1\|+\mu_{21})\{(\|\tilde{E}\|_c+\beta_1)\mu_4 + \|Y\|_c\beta_1\} \leqq \beta,$$

where β_1, μ_{20}, μ_{21} and μ_4 are non-negative constants such that

$$(3.30)\qquad \|E(\hat{Z}, \hat{Z}^{-1})-\tilde{E}\|_c \leqq \beta_1,$$

$$(3.31)\qquad \|f_x(u)-l_0\| \leqq \mu_{20} \qquad \text{for all} \quad u \in D_v \times U_\sigma,$$

$$(3.32)\qquad \|f_z(u)-l_1\| \leqq \mu_{21} \qquad \text{for all} \quad x \in D_v \times U_\sigma,$$

$$(3.33)\qquad \|X_2(u)-Y\|_c \leqq \mu_4 \qquad \text{for all} \quad u \in D_v \times U_\sigma.$$

Then, for the local minimality, we have the following

THEOREM 2. *Suppose* (3.11) *holds for the approximate solution* $u^{(0)}=$

$(x^{(0)}, Z^{(0)}) \in D \times U$ of (3.3) and that there exists a constant α such that

$$e^*\tilde{l}_2 Z^{(0)} e \geqq \alpha \|e\|_* \|e\| \quad \textit{for all} \quad e \in R^n, \tag{3.34}$$

$$\alpha_1 = \|\tilde{l}_2\|\sigma + \beta(\|Z^{(0)}\|_c + \sigma) < \alpha. \tag{3.35}$$

Then $\hat{x}$ *is a solution of* (2.1) *which minimizes* (2.2) *locally.*

PROOF. By (3.32), (3.30) and (3.33) we have

$$\begin{aligned} &\|f_z(\hat{u})E(\hat{Z}, \hat{Z}^{-1})X_2(\hat{u}) - l_1\tilde{E}Y\| \\ &\leqq \|f_z(\hat{u}) - l_1\| \, \|\tilde{E}Y\|_c + \|f_z(\hat{u})\|\{\|E(\hat{Z}, \hat{Z}^{-1})\|_c\|X_2(\hat{u}) - Y\|_c \\ &\quad + \|E(\hat{Z}, \hat{Z}^{-1}) - \tilde{E}\|_c\|Y\|_c\} \\ &\leqq \|\tilde{E}Y\|_c\mu_{21} + (\|l_1\| + \mu_{21})\{(\|\tilde{E}\|_c + \beta_1)\mu_4 + \|Y\|_c\beta_1\}, \end{aligned} \tag{3.36}$$

and by (3.31), (3.36) and (3.29) we see that

$$\begin{aligned} \|f_2(\hat{u}, \hat{Z}, \hat{Z}^{-1}) - \tilde{l}_2\| &\leqq \|f_x(\hat{u}) - l_0\| + \|f_z(\hat{u})E(\hat{Z}, \hat{Z}^{-1})X_2(\hat{u}) - l_1\tilde{E}Y\| \\ &\leqq \beta. \end{aligned} \tag{3.37}$$

Furthermore by (3.11) and (3.37) it follows that

$$\begin{aligned} \|f_2(\hat{u}, \hat{Z}, \hat{Z}^{-1})\hat{Z} - \tilde{l}_2 Z^{(0)}\| &= \|\tilde{l}_2(\hat{Z} - Z^{(0)}) + (f_2(\hat{u}, \hat{Z}, \hat{Z}^{-1}) - \tilde{l}_2)\hat{Z}\| \\ &\leqq \|\tilde{l}_2\|\sigma + \beta\|\hat{Z}\|_c \leqq \alpha_1. \end{aligned} \tag{3.38}$$

By (3.10) we see that

$$\begin{aligned} s''(\hat{c})ee &= e^* f_2(\hat{u}, \hat{Z}, \hat{Z}^{-1})\hat{Z}e \\ &= e^*\tilde{l}_2 Z^{(0)} e + e^*(f_2(\hat{u}, \hat{Z}, \hat{Z}^{-1})\hat{Z} - \tilde{l}_2 Z^{(0)})e \qquad \text{for all } e \in R^n. \end{aligned} \tag{3.39}$$

By (3.38), (3.34), (3.35) and Lemma 1 we have

$$s''(\hat{c})ee > 0 \qquad \text{for all} \quad e \in R^n \ (e \neq 0)$$

and the conclusion of the theorem follows from Theorem 1.

For any real symmetric matrix B, we denote by $\lambda_{\min}(B)$ the least eigenvalue of B. By Corollary 1 to Lemma 1 we have the following

COROLLARY 1. *Suppose the assumptions of Theorem* 2 *hold with* $\|\cdot\|$ *and* (3.34) *replaced by* $\|\cdot\|_2$ *and*

$$\lambda_{\min}(\tilde{l}_2 Z^{(0)} + (\tilde{l}_2 Z^{(0)})^*)/2 \geqq \alpha \tag{3.40}$$

respectively. Then the conclusion of Theorem 2 *is valid.*

By Corollary 2 to Lemma 1, we have the following

COROLLARY 2. Let $l_0=f_x(u^{(0)})$, $l_1=f_z(u^{(0)})$ and $Y=X_2(u^{(0)})$, *and suppose the assumptions of Theorem* 2 *hold with* (3.34) *replaced by*

$$\lambda_{\min}(\tilde{l}_2 Z^{(0)}) \geqq \alpha. \tag{3.41}$$

Then the conclusion of Theorem 2 *is valid.*

In particular we consider the case $X(x, t)=A(t)x$ and $g[x]=\xi[x]-d$, where $A(t)\in M[J]$, $\xi\in L(C[J], R^m)$ and d is a constant m-vector. In this case the theorem yields the following

COROLLARY 3. Let $l_0[\cdot]=(\xi[Z^{(0)}])^*\xi[\cdot]$ *and suppose the following inequality holds*:

$$\lambda_{\min}(l_0[Z^{(0)}]) > \|l_0\|\sigma + (\|Z^{(0)}\|c+\sigma)\mu_{20}.$$

Then $\hat{x}$ *is a solution of* (2.1) *which minimizes* (2.2) *in* D.

4. A posteriori error bounds of $u^{(0)}=(x^{(0)}, Z^{(0)})$

Let $C^1[J]$ be the space of all real n-vector functions continuously differentiable on J with the norm $\|\cdot\|_c$ and denote by $M^1[J]$ the space of all real $n\times n$ matrix functions continuously differentiable on J. Let $W^1[J]=C^1[J]\times M^1[J]$ be the space with the norm

$$\|w\|_w = \max(p^{-1}\|h\|_c, q^{-1}\|V\|_c) \qquad \text{for} \quad w=(h, V)\in W^1[J]$$

and put $D^1=(D\times U)\cap W^1[J]$, where p and q are suitable positive numbers. Let $B=C[J]\times M[J]\times R^n\times M^n$ be the Banach space with the norm

$$\|\varphi\|_b = \max(\|r\|_c, \|P\|_c, \|d\|, \|e\|) \qquad \text{for} \quad \varphi=(r, P, d, e)\in B,$$

where M^n is the space of $n\times n$ real matrices.

Let us define $F\colon D^1\to B$ by

$$Fu = \left(\frac{dx}{dt} - X(x, t),\ \frac{dZ}{dt} - X_x(x, t)Z, f[u],\ Z(t_0)-I\right) \qquad \text{for} \quad u=(x, Z)\in D^1. \tag{4.1}$$

Then the problem (3.3) is equivalent to that of finding the solution $u\in D^1$ of

$$Fu = 0. \tag{4.2}$$

Let A, $\tilde{\Phi}$, $\tilde{\Phi}_I$, Y, l_0, l_1, $\tilde{l}_2$ and $\tilde{E}$ be the matrices and the operators defined in Section 3 and put

(4.3) $$\tilde{G} = \tilde{l}_2[\tilde{\Phi}].$$

When det $\tilde{G} \neq 0$, we define the operators $\tilde{S}_0$, $\tilde{S}_2$, $\tilde{S}_4$ and $\tilde{H}_0$ by

(4.4) $$\tilde{S}_0 = \tilde{\Phi}\tilde{G}^{-1},\ \tilde{S}_2 = \tilde{S}_0 l_1,\ \tilde{S}_4 = I - \tilde{S}_0 \tilde{l}_2,\ \tilde{H}_0 = \tilde{S}_4 \tilde{E},$$

respectively. For any $\varphi=(r, P, d, e) \in B$, let $\tilde{L}_I \in L(B, W^1[J])$ be the operator defined by

(4.5) $$\tilde{L}_I \varphi = w,$$

where $w=(h, V) \in W^1[J]$,

$$h = \tilde{H}_0 r - \tilde{S}_2 \tilde{E} P + \tilde{S}_0 d - \tilde{S}_2 \tilde{\Phi} e, \quad V = \tilde{E} Y h + \tilde{E} P + \tilde{\Phi} e.$$

Let $R_1, R_2 \in L(C[J])$ and the linear operator l: $W^1[J] \to R^n$ be defined as follows:

(4.6) $$R_1 h = \tilde{\Phi} \int_{t_0}^{t} (\tilde{\Phi}_I'(s) + \tilde{\Phi}_I(s) A(s)) h(s) ds + (I - \tilde{\Phi}(t) \tilde{\Phi}_I(t)) h(t),$$

(4.7) $$R_2 h = R_1 h + \tilde{\Phi}(t)(\tilde{\Phi}_I(t_0) - I) h(t_0) \qquad \text{for} \quad h \in C[J],$$

(4.8) $$l[w] = l_0[h] + l_1[V] \qquad \text{for} \quad w = w(h, V) \in W^1[J].$$

For any $w=(h, V) \in W^1[J]$ and $u \in D^1$, let L: $W^1[J] \to B$ and $\tilde{K}$, $\tilde{K}_1$, $\tilde{K}_2$: $D^1 \to W^1[J]$ be the operators defined by

(4.9) $$Lw = \left(\frac{dh}{dt} - A(t)h, \frac{dV}{dt} - A(t)V - Yh, l[w], V(t_0) \right),$$

(4.10) $$\tilde{K}u = u - \tilde{L}_I F u, \quad \tilde{K}_1 u = \tilde{L}_I (Lu - Fu), \quad \tilde{K}_2 u = (I - \tilde{L}_I L) u$$

respectively. Then it holds that

(4.11) $$\tilde{K}u = \tilde{K}_1 u + \tilde{K}_2 u.$$

By (4.1), (4.9), (4.10) and (4.5) we have

(4.12) $$\tilde{K}_1 u = w_1,$$

where $u=(x, Z) \in D^1$, $w_1=(h_1, V_1) \in W^1[J]$,

$$h_1 = \tilde{H}_0(X(x, t) - A(t)x) - \tilde{S}_2 \tilde{E}(T_1(x)Z - Yx) + \tilde{S}_0(l[u] - f[u]) - \tilde{S}_2 \tilde{\Phi},$$

$$V_1 = \tilde{E} Y h_1 + \tilde{E}(T_1(x)Z - Yx) + \tilde{\Phi}, \quad T_1(x) = X_x(x, t) - A(t).$$

Since $\Phi_I' = -\Phi_I A(t)$, by (4.9), (4.10), (4.6) and (4.7), the integration by parts yields

(4.13) $$\tilde{K}_2 u = w_2,$$

where $u=(x, Z)\in D^1$, $w_2=(h_2 V_2)\in W^1[J]$,

$$h_2 = \tilde{S}_4 R_1 x - \tilde{S}_2 R_2 Z, \qquad V_2 = \tilde{E} Y h_2 + R_2 Z.$$

Now we show the following theorem which is an improvement of the results in the previous paper [6, Theorem 8].

THEOREM 3. *Let $u^{(0)}=(x^{(0)}, Z^{(0)})\in D^1$ be an approximate solution of* (4.2) *and suppose there exist an operator $\tilde{L}_I$, a positive constant δ and non-negative constants η, κ, κ_j $(j=0, 1, 2, 3)$ such that*

(i) *$\tilde{L}_I$ is invertible*;
(ii) $D_\delta^1=\{u\in W^1[J] \mid \|u-u^{(0)}\|_w\leqq\delta\}\subset D^1$;
(iii) $\kappa=\max(p^{-1}(\kappa_0+\kappa_2), q^{-1}(\kappa_1+\kappa_3))<1$,

$$p\|\tilde{H}_0\|_c\mu_1 + \|\tilde{S}_0\|\mu_2 + \|\tilde{S}_2\tilde{E}\|_c(q\mu_1+p\mu_4) \leqq \kappa_0, \tag{4.14}$$

$$\|\tilde{E}Y\|_c\kappa_0 + \|\tilde{E}\|_c(q\mu_1+p\mu_4) \leqq \kappa_1, \tag{4.15}$$

$$p\|\tilde{S}_4R_1\|_c + q\|\tilde{S}_2R_2\|_c \leqq \kappa_2, \tag{4.16}$$

$$\|\tilde{E}Y\|_c\kappa_2 + q\|R_2\|_c \leqq \kappa_3, \tag{4.17}$$

where μ_1, μ_2, μ_4 are constants such that

$$\|X_x(x(t), t)-A(t)\|_c \leqq \mu_1 \qquad \text{for all} \quad x\in D_\delta^1\cap D, \tag{4.18}$$

$$\|f'(u)-I\| \leqq \mu_2 \qquad \text{for all} \quad u\in D_\delta^1 \tag{4.19}$$

$$\|X_2(u)-Y\|_c \leqq \mu_4 \qquad \text{for all} \quad u\in D_\delta^1 \tag{4.20}$$

(iv) $\|\tilde{L}_I F u^{(0)}\|_w \leqq \eta$;
(v) $\lambda=\eta/(1-\kappa)\leqq\delta$.

Then the sequence $u^{(k)}$ defined by $u^{(k+1)}=u^{(k)}-\tilde{L}_I F u^{(k)}$ $(k=0, 1,\dots)$ converges to $\hat{u}\in D_\delta^1$ as $k\to\infty$. $\hat{u}$ is the unique solution of (4.2) *in D_δ^1, and*

$$\|\hat{u} - u^{(k)}\|_w \leqq \kappa^k\lambda \qquad (k=0, 1,\dots). \tag{4.21}$$

The proof of this theorem is quite similar to that of [6, Theorem 8] and is omitted.

REMARK 1. A sufficient condition for (i) is given in [6, Lemma 12].

REMARK 2. When the error bound $\lambda(p, q)$ of $u^{(0)}$ can be obtained by applying Theorem 3, since

$$\|\hat{u}-u^{(0)}\|_w = \max(p^{-1}\|\hat{x}-x^{(0)}\|_c, q^{-1}\|\hat{Z}-Z^{(0)}\|_c),$$

we have estimates

(4.22) $$\|\hat{x}-x^{(0)}\|_c \leqq p\lambda(p, q),\ \|\hat{Z}-Z^{(0)}\|_c \leqq q\lambda(p, q).$$

Therefore we can evaluate ν and σ, the bounds of (3.11), as small as possible by choosing the parameters p and q suitably.

5. A numerical illustration

5.1. Chebyshev-series-approximations

In order to obtain an approximation to a solution of the boundary value problem (3.3), we consider finite Chebyshev series

(5.1) $$\begin{cases} x_N(t) = \frac{1}{2}a_0 + \sum_{k=1}^N a_k T_k(t), \\ Z_N(t) = \frac{1}{2}B_0 + \sum_{k=1}^N B_k T_k(t) \end{cases}$$

with undetermined coefficients $a_0, a_1,\dots, a_N$ and $B_0, B_1,\dots, B_N$, where $t \in [-1, 1]$ and $T_k(t)$ is the Chebyshev polynomial of degree k. For (5.1), corresponding to (3.3), we are concerned with the equation

(5.2) $$\begin{cases} \dfrac{dx_N(t)}{dt} = P_{N-1}X(x_N(t), t), \\ \dfrac{dZ_N(t)}{dt} = P_{N-1}(X_x(x_N(t), t)Z_N(t)), \quad Z_N(t_0) = I, \\ f[u_N] \equiv (g'(x_N)[Z_N])^* g[x_N] = 0, \quad u_N = (x_N, Z_N) \in D \times U, \end{cases}$$

where P_{N-1} is the operator which expresses the truncation of a Chebyshev series of the operand by discarding the terms of the order higher than $N-1$. A finite Chebyshev series $u_N(t)$ satisfying (5.2) is called an N-th order Chebyshev-series-approximation to a solution of the given boundary value problem (3.3). For the details of numerical methods refer to [10] and [5].

Throughout this section, coefficients of the Chebyshev series of a function $b(t)$ are called Chebyshev coefficients of $b(t)$ for simplicity.

5.2. A sample problem

Let us consider the differential equation

(5.3) $$\frac{d^2y}{d\tau^2} - 6\frac{dy}{d\tau} - 12y^2 = 0, \qquad 0 \leqq \tau \leqq 1.$$

By the transformation

$$t = 2\tau - 1,$$

the equation (5.3) can be reduced to the following

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 3y^2 = 0, \quad t \in J = [-1, 1].$$

Let $x_1 = y$ and $x_2 = dy/dt$. Then this is reduced to the system

$$\frac{dx}{dt} = X(x, t) \equiv \begin{pmatrix} x_2 \\ 3x_1^2 + 3x_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{5.4}$$

We consider the following least squares condition:

$$g[x] = (Qx(t_1) - d_1, Qx(t_2) - d_2, \ldots, Qx(t_m) - d_m)^*, \tag{5.5}$$

where $Q = (1, 0)$, $m = 11$ and $t_j = 0.2(j-1) - 1$ $(j = 1, 2, \ldots, m)$, that is, the functional $v[x]$ in (2.2) is given by

$$v[x] = (g[x])^* g[x] \equiv \textstyle\sum_{j=1}^{m} (x_1(t_j) - d_j)^2 \tag{5.6}$$

and d_j $(j = 1, 2, \ldots, m)$ are shown in Table 1.

Table 1.

j	1	2	3	4	5	6	7	8	9	10	11
d_j	0.83129	0.74012	0.66559	0.60372	0.55172	0.50764	0.47017	0.43862	0.41305	0.39476	0.38727

In the boundary value problem in Section 3, the functional $f[u]$ in (3.1) can be expressed as follows:

$$f[u] = (g'(x)Z)^* g[x] \equiv \textstyle\sum_{j=1}^{m} (QZ(t_j))^* (Qx(t_j) - d_j). \tag{5.7}$$

In this example we take $t_0 = 0$. Thus

$$\frac{dZ}{dt} = X_1[x]\, Z \equiv \begin{pmatrix} 0 & 1 \\ 6x_1 & 3 \end{pmatrix} Z \tag{5.8}$$

with $Z(t_0) = Z(0) = I$, where $X_1[x] = X_x(x(t), t)$.

Now let $u^{(0)} = (x^{(0)}, Z^{(0)})$ be the approximate solution of this problem obtained by numerical computation such that

$$x_i^{(0)}(t) = \frac{1}{2}\tilde{a}_{0i} + \textstyle\sum_{k=1}^{N} \tilde{a}_{ki} T_k(t) \qquad (i = 1, 2),$$

$$z_{ij}^{(0)}(t) = \frac{1}{2}\tilde{b}_{0ij} + \textstyle\sum_{k=1}^{N} \tilde{b}_{kij} T_k(t) \qquad (i, j = 1, 2),$$

where $N = 27$. Then the Chebyshev coefficients of $u^{(0)}$ are shown in Table 3.

5.3. Estimation of a posteriori error bounds

From now on, let the symbol $\|\cdot\|$ denote the Euclidean norm of vectors or the Frobenius norm of matrices. In applications of Theorem 3, we take $X_1[x^{(0)}]$ as $A(t)$ and $Z^{(0)}$ as $\tilde{\Phi}$.

For $u=(x, Z)\in D^1$ and $w=(h, V)\in W^1[J]$ we have

$$X_2(u)h = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 6z_{11} & 0 & 6z_{12} & 0 \end{array}\right) h,$$

$$f'(u) = f_x(u)h + f_z(u)V,$$

$$f_x(u)h = \textstyle\sum_{j=1}^{m} (QZ(t_j))^*(Qh(t_j)),$$

$$f_z(u)V = \textstyle\sum_{j=1}^{m} (QV(t_j))^*(Qx(t_j)-d_j).$$

We choose the operators Y, l, l_0 and l_1 as follows:

$$Y = X_2(u^{(0)}),\ l[w] = l_0[h] + l_1[V],$$

$$l_0 = f_x(u^{(0)}),\ l_1 = f_z(u^{(0)}).$$

For simplicity put

$$\tilde{\Phi} = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix},\quad \tilde{\Phi}_I = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix},\quad C(t) = \left(\tilde{\Phi}_I(t)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\tilde{\Phi}(t)\right)^*,$$

$$\sigma_1(t, s) = \begin{cases} 1 & t\geqq s, \\ 0 & t<s, \end{cases}\qquad \sigma_2(t, s) = \begin{cases} 1 & t\leqq s, \\ 0 & t>s. \end{cases}$$

Then we have

$$l_0[\tilde{\Phi}] = \textstyle\sum_{j=1}^{m} (Q\tilde{\Phi}(t_j))^*(Q\tilde{\Phi}(t_j)), \tag{5.9}$$

$$l_1\tilde{E}Yh = l_1[\tilde{\Phi}(t)\int_0^t \tilde{\Phi}_I(s)Y(s)h(s)ds],$$

$$l_1\tilde{E}Y[\tilde{\Phi}] = \textstyle\sum_{j=1}^{m} 6\int_0^{t_j} C(s)(\varphi_{11}(s)I,\ \varphi_{12}(s)I)ds(Q\tilde{\Phi}(t_j))^*(x_1^{(0)}(t_j)-d_j). \tag{5.10}$$

Hence by (3.15), (4.3), (5.9) and (5.10) we can obtain $\tilde{G}$. Suppose that $\det \tilde{G}\neq 0$. Then since

$$\tilde{S}_0 = \tilde{\Phi}\tilde{G}^{-1}, \tag{5.11}$$

$$\tilde{S}_2\tilde{E}V = \tilde{S}_0 l_1\tilde{E}V = \tilde{S}_0(t) \textstyle\sum_{j=1}^{m} (Q\tilde{\Phi}(t_j)\int_0^{t_j} \tilde{\Phi}_I(s)V(s)ds)^*(x_1^{(0)}(t_j)-d_j), \tag{5.12}$$

it follows that

$$\|\tilde{S}_2\tilde{E}\|_c \leqq \|\tilde{S}_0\|_c \left\{\sum_{j=1}^m t_j \int_0^{t_j} (b_{1j}(s)^2 + b_{2j}(s)^2)ds\right\}^{1/2}, \tag{5.13}$$

where

$$a_{1j} = \varphi_{11}(t_j)(x_1^{(0)}(t_j) - d_j), \quad a_{2j} = \varphi_{12}(t_j)(x_1^{(0)}(t_j) - d_j),$$

$$b_{1j}(s) = a_{1j}\psi_{11}(s) + a_{2j}\psi_{21}(s), \quad b_{2j}(s) = a_{1j}\psi_{12}(s) + a_{2j}\psi_{22}(s)$$

$$(j=1, 2,\ldots, m).$$

It also follows that

$$\|\tilde{E}\|_c \leqq \max_{t\in J}\left(t\int_0^t \|\tilde{\Phi}(t)\tilde{\Phi}_I(s)\|^2 ds\right)^{1/2}, \tag{5.14}$$

$$\|\tilde{E}Y\|_c \leqq 6\max_{t\in J}\left(t\int_0^t \|\tilde{\Phi}(t)(C(s))^*\|^2 ds\right)^{1/2}. \tag{5.15}$$

By (4.4) we see that

$$\tilde{H}_0 = \tilde{E} - \tilde{S}_0(l_0\tilde{E} + l_1\tilde{E}Y\tilde{E}).$$

Since $t_{12-j} = -t_j$ $(j=7, 8,\ldots, m)$, by some manipulations we have

$$\tilde{H}_0 h = \int_0^1 H_{01}(t, s)h(s)ds + \int_{-1}^0 H_{02}(t, s)h(s)ds \qquad \text{for} \quad h \in C[J], \tag{5.16}$$

where for $t_{k-1} < s \leqq t_k$ $(k=7, 8,\ldots, m)$

$$\begin{aligned} H_{01}(t, s) = \tilde{\Phi}(t)[\sigma_1(t, s)I - \tilde{G}^{-1}\{&\textstyle\sum_{j=1}^m (Q\tilde{\Phi}(t_j))^*(Q\tilde{\Phi}(t_j)) \\ &+ \textstyle\sum_{i=k}^m 6\displaystyle\int_{t_{i-1}}^{t_i} C(\tau)(\textstyle\sum_{j=i}^m (Q\tilde{\Phi}(t_j))^*(x_1^{(0)}(t_j) - d_j))Q\tilde{\Phi}(\tau)d\tau \\ &- 6\displaystyle\int_{t_{k-1}}^{s} C(\tau)(\textstyle\sum_{j=k}^m (Q\tilde{\Phi}(t_j))^*(x_1^{(0)}(t_j) - d_j))Q\tilde{\Phi}(\tau)d\tau\}]\tilde{\Phi}_I(s), \end{aligned}$$

and for $-t_k \leqq s < -t_{k-1}$ $(k=7, 8,\ldots, m)$

$$\begin{aligned} H_{02}(t, s) = -\tilde{\Phi}(t)[\sigma_2(t, s)I + \tilde{G}^{-1}\{&\textstyle\sum_{j=1}^m (Q\tilde{\Phi}(-t_j))^*(Q\tilde{\Phi}(-t_j)) \\ + \textstyle\sum_{i=k}^m 6\displaystyle\int_{-t_{i-1}}^{-t_i} &C(\tau)(\textstyle\sum_{j=i}^m (Q\tilde{\Phi}(-t_j))^*(x_1^{(0)}(-t_j) - d_{12-j}))Q\tilde{\Phi}(\tau)d\tau \\ - 6\displaystyle\int_{-t_{k-1}}^{s} &C(\tau)(\textstyle\sum_{j=k}^m (Q\tilde{\Phi}(-t_j))^*(x_1^{(0)}(-t_j) - d_{12-j}))Q\tilde{\Phi}(\tau)d\tau\}]\tilde{\Phi}_I(s). \end{aligned}$$

Hence it follows that

$$\begin{aligned} \|\tilde{H}_0\|_c \leqq \max_{t\in J}&\left(\int_0^1 \|H_{01}(t, s)\|^2 ds\right)^{1/2} \\ &+ \max_{t\in J}\left(\int_{-1}^0 \|H_{02}(t, s)\|^2 ds\right)^{1/2}. \end{aligned} \tag{5.17}$$

As is readily seen, it follows that

(5.18) $\|X_1[x]-A(t)\|_c \leqq 6\|x-x^{(0)}\|_c \leqq 6p\delta = \mu_1,$

(5.19) $\|f_x(u)-l_0\| \leqq 11\|Z-Z^{(0)}\|_c \leqq 11q\delta = \mu_{20},$

(5.20) $\|f_z(u)-l_1\| \leqq 11\|x-x^{(0)}\|_c \leqq 11p\delta = \mu 21,$

(5.21) $p\mu_{20} + q\mu_{21} = 22pq\delta = \mu_2,$

(5.22) $\|X_2(u)-Y\|_c \leqq 6\|Z-Z^{(0)}\|_c \leqq 6q\delta = \mu_4.$

In (4.5) we take $\varphi = Fu^{(0)}$. Then we see that

$$r = \frac{dx^{(0)}(t)}{dt} - X(x^{(0)}(t), t),\ \ P = \frac{dZ^{(0)}(t)}{dt} - X_1[x^{(0)}]Z^{(0)}(t),$$

$$d = f[u^{(0)}],\ e = Z^{(0)}(t_0) - I.$$

Thus by (5.9)–(5.17) we have

(5.23) $$\tilde{G} = \begin{pmatrix} 88.110421\cdots & 85.460166\cdots \\ 85.460166\cdots & 105.72547\cdots \end{pmatrix},$$

(5.24) $$\det \tilde{G} = 2012.0\cdots \neq 0,$$

$\|\tilde{H}_0\|_c = 10.889,\ \ \|\tilde{S}_0\|_c = 0.43829,\ \ \|\tilde{S}_2\tilde{E}\|_c = 7.1237 \times 10^{-6},$

$\|\tilde{E}\|_c = 15.719,\ \ \|\tilde{E}Y\|_c = 353.06,\ \ \|\tilde{S}_0\|_c\|d\| = 1.1437 \times 10^{-14},$

$\|\tilde{S}_2\tilde{\Phi}e\|_c = 4.1676 \times 10^{-29},\ \ \|r\|_c = 3.7950 \times 10^{-13},\ \ \|P\|_c = 1.1686 \times 10^{-12},$

$\|\tilde{\Phi}e\|_c = 3.3179 \times 10^{-13},\ \ \|\tilde{S}_4\|_c = 15.342,\ \ \|\tilde{S}_2\|_c = 0.065150,$

$\|R_1\|_c = 1.3771 \times 10^{-7},\ \ \|R_2\|_c = 1.3771 \times 10^{-7}.$

Let η_0 and η_1 be the quantities such that

(5.25) $$\begin{cases} \eta_0 = \|\tilde{H}_0\|_c\|r\|_c + \|\tilde{S}_2\tilde{E}\|_c + \|\tilde{S}_0\|_c\|d_0\| + \|\tilde{S}_2\tilde{\Phi}e\|_c, \\ \eta_1 = \|\tilde{E}Y\|_c\eta_0 + \|P\|_c + \|\tilde{\Phi}e\|_c, \end{cases}$$

respectively.
Then for $\tilde{L}_I F^{(0)} = (h, V)$, by (4.5) we have

$$\|h\|_c \leqq \eta_0,\ \|V\|_c \leqq \eta_1.$$

In this case, we obtain

(5.26) $$\eta_0 = 4.1438 \times 10^{-12},\ \ \eta_1 = 1.4817 \times 10^{-9}.$$

If we put $\eta = \max(p^{-1}\eta_0, q^{-1}\eta_1)$, then we have

(5.27) $$\|\tilde{L}_I F u^{(0)}\|_w \leqq \eta.$$

Now we apply Theorem 3 to this problem summarize the results in Table 2. From Table 2 we have the error estimates

(5.28) $$\begin{cases} \|\hat{x}-x^{(0)}\|_c \leqq \lambda(1.0,372.0) = 4.1590 \times 10^{-12}, \\ \|\hat{Z}-Z^{(0)}\|_c \leqq \lambda(0.0028,1.0) = 1.4817 \times 10^{-9}. \end{cases}$$

REMARK. When we choose $p=1$, $q=1$ and $\delta=10^{-4}$, form Table 2 we have $\kappa>1$. Therefore it is impossible to obtain the error bounds.

5.4. A local minimum

For the quantities which are necessary for applying Theorem 2, we have the following values:

$$\|\tilde{\Phi}\|_c = 43.480, \quad \|\tilde{\Phi}_I\|_c = 63.376, \ \|r_1\|_c = 1.1280 \times 10^{-11},$$
$$\|\tilde{E}\|_c = 15.719, \quad \|Y\|_c = 69.908, \quad \|\tilde{E}Y\|_c = 353.06, \quad \|l_0\| = 13.923,$$
$$\|l_1\| = 1.4865 \times 10^{-5}, \ \|\tilde{l}_2\| = 13.928, \quad \|A\|_c = 5.9058,$$
$$\alpha = \lambda_{\min}(\tilde{G}) = 11.005.$$

In (3.11) we choose ν and σ as follows:

(5.29) $$\nu = 10^{-5}, \ \sigma = 10^{-5}.$$

Then by Lemma 2 and (3.29)–(3.35), we have

$$\rho = 1.0, \quad \mu_1 = \mu_4 = 6.0 \times 10^{-5}, \quad \mu_{20} = \mu_{21} = 1.1 \times 10^{-4},$$
$$\mu_3 = 5.9059, \quad \beta_3 = 0.16534, \quad \beta_2 = 10^{-5}, \quad \beta_1 = 10.255,$$
$$\beta = 0.12846, \quad \alpha_1 = 5.5857.$$

Since $\alpha>\alpha_1$, by Corollary 2 to Theorem 2, the exact solution $\hat{x}$ in our example is an isolated one which minimizes (5.6) in D_ν.

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Table 2.

	$\delta=10^{-6}$			$\delta=10^{-4}$		
p	1.0	1.0	0.0028	1.0	1.0	0.0028
q	1.0	372.0	1.0	1.0	372.0	1.0
μ_1	6.0000×10^{-6}	6.0000×10^{-6}	1.6800×10^{-8}	6.0000×10^{-4}	6.0000×10^{-4}	1.6800×10^{-6}
μ_2	2.2000×10^{-5}	8.1840×10^{-3}	6.1600×10^{-8}	2.2000×10^{-3}	8.1840×10^{-1}	6.1600×10^{-6}
μ_4	6.0000×10^{-6}	2.2320×10^{-3}	6.0000×10^{-6}	6.0000×10^{-4}	2.2320×10^{-1}	6.0000×10^{-4}
κ_0	7.4976×10^{-5}	3.6523×10^{-3}	2.7511×10^{-8}	7.4976×10^{-3}	3.6523×10^{-1}	2.7511×10^{-6}
κ_1	2.6660×10^{-2}	1.3597×10^{0}	1.0241×10^{-5}	2.6660×10^{0}	1.3597×10^{2}	1.0241×10^{-3}
κ_2	2.1217×10^{-6}	5.4503×10^{-5}	1.4887×10^{-8}	2.1217×10^{-6}	5.4503×10^{-6}	1.4887×10^{-8}
κ_3	7.4923×10^{-4}	1.9755×10^{-3}	5.3939×10^{-6}	7.4923×10^{-4}	1.9755×10^{-3}	5.3939×10^{-6}
κ	2.7409×10^{-2}	3.6603×10^{-3}	1.5635×10^{-5}	2.6667×10^{0}	3.6551×10^{-1}	1.0295×10^{-3}
η	1.4817×10^{-9}	4.1438×10^{-12}	1.4817×10^{-9}	1.4817×10^{-9}	4.1438×10^{-12}	1.4817×10^{-9}
$\lambda(p, q)$	1.5235×10^{-9}	4.1590×10^{-12}	1.4817×10^{-9}	Undetermined	6.5309×10^{-12}	1.4832×10^{-9}

Table 3.
The Chebyshev coefficients of $u^{(0)}=(x^{(0)}, Z^{(0)})$

n	$\tilde{a}_{n1}$	$\tilde{a}_{n2}$
0	1.111398193054442	−0.460598568498370
1	−0.217631518410970	0.226760729208123
2	0.050660320523359	−0.025335531676430
3	−0.004595698555427	0.024119447114688
4	0.002754838066751	0.002238659656131
5	0.000190936637394	0.002080742580677
6	0.000163023562428	0.000329293282189
7	0.000022137520758	0.000124459831539
8	0.000007443581263	0.000019367991581
9	0.000001030733805	0.000005362531333
10	0.000000257699672	0.000000814783088
11	0.000000035427113	0.000000208537885
12	0.000000008311300	0.000000035386592
13	0.000000001294288	0.000000009066686
14	0.000000000308834	0.000000001735100
15	0.000000000055105	0.000000000419346
16	0.000000000012537	0.000000000081955
17	0.000000000002308	0.000000000018154
18	0.000000000000484	0.000000000003472
19	0.000000000000088	0.000000000000723
20	0.000000000000017	0.000000000000137
21	0.000000000000003	0.000000000000028
22	0.000000000000001	0.000000000000005

Table 3. (Continued)

23	0.000000000000000	0.000000000000001
24	0.000000000000000	0.000000000000000
25	0.000000000000000	0.000000000000000
26	−0.000000000000000	0.000000000000000
27	0.000000000000000	0.000000000000000

n	$\tilde{b}_{n11}$	$\tilde{b}_{n12}$
0	5.042456200446825	3.434548420592477
1	1.775770970216703	3.581548722704508
2	1.817191541478525	2.095483522794581
3	0.744168521784874	1.068491700814619
4	0.323585582416798	0.414304904052059
5	0.100533169018054	0.137165769344503
6	0.029124140848576	0.038068665990039
7	0.006947404231829	0.009361035436103
8	0.001562927105899	0.002053275820570
9	0.000314592693841	0.000422220416536
10	0.000063437214493	0.000083514625184
11	0.000012513040227	0.000016764952245
12	0.000002627589324	0.000003469191064
13	0.000000559146049	0.000000747607217
14	0.000000123256041	0.000000163192653
15	0.000000026609776	0.000000035504631
16	0.000000005691868	0.000000007547266
17	0.000000001175208	0.000000001565815
18	0.000000000238657	0.000000000316670
19	0.000000000047316	0.000000000062991
20	0.000000000009337	0.000000000012395
21	0.000000000001829	0.000000000002434
22	0.000000000000360	0.000000000000479
23	0.000000000000071	0.000000000000095
24	0.000000000000014	0.000000000000019
25	0.000000000000003	0.000000000000004
26	0.000000000000001	0.000000000000001
27	0.000000000000000	0.000000000000000

n	$\tilde{b}_{n21}$	$\tilde{b}_{n22}$
0	9.125101753868796	15.084749194958971
1	10.233282797850009	12.187808104518284
2	5.573559813435393	7.921651749549957

Table 3. (Continued)

3	2.964516631935914	3.805874013339966
4	1.108548682726151	1.510701544662247
5	0.375831972601526	0.491434780923489
6	0.103216992545608	0.139043851217208
7	0.026342282418620	0.034610789043020
8	0.005953333299996	0.007989355111768
9	0.001335448724235	0.001758375913897
10	0.000290664810865	0.000389387614119
11	0.000066704434377	0.000088083410223
12	0.000015377925877	0.000020558664724
13	0.000003642290604	0.000004822824687
14	0.000000840128613	0.000001120877079
15	0.000000191121470	0.000000253430404
16	0.000000041835338	0.000000055738159
17	0.000000008981692	0.000000011917900
18	0.000000001878256	0.000000002500454
19	0.000000000390031	0.000000000517784
20	0.000000000080245	0.000000000106784
21	0.000000000016565	0.000000000022000
22	0.000000000003413	0.000000000004541
23	0.000000000000705	0.000000000000937
24	0.000000000000145	0.000000000000193
25	0.000000000000030	0.000000000000039
26	0.000000000000006	0.000000000000008
27	0.000000000000001	0.000000000000002

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