HIROSHIMA MATH. J.
15 (1985), 173–220

Positive solutions of linear and quasilinear elliptic equations in unbounded domains

Yasuhiro FURUSHO

(Received September 5, 1984)

1. Introduction

Let Ω be an exterior domain in $\boldsymbol{R}^N$, $N \geqq 2$, with smooth boundary $\Gamma = \partial\Omega$ and let $\mathfrak{D}$ and B denote, respectively, an elliptic differential operator and a boundary operator defined by

$$\mathfrak{D} = \sum_{i,j=1}^N a_{ij}(x)\partial^2/\partial x_i \partial x_j + \sum_{i=1}^N b_i(x)\partial/\partial x_i, \quad x \in \Omega, \tag{1.1}$$

and

$$B = \alpha(x)\partial/\partial\beta + (1-\alpha(x))\cdot, \quad x \in \Gamma, \tag{1.2}$$

where $\partial/\partial\beta$ is the directional derivative in the direction of a vector β prescribed on Γ. We are concerned with the following linear and quasilinear boundary value problems:

$$\text{(A)} \qquad -\mathfrak{D}u + c(x)u = \lambda m(x)u \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \Gamma,$$

$$\text{(B)} \qquad -\mathfrak{D}u + c(x)u = \lambda m(x)u^\gamma \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \Gamma,$$

where $c(x)$ and $m(x)$ are given functions, λ is a real parameter and γ is a nonzero constant with $\gamma \neq 1$. We allow Γ to be empty, in which case Ω is the entire space $\boldsymbol{R}^N$ and the boundary condition in (A) or (B) is void.

The objective of this paper is twofold. First, we study the existence and asymptotic behavior of positive functions h which satisfy the differential inequality

$$-\mathfrak{D}h + c(x)h \geqq \lambda m(x)h \quad \text{in } \Omega \tag{1.3}$$

and have minimal order of growth at infinity. Such an h is called a minimal λ-superharmonic function, and the totality of λ-superharmonic functions is denoted by $SH(\lambda)$. An analysis of some particular cases of (1.3) ([10]) shows that the asymptotic behavior of λ-superharmonic functions is in general very complicated. So, we restrict our attention to the situations in which (i) all h in $SH(\lambda)$ converge to zero as $|x| \to \infty$; (ii) all h in $SH(\lambda)$ are bounded both above and below by positive constants; (iii) all h in $SH(\lambda)$ tend to infinity as $|x| \to \infty$, and attempt to obtain conditions for such situations to occur. For this purpose a crucial role is played by the concept and basic properties of the *principal eigenvalue* of the problem (A) which are given in Section 2. Explicit sufficient conditions ensuring that the above cases (i)–(iii) actually hold are developed in Section 3 with the use of results on the existence and asymptotic behavior of positive solutions for second order linear ordinary differential equations, and the recurrence property of the diffusion process with the infinitesimal generator $\mathfrak{D}$. The results in Sections 2 and 3 extend considerably those obtained in [9, 10].

Secondly, we investigate the existence and asymptotic behavior of positive solutions of (A) and (B). We were motivated by the observation that although there is much current interest in positive solutions of semilinear elliptic equations in unbounded domains (see e.g. [8, 9, 18, 19, 25, 28, 32, 33, 36]), most of the literature has been devoted to equations of the form $-\Delta u+c(x)u=\lambda m(x)u^{\gamma}$ and very little is known about general equations of the form (B). We establish existence theorems for (A) and (B) in Sections 4 and 6, respectively; more specifically, we find sufficient conditions under which (A) and (B) possess positive bounded solutions, or positive unbounded solutions with specified order of growth at infinity. The main tool is a generalization of the standard supersolution-subsolution method (Lemma 4.2), which asserts that the existence of a "generalized" supersolution $\hat{u}(x)$ and a "generalized" subsolution $\bar{u}(x)$ of (A) or (B) such that $\bar{u}(x)\leqq\hat{u}(x)$ in Ω implies the existence of a solution $u(x)$ of (A) or (B) satisfying $\bar{u}(x)\leqq u(x)\leqq\hat{u}(x)$ in Ω. In each of the theorems in Sections 4 and 6 suitable "generalized" supersolutions and subsolutions are constructed explicitly with the aid of existence and asymptotic theory of second order ordinary differential equations. Some of the recent results in [18, Theorems 2.3, 2.6], [19, Theorem 1] and [36, Theorem 4.3] are covered by our theory.

In addition it can be shown that the supersolution-subsolution method combined with the results of Section 3 yields various existence theorems for positive solutions of semilinear equations of the form

$$\text{(C)} \qquad -\mathfrak{D}u + c(x)u = \lambda m(x)f(u)$$

defined in some neighborhood of infinity. These byproducts are presented in Section 5.

2. Principal eigenvalues

2.1. Principal eigenvalues

Throughout this paper we assume that Ω is either an exterior domain in $\boldsymbol{R}^N$ of a simply connected bounded domain Ω_0 with boundary $\partial\Omega=\Gamma$ of class $C^{2+\sigma}$, $0<\sigma\leqq 1$, or $\Omega=\boldsymbol{R}^N$, and the following conditions hold for the operators $\mathfrak{D}$ and B defined by (1.1) and (1.2).

(A$_1$) $a_{ij} = a_{ji} \in C^{1+\sigma}_{loc}(\boldsymbol{R}^N)$, $b_i \in C^{\sigma}_{loc}(\boldsymbol{R}^N)$, $i, j = 1, 2,\dots, N$, c, $m \in C^{\sigma}_{loc}(\bar{\Omega})$, $m(x) > 0$, $x \in \bar{\Omega}$.

(A$_2$) $\mathfrak{D}$ is uniformly elliptic in $\boldsymbol{R}^N$, i.e., there exists a constant $\kappa > 0$ such that

$$\textstyle\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqq \kappa|\xi|^2 \qquad \text{for all} \quad x,\ \xi \in \boldsymbol{R}^N,$$

where $|\xi|$ denotes the Euclidean length of $\xi = (\xi_1, \xi_2,\dots, \xi_N)$.

(A$_3$) $\alpha \in C^{2+\sigma}(\Gamma)$, and either $\alpha(x) \equiv 0$ on Γ or $0 < \alpha(x) \leqq 1$ on Γ.

(A$_4$) $\beta = (\beta_1, \beta_2,\dots, \beta_N) \in C^{1+\sigma}(\Gamma)$ is a unit vector satisfying $\sum_{i=1}^{N} \beta_i n_i > 0$ on Γ for the outward (with respect to Ω) unit normal vector $n = (n_1, n_2,\dots, n_N)$ to Γ.

The following notation is employed:

$$B(x_0, R) = \{x\colon |x - x_0| < R\} \quad \text{for } x_0 \in \boldsymbol{R}^N,$$

$$G_R = \boldsymbol{R}^N \backslash \overline{B(0, R)} = \{x\colon |x| > R\} \quad \text{for } R > 0.$$

We now proceed to define the principal eigenvalue of the problem (A). Fix a nonnegative constant ρ such that $\Omega_0 \subset B(0, \rho)$ if $\Omega = \boldsymbol{R}^N \backslash \bar{\Omega}_0$ and $\rho = 0$ if $\Omega = \boldsymbol{R}^N$, and put

$$\Omega_k = \Omega \cap B(0, \rho + k), \quad \Gamma_k = \{x\colon |x| = \rho + k\}$$

for $k \geqq 1$. We first define the principal eigenvalue of the problem

$$\begin{aligned} &-\mathfrak{D}\varphi + c(x)\varphi = \lambda m(x)\varphi \quad \text{in } \Omega_k, \\ &B\varphi = 0 \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \phi), \quad \varphi = 0 \quad \text{on } \Gamma_k. \end{aligned} \tag{2.1}$$

Put $\zeta(x) = c(x)/m(x)$, $x \in \bar{\Omega}$. Let $\underline{\zeta}_k = \min_{x \in \bar{\Omega}_k} \zeta(x)$ and take a constant ζ_k such that $\zeta_k \leqq \underline{\zeta}_k$. We denote by τ_k the principal eigenvalue of the problem

$$\begin{aligned} &-\mathfrak{D}\varphi + (\zeta(x) - \zeta_k)m(x)\varphi = \tau m(x)\varphi \quad \text{in } \Omega_k, \\ &B\varphi = 0 \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset), \quad \varphi = 0 \quad \text{on } \Gamma_k. \end{aligned} \tag{2.2}$$

Since $\zeta(x) - \zeta_k \geqq 0$ in Ω_k, τ_k exists and is positive, and there is a unique normalized positive eigenfuction $\varphi_k \in C^{2+\sigma}(\bar{\Omega}_k)$ corresponding to τ_k (see e.g. [4, Theorem 4.3]). We then define the principal eigenvalue of the problem (2.1) to be the constant $\lambda_k = \zeta_k + \tau_k$, and denote the principal eigenfunction corresponding to λ_k by φ_k. The value of λ_k is independent of the choice of ζ_k. In fact, if we let $\underline{\tau}_k$ be the principal eigenvalue of (2.2) with ζ_k replaced by $\underline{\zeta}_k$, then $\underline{\tau}_k + \underline{\zeta}_k - \zeta_k$ is an eigenvalue of (2.2) with the same positive eigenfunction, so that by the uniqueness of the principal eigenvalue, we have $\underline{\tau}_k + \underline{\zeta}_k - \zeta_k = \tau_k$ and hence $\underline{\tau}_k + \underline{\zeta}_k = \tau_k + \zeta_k$. Letting $k = 1, 2,\dots$, we obtain the sequence $\{\lambda_k\}$ of principal eigenvalues of (2.1) for the bounded domains Ω_k.

We show that $\{\lambda_k\}$ is a strictly decreasing sequence. For this purpose we need the following Lemmas, in which we write $G=\Omega_k$, $\Lambda=\Gamma_k$ and $\lambda_0=\lambda_k$.

LEMMA 2.1. *Suppose that* $f\in C^{\sigma}(\bar{G})$, $g_1\in C^{l+\sigma}(\Gamma)$ *and* $g_2\in C^{2+\sigma}(\Lambda)$, *where* $l=2$ *if* $\alpha(x)\equiv 0$ *on* Γ *and* $l=1$ *if* $\alpha(x)>0$ *on* Γ, *respectively. Then, for every* $\lambda<\lambda_0$, *the problem*

$$-\mathfrak{D}u+(c(x)-\lambda m(x))u=f \quad in\ G,$$
$$Bu=g_1 \quad on\ \Gamma\ (if\ \Gamma\neq\emptyset), \quad u=g_2 \quad on\ \Lambda$$

has exactly one solution $u\in C^{2+\sigma}(\bar{G})$. *Furthermore, if* f, g_1 *and* g_2 *are non-negative and at least one of them is not identically zero, then* $u(x)>0$ *in* G.

LEMMA 2.2. *If there is a function* $w\in C^2(G)\cap C^1(\bar{G})$ *such that* $w(x)>0$ *in* G *and*

$$-\mathfrak{D}w+c(x)w\geqq\lambda m(x)w \quad in\ G,$$
$$Bw\geqq 0 \quad on\ \Gamma\ (if\ \Gamma\neq\emptyset), \quad w\geqq 0 \quad on\ \Lambda,$$

then $\lambda\leqq\lambda_0$.

These lemmas are proved by the same argument as in [4, Theorem 4.4] and [7, Lemma 3.4] (cf. [10, Lemma 3.1]), so the proof will be omitted.

From Lemma 2.2 it follows that $\lambda_{k+1}\leqq\lambda_k$. Suppose that $\lambda_{k+1}=\lambda_k$. Then, $u=\varphi_{k+1}\in C^{2+\sigma}(\bar{\Omega}_k)$ satisfies $u(x)>0$ in Ω_k and

$$\begin{aligned}&-\mathfrak{D}u+(c(x)-\lambda_k m(x))u=0 \quad \text{in } \Omega_k,\\ &Bu=0 \quad \text{on } \Gamma\ (\text{if } \Gamma\neq\emptyset), \quad u=\varphi_{k+1}>0 \quad \text{on } \Gamma_k.\end{aligned} \tag{2.3}$$

Let us now denote by S the solution operator of the problem

$$-\mathfrak{D}u+(\zeta(x)-\underline{\zeta}_k)m(x)u=f \quad \text{in } \Omega_k,$$
$$Bu=0 \quad \text{on } \Gamma\ (\text{if } \Gamma\neq\emptyset),\ u=0 \quad \text{on } \Gamma_k,$$

that is, $u=Sf$ is the unique solution of this problem provided $f\in C^{\sigma}(\bar{\Omega}_k)$. Then, S has a unique extension $\tilde{S}$, which is positive and maps $C(\bar{\Omega}_k)$ compactly into $C^{1+\sigma'}(\bar{\Omega}_k)$, $0<\sigma'<1$ (see e.g. [3, Lemma 5.3]), and the operator T defined by $Tf=\tilde{S}(m(x)f)$ is a positive linear operator on $C(\bar{\Omega}_k)$ with a positive eigenvalue $\underline{\tau}_k^{-1}$ and a positive eigenfunction φ_k (see e.g. the proof of [4, Theorem 4.3]). Let v be the solution of

$$-\mathfrak{D}v+(\zeta(x)-\underline{\zeta}_k)m(x)v=0 \quad \text{in } \Omega_k,$$
$$Bv=0 \quad \text{on } \Gamma\ (\text{if } \Gamma\neq\emptyset), \quad v=\varphi_{k+1} \quad \text{on } \Gamma_k$$

and put $w=\varphi_{k+1}-v$ in $\bar{\Omega}_k$. Then, w is positive by the maximum principle and satisfies the equation

$$\tau_k^{-1}w - Tw = Tv. \tag{2.4}$$

On the other hand, it can be shown ([3, Lemma 5.3]) that $T(Tv)(x) \geqq M\varphi_k(x)$, $x \in \bar{\Omega}_k$, for some constant $M>0$. Hence, by [23, Theorem 2.16], (2.4) has no positive solution. This contradiction implies that $\lambda_k=\lambda_{k+1}$ is impossible. Thus we have $\lambda_{k+1}<\lambda_k$.

Let λ^* denote the limit

$$\lambda^* = \lim_{k\to\infty} \lambda_k. \tag{2.5}$$

It may happen that $\lambda^*=-\infty$. If λ^* is finite, we call it the *principal eigenvalue* of the problem (A) (cf. [8, 9]).

REMARK 2.1. If $\liminf_{|x|\to\infty} \zeta(x)>-\infty$, then the principal eigenvalue λ^* of the problem (A) does exist. For, choose a constant ζ_∞ such that $\zeta_\infty \leqq \underline{\zeta}_k$ for all $k\geqq 1$ and set $\zeta_k=\zeta_\infty$ in (2.2). Then, we have $\lambda_k=\tau_k+\zeta_\infty$, which implies $\lambda^* \geqq \zeta_\infty>-\infty$ since $\tau_k>0$.

A characterization of the principal eigenvalue of the problem (A) is given in the following theorem.

THEOREM 2.1. *Let λ^* be the principal eigenvalue of the problem* (A). *Then the following statements hold.*

(i) *If $\lambda \leqq \lambda^*$, then there exists a solution $u \in C^{2+\sigma}(\bar{\Omega})$ of the problem* (A) *such that $u(x)>0$ in Ω.*

(ii) *If there exists a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that $u(x)>0$ in Ω and*

$$-\mathfrak{D}u + c(x)u \geqq \lambda m(x)u, \quad \text{in } \Omega, \quad Bu \geqq 0 \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset),$$

then $\lambda \leqq \lambda^*$.

PROOF. The proof is similar to that of Proposition 4.1 of [9].

(i) We prove the assertion in the case where Ω is an exterior domain; the case where $\Omega=\boldsymbol{R}^N$ can be treated similarly.

Step 1. Since $\lambda<\lambda_k$, by Lemma 2.1 there exists a unique $u_k \in C^{2+\sigma}(\bar{\Omega}_k)$ satisfying $u_k(x)>0$ in Ω_k and

$$\begin{aligned} -\mathfrak{D}u_k + c(x)u_k &= \lambda m(x)u_k \quad \text{in } \Omega_k, \\ Bu_k = 0 \quad \text{on } \Gamma, &\quad u_k = 1 \quad \text{on } \Gamma_k \end{aligned} \tag{2.6}$$

for every $k\geqq 1$. Furthermore, for each compact subset $K \subset \Omega$ there exist an integer $k_0 \geqq 1$ and a constant $M_1 \geqq 1$ such that

$$(2.7) \qquad M_1^{-1}u_k(x') \leqq u_k(x'') \leqq M_1 u_k(x'), \quad x', x'' \in K, \quad k \geqq k_0.$$

(See Step 2 of the proof of [9, Proposition 4.1].)

Step 2. Fix an $x_1 \in \Omega_1$ and define the functions $\tilde{u}_k$ by

$$\tilde{u}_k(x) = u_k(x)/u_k(x_1) \quad \text{for } x \in \Omega_k;\ \tilde{u}_k(x) = 0 \quad \text{for } x \in \Omega \backslash \Omega_k.$$

We will show that, for any bounded subdomain G of $\bar{\Omega}$, there exist an integer $k_1 \geqq 1$ and a constant $M_2 > 0$ such that

$$(2.8) \qquad \|\tilde{u}_k\|_{2+\sigma,\bar{G}} \leqq M_2, \quad k \geqq k_1,$$

where $\|\cdot\|_{2+\sigma,\bar{G}}$ is the usual norm of $C^{2+\sigma}(\bar{G})$ and similar Hölder norms are used throughout this paper.

We first prove (2.8) in the case where $\bar{G} \subset \Omega$. Take $k_1 \geqq 1$ such that $\bar{G} \subset \Omega_{k_1}$ and let $x_0 \in G$ and choose $\delta_1 > 0$ so small that $B(x_0, 3\delta_1) \subset \Omega_{k_1}$ and the Dirichlet problem

$$-\mathfrak{D}u + c(x)u = \lambda m(x)u \quad \text{in } B(x_0, 3\delta_1), \quad u = 0 \quad \text{on } \partial B(x_0, 3\delta_1)$$

has no nontrivial solution ([30, p. 77]). Then, applying interior L^p estimates to $\tilde{u}_k$ regarded as a solution of the problem

$$(2.9) \quad -\mathfrak{D}u + (c(x) - \lambda m(x))u = 0 \quad \text{in } B(x_0, 3\delta_1), \quad u = \tilde{u}_k \quad \text{on } \partial B(x_0, 3\delta_1),$$

we have

$$(2.10) \qquad \|\tilde{u}_k\|_{2,p,B(x_0,2\delta_1)} \leqq M_3 \|\tilde{u}_k\|_{0,p,B(x_0,3\delta_1)}$$

for some constant $M_3 > 0$ independent of k([1]), where $\|\cdot\|_{2,p,G'}$ and $\|\cdot\|_{0,p,G'}$ denote the norms of $W^{2,p}(G')$ and $L^p(G')$, $p \geqq 1$, respectively. In what follows we continue to use M_i, $i \geqq 4$, to denote positive constants which are independent of k. By (2.7) with $x' = x_1$ and the definition of $\tilde{u}_k$ we have

$$\sup_{k \geqq k_0} \max\{\tilde{u}_k(x) : x \in \overline{B(x_0, 3\delta_1)}\} = M_4 < \infty,$$

which combined with (2.10) yields $\|\tilde{u}_k\|_{2,p,B(x_0,2\delta_1)} \leqq M_5$ for all $k \geqq k_0$. From this with $p > N$ and Sobolev's imbedding theorem it follows that $\|\tilde{u}_k\|_{1,\overline{B(x_0,2\delta_1)}} \leqq M_6$ for $k \geqq k_0$. Using this and the interior Schauder estimates for the solution of (2.9), we have

$$(2.11) \qquad \|\tilde{u}_k\|_{2+\sigma,\overline{B(x_0,\delta_1)}} \leqq M_7 \|\tilde{u}_k\|_{0,\overline{B(x_0,2\delta_1)}} \leqq M_8$$

for $k \geqq k_0$. Since $\bar{G}$ is compact, (2.11) implies (2.8).

Next, suppose that $\bar{G} \cap \Gamma \neq \emptyset$. Without loss of generality we may assume that $G \subset \Omega_1$. Since $\{\tilde{u}_k\}$ is uniformly bounded on Γ_1 as proved just above, we put

$$M_9 = \sup\{\tilde{u}_k(x) : x \in \Gamma_1,\ k=1, 2, \dots\}.$$

Now, we take a solution $w \in C^{2+\sigma}(\bar{\Omega}_1)$ of the problem

$$-\mathfrak{D}w + c(x)w = \lambda m(x)w \quad \text{in } \Omega_1,$$
$$Bw = 0 \quad \text{on } \Gamma, \quad w = M_9 \quad \text{on } \Gamma_1.$$

As $\lambda < \lambda_1$ and $M_9 > 0$, w exists and is positive in Ω_1 by Lemma 2.1. It is easy to check that $v_k = w - \tilde{u}_k$ satisfies

$$-\mathfrak{D}v_k + c(x)v_k = \lambda m(x)v_k \quad \text{in } \Omega_1,$$
$$Bv_k = 0 \quad \text{on } \Gamma, \quad v_k \geqq 0 \quad \text{on } \Gamma_1, \quad k \geqq 1.$$

Hence by the maximum principle we have $v_k(x) \geqq 0$, i.e.,

$$0 < \tilde{u}_k(x) \leqq w(x) \quad \text{on } \bar{\Omega}_1, \quad k \geqq 1. \tag{2.12}$$

Using the boundary Schauder estimates ([1, Theorem 7.3]) for $\tilde{u}_k$ considered as a solution of the problem

$$-\mathfrak{D}u + (c(x) - \lambda m(x))u = 0 \quad \text{in } \Omega_1,$$
$$Bu = 0 \quad \text{on } \Gamma, \quad u = \tilde{u}_k \quad \text{on } \Gamma_1,$$

we have

$$\|\tilde{u}_k\|_{2+\sigma,\bar{\Omega}_1} \leqq M_{10}(\|\tilde{u}_k\|_{0,\bar{\Omega}_1} + \|\tilde{u}_k\|_{2+\sigma,\Gamma_1}). \tag{2.13}$$

Since $\|\tilde{u}_k\|_{2+\sigma,\Gamma_1}$ is uniformly bounded as we have seen above, we see from (2.12) and (2.13) that $\|\tilde{u}_k\|_{2+\sigma,\bar{\Omega}_1} \leqq M_{11}$. Thus (2.8) is proved.

Step 3. Since the sequence $\{\tilde{u}_k\}$ is bounded in $C^{2+\sigma}(\bar{G})$ for any bounded subdomain G of Ω by Step 2, using the Ascoli–Arzelà theroem and the standard diagonal process, we can find a subsequence $\{\tilde{u}_{k_j}\}$ of $\{\tilde{u}_k\}$ and a function $u \in C^{2+\sigma'}(\bar{\Omega})$, $0 < \sigma' < \sigma$, such that

$$\|\tilde{u}_{k_j} - u\|_{2+\sigma',K} \to 0 \quad \text{as } j \to \infty$$

for any compact set $K \subset \bar{\Omega}$. From (2.6) it follows that u is a solution of (A). The proof of the positivity of u is as follows. Since u satisfies

$$-\mathfrak{D}u + (c - \lambda m)^+(x)u = (c - \lambda m)^-(x)u \geqq 0, \quad u(x) \geqq 0 \quad \text{in } \Omega_k,$$

where $(c - \lambda m)^+(x) = \max\{c(x) - \lambda m(x),\ 0\}$, $(c - \lambda m)^-(x) = \max\{-(c(x) - \lambda m(x)),\ 0\}$, the maximum principle implies that either $u(x) \equiv 0$ or $u(x) > 0$ in Ω_k for every $k \geqq 1$. The former case cannot occur, since $\tilde{u}_k(x_1) = 1$.

(ii) Since u satisfies all conditions of Lemma 2.2, we obtain $\lambda \leqq \lambda_k$ for all

$k \geqq 1$. Hence $\lambda \leqq \lambda^*$ follows from the definition of λ^*. Thus the proof finishes.

2.2. Relation to elliptic oscillation

In this subsection we investigate the relation of the principal eigenvalue to the oscillation theory ([2], [31], [38]) for the equation

$$-\mathfrak{D}u + c(x)u = \lambda m(x)u \quad \text{in } \Omega. \tag{2.14}$$

In what follows we denote by $\lambda^*(R)$ the principal eigenvalue of the particular exterior Dirichlet problem

$$-\mathfrak{D}u + c(x)u = \lambda m(x)u \quad \text{in } G_R, \quad u = 0 \quad \text{on } \Gamma_R \tag{2.15}$$

for every $R>0$ and define $\lambda^*(\infty)$ by $\lambda^*(\infty) = \lim_{R \to \infty} \lambda^*(R)$. Since $\lambda^*(R)$ is nondecreasing in R, the above limit exists. When $\lambda^*(R)$ cannot be defined, that is, when $\lim_{k \to \infty} \lambda_k = -\infty$ for the problem (2.15), we write conventionally $\lambda^*(R) = \lambda^*(\infty) = -\infty$.

The equation (2.14) is said to be *nonoscillatory* at ∞ if there is no nodal domain in G_R for any $R>0$, or equivalently, if there is a positive solution $u \in C^2(G_R)$ of

$$-\mathfrak{D}u + c(x)u = \lambda m(x)u \quad \text{in } G_R \tag{2.16}$$

for some $R>0$; see e.g. [2], [31], [38]. Equation (2.14) is said to be *oscillatory* at ∞ if every nontrivial solution of (2.14) has always zeros in any neighborhood of infinity.

THEOREM 2.2. *Equation* (2.14) *is nonoscillatory at* ∞ *if and only if* $\lambda \leqq \lambda^*(\infty)$, *where the equality can occur only in the case where* $\lambda^*(\infty) = \lambda^*(R)$ *for some* $R>0$.

PROOF. Suppose that (2.14) is nonoscillatory at ∞. Then, there is $u \in C^2(\bar{G}_R)$ satisfying (2.16) and $u(x)>0$ in $\bar{G}_R$ for some $R>0$. From (ii) of Theorem 2.1 it follows that $\lambda \leqq \lambda^*(R) \leqq \lambda^*(\infty)$. If $\lambda = \lambda^*(\infty)$, then (2.16) with $\lambda = \lambda^*(\infty)$ is nonoscillatory at ∞, so that $\lambda = \lambda^*(\infty) \leqq \lambda^*(R)$ for some $R>0$ as shown just above. This implies that $\lambda^*(\infty) = \lambda^*(R)$.

Conversely, if $\lambda < \lambda^*(\infty)$, we can choose $R>0$ so that $\lambda < \lambda^*(R) \leqq \lambda^*(\infty)$. Hence the prolbem (2.15) has a positive solution $u \in C^2(\bar{G}_R)$ by (i) of Theorem 2.1. This implies that (2.16) is nonoscillatory at ∞. The same argument holds if we assume that $\lambda = \lambda^*(\infty) = \lambda^*(R)$ for some $R>0$.

COROLLARY 2.1. *Equation* (2.14) *is oscillatory at* ∞ *for any* $\lambda \in \boldsymbol{R}$ *if and only if* $\lambda^*(\infty) = -\infty$.

3. Minimal λ-superharmonic functions

3.1. Definition of minimal λ-superharmonic functions

In this section we introduce the concept of λ-superharmonic functions minimal at ∞ (cf. [8], [10]).

DEFINITION 3.1. For $\lambda \in \boldsymbol{R}$ we denote by $SH(\lambda; \Omega)$ the class of all functions $h \in C^2(\Omega)$ satisfying the following conditions:

(i) h is positive and λ-*superharmonic* in Ω, i.e.,

$$-\mathfrak{D}h(x) + c(x)h \geqq \lambda m(x)h, \quad h > 0 \quad \text{in } \Omega; \tag{3.1}$$

(ii) h is *minimal* at ∞, i.e.,

$$h(x) = O(\varphi(x)) \quad \text{as } |x| \to \infty \tag{3.2}$$

for any positive λ-superharmonic function φ in Ω.

We employ the notation:

$$H(\lambda; \Omega) = \{h \in SH(\lambda; \Omega) \colon -\mathfrak{D}h + c(x)h = \lambda m(x)h \text{ in } \Omega\},$$
$$SH(\lambda) = \cup_{R>0} SH(\lambda; G_R).$$

Throughout this section we assume that the coefficients a_{ij} and b_i of $\mathfrak{D}$ are bounded on $\boldsymbol{R}^N$.

Next, we denote by $X = (x(t), \mathscr{F}_t, P_x)$, $x \in \boldsymbol{R}^N$, the diffusion process on $\boldsymbol{R}^N$ with the infinitesimal generator $\mathfrak{D}$ (see [16]). For a subset E of $\boldsymbol{R}^N$, let $\sigma(E)$ be the first hitting time for the set E ($\sigma(E) = \infty$ if the set E is never hit).

THEOREM 3.1. (i) *$\{h \in SH(\lambda; \Omega) \cap C^1(\bar{\Omega}) \colon Bh \geqq 0$ on $\Gamma\} \neq \emptyset$ if and only if $\lambda \leqq \lambda^*$.*

(ii) *For any $\lambda \leqq \lambda^*$ and any subdomain Ω' of Ω such that $\bar{\Omega}' \subset \Omega$,*

$$\{h \colon h \in SH(\lambda; \Omega) \cap H(\lambda; \Omega'),\ Bh \geqq 0 \text{ on } \Gamma\} \neq \emptyset.$$

Furthermore, for each $h \in SH(\lambda; \Omega) \cap H(\lambda; \Omega')$ we have

$$h(x) = E_x\left[h(x(\sigma(\partial\Omega'))) \exp\left(\int_0^{\sigma(\partial\Omega')} (\lambda m(x(s)) - c(x(s)))ds\right) \colon \sigma(\partial\Omega') < \infty\right], \quad x \in \Omega'. \tag{3.3}$$

The proof of this theorem is essentially the same as that of [10, Theorem 2.1].

COROLLARY 3.1. *$SH(\lambda) \neq \emptyset$ if and only if $\lambda \leqq \lambda^*(R)$ for some $R > 0$.*

PROOF. (The "only if" part). If $SH(\lambda) \neq \emptyset$, then there exists an $R>0$ with the property $SH(\lambda; G_R) \neq \emptyset$. Hence $\lambda \leqq \lambda^*(R)$ by (i) of Theorem 3.1.

(The "if" part). Since $SH(\lambda; G_R) \neq \emptyset$ by (i) of Theorem 3.1, we have $SH(\lambda) \neq \emptyset$.

The following theorem is a partial extension of [10, Theorem 2.2].

THEOREM 3.2. *Let* $f \in C^\sigma(\bar{\Omega})$ *and* $g \in C^{l+\sigma}(\Gamma)$, *where* $l=2$ *if* $\alpha(x) \equiv 0$ *and* $l=1$ *if* $\alpha(x)>0$ *on* Γ. *If* $\lambda<\lambda^*$ *and* f *satisfies*

$$f(x)/m(x) = O(h(x)) \quad \text{as } |x| \to \infty \tag{3.4}$$

for some $h \in SH(\lambda; \Omega)$, *then for any* $h' \in SH(\lambda'; \Omega)$ *with* $\lambda<\lambda'<\lambda^*$ *there is a solution* $u \in C^{2+\sigma}(\bar{\Omega})$ *of the problem*

$$-\mathfrak{D}u + (c(x)-\lambda m(x))u = f(x) \quad \text{in } \Omega, \quad Bu = g(x) \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset) \tag{3.5}$$

satisfying

$$u(x) = O(h'(x)) \quad \text{as } |x| \to \infty. \tag{3.6}$$

PROOF. The proof is similar to that of [10, (1) of Theorem 2.2] and will be omitted.

3.2. Asymptotic behavior of minimal λ-superharmonic functions

This subsection is devoted to the study of asymptotic behavior of minimal λ-superharmonic functions.

DEFINITION 3.2. The class $SH(\lambda)$ is said to be of type I, type II or type III, according to whether any function $h \in SH(\lambda)$ satisfies

$$\lim_{|x|\to\infty} h(x) = 0, \tag{I}$$

$$0 < \liminf_{|x|\to\infty} h(x) \leqq \limsup_{|x|\to\infty} h(x) < \infty \tag{II}$$

or

$$\lim_{|x|\to\infty} h(x) = \infty. \tag{III}$$

Our purpose here is to give sufficient conditions for $SH(\lambda)$ to be of one of the types I, II and III. We note that there is a class $SH(\lambda)$ whose type is different from these three types (see e.g. [10, Proposition 5.3]).

Before stating the main theorems we prepare two lemmas on the existence and asymptotic behavior of solutions of the ordinary differential equation

$$(p(r)y')' + \lambda q(r)y = 0, \quad r > r_0, \tag{3.7}$$

where $' = d/dr$, $\lambda \in \boldsymbol{R}$ and p and q satisfy the conditions

$$p \in C^1[r_0, \infty) \quad \text{and} \quad p(r) > 0 \quad \text{on } [r_0, \infty),$$
$$q \in C[r_0, \infty) \quad \text{and} \quad q(r) > 0 \quad \text{on } [r_0, \infty).$$

LEMMA 3.1. (i) *Suppose that*

$$\int_{r_0}^{\infty} dr/p(r) = \infty, \tag{3.8}$$

and

$$\int_{r_0}^{\infty}\left(\int_{r_0}^{r} ds/p(s)\right)q(r)dr < \infty.$$

Then, the following statements hold.

(a) *For any* $\xi \geqq 0$ *and* $\eta > 0$, *there is a constant* $K > 0$ *such that if*

$$\int_{r_0}^{\infty} \max\left\{1, \int_{r_0}^{r} ds/p(s)\right\}q(r)dr < K,$$

then (3.7) *has a solution* y *satisfying*

$$y(r_0) = \xi, \quad y'(r_0) = \eta, \quad y'(r) > 0, \quad r > r_0,$$
$$\lim_{r\to\infty} y(r) \Big/ \int_{r_0}^{r} ds/p(s) = \text{constant} > 0.$$

In the particular case where $\lambda < 0$, (3.7) *has a solution with the above properties even for* $\xi > 0$ *and* $\eta = 0$.

(b) *For any* $\lambda \neq 0$, (3.7) *has a solution* y *satisfying* $\lambda y'(r) > 0$, $r \geqq \tilde{r}_0$ *for some* $\tilde{r}_0 \geqq r_0$, *and* $\lim_{r\to\infty} y(r) = \text{constant} > 0$.

(ii) *In addition to* (3.8) *suppose that*

$$\int_{r_0}^{\infty}\left(\int_{r_0}^{r} ds/p(s)\right)q(r)dr = \infty.$$

Then, for any $\lambda < 0$, (3.7) *has a solution* y *satisfying* $y'(r) < 0$ *for* $r \geqq \tilde{r}_0$ *for some* $\tilde{r}_0 \geqq r_0$ *and* $\lim_{r\to\infty} y(r) = 0$.

LEMMA 3.2. (i) *Suppose that*

$$\int_{r_0}^{\infty} dr/p(r) < \infty. \tag{3.9}$$

If (3.7) *is nonoscillatory at* ∞, *then* (3.7) *has a positive solution* y *on* $[\tilde{r}_0, \infty)$ *for some* $\tilde{r}_0 \geqq r_0$ *such that* $y'(r) < 0$ *for* $r \geqq \tilde{r}_0$ *and* $\lim_{r\to\infty} y(r) = 0$.

(ii) *In addition to* (3.9) *suppose that*

$$\int_{r_0}^{\infty}\left(\int_{r}^{\infty} ds/p(s)\right)q(r)dr < \infty.$$

Then the following statements hold.

(a) *For* $\lambda>0$, *there are* $\tilde{r}_0>0$ *and* $\eta_0>0$ *such that the solution* y *of* (3.7) *with initial data*

$$y(\tilde{r}_0)=1,\quad y'(\tilde{r}_0)=-\eta,\quad \eta\in(0,\eta_0]$$

satisfies $y'(r)<0$ *for* $r\geqq\tilde{r}_0$ *and* $\lim_{r\to\infty}y(r)=\text{constant}>0$.

(b) *For* $\lambda\leqq 0$, *the solution* y *of* (3.7) *with initial data* $y(r_0)=0$, $y'(r_0)=1$ *satisfies* $y'(r)>0$ *for* $r\geqq r_0$ *and* $\lim_{r\to\infty}y(r)=\text{constant}>0$.

The proofs of Lemmas 3.1 and 3.2 are easy and will be deleted (see e.g. [14], [15]).

The following notation is used throughout this paper:

$$a(x)=\textstyle\sum_{i,j=1}^N a_{ij}(x)x_ix_j/|x|^2,$$
$$b(x)=\left(\textstyle\sum_{i=1}^N b_i(x)x_i+\sum_{i=1}^N a_{ii}(x)-a(x)\right)/|x|$$
$$b_*(r)=\min_{|x|=r}b(x)/a(x),\quad b^*(r)=\max_{|x|=r}b(x)/a(x),$$
$$c_*(r)=\min_{|x|=r}c(x)/a(x),\quad c^*(r)=\max_{|x|=r}c(x)/a(x),$$
$$m_*(r)=\min_{|x|=r}m(x)/a(x),\quad m^*(r)=\max_{|x|=r}m(x)/a(x),$$
$$p_*(r)=\exp\left(\int_{r_0}^r b_*(s)ds\right),\quad p^*(r)=\exp\left(\int_{r_0}^r b^*(s)ds\right),$$
$$\zeta(x)=c(x)/m(x),$$
$$\zeta_*=\liminf_{|x|\to\infty}\zeta(x),\quad \zeta^*=\limsup_{|x|\to\infty}\zeta(x).$$

In what follows we treat the operator $\mathfrak{D}$ satisfying one of the following conditions:

$$(\mathrm{H}_1)\qquad \int_{r_0}^\infty dr/p^*(r)=\infty,$$

$$(\mathrm{H}_2)\qquad \int_{r_0}^\infty dr/p_*(r)<\infty.$$

We study the asymptotic behavior of $h\in SH(\lambda)$ by distinguishing the following three cases:

(***a***) $-\infty<\liminf_{|x|\to\infty}\zeta(x)\leqq\limsup_{|x|\to\infty}\zeta(x)<\infty$;

(***b***) $\lim_{|x|\to\infty}\zeta(x)=\infty$;

(***c***) $\lim_{|x|\to\infty}\zeta(x)=-\infty$.

The case (***a***). In this case, we have $\lambda^*(\infty)\geqq\zeta_*$. In fact, for any $\lambda<\zeta_*$, one can choose a constant $R>0$ such that

$$c(x) - \lambda m(x) = (\zeta(x) - \lambda)m(x) \geqq 0 \quad \text{in } G_R .$$

Then the function $u(x) \equiv 1$ is obviously λ-superharmonic in G_R, and so by (ii) of Theorem 2.1 we have $\lambda \leqq \lambda^*(R) \leqq \lambda^*(\infty)$. This yields $\zeta_* \leqq \lambda^*(\infty)$ as desired.

THEOREM 3.3. *Suppose that* ($\boldsymbol{a}$) *holds. Then the following statements hold.*

(i) *Assume that* (H_1) *holds and*

$$\int_{r_0}^{\infty} \left(\int_{r_0}^{r} ds/p^*(s) \right) p^*(r) m^*(r) dr < \infty . \tag{3.10}$$

Then $\lambda^*(\infty) = \infty$ *and* $SH(\lambda)$ *is of type* II *for every* $\lambda \in \boldsymbol{R}$.

(ii) *Assume that* (H_1) *holds and*

$$\int_{r_0}^{\infty} \left(\int_{r_0}^{r} ds/p_*(s) \right) p_*(r) m^*(r) dr = \infty . \tag{3.11}$$

Then $SH(\lambda)$ *is of type* I *for* $\lambda < \zeta_*$ *and of type* III *for* $\zeta^* < \lambda < \lambda^*(\infty)$ *provided* $\zeta^* < \lambda^*(\infty)$.

(iii) *Assume that* (H_2) *holds and*

$$\int_{r_0}^{\infty} \left(\int_{r}^{\infty} ds/p_*(s) \right) p_*(r) m^*(r) dr < \infty . \tag{3.12}$$

Then $\lambda^*(\infty) = \infty$ *and* $SH(\lambda)$ *is of type* I *for every* $\lambda \in \boldsymbol{R}$.

PROOF. In what follows let $R_0 > 0$ be a fixed constant such that

$$\zeta_* - 1 < \zeta(x) < \zeta^* + 1 \quad \text{in } G_{R_0} .$$

(i) Let $\lambda > \zeta_*$. By (H_1), (3.10) and (i–b) of Lemma 3.1, for some $R > R_0$ the equation

$$(p^*(r) y')' + (\lambda + 1 - \zeta_*) p^*(r) m^*(r) y = 0 , \tag{3.13}$$

has a solution φ such that $\varphi(r) > 0$ and $\varphi'(r) > 0$ on $[R, \infty)$ and $\varphi(r) \to 1$ as $r \to \infty$. The function $u(x) = \varphi(|x|)$ on $\bar{G}_R$ is positive and λ-superharmonic in G_R. For, since φ satisfies

$$\varphi''(r) + b^*(r)\varphi'(r) + (\lambda + 1 - \zeta_*) m^*(r) \varphi(r) = 0, \quad r > R,$$

by (3.13), $\varphi'(r) > 0$ and $\lambda - \zeta(x) < \lambda + 1 - \zeta_*$ in G_R, we have

$$\begin{aligned}
&- \mathfrak{D} u(x) + (c(x) - \lambda m(x)) u(x) \\
&= - a(x)\varphi''(|x|) - b(x)\varphi'(|x|) - (\lambda - \zeta(x)) m(x) \varphi(|x|) \\
&= - a(x)(\varphi''(|x|) + b^*(|x|)\varphi'(|x|)) + (a(x) b^*(|x|) - b(x))\varphi'(|x|) \\
&\qquad\qquad\qquad\qquad - (\lambda - \zeta(x)) m(x) \varphi(|x|)
\end{aligned}$$

$$\geqq \{(\lambda+1-\zeta_*)a(x)m^*(|x|)-(\lambda-\zeta(x))m(x)\}\varphi(|x|)$$
$$\geqq (\lambda+1-\zeta_*)(a(x)m^*(|x|)-m(x))\varphi(|x|) \geqq 0 \quad \text{in } G_R.$$

From this and (ii) of Theorem 2.1 it follows that $\lambda \leqq \lambda^*(R) \leqq \lambda^*(\infty)$. Hence we have $\lambda^*(\infty)=\infty$ by the arbitrariness of λ.

Now, we show that $SH(\lambda)$ is of type II. First note that, using the fact that $\lambda^*(\infty)=\infty$ and (ii) of Theorem 3.1, for any $\lambda_1>0$ we can choose a constant R with the property

$$SH(\lambda;\, G_R) \cap H(\lambda;\, G_{R'}) \neq \emptyset \quad \text{for } R' > R \text{ and } \lambda < \lambda_1.$$

Let $\lambda'>\zeta_*$ and $h' \in SH(\lambda';\, G_R)$. Take the function u mentioned in the above proof with $\lambda=\lambda'$. This u is bounded, positive and λ'-superharmonic in G_R. Hence by (ii) of Definition 3.1 there is a constant $M_1>0$ such that

$$0 < h'(x) \leqq M_1 \quad \text{in } G_R. \tag{3.14}$$

Next, let $\lambda''<\zeta^*$ and $h'' \in SH(\lambda'';\, G_R) \cap H(\lambda'';\, G_{R'})$. By (ii) of Theorem 3.1 we have for $x \in G_{R'}$

$$\begin{aligned} &h''(x) \\ &\geqq (\min\nolimits_{x\in\Gamma_{R'}} h''(x))E_x\left[\exp\left(\int_0^{\sigma(\Gamma_{R'})} (\lambda''-\zeta(x(s))m(x(s))\,ds\right):\ \sigma(\Gamma_{R'})<\infty\right]. \end{aligned} \tag{3.15}$$

Now, we put for $\mu \in \boldsymbol{R}$

$$\tilde{h}_\mu(x) = E_x\left[\exp\left(\int_0^{\sigma(\Gamma_{R'})} \mu m(x(s))ds\right):\ \sigma((\Gamma_{R'}) < \infty\right], \quad x \in G_{R'}, \tag{3.16}$$

if the right hand side is finite. Since $\lambda''-\zeta(x)>\lambda''-\zeta^*-1$ in $G_{R'}$, (3.15) gives

$$h''(x) \geqq (\min\nolimits_{x\in\Gamma_{R'}} h''(x))\tilde{h}_{\lambda''-\zeta^*-1}(x), \quad x \in G_{R'}. \tag{3.17}$$

We note that the diffusion process X with the infinitesimal generator $\mathfrak{D}$ is recurrent by [16, Lemma 8.1]. So, using Schwarz's inequality, we have for $\mu \in \boldsymbol{R}$

$$\begin{aligned} 1 &\leqq \left(E_x\left[\exp\left(\int_0^{\sigma(\Gamma_{R'})} \mu m(x(s))ds\right):\ \sigma(\Gamma_{R'}) < \infty\right]\right)^{1/2} \\ &\quad \times \left(E_x\left[\exp\left(\int_0^{\sigma(\Gamma_{R'})} -\mu m(x(s))ds\right):\ \sigma(\Gamma_{R'}) < \infty\right]\right)^{1/2} \\ &= (\tilde{h}_\mu(x))^{1/2}(\tilde{h}_{-\mu}(x))^{1/2}, \quad x \in G_{R'}. \end{aligned} \tag{3.18}$$

Put $\bar{\lambda}=2(\zeta^*+1)-\lambda''$. Then, $\bar{\lambda}>\zeta_*$ and $\bar{\lambda}-\zeta(x) \geqq \zeta^*+1-\lambda''>0$ in G_R, so that by (3.14) with $\lambda'=\bar{\lambda}$ we have

$$\tilde{h}_{-(\lambda''-\zeta^*-1)}(x) \leqq E_x\left[\exp\left(\int_0^{\sigma(\Gamma_{R'})} (\bar{\lambda}-\zeta(x(s)))m(x(s))ds\right) : \sigma(\Gamma_{R'})<\infty\right]$$
$$\leqq M_2, \quad x \in G_{R'},$$

which, in view of (3.18), leads to

$$\tilde{h}_{\lambda''-\zeta^*-1}(x) \geqq (\tilde{h}_{-(\lambda''-\zeta^*-1)}(x))^{-1} \geqq M_2^{-1}, \quad x \in G_{R'}.$$

This combined with (3.17) yields

$$h''(x) \geqq M_3, \quad x \in G_{R'} \tag{3.19}$$

for some $M_3>0$.

Let now $\lambda \in \boldsymbol{R}$ and $h \in SH(\lambda; G_R)$. Take λ' and λ'' such that

$$\lambda' > \max\{\zeta_*, \lambda\}, \quad \lambda'' < \min\{\zeta^*, \lambda\}$$

and choose $h' \in SH(\lambda'; G_R)$ and $h'' \in SH(\lambda''; G_R) \cap H(\lambda''; G_{R'})$. Then,

$$M_4 h''(x) \leqq h(x) \leqq M_5 h'(x) \quad \text{in } G_{R'}.$$

Combining this with (3.14) and (3.19), we have

$$0 < \lim_{|x|\to\infty} h(x) \leqq \limsup_{|x|\to\infty} h(x) < \infty,$$

showing that $SH(\lambda)$ is of type II.

(ii) Let $\lambda<\zeta_*$ and choose a constant $R_0>0$ such that

$$\lambda - \zeta(x) < (\lambda-\zeta_*)/2 < 0 \quad \text{in } G_{R_0}.$$

By (H_1), (3.11) and (ii) of Lemma 3.1, the equation

$$(p_*(r)y')' + \frac{1}{2}(\lambda-\zeta_*)p_*(r)m_*(r)y = 0, \quad r > R_0, \tag{3.20}$$

has a positive solution φ satisfying $\varphi'(r)<0$ on $[R_0, \infty)$ and $\varphi(r)\to 0$ as $r\to\infty$. The function u defined by $u(x)=\varphi(|x|)$, $x \in G_{R_0}$, is λ-superharmonic in G_{R_0}, because

$$\begin{aligned}
&-\mathfrak{D}u(x) + (c(x)-\lambda m(x))u(x) \\
&= -a(x)(\varphi''(|x|)+b_*(|x|)\varphi'(|x|)) \\
&\qquad\qquad + (a(x)b_*(|x|)-b(x))\varphi'(|x|) - (\lambda-\zeta(x))m(x)\varphi(|x|) \\
&\geqq \frac{1}{2}(\lambda-\zeta_*)(a(x)m_*(|x|)-m(x))\varphi(|x|) \geqq 0, \quad x \in G_{R_0}.
\end{aligned}$$

Here, we have used the relation $\varphi''+b_*(r)\varphi' = -2^{-1}(\lambda-\zeta_*)m_*(r)\varphi$ that follows from (3.20). Therefore, for sufficiently large $R \geqq R_0$ and for any $h \in SH(\lambda; G_R)$, we have

$$0 < h(x) \leqq M_6 u(x) \quad \text{in } G_R,$$

where $M_6 > 0$ is a constant. Since $u(x) \to 0$ as $|x| \to \infty$, $h(x) \to 0$ as $|x| \to \infty$ and hence $SH(\lambda)$ is of type I.

Next, let $\zeta^* < \lambda < \lambda^*(\infty)$, $h \in SH(\lambda; G_R)$ and put $\mu = (\lambda - \zeta^*)/2$. Let $\tilde{h}_{\pm\mu}$ be the function defined by (3.16). Then, from the fact that $\tilde{h}_{-\mu}$ is minimal $-\mu$-superharmonic in G_{R_0} with $c(x) \equiv 0$ and $\mu > 0$, it follows that $\tilde{h}_{-\mu}(x) \to 0$ as $|x| \to \infty$ as proved just above. This combined with (3.18) shows that $\tilde{h}_\mu(x) \to \infty$ as $|x| \to \infty$. Choosing a constant $R > 0$ such that

$$\lambda - \zeta(x) \geqq (\lambda - \zeta^*)/2 \quad \text{in } G_R,$$

we have by (ii) of Theorem 3.1

$$0 < \tilde{h}_\mu(x) \leqq M_7 h(x), \quad x \in G_{R'}$$

for some $R' > R$. This implies that $h(x) \to \infty$ as $|x| \to \infty$. Thus $SH(\lambda)$ is of type III.

(iii) Let $\lambda > \zeta_*$ and choose a constant $R_0 > 0$ such that

$$\lambda - \zeta(x) < \lambda - \zeta_* \quad \text{in } G_{R_0}.$$

By (H_2) and (i) of Lemma 3.2, for some $R > R_0$ the equation

$$(p_*(r)y')' + (\lambda - \zeta_*)p_*(r)m^*(r)y = 0, \quad r > R,$$

has a positive solution φ satisfying $\varphi'(r) < 0$ on $[R, \infty)$ and $\varphi(r) \to 0$ as $r \to \infty$, since this equation is nonoscillatory at ∞ by (3.12). Then, the function $u(x) = \varphi(|x|)$, $x \in \bar{G}_R$ is λ-superharmonic in G_R. From this fact and the arbitrariness of $\lambda > \zeta_*$, it follows that $\lambda^*(\infty) = \infty$. It is obvious that $SH(\lambda)$ is of type I. Thus the proof is complete.

The case (***b***). In this case we see that $\lambda^*(\infty) = \infty$ as in the case (***a***).

THEOREM 3.4. *Suppose that* (***b***) *holds. Then the following statements hold.*

(i) *Assume that* (H_1) *holds and*

$$\int_{r_0}^{\infty} \left(\int_{r_0}^{r} ds/p^*(s) \right) p^*(r)c^*(r)dr < \infty. \tag{3.21}$$

Then $SH(\lambda)$ *is of type* II *for every* $\lambda \in \boldsymbol{R}$.

(ii) *Assume that* (H_1) *holds and*

$$\int_{r_0}^{\infty} \left(\int_{r_0}^{r} ds/p_*(s) \right) p_*(r)c_*(r)dr = \infty. \tag{3.22}$$

Then $SH(\lambda)$ is of type I *for every $\lambda \in \boldsymbol{R}$.*

(iii) *If* (H_2) *holds, then $SH(\lambda)$ is of type* I *for every $\lambda \in \boldsymbol{R}$.*

PROOF. By the assumption, for any $\lambda \in \boldsymbol{R}$ there is an $R>0$ such that $SH(\lambda; G_R) \neq \emptyset$ and

$$-2c(x) \leqq (\lambda - \zeta(x))m(x) \leqq -c(x)/2, \quad x \in G_R.$$

(i) Let $h \in SH(\lambda; G_R)$ and $h(x)>0$ in $\bar{G}_R$. The function $h_0(x) \equiv 1$ on $\bar{G}_R$ is λ-superharmonic in G_R, so that we have

$$0 < h(x) \leqq M_1 h_0(x) = M_1, \quad x \in G_R,$$

for some $M_1>0$. On the other hand, proceeding as in the proof of (i) of Theorem 3.3, we have $h(x) \geqq M_2$ in G_R for some $M_2>0$. This shows that $SH(\lambda)$ is of type II.

(ii) Statement (ii) is proved in exactly the same way as in the proof of (ii) of Theorem 3.3.

(iii) Since $c^*(r)>0$, the equation

$$(3.23) \qquad (p_*(r)y')' - \frac{1}{2} p_*(r)c^*(r)y = 0, \quad r > R,$$

is nonoscillatory at ∞. By (H_2) and (i) of Lemma 3.2, (3.23) has a positive solution φ satisfying $\varphi'(r)<0$ on $[R, \infty)$ and $\varphi(r) \to 0$ as $r \to \infty$. The function $u(x) = \varphi(|x|)$ on $\bar{G}_R$ is λ-superharmonic in G_R and tends to 0 as $|x| \to \infty$. It follows that $SH(\lambda)$ is of type I. This completes the proof.

The case ($\boldsymbol{c}$). In this case, since $c(x)<0$ on $\bar{G}_{R_0}$ for large $R_0>0$, the principal eigenvalue $\mu^*(R)$ for the problem

$$(3.24) \qquad -\mathfrak{D}\psi = -\mu c(x)\psi \quad \text{in } G_R, \quad \psi = 0 \quad \text{on } \Gamma_R$$

exists and $\mu^*(R) \geqq 0$ for $R \geqq R_0$. Hence $\mu^*(\infty) = \lim_{R \to \infty} \mu^*(R)$ is well defined.

LEMMA 3.3. *Let* $\lim_{|x| \to \infty} \zeta(x) = -\infty$. *Then the following statements hold.*

(i) *If $\mu^*(\infty)>1$, then $\lambda^*(\infty)=\infty$.*

(ii) *If $\mu^*(\infty)=\mu^*(R)=1$ for some $R>0$, then $\lambda^*(\infty) \geqq 0$, and if $\mu^*(R) < \mu^*(\infty)=1$ for any large $R>0$, then $\lambda^*(\infty) \leqq 0$.*

(iii) *If $\mu^*(\infty)<1$, then $\lambda^*(\infty) = -\infty$ in the sense of the remark mentioned in Section* 2.2.

PROOF. (i) Let $1<\mu<\mu^*(\infty)$. Then $\mu<\mu^*(R)$ for some $R>0$. By (i) of Theorem 2.1 there is a function $v \in C^2(\bar{G}_R)$ such that $v(x)>0$ in G_R and $-\mathfrak{D}v + \mu c(x)v = 0$ in G_R. Let $\lambda>0$. Since $\mu>1$, we can choose a constant $R'>R$ such that

$$\lambda - \zeta(x) \leqq -\mu\zeta(x) \quad \text{in } G_{R'}.$$

From this it follows that v is λ-superharmonic in $G_{R'}$. Hence we have $\lambda \leqq \lambda^*(R') \leqq \lambda^*(\infty)$ by (ii) of Theorem 2.1. Since λ is arbitrary, we obtain $\lambda^*(\infty)=\infty$.

(ii) Suppose that $\mu^*(R)=\mu^*(\infty)=1$ for some $R>0$. Then, there is a function $v \in C^2(\bar{G}_R)$ satisfying $v(x)>0$ in $\bar{G}_R$ and

$$-\mathfrak{D}v + c(x)v = 0 \quad \text{in } G_R. \tag{3.25}$$

That $0 \leqq \lambda^*(R) \leqq \lambda^*(\infty)$ now follows from (ii) of Theorem 2.1.

Next, if $\mu^*(R)<\mu^*(\infty)=1$ for any large $R>0$, then (3.25) and hence the equation

$$-\mathfrak{D}u + c(x)u = \lambda m(x)u \quad \text{in } G_R \tag{3.26}$$

is oscillatory at ∞ for any $\lambda \geqq 0$. For, otherwise there is a positive function u satisfying (3.26), so that we have $\lambda^*(R) \geqq \lambda \geqq 0$. Furthermore, since this u satisfies $-\mathfrak{D}u \geqq -c(x)u$ in G_R, (ii) of Theorem 2.1 implies that $\mu^*(R) \geqq 1$, a contradiction.

(iii) Assume that $\lambda^*(\infty)>-\infty$. Then, for any $\lambda<\lambda^*(\infty)$ and $0<\mu<1$ there is an $R_0>0$ such that $\lambda-\zeta(x)>-\mu\zeta(x)$ in G_{R_0}. Choose $R \geqq R_0$ satisfying $\lambda<\lambda^*(R) \leqq \lambda^*(\infty)$ and $h \in SH(\lambda; G_R)$. We then have

$$-\mathfrak{D}h \geqq (\lambda-\zeta(x))m(x)h \geqq -\mu\zeta(x)m(x)h = -\mu c(x)h \quad \text{in } G_R,$$

from which, in view of (ii) of Theorem 2.1, it follows that $\mu \leqq \mu^*(\infty)$, which leads to $\mu^*(\infty) \geqq 1$, since $\mu \in (0, 1)$ is arbitrary. This contradiction shows that $\lambda^*(\infty)=-\infty$.

THEOREM 3.5. *Suppose that* (**c**) *holds. Then the following statements hold.*

(i) *Assume that* (H_1) *holds and*

$$\int_{r_0}^{\infty}\left(\int_{r_0}^{r} ds/p^*(s)\right)p^*(r)c_*(r)dr > -\infty. \tag{3.27}$$

Then $\lambda^*(\infty)=\infty$ *and* $SH(\lambda)$ *is of type* II *for every* $\lambda \in \boldsymbol{R}$.

(ii) *Assume that* (H_1) *holds and*

$$\int_{r_0}^{\infty}\left(\int_{r_0}^{r} ds/p_*(s)\right)p_*(r)c^*(r)dr = -\infty. \tag{3.28}$$

Then $SH(\lambda)$ *is of type* III *for* $\lambda<\lambda^*(\infty)$.

(iii) *Assume that* (H_2) *holds and the equation*

$$y'' + b_*(r)y' - \mu c_*(r)y = 0 \tag{3.29}$$

is nonoscillatory at ∞ *for some* $\mu>1$. *Then* $\lambda^*(\infty)=\infty$ *and* $SH(\lambda)$ *is of type* I *for every* $\lambda \in \boldsymbol{R}$.

PROOF. By the assumption, for any $\lambda \in \boldsymbol{R}$ there is an $R_0>0$ such that

$$-c(x)/2 \leqq \lambda m(x) - c(x) \leqq -2c(x) \quad \text{in } G_{R_0}. \tag{3.30}$$

(i) Statement (i) can be proved as in the proof of (i) of Theorem 3.4 by using (3.30)

(ii) Statement (ii) can be proved as in the proof of (ii) of Theorem 3.3 by using (3.30) and the fact that $-c(x)/2>0$ in G_{R_0}.

(iii) For $\lambda>0$, take an $R>0$ such that

$$-c(x) < (\lambda-\zeta(x))m(x) \leqq -\mu c(x) \quad \text{in } \bar{G}_R$$

and (3.29) has a positive solution φ satisfying $\varphi'(r)<0$ on $[R, \infty)$ and $\varphi(r)\to 0$ as $r\to\infty$. Then, since $u(x)=\varphi(|x|)$ is λ-superharmonic in G_R, we have $\lambda\leqq\lambda^*(R)\leqq\lambda^*(\infty)$. This means $\lambda^*(\infty)=\infty$ by the arbitrariness of λ. Since $u(x)$ tends to 0 as $|x|\to\infty$, $SH(\lambda)$ is of type I. This finishes the proof.

REMARK 3.1. Let B_*, B^*, C_*, C^*, M_* and M^* be locally Hölder continuous functions on $(0, \infty)$ with the properties

$$B_*(r) \leqq b_*(r) \leqq b^*(r) \leqq B^*(r),$$
$$C_*(r) \leqq c_*(r) \leqq c^*(r) \leqq C^*(r),$$
$$M_*(r) \leqq m_*(r) \leqq m^*(r) \leqq M^*(r),$$

and put

$$P_*(r) = \exp\left(\int_{r_0}^r B_*(s)ds\right), \quad P^*(r) = \exp\left(\int_{r_0}^r B^*(s)ds\right).$$

It is easy to check that all the conclusions of Theorems 3.3, 3.4 and 3.5 remain true if in the hypotheses of these theorems the functions p_*, p^*, c_*, c^*, m_* and m^* are replaced by P_*, P^*, C_*, C^*, M_* and M^*, respectively.

3.3. Examples

We present some examples illustrating the results obtained in the preceding subsection. The following is a direct consequence of Theorem 3.3.

EXAMPLE 3.1. Consider the equation

$$-\Delta u + c(x)u = \lambda m(x)u \quad \text{in } G_R, \tag{3.31}$$

where Δ is the N-dimensional Laplacian. Suppose that
$-\infty < \zeta_* = \liminf_{|x|\to\infty} \zeta(x) \leqq \limsup_{|x|\to\infty} \zeta(x) = \zeta^* < \infty$, $\zeta(x) = c(x)/m(x)$.

(i) Suppose that $N=2$. If

$$\int_{r_0}^{\infty} r \log r\, m^*(r)dr < \infty,$$

then $\lambda^*(\infty)=\infty$ and $SH(\lambda)$ is of type II for every $\lambda\in \boldsymbol{R}$.

(ii) Suppose that $N=2$. If

$$\int_{r_0}^{\infty} r\log r\, m^*(r)dr = \infty,$$

then $\lambda_*(\infty)\geqq\zeta_*$ and $SH(\lambda)$ is of type I for $\lambda<\zeta_*$, and of type III for $\zeta^*<\lambda<\lambda^*(\infty)$ provided $\zeta^*<\lambda^*(\infty)$.

(iii) Suppose that $N\geqq 3$. If

$$\int_{r_0}^{\infty} rm^*(r)dr < \infty,$$

then $\lambda^*(\infty)=\infty$ and $SH(\lambda)$ is of type I for every $\lambda\in \boldsymbol{R}$.

EXAMPLE 3.2. Consider the equation

$$-\Delta u + \mu c(x)u = \lambda m(x)u \quad \text{in } G_R, \quad (R\geqq e^e), \tag{3.32}$$

where

$$c(x) = \begin{cases} -(2|x|\log|x|)^{-2}, & \text{if } N=2, \\ -(N-2)^2/4|x|^2, & \text{if } N\geqq 3, \end{cases}$$

$$m(x) = \begin{cases} (|x|\log|x|(\log(\log|x|)))^{-2}, & \text{if } N=2, \\ (|x|\log|x|)^{-2}, & \text{if } N\geqq 3. \end{cases}$$

In this case it is obvious that $\lim_{|x|\to\infty}\zeta(x)=-\infty$, where $\zeta(x)=c(x)/m(x)$. Applying the oscillation theory ([17], [38]) and Theorem 2.2 to the equation

$$-\Delta\psi = -\mu c(x)\psi \quad \text{in } G_R, \quad (R\geqq e^e),$$

we see that $\mu^*(\infty)=1$. Therefore, by Lemma 3.3 and the oscillation theory, we obtain $\lambda^*(\infty)=\infty$ if $\mu<1$, $\lambda^*(\infty)=1/4$ if $\mu=1$ and $\lambda^*(\infty)=-\infty$ if $\mu>1$. Furthermore, from Theorems 3.3–3.5 it follows that if $N\geqq 3$, $SH(\lambda)$ is of type I for $\lambda<\lambda^*(\infty)$ and that if $N=2$, $SH(\lambda)$ is of type I, II or III for $\lambda<\lambda^*(\infty)$ according to whether $\mu<0$, $\mu=0$ or $0<\mu\leqq 1$.

EXAMPLE 3.3. Consider the equation

$$-\mathfrak{D}u = \lambda m(x)u \quad \text{in } G_R, \tag{3.33}$$

where $\mathfrak{D}$ is as in Section 2 with bounded coefficients. Assume that there exist M^*, $M_*\in C^{\sigma}_{\mathrm{loc}}(0,\infty)$ such that $0<M_*(|x|)\leqq m(x)\leqq M^*(|x|)$, $x\in G_R$.

(i) Suppose that

$$\limsup_{|x|\to\infty} \sum_{i=1}^N b_i(x)x_i/|x| < 0. \tag{3.34}$$

If

$$\int_{r_0}^{\infty} M^*(r)dr < \infty, \tag{3.35}$$

then $\lambda^*(\infty)=\infty$ and $SH(\lambda)$ is of type II for every $\lambda \in \boldsymbol{R}$.

(ii) Suppose that (3.34) holds and

$$\int_{r_0}^{\infty} M_*(r)dr = \infty.$$

Then $SH(\lambda)$ is of type I or II acoding as $\lambda<0$ or $\lambda=0$. If $\lambda^*(\infty)>0$, then $SH(\lambda)$ is of type III for $0<\lambda<\lambda^*(\infty)$.

(iii) Suppose that

$$\liminf_{|x|\to\infty} \textstyle\sum_{i=1}^{N} b_i(x)x_i/|x| > 0.$$

If M^* satisfies (3.35), then $\lambda^*(\infty)=\infty$ and $SH(\lambda)$ is of type I for every $\lambda \in \boldsymbol{R}$.

PROOF. (i) In view of (3.34) and the boundedness of a_{ij}, one can take a constant $\kappa^*>0$ such that the function $B^*(r)=-\kappa^*$ satisfies $b^*(r)\leqq B^*(r)$ for $r\geqq r_0$, provided $r_0>0$ is sufficiently large. Since the function

$$P^*(r) = \exp\left(\int_{r_0}^{r} B^*(s)ds\right) = ce^{-\kappa^* r} \quad (c=e^{\kappa^* r_0})$$

satisfies by (3.35)

$$\int_{r_0}^{\infty} dr/P^*(r) = \infty, \quad \int_{r_0}^{\infty}\left(\int_{r_0}^{r} ds/P^*(s)\right)P^*(r)M^*(r)dr < \infty,$$

the assertion follows from (i) of Theorem 3.3 and Remark 3.1.

Statements (ii) and (iii) can be proved similarly.

4. Global positive solutions of linear equations

4.1. Supersolutions and subsolutions

In this section we study the existence of positive solutions, with specified asymptotic behavior, of the problem

$$-\mathfrak{D}u + c(x)u = \lambda m(x)u \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset). \tag{A}$$

More precisely, we want to obtain conditions guaranteeing the existence of unbounded positive solutions satisfying

$$0 < \liminf_{|x|\to\infty} u(x)\Big/\int_{r_0}^{|x|} dr/p^*(r) \leqq \limsup_{|x|\to\infty} u(x)\Big/\int_{r_0}^{|x|} dr/p^*(r) < \infty, \tag{4.1}$$

if the condition (H_1) holds, and the existence of bounded positive solutions satisfying

$$(4.2) \qquad 0 < \liminf_{|x|\to\infty} u(x) \leqq \limsup_{|x|\to\infty} u(x) < \infty,$$

if the condition (H_2) holds.

For this purpose we prepare two lemmas. In the first lemma we let $G=\Omega_k$ and $\Lambda=\Gamma_k$ for some k, and suppose that G is separated by an $(N-1)$-dimensional $C^{2+\sigma}$-manifold Λ_0 in two subdomains G_l, $l=1, 2$, as follows:

$$G = G_1 \cup G_2 \cup \Lambda_0,\ \Gamma \cap \Lambda_0 = \Lambda \cap \Lambda_0 = \emptyset,\ \partial G_1 = \Gamma \cup \Lambda_0,\ \partial G_2 = \Lambda \cup \Lambda_0.$$

Let us denote by $\nu=(\nu_1, \nu_2,\dots, \nu_N)$ the outward (with respect to G_1) unit conormal vector to Λ_0:

$$\nu_i = (\textstyle\sum_{j=1}^N a_{ij}(x)n_j)(\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)n_j)^2)^{-1/2}, \quad i = 1, 2,\dots, N,$$

where $n=(n_1, n_2,\dots, n_N)$ is the outward (with respect to G_1) unit normal vector to Λ_0. Furthermore, λ_0 denotes the principal eigenvalue of the problem

$$(4.3) \qquad \begin{aligned} &-\mathfrak{D}\varphi + c(x)\varphi = \lambda m(x)\varphi \quad \text{in } G, \\ &B\varphi = 0 \quad \text{on } \Gamma, \quad \varphi = 0 \quad \text{on } \Lambda. \end{aligned}$$

LEMMA 4.1. *Let $u\in C(\bar{G})$ and let $u_l=u|_{\bar{G}_l}$ be the restriction of u on $\bar{G}_l$. Suppose that $u_1\in C^2(G_1)\cap C^1(\bar{G}_1)$, $u_2\in C^2(G_2)\cap C^1(G_2\cup\Lambda_0)$ and u_l, $l=1, 2$, satisfy*

$$(4.4) \qquad -\mathfrak{D}u_l + c(x)u_l \geqq \lambda m(x)u_l \quad \text{in } G_l, \quad l = 1, 2,$$

$$(4.5) \qquad \partial u_1(x)/\partial\nu - \partial u_2(x)/\partial\nu \geqq 0 \quad \text{on } \Lambda_0,$$

$$(4.6) \qquad Bu_1 \geqq 0 \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset), \quad u_2 \geqq 0 \quad \text{on } \Lambda.$$

If $\lambda<\lambda_0$, then $u(x)>0$ in G unless $u(x)\equiv 0$ in G.

In case $\alpha(x)\equiv 0$ on Γ the regularity condition on u_1 may be replaced by the weaker condition $u_1\in C^2(G_1)\cap C^1(G_1\cup\Lambda_0)$.

PROOF. Suppose that $u(x)\not\equiv 0$ in G. Since $\lambda<\lambda_0$, the problem

$$\begin{aligned} &-\mathfrak{D}w + c(x)w = \lambda m(x)w \quad \text{in } G, \\ &Bw = 0 \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset \text{ and } \alpha(x)>0),\ w = 1 \quad \text{on } \Lambda, \\ &w = 1 \quad \text{on } \Gamma \cup \Lambda \ (\text{if } \alpha(x)\equiv 0) \end{aligned}$$

has a solution $w\in C^{2+\sigma}(\bar{G})$ satisfying $w(x)>0$ on $\bar{G}$ (cf. Lemma 2.1). Set $v=u/w$ and $v_l=v|_{\bar{G}_l}$, $l=1, 2$. Then, we have

$$(4.7) \qquad -\mathfrak{D}v_l - 2w^{-1}\textstyle\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)\partial w/\partial x_j)\partial v_l/\partial x_i \geqq 0 \quad \text{in } G_l,\ l = 1, 2,$$

(4.8) $\quad \partial v_1(x)/\partial \nu - \partial v_2(x)/\partial \nu = w(x)^{-1}(\partial u_1(x)/\partial \nu - \partial u_2(x)/\partial \nu) \geqq 0 \quad \text{on } \Lambda_0,$

(4.9) $\quad \partial v_1/\partial \beta \geqq 0 \quad \text{on } \Gamma$ (if $\Gamma \neq \emptyset$ and $\alpha(x)>0$),

(4.10) $\quad v_1 \geqq 0 \quad \text{on } \Gamma$ (if $\Gamma \neq \emptyset$ and $\alpha(x)\equiv 0$),

(4.11) $\quad v_2 \geqq 0 \quad \text{on } \Lambda.$

There is a point $x_0 \in \bar{G}$ such that $v(x_0)=\min_{x\in \bar{G}} v(x)$. Suppose that x_0 is an interior point of G. Then, applying the maximum principle to v_l, $l=1, 2$, we have $x_0 \in \Lambda_0$,

$$\partial v_1(x_0)/\partial \nu \leqq 0 \quad \text{and} \quad \partial v_2(x_0)/\partial \nu \geqq 0,$$

and since $v \not\equiv 0$, at least one of the inequalities must be strict. This, however, contradicts (4.8). Hence we have $x_0 \in \Gamma$ or $x_0 \in \Lambda$. In the case where $\alpha(x)\equiv 0$ on Γ, we have $v(x_0)\geqq 0$ by (4.10) and (4.11). In the case where $\alpha(x)>0$ on Γ, by (4.7), (4.9) and the maximum principle, we see that $x_0 \notin \Gamma$. Thus, $x_0 \in \Lambda$ and so $v(x_0)\geqq 0$ by (4.11). Since v cannot take the minimum in G, we have $v(x)>0$, and hence $u(x)=v(x)w(x)>0$ in G. Thus the proof is complete.

We now consider a semilinear boundary value problem

(4.12) $\qquad -\mathfrak{D}u + c(x)u = f(x, u) \text{ in } \Omega, \quad Bu = g(x) \text{ on } \Gamma$ (if $\Gamma \neq \emptyset$).

First, we introduce the definition of a supersolution and a subsolution of (4.12).

Let G_1 be a bounded subdomain of Ω with boundary $\partial G_1 = \Gamma \cup \Lambda_0 \in C^{2+\sigma}$ such that $\Gamma \cap \Lambda_0 = \emptyset$ and put $G_2 = \Omega \backslash \bar{G}_1$, i.e., $\Omega = G_1 \cup G_2 \cup \Lambda_0$.

A function $\hat{u} \in C(\bar{\Omega})$ is said to be a *supersolution* of (4.12) if it satisfies the following relations.

$\hat{u}_l = \hat{u}|_{\bar{G}_l}$, $l=1, 2$, satisfy $\hat{u}_l \in C^{2+\sigma}(G_l) \cap C^1(\bar{G}_l)$ and

(4.13) $\qquad -\mathfrak{D}\hat{u}_l + c(x)\hat{u}_l \geqq f(x, \hat{u}_l) \quad \text{in } G_l, \quad l = 1, 2,$

(4.14) $\qquad \partial \hat{u}_1/\partial \nu - \partial \hat{u}_2/\partial \nu \geqq 0 \quad \text{on } \Lambda_0,$

(4.15) $\qquad B\hat{u}_1 \geqq g(x) \quad \text{on } \Gamma$ (if $\Gamma \neq \emptyset$),

where ν is the outward (with respect to G_1) unit conormal vector to Λ_0.

A *subsolution* $\bar{u} \in C(\bar{\Omega})$ of (4.12) is defined by reversing the inequality signs in the above definition.

LEMMA 4.2. *Assume that $f \in C^{\sigma}_{loc}(\bar{\Omega} \times \boldsymbol{R})$ and $g \in C^{l+\sigma}(\Gamma)$, where $l=2$ if $\alpha(x)\equiv 0$ on Γ and $l=1$ if $\alpha(x)>0$ on Γ. Moreover, assume that for any compact set $G \subset \bar{\Omega}$ and finite interval $I \subset \boldsymbol{R}$, there is a positive constant K such that $f(x, u)+Ku$ is nondecreasing in $u \in I$ for any fixed $x \in G$.*

If there exist a supersolution $\hat{u}$ and a subsolution $\bar{u}$ of (4.12) *such that*

$$\bar{u}(x) \leqq \hat{u}(x) \quad in\ \Omega,$$

then (4.12) *has a solution* $u \in C^2(\bar{\Omega})$ *satisfying*

$$\bar{u}(x) \leqq u(x) \leqq \hat{u}(x) \quad in\ \Omega. \tag{4.16}$$

PROOF. Let $\{\Omega_k\}$ be a sequence of subdomains of Ω as mentioned in Section 2 such that $G_1 \subset \Omega_1$ and $\Lambda_0 \cap \Gamma_1 = \emptyset$.

For every $k \geqq 1$, choose a constant c_k such that $c_k \geqq \max\{|c(x)|: x \in \bar{\Omega}_k\} + 1$ and $f(x, u) + c_k u$ is nondecreasing in $\min_{\bar{\Omega}_k} \bar{u} \leqq u \leqq \max_{\bar{\Omega}} \hat{u}_k$ for any fixed $x \in \bar{\Omega}_k$.

Consider the following iteration scheme:

$$\begin{aligned} &-\mathfrak{D}v_j + (c(x)+c_k)v_j = f(x, v_{j-1}(x)) + c_k v_{j-1}(x) \quad \text{in } \Omega_k, \\ &Bv_j = g(x) \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset), \quad v_j = \hat{u}(x) \quad \text{on } \Gamma_k, \\ &v_0(x) = \hat{u}(x) \quad \text{on } \bar{\Omega}_k. \end{aligned} \tag{4.17}$$

Since $c(x) + c_k \geqq 1$ on $\bar{\Omega}_k$, (4.17) has a unique solution $v_j \in C^{2+\sigma'}(\bar{\Omega}_k)$, $0 < \sigma' < \sigma$, with the property

$$\bar{u}(x) \leqq v_{j+1}(x) \leqq v_j(x) \leqq \hat{u}(x), \quad x \in \Omega_k, \quad j = 1, 2, \dots. \tag{4.18}$$

In fact, the existence of $\{v_j\}$ is well known (see e.g. [13, Theorem 6.31]) and (4.18) is a consequence of Lemma 4.1 as follows. First note that the principal eigenvalue of the problem (4.3) with replaced $c(x)$ by $c(x) + c_k$ is positive. Put $w(x) = v_0(x) - v_1(x)$ on $\bar{\Omega}_k$ and $w_l = w|_{\bar{G}_l}$, $l = 1, 2$, where $G_2 = \Omega_k \backslash \bar{G}_1$. Then, $w \in C(\bar{\Omega}_k)$, $w_l \in C^{2+\sigma'}(G_l) \cap C^1(\bar{G}_l)$, $l = 1, 2$, and w_l satisfy

$$\begin{aligned} &-\mathfrak{D}w_l + (c(x)+c_k)w_l \geqq 0 \quad \text{in } G_l, \quad l = 1, 2, \\ &\partial w_1/\partial \nu - \partial w_2/\partial \nu = \partial \hat{u}/\partial \nu - \partial \hat{u}/\partial \nu \geqq 0 \quad \text{on } \Lambda_0, \\ &Bw_1 \geqq 0 \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset), \quad w_2 = 0 \quad \text{on } \Gamma_k. \end{aligned}$$

From this and Lemma 4.1 it follows that

$$w(x) \geqq 0, \quad \text{i.e., } v_0(x) \geqq v_1(x) \quad \text{in } \Omega_k.$$

The inequality $\bar{u}(x) \leqq v_1(x)$ in Ω_k is proved similarly. An induction shows that (4.18) holds.

By (4.18) the sequence $\{v_j\}$ is uniformly bounded on $\bar{\Omega}_k$, and so proceeding as in the proof of (i) of Theorem 2.1, we have a function $u_k \in C^{2+\sigma''}(\bar{\Omega}_k)$, $0 < \sigma'' < \sigma'$, satisfying

$$\begin{aligned} &-\mathfrak{D}u_k + c(x)u_k = f(x, u_k) \quad \text{in } \Omega_k, \\ &Bu_k = g(x) \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset), \quad u_k = \hat{u}(x) \quad \text{on } \Gamma_k, \\ &\bar{u}(x) \leqq u_k(x) \leqq \hat{u}(x) \quad \text{in } \Omega_k. \end{aligned}$$

Using the usual compactness argument for the sequence $\{u_k\}$, we obtain a desired solution of (4.12). The details may be omitted. This completes the proof.

In what follows by a positive solution of (A) we mean a function $u \in C^2(\bar{\Omega})$ which is positive throughout Ω and satisfies (A).

Let us list up the conditions which are assumed from now on.

(H_1) and (H_2) are stated in Section 3.

(H_3) $\displaystyle\int_{r_0}^{\infty}\left(\int_{r_0}^{r} ds/p^*(s)\right) p^*(r)(c^*(r)+m^*(r))dr < \infty.$

(H_4) $\displaystyle\int_{r_0}^{\infty}(b^*(r)-b_*(r))dr < \infty.$

(H_5) $\displaystyle\int_{r_0}^{\infty}\left(\int_{r}^{\infty} ds/p_*(s)\right) p_*(r)(c^*(r)+m^*(r))dr < \infty.$

(H_5') There exists a function $M \in C^{\sigma}(\bar{\Omega})$ such that $M(x)>0$, $|m(x)| \leqq M(x)$, $x \in \bar{\Omega}$, and

$$\int_{r_0}^{\infty}\left(\int_{r}^{\infty} ds/p_*(s)\right) p_*(r)(c^*(r)+M^*(r))dr < \infty,$$

where $M^*(r)=\max_{|x|=r} M(x)/a(x)$.

4.2. Existence of unbounded positive solutions

THEOREM 4.1. *Suppose that* (H_1), (H_3) *and* (H_4) *hold and* $c(x) \geqq 0$ *in* Ω. *Then, the problem* (A) *has a positive solution* u *satisfying* (4.1) *for every* $\lambda<0$.

PROOF. Taking $r_0>0$ so that $\{x: |x|>r_0\} \subset \Omega$, we put

$$G_1 = \Omega \cap B(0, r_0), \quad G_2 = \Omega \backslash \bar{G}_1, \quad \Lambda_0 = \{x: |x|=r_0\}.$$

Since (H_1) and (H_3) hold and $p^*(r)(c_*(r)-\lambda m_*(r))>0$, by (i–a) of Lemma 3.1 the problem

$$(4.19) \qquad \begin{aligned} &(p^*(r)y')' - p^*(r)(c_*(r)-\lambda m_*(r))y = 0, \quad r > r_0, \\ &y(r_0) = 1, \quad y'(r_0) = 0 \end{aligned}$$

has a positive solution φ such that $\varphi'(r)>0$ on $[r_0, \infty)$ and

$$(4.20) \qquad \lim_{r\to\infty} \varphi(r) \Big/ \int_{r_0}^{r} ds/p^*(s) = \text{constant} > 0.$$

Now, define a function $\hat{u}$ by

$$\hat{u}(x) = 1 \quad \text{for} \quad x \in \bar{G}_1; \; \hat{u}(x) = \varphi(|x|) \quad \text{for} \quad x \in \bar{G}_2.$$

Then we see that $\hat{u}$ is a supersolution of (A). In fact, it is obvious that $\hat{u} \in C(\bar{\Omega})$, $\hat{u}_l = \hat{u}|_{\bar{G}_l} \in C^2(G_l) \cap C^1(\bar{G}_l)$, $l=1, 2$, $B\hat{u}_1 \geqq 0$ on Γ and $\partial\hat{u}_1/\partial\nu = \partial\hat{u}_2/\partial\nu = 0$ on Λ_0. So, we prove that

$$-\mathfrak{D}\hat{u}_l + (c(x)-\lambda m(x))\hat{u}_l \geqq 0 \quad \text{in } G_l, \quad l = 1, 2.$$

For $\hat{u}_1$, this is obvious, and for $\hat{u}_2$, this is verified as follows:

$$\begin{aligned}
&-\mathfrak{D}\hat{u}_2 + (c(x)-\lambda m(x))\hat{u}_2 \\
&\qquad = -a(x)\varphi''(|x|) - b(x)\varphi'(|x|) + (c(x)-\lambda m(x))\varphi(|x|) \\
&= -a(x)(\varphi''(|x|) + b^*(|x|)\varphi'(|x|)) \\
&\qquad + (a(x)b^*(|x|) - b(x))\varphi'(|x|) + (c(x)-\lambda m(x))\varphi(|x|) \\
&\geqq (c(x)-\lambda m(x)-a(x)(c_*(|x|)-\lambda m_*(|x|)))\varphi(|x|) \geqq 0, \quad x \in G_2,
\end{aligned}$$

where we have used the relations $\varphi'(r)>0$ and

$$\varphi''(r) + b^*(r)\varphi'(r) - (c_*(r)-\lambda m_*(r))\varphi(r) = 0$$

on (r_0, ∞), the latter being a reformulation of (4.19). Thus $\hat{u}$ is a supersolution of the problem (A).

Before constructing a subsolution $\bar{u}$ of (A) satisfying $\bar{u}(x) \leqq \hat{u}(x)$ in Ω, we note that (H_1) and (H_4) yield

$$\begin{aligned}
&\lim_{r\to\infty}\left(\int_{r_0}^{r} ds/p_*(s)\right)\left(\int_{r_0}^{r} ds/p^*(s)\right)^{-1} \\
&\quad = \lim_{r\to\infty} p^*(r)/p_*(r) = \exp\left(\int_{r_0}^{\infty}(b^*(s)-b_*(s))ds\right) < \infty,
\end{aligned}$$

and for some constant $M_1>0$

$$\int_{r_0}^{r} ds/p^*(s) \leqq \int_{r_0}^{r} ds/p_*(s) \leqq M_1\int_{r_0}^{r} ds/p^*(s), \quad r \geqq r_0. \tag{4.21}$$

We take the solution ψ of the problem

$$\begin{aligned}
&z'' + b_*(r)z' - (c^*(r)-\lambda m^*(r))z = 0, \quad r > r_0, \\
&z(r_0) = 0, \quad z'(r_0) = 1.
\end{aligned}$$

Since (H_1), (4.21) and (H_3) hold, by (i–a) of Lemma 3.1, ψ satisfies $\psi'(r)>0$ on $[r_0, \infty)$ and

$$\lim_{r\to\infty}\psi(r)\Big/\int_{r_0}^{r} ds/p_*(s) = \text{constant} > 0.$$

Combining this with (4.20) and (4.21) we have

$$M_2 \int_{r_0}^{r} ds/p^*(s) \leqq M_3\psi(r) \leqq \varphi(r), \quad r \geqq r_0 \tag{4.22}$$

for some positive constants M_2 and M_3. It is easy to see that the function $\bar{u}$ defined by

$$\bar{u}(x) = 0 \quad \text{for } x \in \bar{G}_1; \quad \bar{u}(x) = M_3\psi(|x|) \quad \text{for } x \in \bar{G}_2$$

is a subsolution of (A) which satisfies $\bar{u}(x) \leqq \hat{u}(x)$ in Ω by (4.22). From Lemma 4.2 it follows that the problem (A) has a solution u such that

$$\bar{u}(x) \leqq u(x) \leqq \hat{u}(x) \quad \text{in } \Omega. \tag{4.23}$$

The positivity of u follows from the maximum principle, and the relation (4.1) is a consequence of (4.21)–(4.23). This completes the proof.

As easily seen, Theorem 4.1 applies to the case where $\Omega = \boldsymbol{R}^N$ and guarantees the existence of unbounded entire solutions of (A) with $\lambda<0$ (and with the boundary condition deleted). However, the situation is different for (A) with $\lambda \geqq 0$. In fact, if $\lambda \geqq 0$ and $c(x) \equiv 0$, then there is no positive entire solution of (A) satisfying (4.1) in $\boldsymbol{R}^N$, because such a solution becomes a positive constant by the maximum principle, which is impossible. The existence of a positive solution of (A) with $\lambda>0$ in exterior domain Ω is given in the following theorem.

THEOREM 4.2. *Suppose that Ω is an exterior domain such that $0 \in \Omega_0 = \boldsymbol{R}^N \backslash \bar{\Omega}$. Moreover, suppose that* (H_1), (H_3) *and* (H_4) *hold, $c(x) \geqq 0$ in Ω and $0 \leqq \alpha(x) < 1$ on Γ. Then, there is a constant $\tilde{\lambda}>0$ such that for every $\lambda<\tilde{\lambda}$, the problem* (A) *has a positive solution u satisfying* (4.1).

PROOF. We need only to consider the case where $\lambda \geqq 0$.

Let $r_0>0$, G_1 and G_2 be as in the proof of Theorem 4.1. We shall construct a supersolution of (A). First, we can choose $\lambda'>0$ and $\hat{u}_1 \in C^{2+\sigma}(\bar{G}_1)$ such that $\hat{u}_1(x)>0$ on $\bar{G}_1$ and

$$-\mathfrak{D}\hat{u}_1 + c(x)\hat{u}_1 \geqq \lambda' m(x)\hat{u}_1 \quad \text{in } G_1, \tag{4.24}$$

$$B\hat{u}_1 \geqq 0 \quad \text{on } \Gamma, \quad \partial\hat{u}_1/\partial\nu > 0 \quad \text{on } \Lambda_0. \tag{4.25}$$

In fact, put

$$a_* = \min_{x\in\bar{G}_1} a(x), \quad \bar{b} = \max\{\max_{x\in\bar{G}_1} b(x),\, 1\}, \quad \hat{b} = \bar{b}/2a_*,$$
$$\bar{\alpha} = \max_{x\in\Gamma} \alpha(x), \quad \kappa = (1-\bar{\alpha})/\bar{\alpha}, \quad \bar{m} = \max_{x\in\bar{G}_1} m(x)/a(x),$$

and for $\delta = (\hat{b}^2 - \lambda\bar{m})^{1/2}$, $\lambda \in (0, \hat{b}^2/\bar{m})$, define φ_1 by

$$\varphi_1(r) = e^{-\hat{b}r}(\delta\cosh\delta r + (\hat{b}+\kappa)\sinh\delta r)/\delta.$$

Then, φ_1 satisfies

$$\varphi_1'' + 2\hat{b}\varphi_1' + \lambda\bar{m}\varphi_1 = 0, \quad r > 0,$$
$$\varphi_1(0) = 1, \quad \varphi_1'(0) = \kappa.$$

Noting that

$$\lim_{\lambda\to 0} \kappa\delta/(\hat{b}^2+\kappa\hat{b}-\delta^2) = 1,$$
$$\lim_{\lambda\to 0} (\hat{b}^2\delta+2\hat{b}\delta\kappa-\delta^3)/(\hat{b}^3+\hat{b}^2\kappa-\hat{b}^2\delta+\delta^2\kappa) = 1,$$

we can choose $\lambda=\lambda'$ so that

$$\varphi_1'(r) > 0, \quad \varphi_1''(r) < 0, \quad 0 < r \leqq r_0.$$

Define $\hat{u}_1(x)=\varphi_1(|x|)$, $x\in\bar{G}_1$. We then see that $\hat{u}_1\in C^{2+\sigma}(\bar{G}_1)$, and

$$\begin{aligned}
&-\mathfrak{D}\hat{u}_1(x) + c(x)\hat{u}_1(x)\\
&= -a(x)(\varphi_1''(|x|)+2\hat{b}\varphi_1'(|x|)) + (2\hat{b}a(x)-b(x))\varphi_1'(|x|) + c(x)\varphi_1(|x|)\\
&\geqq \lambda'\bar{m}a(x)\varphi_1(|x|) \geqq \lambda' m(x)\hat{u}_1(x) \quad \text{in } G_1,\\
B\hat{u}_1(x) &= \alpha(x)(\varphi_1'(|x|)\textstyle\sum_{i=1}^N \beta_i x_i/|x| + (1-\alpha(x))\varphi_1(|x|)/\alpha(x))\\
&\geqq \alpha(x)(-\varphi_1'(|x|) + (1-\bar{\alpha})\varphi_1(|x|)/\bar{\alpha})\\
&= \alpha(x)\varphi_1(|x|)(-\varphi_1'(|x|)/\varphi_1(|x|)+\kappa) \geqq 0 \quad \text{on } \Gamma \text{ (if } \alpha(x)>0),\\
B\hat{u}_1(x) &= \varphi_1(|x|) > 0 \quad \text{on } \Gamma \text{ (if } \alpha(x)\equiv 0),\\
\partial\hat{u}_1(x)/\partial\nu &= \varphi_1'(r_0)a(x)(\textstyle\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)x_j/|x|)^2)^{-1/2} > 0 \quad \text{on } \Lambda_0,
\end{aligned}$$

where we have used the relation $0<\varphi_1'(r)/\varphi_1(r)\leqq\varphi_1'(0)/\varphi_1(0)=\kappa$ for $r\leqq r_0$. Thus, the existence of $\lambda'>0$ and $\hat{u}_1\in C^2(\bar{G}_1)$ satisfying (4.24) and (4.25) is proved.

Next, by (H_1), (H_3) and (i–a) of Lemma 3.1, we can choose $\lambda''>0$ so small that

$$y'' + b^*(r)y' + \lambda'' m^*(r)y = 0, \quad r \geqq r_0,$$
$$y(r_0) = \varphi_1(r_0), \quad y'(r_0) = \varphi_1'(r_0)$$

has a positive solution φ_2 satisfying $\varphi_2'(r)>0$ on $[r_0, \infty)$ and (4.20) with $\varphi=\varphi_2$. It is easy to see that $\hat{u}_2(x)=\varphi_2(|x|)$ satisfies $\hat{u}_2\in C^{2+\sigma}(\bar{G}_2)$ and

$$-\mathfrak{D}\hat{u}_2 + c(x)\hat{u}_2 \geqq \lambda'' m(x)\hat{u}_2 \quad \text{in } G_2,$$
$$\hat{u}_2(x) = \hat{u}_1(x), \quad \partial\hat{u}_2(x)/\partial\nu = \partial\hat{u}_1(x)/\partial\nu \quad \text{on } \Lambda_0.$$

Hence, the function $\hat{u}$ defined by

$$\hat{u}(x) = \hat{u}_1(x) \quad \text{for } x\in\bar{G}_1; \ \hat{u}(x) = \hat{u}_2(x) \quad \text{for } x\in\bar{G}_2,$$

is a supersolution of (A) for $0<\lambda<\tilde{\lambda}$, where $\tilde{\lambda}=\min\{\lambda', \lambda''\}$, and satisfies

$$\lim_{|x|\to\infty} \hat{u}(x)\Big/\int_{r_0}^{|x|} ds/p^*(s) = \text{constant} > 0.$$

To construct a subsolution of (A) for $0<\lambda<\tilde{\lambda}$, we take a positive solution v of the problem

$$-\mathfrak{D}v + c(x)v = -m(x)v \quad \text{in } \Omega, \quad Bv = 0 \quad \text{on } \Gamma,$$

satisfying (4.1). This is possible by Theorem 4.1. Since both $\hat{u}$ and v satisfy (4.1), we can choose a constant $M>0$ so that

$$Mv(x) \leqq \hat{u}(x) \quad \text{in } \Omega.$$

The function $\bar{u}=Mv(x)$ is clearly a subsolution of (A) with the property $\bar{u}(x)\leqq\hat{u}(x)$ in Ω, and so, by Lemma 4.2 (A) has a desired solution u for every $\lambda\in[0, \tilde{\lambda})$. This completes the proof.

4.2. Existence of bounded positive solutions

THEOREM 4.3. *Assume that* (H_2) *and* (H_5) *hold and* $c(x)\geqq 0$ *in* Ω. *Then, there is a constant* $\tilde{\lambda}>0$ *such that for every* $\lambda<\tilde{\lambda}$, *the problem* (A) *has positive solutions satisfying* (4.2).

PROOF. First we consider the case where $\lambda<0$. Let $r_0>0$, G_1 and G_2 be as in the proof of Theorem 4.1. By (ii–b) of Lemma 3.2, the initial value problem

$$z'' + b_*(r)z' - (c^*(r) - \lambda m^*(r))z = 0, \quad r > r_0,$$
$$z(r_0) = 0, \quad z'(r_0) = 1$$

has a positive solution ψ satisfying $\psi'(r)>0$ on $[r_0, \infty)$ and $\lim_{r\to\infty}\psi(r)=\text{constant}>0$. Define a function $\bar{u}$ by

$$\bar{u}(x) = 0 \quad \text{for } x\in\bar{G}_1; \ \bar{u}(x) = \psi(|x|) \quad \text{for } x\in\bar{G}_2.$$

Then, $\bar{u}$ is a subsolution of (A) and $\hat{u}(x)\equiv\lim_{r\to\infty}\psi(r)$ on $\bar{\Omega}$ is a supersolution of (A) satisfying $\bar{u}(x)\leqq\hat{u}(x)$ in Ω. Lemma 4.2 then implies the existence of a solution u of (A) lying between $\bar{u}(x)$ and $\hat{u}(x)$ in Ω.

Next we consider the case $\lambda\geqq 0$. By (ii–a) of Lemma 3.2, there exist $\tilde{r}_0>0$ and $\eta_0>0$ such that $\Omega\supset\{x: |x|>\tilde{r}_0\}$ and

$$y'' + b_*(r)y' + m^*(r)y = 0, \quad r > \tilde{r}_0,$$
$$y(\tilde{r}_0) = 1, \quad y'(\tilde{r}_0) = -\eta_0$$

has a positive solution φ satisfying $\varphi'(r)<0$ on $[\tilde{r}_0, \infty)$ and $\lim_{r\to\infty}\varphi(r)=\text{constant}$

>0. Put $G_1'=\Omega\cap B(0,\tilde{r}_0)$ and let $\lambda_1>0$ be the principal eigenvalue of the problem

$$-\mathfrak{D}\varphi=\lambda m(x)\varphi \quad \text{in } G_1',$$
$$\partial\varphi/\partial\beta=0 \quad \text{on } \Gamma \text{ (if } \Gamma\neq\emptyset), \quad \varphi=0 \quad \text{on } \Lambda_0'=\{x\colon |x|=\tilde{r}_0\}.$$

For $0\leqq\lambda<\lambda_1$, the problem

$$-\mathfrak{D}u=\lambda m(x)u \quad \text{in} \quad G_1',$$
$$\partial u/\partial\beta=0 \quad \text{on } \Gamma \text{ (if } \Gamma\neq\emptyset), \quad u=1 \quad \text{on } \Lambda_0'$$

has a unique positive solution $u=u(x;\lambda)\in C^{2+\sigma}(\bar{G}_1')$. Since, for $0\leqq\lambda'<\lambda''<\lambda_1$, $v(x)=u(x;\lambda'')-u(x;\lambda')$ satisfies

$$-\mathfrak{D}v-\lambda'' m(x)v=(\lambda''-\lambda')m(x)u(x;\lambda')>0 \quad \text{in } G_1',$$
$$\partial v/\partial\beta=0, \quad \text{on } \Gamma \text{ (if } \Gamma\neq\emptyset), \quad v=0 \quad \text{on } \Lambda_0',$$

we have $v(x)\geqq 0$ by Lemma 4.1, and so

$$1\leqq u(x;\lambda')\leqq u(x;\lambda''), \quad x\in G_1', \quad 0\leqq\lambda'<\lambda''<\lambda_1.$$

Using L^p estimates $(p>N)$ and Sobolev's imbedding theorem for the solution $w(x;\lambda)=u(x;\lambda)-1$ of the problem

$$-\mathfrak{D}w=\lambda m(x)u(x;\lambda) \quad \text{in } G_1',$$
$$\partial w/\partial\beta=0 \quad \text{on } \Gamma \text{ (if } \Gamma\neq\emptyset),\ w=0 \quad \text{on } \Lambda_0',$$

we have

$$\begin{aligned}\|w(\cdot\,;\lambda)\|_{1\,G_1'}&\leqq M_1\|w(\cdot\,;\lambda)\|_{2,p,G_1'}\leqq \lambda M_2\|mw(\cdot\,;\lambda)\|_{0,p,G_1'}\\&\leqq \lambda M_2\|mw(\cdot\,;\lambda'')\|_{0,p,G_1'}\leqq M_3\lambda, \quad 0<\lambda<\lambda'',\end{aligned}$$

where the constants M_j are independent of λ for $0<\lambda<\lambda''$. Hence for any $\varepsilon>0$, we can choose a constant $\lambda_\varepsilon>0$ such that $\max_{x\in\Lambda_0'}|\partial u(x;\lambda)/\partial\nu|\leqq\varepsilon$ for any $0\leqq\lambda\leqq\lambda_\varepsilon$. For the above $\eta_0>0$, let

$$\varepsilon=\eta_0\min_{x\in\Lambda_0'}\{a(x)(\textstyle\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)x_j/|x|)^2)^{-1/2}\},$$

and define a function $\hat{u}$ by

$$\hat{u}(x)=u(x;\lambda_\varepsilon) \quad \text{for } x\in\bar{G}_1';\ \hat{u}(x)=\varphi(|x|) \quad \text{for } x\in\bar{G}_2',$$

where $G_2'=\Omega\backslash\bar{G}_1'$. Then, for every $0<\lambda<\tilde{\lambda}=\min\{\lambda_\varepsilon,1\}$, the function $\hat{u}$ is a supersolution of (A). To see this, we need only check the required relation on Λ_0', since the other relations can be checked by the same argument as in Theorem 4.1. We obtain

$$\partial\hat{u}_1/\partial\nu \geqq -\varepsilon,$$

$$\begin{aligned}\partial\hat{u}_2/\partial\nu &= \varphi'(r_0)a(x)(\textstyle\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)x_j/|x|)^2)^{-1/2}\\ &= -\eta_0 a(x)(\textstyle\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)x_j/|x|)^2)^{-1/2} \leqq -\varepsilon \quad \text{on } \Lambda_0'.\end{aligned}$$

This implies $\partial\hat{u}_1/\partial\nu - \partial\hat{u}_2/\partial\nu \geqq -\varepsilon+\varepsilon=0$ on Λ_0'.

As a subsolution $\bar{u}$ of (A) we take a solution of (A) with $\lambda=-1$ satisfying (4.2) and $\bar{u}(x)\leqq\hat{u}(x)$ in Ω; such a $\bar{u}$ exists from the first part of the above proof. Consequently, for every $0\leqq\lambda<\tilde{\lambda}$, there exists a solution u of (A) satisfying (4.2) by Lemma 4.2. The proof is thus complete.

COROLLARY 4.1. *Suppose that* (H_2), (H_5') *hold and* $c(x)\geqq 0$ *in* Ω. *Then, there is a constant* $\tilde{\lambda}>0$ *such that* (A) *has a positive solution u satisfying* (4.2) *provided* $|\lambda|<\tilde{\lambda}$.

PROOF. In view of Theorem 4.3, there is a $\tilde{\lambda}>0$ such that for $\mu<\tilde{\lambda}$

$$(4.26) \qquad -\mathfrak{D}v + c(x)v = \mu M(x)v \quad \text{in } \Omega, \quad Bv = 0 \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset)$$

has a solution v with the property (4.2). For $|\lambda|<\tilde{\lambda}$, let $\hat{u}$ and $\bar{u}$ be positive solutions of (4.26) with $\mu=|\lambda|$ and $\mu=-|\lambda|$, respectively, which satisfy (4.2) and $\bar{u}(x)\leqq\hat{u}(x)$ in Ω. Then, since $\hat{u}$ and $\bar{u}$ are, respectively, a supersolution and a subsolution of (A) with $\lambda\in(-\tilde{\lambda},\tilde{\lambda})$, the assertion follows from Lemma 4.2.

4.4. Example

EXAMPLE 4.1. We consider the problem:

$$(4.27) \qquad -\Delta u + c(x)u = \lambda m(x)u \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset),$$

where $c, m \in C^{\theta}_{loc}(\bar{\Omega})$, and $c(x)\geqq 0$, $m(x)>0$ on $\bar{\Omega}$. We note that

$$c^*(r) = \max_{|x|=r} c(x), \quad m^*(r) = \max_{|x|=r} m(x).$$

(i) Suppose that Ω is an exterior domain in $\boldsymbol{R}^2$ such that $0\in\boldsymbol{R}^2\backslash\bar{\Omega}$ and $\alpha(x)<1$ on Γ. If

$$\int_{r_0}^{\infty} r\log r\,(c^*(r)+m^*(r))dr < \infty,$$

then by Theorem 4.2 there is a constant $\tilde{\lambda}>0$ such that for $\lambda<\tilde{\lambda}$, (4.27) has a positive solution u with the property

$$(4.28) \qquad 0 < \liminf_{|x|\to\infty} u(x)/\log|x| \leqq \limsup_{|x|\to\infty} u(x)/\log|x| < \infty.$$

In the case where $\lambda<0$, the same assertion holds for $\Omega=\boldsymbol{R}^2$.

(ii) Suppose that $N\geqq 3$. If

$$\int_{r_0}^{\infty} r(c^*(r) + m^*(r))dr < \infty,$$

then by Theorem 4.3, there is a constant $\tilde{\lambda}>0$ such that for $\lambda<\tilde{\lambda}$, (4.27) has a solution u satisfying (4.2).

REMARK. The special case of (4.27) in which $\Omega=\boldsymbol{R}^N$, $N\geqq 3$ and $c(x)\equiv 0$ has recently been studied by Kawano [18].

5. Local existence of positive solutions of semilinear equations

In this section we consider the semilinear elliptic equation

$$\text{(C)} \qquad -\mathfrak{D}u + c(x)u = \lambda m(x)f(u) \quad \text{in } G_R,\ G_R = \{x\in \boldsymbol{R}^N\colon |x|>R\},$$

where $\mathfrak{D}$, $c(x)$ and $m(x)$ are as in the preceding sections, λ is a real parameter, anf f is continuous on $(0, \infty)$. Our purpose here is to develop local existence theorems for positive solutions of (C) on the basis of the results known for the associated linear equation

$$\text{(2.16)} \qquad -\mathfrak{D}u + c(x)u = \lambda m(x)u \quad \text{in } G_R.$$

By a local existence theorem for (C) we mean a theorem which guarantees the existence of solution of (C), with specific properties, in a "small" neighborhood of infinity, that is, in a domain G_R for R sufficiently large.

We assume without further mention that the value of $\lambda^*(\infty)$ associated with (2.16) is positive. Conditions on f are selected from the following list:

(F_1) $f\in C^{\theta}_{\mathrm{loc}}(0, \infty)$, $0<\theta<1$, $f(u)>0$ on $(0, \infty)$, and for any finite subinterval I of $(0, \infty)$ there is a $K>0$ such that $f(u)+Ku$ is nondecreasing on I.

(F_2) $f(+0)=0$ and $\hat{f}_0 \equiv \limsup_{u\to+0} f(u)/u < \infty$.

(F_3) $\hat{f}_\infty \equiv \limsup_{u\to\infty} f(u)/u < \infty$.

THEOREM 5.1. *Assume that* (F_1) *and* (F_2) *are satisfied. Then, the following statements hold.*

(i) *Suppose that* $0<\lambda<\lambda^*(\infty)/\hat{f}_0$ *and* $SH(\mu)$ *are of type* I *for all* $\mu<\lambda^*(\infty)$. *Then, for any* $h\in SH(\lambda')$ *with* $\lambda'\in(\lambda\hat{f}_0, \lambda^*(\infty))$, (C) *has a solution* u *satisfying*

$$\text{(5.1)} \qquad 0 < u(x) \leqq Mh(x) \quad \textit{in } G_R$$

for some constants $M>0$ *and* $R>0$.

(ii) *Suppose that* $0<\lambda<\lambda^*(\infty)/\hat{f}_0$ *and* $SH(\mu)$ *are of type* II *for all* $\mu\in [0, \lambda^*(\infty))$. *Then,* (C) *has a solution* u *satisfying*

$$(5.2) \qquad M \leqq u(x) \leqq M^{-1} \quad in \ G_R$$

for some constants $M>0$ *and* $R>0$.

(iii) *Suppose that* $SH(\mu)$ *are of type* III *for all* $\mu \in (0, \lambda^*(\infty))$. *Then, for any* $\lambda>0$, (C) *has no solution* u *satisfying* (5.2).

(iv) *Suppose that* $SH(\mu)$ *are of type* I *or* II *for all* $\mu<0$. *Then, for every* $\lambda<0$ *and* $h \in SH(\lambda')$ *with* $\lambda'<\lambda\hat{f}_0$, (C) *has a solution* u *satisfying*

$$(5.3) \qquad Mh(x) \leqq u(x) \leqq M^{-1} \quad in \ G_R$$

for some constants $M>0$ *and* $R>0$.

PROOF. (i) and (ii) Let $\lambda\hat{f}_0<\lambda'<\lambda^*(\infty)$ and $h \in SH(\lambda')$. Then there is an $R>0$ such that $h \in SH(\lambda'; G_R)$ and $h(x)>0$ on $\bar{G}_R$. Choose a constant $u_*>0$ so that

$$0 < f(u)/u < \lambda'/\lambda \quad \text{for } 0 < u \leqq u_*$$

and put $M=u_*/\sup_{x \in G_R} h(x)$. Then, the function $\hat{u}$ defined by $\hat{u}(x)=Mh(x)$ on $\bar{G}_R$ becomes a supersolution of (C) with boundary values $Mh(x)$ on Γ_R. Next, take a function $\bar{u} \in H(0; G_R)$ such that

$$0 < \bar{u}(x) \leqq Mh(x) \quad \text{in } \bar{G}_R.$$

This $\bar{u}$ is obviously a subsolution of (C) and satisfies $\bar{u}(x) \leqq \hat{u}(x)$ in G_R. From Lemma 4.2 there exists a solution u of (C) satisfying $\bar{u}(x) \leqq u(x) \leqq \hat{u}(x)$ in G_R.

(iii) Assume that u is a solution of (C) with the property (5.2), and put $\tilde{\lambda}=\lambda \inf \{f(u)/u\colon M \leqq u \leqq M^{-1}\}>0$. Then, since u is positive and $\tilde{\lambda}$-superharmonic in G_R, we see that $\tilde{\lambda} \leqq \lambda^*(R)$ and for any $h \in SH(\tilde{\lambda}; G_R)$, $h(x) \leqq M_1 u(x)$ in G_R for some $M_1>0$. This contradicts the hypothesis that $SH(\tilde{\lambda})$ is of type III.

(iv) Let $\lambda<0$ and $h \in H(\lambda'; G_R)$ with $\lambda'<\lambda\hat{f}_0$. Letting u_* and M be as in the proof of (i) and (ii), we have a subsolution $\bar{u}(x) \equiv Mh(x)$ of (C) with boundary values $Mh(x)$ on Γ_R. Furthermore, as a supersolution $\hat{u}$ of (C) we can take $\hat{u} \in SH(0; G_{R_1})$ such that

$$Mh(x) \leqq \hat{u}(x) \quad \text{in } G_{R_1}$$

for some $R_1>R$. Hence the assertion follows from Lemma 4.2. This finishes the proof.

THEOREM 5.2. *Assume that* (F_1) *and* (F_3) *are satisfied. Then, the following statements hold.*

(i) *Suppose that* $0<\lambda<\lambda^*(\infty)/\hat{f}_\infty$ *and* $SH(\mu)$ *are of type* II *or* III *for all* $\mu \in (0, \lambda^*(\infty))$. *Then, for any* $h \in SH(\lambda')$ *with* $\lambda' \in (\lambda\hat{f}_\infty, \lambda^*(\infty))$, (C) *has a solution* u *satisfying* (5.1).

(ii) *Suppose that* $\liminf_{u\to 0} f(u)/u>0$ *and* $SH(\mu)$ *are of type* II (*or of type* III) *for all* $\mu\in(0, \lambda^*(\infty))$. *Then, for any* $\lambda>0$, (C) *has no positive solution* u *satisfying*

$$\lim_{|x|\to\infty} u(x) = 0 \quad (\textit{resp.}\ \limsup_{|x|\to\infty} u(x) < \infty).$$

(iii) *Suppose that* $\lambda<0$ *and* $SH(\mu)$ *are of type* II *for all* $\mu\leqq 0$. *Then,* (C) *has a solution* u *satisfying* (5.2).

PROOF. (i) Let $\lambda\hat{f}_\infty<\lambda'<\lambda^*(\infty)$ and choose an $h\in SH(\lambda'; G_R)$ such that $h(x)>0$ on $\bar{G}_R$. Let $u^*>0$ be a constant such that

$$0 < f(u)/u < \lambda'/\lambda \quad \text{for } u \geqq u^*.$$

Set $M=u^*/\inf_{x\in G_R}h(x)$ and $\hat{u}(x)=Mh(x)$ on $\bar{G}_R$, and choose a function $\bar{u}\in H(0; G_R)$ satisfying $\bar{u}(x)\leqq\hat{u}(x)$ in G_R. Then, since $\hat{u}$ and $\bar{u}$ become, respectively, a supersolution and a subsolution of (C), the assertion follows from Lemma 4.2.

(ii) Let $SH(\mu)$ be of type II (or of type III) for all $\mu\in(0, \lambda^*(\infty))$ and assume that (C) has a positive solution u satisfying

$$\lim_{|x|\to\infty} u(x) = 0 \ (\text{resp. } \limsup_{|x|\to\infty} u(x)<\infty).$$

Putting $\tau=\sup_{x\in G_R} u(x)$ and $\tilde{\lambda}=\lambda\inf\{f(u)/u\colon 0<u\leqq\tau\}>0$, we see that u is positive and $\tilde{\lambda}$-superharmonic in G_R, and so we have $\tilde{\lambda}\leqq\lambda^*(R)\leqq\lambda^*(\infty)$ and $SH(\tilde{\lambda})$ is of type I (resp. of type II). This contradicts the hypothesis.

(iii) Take $\lambda'<\lambda\hat{f}_\infty$ and $h\in H(\lambda'; G_R)$. Then, the function $\bar{u}(x)=u^*h(x)/\inf_{x\in G_R} h(x)$ is a subsolution of (C) with boundary values $u^*h(x)/\inf_{x\in G_R} h(x)$ on Γ_R, where u^* is the same constant as in the proof of (i). Furthermore, choosing $\hat{u}\in SH(0; G_R)$ with the property $\bar{u}(x)\leqq\hat{u}(x)$ in G_R, we have a supersolution $\hat{u}$. The desired assertion now follows from Lemma 4.2 and the hypothesis that $SH(\mu)$ are of type II for all $\mu\leqq 0$. This completes the proof.

As an application of Theorems 5.1 and 5.2, we consider

$$(\mathrm{C}') \qquad -\mathfrak{D}u + c(x)u = \lambda m(x)u^\gamma \quad \text{in } G_R, \quad \gamma \neq 0, 1.$$

In what follows we assume that the coefficients a_{ij} and b_i of $\mathfrak{D}$ are bounded and $c(x)\geqq 0$ on $\boldsymbol{R}^N$

In addition to $(\mathrm{H}_1)-(\mathrm{H}_5')$ mentioned in the preceding section, we need the following conditions:

$$(\mathrm{H}_6) \qquad \int_{r_0}^{\infty}\left(\int_{r_0}^{r} ds/p_*(s)\right)p_*(r)(c_*(r) + m_*(r))dr = \infty.$$

$$(\mathrm{H}_7) \qquad \int_{r_0}^{\infty}\left(\int_{r}^{\infty} ds/p_*(s)\right)^{\gamma} p_*(r)m^*(r)dr < \infty.$$

EXAMPLE 5.1. Let $\gamma > 1$. Then the following statements hold.

(i) Suppose that $\lambda > 0$.

(a) If (H_1) and (H_3) hold, then (C′) has a solution u satisfying (5.2).

(b) If (H_1) and (H_6) hold, then (C′) has no positive solution u satisfying (5.2) for any $\lambda \in (\zeta^*, \lambda^*(\infty))$ provided $\zeta^* < \lambda^*(\infty)$.

(c) If (H_2) and (H_5) hold, then for any $h \in SH(\lambda)$, (C′) has a solution u satisfying (5.1).

(ii) Suppose that $\lambda < 0$. For any $h \in SH(\lambda)$, (C′) has a solution u satisfying (5.3). Moreover, if (H_1) and (H_3) hold, then the above solution u satisfies (5.2).

PROOF. (i–a) By (H_1), (H_3) and (i) of Theorem 3.3 or (i) of Theorem 3.4, $SH(\mu)$ are of type II for all $\mu \in \boldsymbol{R}$. Hence the assertion follows from (ii) of Theorem 5.1.

(i–b) Since, by (ii) of Theorem 3.3, $SH(\mu)$ are of type III for all $\mu \in (\zeta^*, \lambda^*(\infty))$ provided $\zeta^* < \lambda^*(\infty)$, (iii) of Theorem 5.1 implies the assertion.

(i–c) Since, by (iii) of Theorem 3.3 or (iii) of Theorem 3.4, $SH(\mu)$ are of type I for all $\mu \in \boldsymbol{R}$, the assertion is a consequence of (i) of Theorem 5.1.

(ii) Noting that $\sup_{x \in G_R} h(x) < \infty$ for $h \in SH(\lambda; G_R)$, we have the first assertion by (iv) of Theorem 5.1. Since $SH(\lambda)$ is of type II by (H_1), (H_3) and (i) of Theorem 3.3, the second part is obvious.

EXAMPLE 5.2. Let $\gamma < 1$. Then the following statements hold.

(i) Suppose that $\lambda > 0$.

(a) If (H_1) and (H_3) hold, then (C′) has a solution u satisfying (5.2). Furthermore, (C′) has no positive solution tending to 0 as $|x| \to \infty$.

(b) Suppose that $\zeta^* < \lambda < \lambda^*(\infty)$. If (H_1) and (H_6) hold, then for any $h \in SH(\lambda)$, (C′) has a solution u satisfying (5.1). Furthermore, any positive solution of (C′) cannot be bounded.

(c) Let $0 < \gamma < 1$. If (H_2) and (H_7) hold, then for any $h \in SH(\lambda)$, (C′) has a solution u satisfying

$$Mh(x) \leqq u(x) \leqq M^{-1} \int_{|x|}^{\infty} ds/p_*(s) \quad \text{in } G_R \tag{5.4}$$

for some constants $M > 0$ and $R > 0$.

(ii) Suppose that $\lambda < 0$. If (H_1) and (H_3) hold, then (C′) has a solution u satisfying (5.2).

PROOF. (i–a) Since we see that $SH(\mu)$ are of type II for all $\mu \in \boldsymbol{R}$ by (i) of Theorem 3.3 or (i) of Theorem 3.4, the assertion follows from the proofs of (i) and (ii) of Theorem 5.2.

(i–b) This follows from (ii) of Theorem 3.3 and (i) and (ii) of Theorem 5.2.

(i–c) In view of (H_2) and (H_7), for some $R > 0$ the equation

$$y'' + b_*(r)y' + \lambda m^*(r)y^\gamma = 0, \quad \lambda > 0,$$

has a solution φ satisfying

$$(5.5) \qquad \begin{aligned} &0 < \varphi \leqq 1, \quad \varphi'(r) \leqq 0, \quad r \in [R, \infty), \\ &\lim_{r\to\infty} \varphi(r) \Big/ \int_r^\infty ds/p_*(r) = \text{constant} > 0 \end{aligned}$$

(see e.g. [4]). The function $\hat{u}(x) = \varphi(|x|)$ is a supersolution of (C′) and is λ-superharmonic in G_R. Hence, for $h \in H(\lambda; G_R)$ we can choose a constant $M > 0$ so that $Mh(x) \leqq \hat{u}(x)$ in G_R. Putting $\bar{u}(x) = Mh(x)$, we have a subsolution $\bar{u}$ of (C′) such that $\bar{u}(x) \leqq \hat{u}(x)$ in G_R. The assertion now follows from Lemma 4.2 and (5.5).

The statement (ii) is obvious (cf. (i–a)).

6. Global existence of positive solutions of quasilinear equations

The final section is devoted to the study of the existence of positive solutions of the problem

$$\text{(B)} \qquad -\mathfrak{D}u + c(x)u = \lambda m(x)u^\gamma \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \Gamma \ (\text{if } \Gamma \neq \emptyset),$$

where γ is a nonzero constant with $\gamma \neq 1$. We want to obtain explicit conditions for (B) to have solutions satisfying (4.1) or (4.2). For this purpose we make extensive use of results concerning the existence and asymptotic behavior of solutions of second order ordinary differential equations. So, we begin with an analysis of the ordinary differential equations associated with (B).

6.1. Preliminaries for ordinary differential equations

We consider the initial value problem

$$(6.1) \qquad \begin{aligned} &(p(r)y')' - q_1(r)y + q_2(r)y^\gamma = 0, \quad r > r_0, \quad \gamma \neq 0, 1, \\ &y(r_0) = \xi, \quad y'(r_0) = \eta, \end{aligned}$$

where p, q_1 and q_2 satisfy the conditions

$p \in C^1[r_0, \infty)$, $p(r) > 0$ on $[r_0, \infty)$, $q_1, q_2 \in C[r_0, \infty)$, $q_1(r) \geqq 0$ on $[r_0, \infty)$, and either $q_2(r) > 0$ or $q_2(r) < 0$ on $[r_0, \infty)$.

In addition we need some or all of the following conditions:

$$(6.2) \qquad \int_{r_0}^\infty dr/p(r) = \infty,$$

$$\int_{r_0}^{\infty} \max\{1, \int_{r_0}^{r} ds/p(s)\} q_1(r)dr < 1, \tag{6.3}$$

$$\int_{r_0}^{\infty} \left(\int_{r_0}^{r} ds/p(s)\right)^{\gamma} |q_2(r)| dr < \infty. \tag{6.4}$$

LEMMA 6.1. *Suppose that* $q_2(r)<0$ *on* $[r_0, \infty)$ *and* (6.2)–(6.4) *hold, and let* $\gamma>1$ *(resp.* $0<\gamma<1$*). Then, there are positive constants* ξ_0 *and* η_0 *such that for any* $\xi \in [0, \xi_0]$ *and* $\eta \in [0, \eta_0]$ *(resp.* $\xi \in [\xi_0, \infty)$ *and* $\eta \in [\eta_0, \infty) \cup \{0\}$*), the problem* (6.1) *has a solution* y *satisfying* $y'(r) \geqq 0$ *and*

$$\xi + p(r_0)\eta \int_{r_0}^{r} ds/p(s) \leqq y(r) \leqq M(\xi + p(r_0)\eta) \max\left\{1, \int_{r_0}^{r} ds/p(s)\right\}, \ r \geqq r_0, \tag{6.5}$$

where M *is a positive constant.*

PROOF. Suppose that $\gamma>1$. Let $\mathscr{C}$ denote the locally convex vector space of all continuous funjtions on $[r_0, \infty)$ with usual metric topology. In what follows we use the notation:

$$P(r) = \int_{r_0}^{r} ds/p(s), \quad \Phi(r) = \max\{1, P(r)\}, \quad r \geqq r_0,$$

$$M_1 = \int_{r_0}^{\infty} \Phi(r) q_1(r) dr, \quad \tilde{M}_1 = (1-M_1)^{-1}, \quad M_2 = \int_{r_0}^{\infty} \Phi(r)^{\gamma} |q_2(r)| dr,$$

$$\xi_0 = (2^{2\gamma} \tilde{M}_1^{\gamma} M_2)^{-1/(\gamma-1)}, \quad \eta_0 = p(r_0)^{-1}\xi_0. \tag{6.6}$$

For any fixed $\xi \in [0, \xi_0]$ and $\eta \in [0, \eta_0]$, we consider the set

$$\mathscr{Y} = \{y \in \mathscr{C} : \xi + p(r_0)\eta P(r) \leqq y(r) \leqq 2\tilde{M}_1(\xi + p(r_0)\eta)\Phi(r), \quad r \geqq r_0\}.$$

Clearly $\mathscr{Y}$ is a closed convex subset of $\mathscr{C}$. Define a mapping $F: \mathscr{Y} \to \mathscr{C}$ by

$$\begin{aligned} Fy(r) = \xi + p(r_0)\eta P(r) + \int_{r_0}^{r} \left(\int_{\tau}^{r} ds/p(s)\right) q_1(\tau) y(\tau) d\tau \\ - \int_{r_0}^{r} \left(\int_{\tau}^{r} ds/p(s)\right) q_2(\tau) y(\tau)^{\gamma} d\tau. \end{aligned} \tag{6.7}$$

It is verified that (i) F maps $\mathscr{Y}$ into itself; (ii) F is continuous on $\mathscr{Y}$; (iii) $F\mathscr{Y}$ is relatively compact in $\mathscr{C}$. Since $q_1(r) \geqq 0$ and $q_2(r)<0$ on $[r_0, \infty)$, it is obvious that

$$Fy(r) \geqq \xi + p(r_0)\eta P(r), \quad y \in \mathscr{Y}.$$

Next, setting $\tilde{\eta} = p(r_0)\eta$ for simplicity, by (6.6) and (6.7), we have for $y \in \mathscr{Y}$

$$\begin{aligned}(6.8)\qquad Fy(r) &\leqq \xi + \tilde{\eta}\Phi(r) + 2\tilde{M}_1(\xi+\tilde{\eta})\int_{r_0}^{r}\left(\int_{\tau}^{r} ds/p(s)\right) q_1(\tau)\Phi(\tau)d\tau \\ &\quad - (2\tilde{M}_1)^{\gamma}(\xi+\tilde{\eta})^{\gamma}\int_{r_0}^{r}\left(\int_{\tau}^{r} ds/p(s)\right) q_2(\tau)\Phi(\tau)^{\gamma}d\tau \\ &\leqq [\xi+\tilde{\eta}+2\tilde{M}_1M_1(\xi+\tilde{\eta})+M_2(2\tilde{M}_1)^{\gamma}(\xi+\tilde{\eta})^{\gamma}]\Phi(r) \\ &\leqq [(1+2\tilde{M}_1M_1)(\xi+\tilde{\eta}) + 2^{2\gamma}\tilde{M}_1^{\gamma}M_2(\xi^{\gamma}+\tilde{\eta}^{\gamma})]\Phi(r) \\ &\leqq 2(1+\tilde{M}_1M_1)(\xi+\tilde{\eta})\Phi(r) = 2\tilde{M}_1(\xi+\tilde{\eta})\Phi(r).\end{aligned}$$

Thus, F maps $\mathscr{Y}$ into itself. The verification of (ii) and (iii) is routine, so we omit it. By virtue of Schauder-Tychonoff's fixed point theorem, F has a fixed point $y \in \mathscr{Y}$: $(Fy)(r) = y(r)$, $r \geqq r_0$. This y is clearly a solution of (6.1) with the desired properties.

To prove the assertion in the case where $0<\gamma<1$, we need only to note that (6.8) is valid for $\xi \in [\xi_0, \infty)$ and $\eta \in [\eta_0, \infty) \cup \{0\}$ with the same constants as (6.6). This completes the proof.

LEMMA 6.2. *Suppose that* $q_1(r) \equiv 0$, $q_2(r) > 0$ *on* $[r_0, \infty)$ *and* (6.2), (6.4) *hold, and let* $\gamma > 1$ *(resp.* $0<\gamma<1$*). If* $\eta_0 > 0$ *is sufficiently small (resp. sufficiently large), then for* $\eta \in (0, \eta_0]$ *(resp.* $\eta \in [\eta_0, \infty)$*), there is a positive constant* $\xi_0 = \xi_0(\eta)$ *such that for* $\xi \in [0, \xi_0]$ *(resp.* $\xi \in (0, \xi_0]$*) the problem* (6.1) *has a solution* y *satisfying* $y'(r) \geqq 0$ *and*

$$(6.9)\qquad \xi + 2^{-1}p(r_0)\eta\int_{r_0}^{r} ds/p(s) \leqq y(r) \leqq \xi + p(r_0)\eta\int_{r_0}^{r} ds/p(s), \quad r \geqq r_0.$$

PROOF. Suppose that $\gamma > 1$. For fixed $\xi \geqq 0$, $\eta > 0$ let y be a local solution of (6.1), and set

$$r^* = \sup\{\tilde{r}: 0 < y(r) < \infty \quad \text{in } r_0 < r < \tilde{r}\}.$$

We claim that $r^* = \infty$. Integrating (6.1), we obtain

$$(6.10)\qquad y(r) = \xi + \tilde{\eta}\int_{r_0}^{r} ds/p(s) - \int_{r_0}^{r}\left(\int_{\tau}^{r} ds/p(s)\right) q_2(\tau)y(\tau)^{\gamma}d\tau,$$

which implies

$$(6.11)\qquad y(r) \leqq \xi + \tilde{\eta}P(r), \quad r_0 \leqq r < r^*,$$

where $P(r)$ and $\tilde{\eta}$ are as in the proof of Lemma 6.1. On the other hand, using (6.11) in (6.10), we have for $r_0 \leqq r < r^*$,

$$(6.12)\quad y(r) \geqq \xi + [\tilde{\eta} - 2^{\gamma}\xi^{\gamma}\int_{r_0}^{\infty} q_2(\tau)d\tau - 2^{\gamma}\tilde{\eta}^{\gamma}\int_{r_0}^{\infty}\left(\int_{r_0}^{\tau} ds/p(s)\right)^{\gamma} q_2(\tau)d\tau]P(r).$$

Define

$$\eta_0 = p(r_0)^{-1}(2^{\gamma+2}M_1)^{-1/(\gamma-1)}, \quad \xi_0(\eta) = (2^{-(\gamma+2)}M_2^{-1}p(r_0)\eta)^{1/\gamma}, \quad \eta > 0,$$

where

$$M_1 = \int_{r_0}^{\infty}\left(\int_{r_0}^{\tau} ds/p(s)\right)^{\gamma} q_2(\tau)d\tau, \quad M_2 = \int_{r_0}^{\infty} q_2(s)ds.$$

From (6.12) we see that if $\xi \in [0, \xi_0(\eta)]$, $\eta \in (0, \eta_0]$, then

$$y(r) \geqq \xi + \tilde{\eta}P(r)/2, \quad r_0 \leqq r < r^*. \tag{6.13}$$

The relations (6.11) and (6.13) shows that $r^* = \infty$ and (6.9) holds. It is easy to see that $y'(r) > 0$ for $r \geqq r_0$.

A similar argument holds if we assume that $0 < \gamma < 1$.

Remark 6.1. In the case where $\gamma > 1$, from the above choice of $\xi_0(\eta)$ we may assume that $\lim_{\eta \to 0} \eta/\xi_0(\eta) = 0$. This fact will be used in the proof of Theorem 6.2 below.

6.2. Superlinear equations

In this subsection we establish the existence of positive solutions of the problem (B) in the case where $\gamma > 1$.

In addition to the conditions (H_1)–(H_5') mentioned in Section 4, the following conditions are employed:

$$(H_8) \qquad \int_{r_0}^{\infty}\left(\int_{r_0}^{r} ds/p_*(s)\right)p_*(r)c^*(r)dr < \infty;$$

$$(H_9) \qquad \int_{r_0}^{\infty}\left(\int_{r_0}^{r} ds/p^*(s)\right)^{\gamma} p^*(r)m^*(r)dr < \infty.$$

Theorem 6.1. *Let $\gamma > 1$. Suppose that (H_1), (H_4), (H_8) and (H_9) hold and $c(x) \geqq 0$ in Ω. Then, for any $\lambda < 0$, the problem (B) has infinitely many positive solutions satisfying (4.1).*

Proof. First we choose $r_1 > 0$ such that $\Omega \supset \{x: |x| > r_1\}$ and

$$\int_{r_1}^{\infty} \max\left\{1, \int_{r_1}^{r} ds/p_*(s)\right\} p_*(r)c^*(r)dr < 1. \tag{6.14}$$

Since $\lambda < 0$, by (H_1), (H_9) and Lemma 6.1, for sufficiently small $\xi > 0$, the problem

$$(p^*(r)y')' + \lambda p^*(r)m_*(r)y^{\gamma} = 0, \quad r > r_1,$$
$$y(r_1) = \xi, \quad y'(r_1) = 0$$

has a solution φ satisfying $\varphi'(r) \geqq 0$ and

$$(6.15) \qquad \xi \leqq \varphi(r) \leqq M_1 \xi \max \left\{1, \int_{r_1}^{r} ds/p^*(s)\right\}, \quad r \geqq r_1,$$

for some constant $M_1 > 0$. We now define the function $\hat{u}$ by

$$\hat{u}(x) = \xi, \quad \text{for } x \in \bar{G}_1; \ \hat{u}(x) = \varphi(|x|) \quad \text{for } x \in \bar{G}_2,$$

where $G_1 = \Omega \cap B(0, r_1)$, $G_2 = \Omega \backslash \bar{G}_1$. In essentially the same way as in the proof of Theorem 4.1, we see that $\hat{u}$ is a supersolution of (B).

To construct a subsolution, we note that φ satisfies $\varphi'(r_1+1) > 0$ and

$$(6.16) \qquad \xi + p^*(r_1+1)\varphi'(r_1+1)\int_{r_1+1}^{r} ds/p^*(s) \leqq \varphi(r), \quad r \geqq r_1 + 1.$$

Applying Lemma 6.1 again, we see that the problem

$$(p_*(r)z')' - p_*(r)c^*(r)z + \lambda p_*(r)m^*(r)z^\gamma = 0, \quad r > r_1 + 1,$$
$$z(r_1+1) = 0, \quad z'(r_1+1) = \eta$$

has a solution ψ satisfying $\psi'(r) > 0$ and

$$p_*(r_1+1)\eta \int_{r_1+1}^{r} ds/p_*(s) \leqq \psi(r) \leqq M_2 p_*(r_1+1)\eta \max\left\{1, \int_{r_1+1}^{r} ds/p_*(s)\right\}, \ r \geqq r_1 + 1,$$

for some constant $M_2 > 0$, provided $\eta > 0$ is sufficiently small. Combining this with (6.15), (6.16) and using (4.21), we can take η small enough so that for $r \geqq r_1 + 1$,

$$(6.17) \quad p_*(r_1+1)\eta \int_{r_1+1}^{r} ds/p_*(s) \leqq \psi(r) \leqq \varphi(r) \leqq M_1 \xi \max\left\{1, \int_{r_1}^{r} ds/p^*(s)\right\}.$$

Then, the function $\bar{u}$ defined by

$$\bar{u}(x) \equiv 0 \quad \text{for } x \in \bar{G}_1' = \overline{\Omega \cap B(0, r_1+1)}; \ \bar{u}(x) = \psi(|x|) \quad \text{for } x \in \overline{\Omega \backslash G_1'}$$

is a subsolution of (B) which satisfies $\bar{u}(x) \leqq \hat{u}(x)$ in Ω. From Lemma 4.2 it follows that (B) has a solution u such that $\bar{u}(x) \leqq u(x) \leqq \hat{u}(x)$ in Ω. That u satisfies (4.1) follows from (6.17) and (4.21), and that u is positive is a consequence of the maximum principle.

From the above proof it is easily seen that there exist infinitely many positive solutions of (B) satisfying (4.1). This completes the proof.

We now give an existence theorem which applies to the case $\lambda \geqq 0$ in (B).

THEOREM 6.2. *Let $\gamma > 1$. Suppose that Ω is an exterior domain such that $0 \in \Omega_0 = \boldsymbol{R}^N \backslash \bar{\Omega}$. Suppose that* ($H_1$), ($H_4$), ($H_8$) *and* ($H_9$) *hold, $c(x) \geqq 0$ in Ω and $\alpha(x) < 1$ on Γ. Then, for any $\lambda \in \boldsymbol{R}$, the problem* (B) *has infinitely many positive solutions satisfying* (4.1).

PROOF. It is enough to show the assertion for the case $\lambda>0$. Furthermore, replacing $\lambda m(x)$ by $m(x)$, we may assume that $\lambda=1$.

Let r_0 and G_j, $j=1, 2$ be as in the proof of Theorem 4.1. We first construct a supersolution of (B). As in the proof of Theorem 4.2, we can take a constant $\lambda'>0$ and a function $\hat{u}_1\in C^{2+\sigma}(\bar{G}_1)$ such that

$$
\begin{aligned}
&-\mathfrak{D}\hat{u}_1 + c(x)\hat{u}_1 \geqq \lambda' m(x)\hat{u}_1, \quad \hat{u}_1(x) > 0 \quad \text{in } G_1,\\
&B\hat{u}_1 \geqq 0 \quad \text{on } \Gamma, \quad \hat{u}_1(x) > 0, \quad \partial\hat{u}_1(x)/\partial\nu > 0 \quad \text{on } \Lambda_0, \qquad (6.18)\\
&\hat{u}_1(x) = \varphi_1(|x|),\ x\in\bar{G}_1 \ \text{ for some } \varphi_1 \text{ such that } \varphi_1'(r) > 0 \text{ on } [0, r_0].
\end{aligned}
$$

Next, consider the problem

$$
\begin{aligned}
&(p^*(r)y')' + p^*(r)m^*(r)y^\gamma = 0, \quad r > r_0,\\
&y(r_0) = \xi, \quad y'(r_0) = \eta.
\end{aligned} \qquad (6.19)
$$

By Lemma 6.2 we can choose an η_0 in such a way that for every $\eta\in(0, \eta_0]$ there is a $\xi_0(\eta)$ such that for η and $\xi\in[0, \xi_0(\eta)]$ problem (6.19) has a solution φ_2 satisfying $\varphi_2'(r)>0$ and

$$
\xi + \frac{1}{2} p^*(r_0)\eta\int_{r_0}^{r} ds/p^*(s) \leqq \varphi_2(r) \leqq \xi + p^*(r_0)\eta\int_{r_0}^{r} ds/p^*(s), \quad r\geqq r_0. \qquad (6.20)
$$

Put now $\theta=\varphi_1'(r_0)/\varphi_1(r_0)$ and choose $\eta_1\in(0, \eta_0]$ satisfying

$$
\xi_0(\eta_1) \leqq (\lambda')^{1/(\gamma-1)} \quad \text{and} \quad \eta_1/\xi_0(\eta_1) < \theta.
$$

This is possible, since $\xi_0(\eta)\to 0$ and $\eta/\xi_0(\eta)\to 0$ as $\eta\to 0$ as noted in Remark 6.1. Denoting by $\varphi_2(r)$ the solution of (6.19) with $\xi=\xi_1=\eta_1/\theta$ and $\eta=\eta_1$, we define

$$
\hat{u}(x) = \xi_1\varphi_1(|x|)/\varphi_1(r_0) \quad \text{for } x\in\bar{G}_1;\ \hat{u}(x) = \varphi_2(|x|) \quad \text{for } x\in\bar{G}_2.
$$

This $\hat{u}$ is a supersolution of (B). In fact, noting that $0<\hat{u}(x)\leqq\xi_1\leqq(\lambda')^{1/(\gamma-1)}$ and so $\hat{u}(x)^\gamma\leqq\lambda'\hat{u}(x)$ in G_1, we have by (6.18)

$$
-\mathfrak{D}\hat{u}(x) + c(x)\hat{u} \geqq \lambda' m(x)\hat{u} \geqq m(x)\hat{u}^\gamma \quad \text{in } G_1,
$$

and by (6.19) we have

$$
-\mathfrak{D}\hat{u} + c(x)\hat{u} \geqq m(x)\hat{u}^\gamma \quad \text{in } G_2.
$$

Furthermore, an easy calculation shows that

$$
\begin{aligned}
\partial\hat{u}_1/\partial\nu &= \xi_1\theta a(x)(\textstyle\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)x_j/|x|)^2)^{-1/2}\\
&= \eta_1 a(x)(\textstyle\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)x_j/|x|)^2)^{-1/2}\\
&= \varphi_2'(r_0)a(x)(\textstyle\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)x_j/|x|)^2)^{-1/2} = \partial\hat{u}_2(x)/\partial\nu \quad \text{on } \Lambda_0,
\end{aligned}
$$

where $\hat{u}_l = \hat{u}|_{\bar{G}_l}$, $l=1, 2$. Thus $\hat{u}$ is a supersolution of (B).

To construct a subsolution, we consider the linear problem

$$-\mathfrak{D}v + c(x)v = 0 \quad \text{in } \Omega, \quad Bv = 0 \quad \text{on } \Gamma.$$

Theorem 4.2 ensures the existence of a solution $v \in C^{2+\sigma}(\bar{\Omega})$ of this problem satisfying (4.1). In view of (4.1), (6.20) and the boundary condition on Γ, we can choose a constant $M>0$ such that $Mv(x) \leqq \hat{u}(x)$ in Ω. Obviously, $\bar{u} = Mv$ is a subsolution of (B) such that $\bar{u}(x) \leqq \hat{u}(x)$ in Ω, and the existence of the desired solution follows from Lemma 4.2. The existence of infinitely many solutions is easily verified. This finishes the proof.

In the following theorem we indicate a situation in which (B) possesses bounded positive solutions.

THEOREM 6.3. *Let* $\gamma > 1$. *Suppose that* (H_2) *and* (H_5) *hold and* $c(x) \geqq 0$ *in* Ω. *Then, for any* $\lambda \in \boldsymbol{R}$, *the problem* (B) *has infinitely many positive solutions satisfying* (4.2).

PROOF. We may assume that $\lambda \neq 0$. We first note that by Theorem 4.3, there is a constant $\tilde{\mu} > 0$ such that for every $\mu \leqq \tilde{\mu}$

$$-\mathfrak{D}u + c(x)u = \mu m(x)u \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \Gamma \text{ (if } \Gamma \neq \emptyset) \tag{6.21}$$

has a positive solution satisfying (4.2).

If $\lambda > 0$, let $\hat{u}$ and $\bar{u}$ be positive solutions of (6.21) with $\mu = \tilde{\mu}$ and $\mu = 0$, respectively, which satisfy (4.2) and

$$0 < \bar{u}(x) \leqq \hat{u}(x) \leqq (\tilde{\mu}/\lambda)^{1/(\gamma-1)} \quad \text{in } \Omega.$$

Then $\hat{u}$ and $\bar{u}$ are, respectively, a supersolution and a subsolution of (B), and the assertion follows from Lemma 4.2.

If $\lambda < 0$, we set $\hat{u}(x) \equiv 1$ on $\bar{\Omega}$ and let $\bar{u}$ be a solution of (6.21) with $\mu = \lambda$ satisfying (4.2) and $0 < \bar{u}(x) \leqq 1$ on $\bar{\Omega}$. Since $\hat{u}$ is a supersolution and $\bar{u}$ is a subsolution of (B), we have a desired solution of (B) by Lemma 4.2.

6.3. Sublinear equations

THEOREM 6.4. *Let* $0 < \gamma < 1$, $\Omega = \boldsymbol{R}^N$ *and suppose that* (H_1), (H_4), (H_8) *and* (H_9) *hold and* $c(x) \geqq 0$ *on* $\boldsymbol{R}^N$. *Then, for every* $\lambda < 0$, *the problem* (B) (*with the boundary condition deleted*) *has infinitely many positive solutions satisfying* (4.1).

PROOF. Without loss of generality we may assume that $\lambda = -1$. Let $r_1 > 0$ be such that (6.14) holds, and put

$$G_1 = B(0, r_1),\quad G_2 = \mathbf{R}^N\backslash\bar{G}_1;\ G_1' = B(0, r_1+1),\quad G_2' = \mathbf{R}^N\backslash\bar{G}_1'.$$

We take a unique solution $u_1 \in C^{2+\sigma}(\bar{G}_1')$ of

$$(6.22)\qquad -\mathfrak{D}u + (c(x)+m(x))u = 0 \quad \text{in } G_1',\quad u = 1 \quad \text{on } \partial G_1' = \Lambda_1.$$

From the maximum principle it follows that $0<u_1(x)\leqq 1$ on $\bar{G}_1'$. Next, by Lemma 6.1, there exist $\bar{\xi}_0$ and $\bar{\eta}_0>0$ such that for $\bar{\xi}\in[\bar{\xi}_0, \infty)$ and $\bar{\eta}\in[\bar{\eta}_0, \infty)$ the problem

$$(6.23)\qquad \begin{aligned} &(p_*(r)z')' - p_*(r)c^*(r)z - p_*(r)m^*(r)z^\gamma = 0, \quad r > r_1 + 1,\\ &z(r_1+1) = \bar{\xi},\quad z'(r_1+1) = \bar{\eta}\end{aligned}$$

has a positive solution z satisfying

$$(6.24)\qquad \begin{aligned}&\bar{\xi} + p_*(r_1+1)\bar{\eta}\int_{r_1+1}^r ds/p_*(s) \leqq z(r)\\ &\leqq M_1(\bar{\xi}+p_*(r_1+1)\bar{\eta}) \max\left\{1, \int_{r_1+1}^r ds/p_*(s)\right\},\quad r \geqq r_1 + 1,\end{aligned}$$

for some constant $M_1>0$. Put $M_2=\max\{\bar{\xi}_0, (\min_{x\in\bar{G}_1} u_1(x))^{-1}\}$, and $\bar{u}_1(x)=M_2u_1(x)$ on $\bar{G}_1$. Let ψ be a solution of (6.23) with $\bar{\xi}=M_2$, $\bar{\eta}=\bar{\eta}_1$, which satisfies (6.24), where

$$\bar{\eta}_1 = \max\{\bar{\eta}_0, \max_{x\in\Lambda_1}\{|\partial\bar{u}_1(x)/\partial\nu|a(x)^{-1}(\textstyle\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)x_j/|x|)^2)^{1/2}\}\},$$

and define the function $\bar{u}$ by

$$\bar{u}(x) = \bar{u}_1(x) \quad \text{for } x\in\bar{G}_1';\ \bar{u}(x) = \psi(|x|) \quad \text{for } x\in\bar{G}_2'.$$

Denote by $\bar{u}_j$ the restriction of $\bar{u}$ on $\bar{G}_j'$, $j=1, 2$. Then,

$$(6.25)\qquad -\mathfrak{D}\bar{u}_j + c(x)\bar{u}_j + m(x)\bar{u}_j^\gamma \leqq 0 \quad \text{in } G_j',\quad j = 1, 2;$$

(6.25) for $j=1$ follows from (6.22) and the relation $\bar{u}_1(x)\geqq 1$ on G_1', and (6.25) for $j=2$ from (6.23). Furthermore, we have

$$\begin{aligned}&\partial\bar{u}_1(x)/\partial\nu - \partial\bar{u}_2(x)/\partial\nu\\ &= \partial\bar{u}_1(x)/\partial\nu - \psi'(r_1+1)a(x)(\textstyle\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)x_j/|x|)^2)^{-1/2}\\ &= \partial\bar{u}_1(x)/\partial\nu - \bar{\eta}_1 a(x)(\textstyle\sum_{i=1}^N(\sum_{j=1}^N a_{ij}(x)x_j/|x|)^2)^{-1/2} \leqq 0 \text{ on } \Lambda_1.\end{aligned}$$

It follows that $\bar{u}$ is a subsolution of (B). Obviously, $\bar{u}$ satisfies (4.1) by (6.24).

We now choose $\xi_0>0$ so that for $\xi\in[\xi_0, \infty)$

$$(6.26)\qquad \begin{aligned}&(p^*(r)y')' - p^*(r)m_*(r)y^\gamma = 0, \quad r > r_1,\\ &y(r_1) = \xi,\quad y'(r_1) = 0\end{aligned}$$

has a solution φ satisfying

$$\xi \leqq \varphi(r) \leqq M_3 \xi \max \left\{1, \int_{r_1}^{r} ds/p^*(s)\right\}, \quad r \geqq r_1 .$$

An integration of (6.26) from r_1 to r_1+1 yields

$$p^*(r_1+1)\varphi'(r_1+1) \geqq \xi^\gamma \int_{r_1}^{r_1+1} p^*(s)m_*(s)ds \equiv M_4\xi^\gamma,$$

and so from (6.26) we obtain

$$\varphi(r) \geqq \xi + M_4\xi^\gamma \int_{r_1+1}^{r} ds/p^*(s), \quad r \geqq r_1 + 1 \tag{6.27}$$

(see (6.5)). In view of (4.21), for $\bar{\xi}=M_2$, $\bar{\eta}=\bar{\eta}_1$ we can choose $\tilde{\xi}_0 \geqq \xi_0$ such that for $\xi \geqq \tilde{\xi}_0$ and $r \geqq r_1+1$,

$$M_1(\bar{\xi}+p_*(r_1+1)\bar{\eta}) \max \left\{1, \int_{r_1+1}^{r} ds/p_*(s)\right\} \leqq \xi + M_4\xi^\gamma \int_{r_1}^{r} ds/p^*(s).$$

From this, (6.24) and (6.27) it follows that $\psi(r) \leqq \varphi(r)$ for $r \geqq r_1+1$, provided $\xi \geqq \tilde{\xi}_0$. Let φ be the above solution of (6.26) with ξ such that

$$\xi \geqq \max\{\tilde{\xi}_0, \max\{\bar{u}_1(x): x \in \bar{G}_1'\}\}$$

and define a function $\hat{u}$ by

$$\hat{u}(x) \equiv \xi \quad \text{for } x \in \bar{G}_1; \ \hat{u}(x) = \varphi(|x|) \quad \text{for } x \in \bar{G}_2.$$

It is easily verified that $\hat{u}$ is a supersolution of (B) satisfying (4.1) and $\bar{u}(x) \leqq \hat{u}(x)$ in Ω. Therefore, by Lemma 4.2, (B) has a solution satisfying $\bar{u}(x) \leqq u(x) \leqq \hat{u}(x)$ in Ω. It is not to hard to show that there exist infinitely many such solutions $u(x)$ of (B). This completes the proof.

THEOREM 6.5. *Let* $0<\gamma<1$ *and* Ω *be an exterior domain such that* $0 \in \boldsymbol{R}^N \backslash \bar{\Omega}$, *and suppose that* (H_1), (H_3) *and* (H_4) *hold*, $c(x) \geqq 0$ *in* Ω *and* $\alpha(x)<1$ *on* Γ. *Then, for every* $\lambda>0$, *the problem* (B) *has infinitely many positive solutions satisfying* (4.1).

PROOF. From the proof of Theorem 4.1 there exists a constant $\tilde{\lambda}>0$ and a supersolution $\hat{v}$ of (6.21) with $\mu=\tilde{\lambda}$ which is positive throughout Ω and satisfies (4.1). The function $\hat{u}(x)=(\tilde{\lambda}/\lambda)^{1/(1-\gamma)}\hat{v}(x)/\inf_{x\in\Omega}\hat{v}(x)$ is a supersolution of (B) with the property (4.1). Next, let u be a solution of (6.21) with $\mu=0$ satisfying (4.1), and take a constant $M>0$ such that $Mu(x) \leqq \hat{u}(x)$ in Ω. Then, $\bar{u}=Mu$ is a subsolution of (B) which satisfies (4.1) and $\bar{u}(x) \leqq \hat{u}(x)$ in Ω, and so (B) has a desired solution. Moreover, it can be shown that there exist infinitely many such solutions of (B).

The following results establish the existence of bounded positive solutions.

THEOREM 6.6. *Let $\gamma<1$ and $\Omega=\boldsymbol{R}^N$, and suppose that* (H_2), (H_5) *hold and* $c(x)\geqq 0$ *on* $\boldsymbol{R}^N$. *Then, for every* $\lambda\in\boldsymbol{R}$, *the problem* (B) *has infinitely many positive solutions satisfying* (4.2).

The proof is analogous to that of Theorem 6.3 and will be omitted.

COROLLARY 6.1. *Let $\gamma\neq 0$ and $\Omega=\boldsymbol{R}^N$, and suppose that* (H_2), (H'_5) *hold and* $c(x)\geqq 0$ *on* $\boldsymbol{R}^N$. *Then, for every* $\lambda\in\boldsymbol{R}$, *the problem* (B) *has infinitely many positive solutions satisfying* (4.2).

If $\gamma>1$, *then the same conclusion holds even in the case where Ω is an exterior domain.*

PROOF. It suffices to consider the problems

$$-\mathfrak{D}u+c(x)u=\mu M(x)u^{\gamma}\quad\text{in } \boldsymbol{R}^N,\quad \text{for } \gamma<1,$$

and

$$-\mathfrak{D}u+c(x)u=\mu M(x)u^{\gamma}\quad\text{in } \Omega,\quad Bu=0\quad\text{on } \Gamma,\quad \text{for } \gamma>1,$$

with $\mu=\pm|\lambda|$, where $M(x)$ is the function in the condition (H'_5). The assertion for $\gamma>1$ or $\gamma<1$ follows from Theorem 6.3 or Theorem 6.6.

THEOREM 6.7. *Let $0<\gamma<1$ and Ω be an exterior domain, and suppose that* (H_2) *and* (H_5) *hold and* $c(x)\geqq 0$ *on* $\boldsymbol{R}^N$. *Then, for every* $\lambda>0$ *the problem* (B) *has infinitely many positive solutions satisfying* (4.2).

The proof is omitted, since it is essentially the same as that of Theorem 6.5 except that Theorem 4.3 is used in place of Theorem 4.1.

6.4. Example

Consider the equation

$$-\Delta u+c(x)u=\lambda m(x)u^{\gamma}\quad\text{in } \Omega,\quad Bu=0\quad\text{on } \Gamma\ (\text{if } \Gamma\neq\emptyset), \tag{6.28}$$

where c, $m\in C^{\sigma}_{\mathrm{loc}}(\bar{\Omega})$ and $c(x)\geqq 0$, $m(x)>0$ on $\bar{\Omega}$, $\lambda\in\boldsymbol{R}$ and $\gamma\neq 0, 1$.

EXAMPLE 6.1. (i) Let $N=2$ and $\Omega=\boldsymbol{R}^2$. If

$$\int_{r_0}^{\infty} r\log r\, c^*(r)dr<\infty,\quad \int_{r_0}^{\infty} r(\log r)^{\gamma} m_*(r)dr<\infty, \tag{6.29}$$

then for every $\lambda<0$ and $\gamma>0$, (6.28) has infinitely many positive solutions u satisfying

$$0<\liminf_{|x|\to\infty} u(x)/\log|x|\leqq \limsup_{|x|\to\infty} u(x)/\log|x|<\infty. \tag{4.28}$$

(ii) Let $N\geqq 3$ and $\Omega=\boldsymbol{R}^N$. If

$$\int_{r_0}^{\infty} r(c^*(r)+m^*(r))dr < \infty, \tag{6.30}$$

then for every $\lambda \in \boldsymbol{R}$ and $\gamma \neq 0$, (6.28) has infinitely many positive solutions satisfying (4.2).

(iii) Let $N=2$ and let Ω be an exterior domain such that $0 \in \boldsymbol{R}^N \backslash \bar{\Omega}$. Suppose that $\alpha(x)<1$ on Γ.

(a) If $\gamma>1$ and (6.29) holds, then for every $\lambda \in \boldsymbol{R}$, (6.28) has infinitely many positive solutions satisfying (4.28).

(b) If $0<\gamma<1$ and

$$\int_{r_0}^{\infty} r \log r\,(c^*(r)+m^*(r))dr < \infty,$$

then for every $\lambda>0$, (6.28) has infinitely many positive solutions satisfying (4.28).

(iv) Let $N \geqq 3$ and let Ω be an exterior domain. Suppose that (6.30) holds. Then, (6.28) has infinitely many positive solutions satisfying (4.2) for every $\lambda \in \boldsymbol{R}$ or for $\lambda>0$, according as $\gamma>1$ or $0<\gamma<1$.

PROOF. (i) The assertion follows from Theorems 6.1 and 6.4.

(ii) This is a consequence of Theorems 6.3 and 6.6.

The statements (iii–a) and (iii–b) follow from Theorem 6.2 and Theorem 6.5, respectively.

(iv) Theorems 6.3 and 6.7 yield the assertion.

REMARK 6.1. The condition that $m(x)>0$ on $\bar{\Omega}$ may be replaced by the requirement that there exists a function $M \in C^{\sigma}_{loc}[r_0, \infty)$ such that $M(r)>0$ and $|m(x)| \leqq M(|x|)$ on $\bar{\Omega}$, in which case (6.30) should be replaced by

$$\int_{r_0}^{\infty} r(c^*(r)+M(r))dr < \infty.$$

REMARK 6.2. When $c(x) \equiv 0$ on $\boldsymbol{R}^N$, the assertions (i) and (ii) reduce to recent results of Kawano, Kusano and Naito [19, Theorem 1] and Kawano [18, Theorems 2.3 and 2.6], respectively. The assertions (iii–a) and (iv) include Theorem 4.3 of Noussair and Swanson [36] for the case where $\gamma>1$, $c(x) \equiv 0$ in Ω and $\alpha(x) \equiv 0$ on Γ.

ACKNOWLEDGEMENT. The author would like to express his sincere gratitude to Professor Takaŝi Kusano for many available suggestions and comments during the preparation of this paper. The author wishes to thank Professor Akira Ono for his constant encouragement and Professor Yukio Ogura for stimulating discussions.

References

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I., Comm. Pure Appl. Math., **12** (1959), 623–727.

[2] W. Allegretto, On the equivalence of two types of oscillation for elliptic operators, Pacific J. Math., **55** (1974), 319–328.

[3] H. Amann, On the number of solutions of nonlinear equations in ordered Banach spaces, J. Funct. Anal., **11** (1972), 346–384.

[4] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev., **18** (1976), 620–709.

[5] C. V. Coffman and J. S. Wong, Oscillation and nonoscillation theorems for second order ordinary differential equations, Funkcial. Ekvac., **15** (1972), 119–130.

[6] A. Friedman, Bounded entire solutions of elliptic equations, Pacific J. Math., **44** (1973), 497–507.

[7] A. Friedman, Reinforcement of the principal eigenvalue of an elliptic operator, Arch. Rational Mech. Anal., **73** (1980), 1–17.

[8] Y. Furusho, Existence and bifurcation of positive solutions of asymptotically linear elliptic equations in exterior domains, Nonlinear Anal., **8** (1984), 583–593.

[9] Y. Furusho and Y. Ogura, On the existence of bounded positive solutions of semilinear elliptic equations in exterior domains, Duke Math. J., **48** (1981), 497–521.

[10] Y. Furusho and Y. Ogura, On the solutions of linear elliptic equations in exterior domains, J. Differential Equations, **50** (1983), 348–374.

[11] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math., **34** (1981), 525–598.

[12] D. Gilbarg and J. Serrin, On isolated singularities of solutions of second order elliptic differential equations, J. Analyse. Math., (1956), 309–340.

[13] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[14] P. Hartman, Ordinary Differential Equations, Second edition, Birkhäuser, Boston-Basel-Stuttgart, 1982.

[15] P. Hartman and A. Wintner, On nonoscillatory linear differential equations, Amer. J. Math., **75** (1953), 717–730.

[16] R. Z. Has'minskii, Stochastic Stability of Differential Equations, Sijthoff & Noordhoff, The Netherlands, 1980, (English translation).

[17] V. B. Headley, Some oscillation properties of self-adjoint elliptic equations, Proc. Amer. Math. Soc., **25** (1970), 824–829.

[18] N. Kawano, On bounded entire solutions of semilinear elliptic equations, Hiroshima Math. J., **14** (1984), 125–158.

[19] N. Kawano, T. Kusano and M. Naito, On the elliptic equation $\Delta u = \varphi(x)u^{\gamma}$ in $\boldsymbol{R}^2$, Proc. Amer. Math. Soc., **93** (1985), 73–78.

[20] N. Kawano and M. Naito, Positive solutions of semilinear second order equations in exterior domains, Hiroshima Math. J., **12** (1982), 142–149.

[21] C. E. Kenig and W. M. Ni, An exterior Dirichlet problem with applications to some nonlinear equations arising in geometry, Amer. J. Math., **106** (1984), 689–702.

[22] R. Z. Khas'minskii, Ergodic properties of recurrent diffusion processes and stabilization of solution to Cauchy problems for parabolic equations, Theory Probab. Appl., **5** (1960),

179–196, (English translation).

[23] M. A. Krasnosel'skii, Positive Solutions of Operator Equations, Noordhoff, Groningen, The Netherlands, 1964, (English translation).

[24] T. Kusano, On bounded solutions of exterior boundary value problems for linear and quasilinear elliptic differential equations, Japan. J. Math., **35** (1965), 35–59.

[25] T. Kusano and S. Oharu, Bounded entire solutions of second order semilinear elliptic equations with application to a parabolic initial value problem, Indiana Univ. Math. J., **34** (1985), 85–95.

[26] T. Kusano and C. A. Swanson, Asymptotic properties of semilinear elliptic equations, Funkcial. Ekvac., **26** (1983), 115–129.

[27] T. Kusano and C. A. Swanson, Asymptotic theory of singular semilinear elliptic equations, Canad. Math. Bull., **27** (1984), 223–232.

[28] T. Kusano and H. Usami, Positive solutions of class of second order elliptic euqtions in the plane, Math. Ann., **268** (1984), 255–264.

[29] N. Meyers and J. Serrin, The exterior Dirichlet problem for second order elliptic partial differential equations, J. Math. Mech., **9** (1960), 513–538.

[30] G. Miranda, Partial Differential Equations of Elliptic Type, Second edition, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[31] W. F. Moss and J. Piepenbrink, Positive solutions of elliptic equations, Pacific J. Math., **75** (1978), 219–226.

[32] M. Naito, A note on bounded positive entire solutions of semilinear elliptic equations, Hiroshima Math. J., **14** (1984), 211–214.

[33] W. M. Ni, On the elliptic equation $\Delta u+K(x)u^{(n+2)/(n-2)}=0$, its generalizations, and applications in geometry, Indiana Univ. Math. J., **31** (1982), 493–529.

[34] E S. Noussair, On the existence of solutions of semilinear elliptic boundary value problems, J. Differential Equations, **34** (1979), 482–495.

[35] E. S. Noussair and C. A. Swanson, Positive solutions of quasilinear elliptic equation in exterior domains, J. Math. Anal. Appl., **75** (1980), 121–133.

[36] E. S. Noussair and C. A. Swanson, Global positive solutions of semilinear elliptic equations, Canad. J. Math., **35** (1983), 839–861.

[37] E. S. Noussair and C. A. Swanson, Global positive solutions of semilinear elliptic problems, Pacific J. Math., **115** (1984), 177–192.

[38] J. Piepenbrink, Nonoscillatory elliptic equations, J. Differential Equations, **15** (1974), 541–550.

[39] M. H. Protter and H. F. Weinberger, Maximum Principle in Differential Equations, Prentice-Hall, Englewood Cliffs, New Jersey, 1967.

[40] J. Serrin, A remark on the preceding paper of Amann, Arch. Rational Mech. Anal., **44** (1972), 182–186.

[41] C. A. Swanson, Semilinear second order elliptic oscillation, Canad. Math. Bull., **22** (1979), 139–157.

[42] C. A. Swanson, Extremal positive solutions of semilinear Schrödinger equations, Canad. Math. Bull., **26** (1983), 171–178.

Department of Mathmatics
Faculty of Science and Engineering
Saga University