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Remarks on the separation of the Aa-adic topology and permutations of M-sequences

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1. Introduction

Let M be a non-zero, finite module over a noetherian ring A. It is well known that if A is a local ring with the maximal ideal $\mathfrak{m}$, then every permutation of an M-sequence is an M-sequence. It seems to the author that this property arises from the fact that the $\mathfrak{m}$-adic topology on M is a Hausdorff space. In this paper we study modules M which satisfy the condition that the Aa-adic topology on M is separated for every M-regular element a. As a tool in this investigation we consider the subset $\mathscr{K}(M)$ of A which consists of those elements a with separated Aa-adic topology.

In section 2 we study some inclusion relations among the set $\mathscr{K}(M)$, the set of all zero-divisors of M and the set of all M-regular elements. In section 3 we establish a method of constructing modules M such that the sets $\mathscr{K}(M)$ are as large as possible. In section 4 we give some conditions equivalent to the assertion that the sequence $\{b, a\}$ is an M-sequence for every M-sequence $\{a, b\}$.

All rings are assumed to be noetherian, commutative, with unity, and all modules are assumed to be of finite type, unitary.

Let A be a ring and M an A-module. We write $\mathscr{Z}(M)$ for the set of zero-divisors on M. Let a be an element of A and let f_a be the homomorphism $M \xrightarrow{a} M$, where $f_a(m)=am$ for $m\in M$. Then $a\in\mathscr{Z}(M)$ if and only if f_a is not injective. We denote by $\mathscr{R}(M)$ the set of M-regular elements. Note that $a\in \mathscr{R}(M)$ if and only if f_a is injective but not surjective. We let $\mathscr{U}(M)$ denote the set of all elements a in A such that f_a are isomorphims. If M is a non-zero module, it is clear that A is a disjoint union of the subsets $\mathscr{Z}(M)$, $\mathscr{R}(M)$ and $\mathscr{U}(M)$. Further we use freely the terminologies in [2].

2. The set $\mathscr{K}(M)$

DEFINITION. Let A be a ring, M an A-module. Then the set $\mathscr{K}(M)$ is defined to be the set of those elements a of A such that $\bigcap_{n=1}^{\infty} a^nM=0$.

It follows easily from our definition that $\mathscr{K}(M)\subset\mathscr{Z}(M)\cup\mathscr{R}(M)$ for a non-zero A-module M. In general $\mathscr{K}(M)$ is not an ideal. Applying Krull's intersection theorem, we have a basic proposition about $\mathscr{K}(M)$.

PROPOSITION 2.1. *Let A be a ring, M an A-module. Let a be an element of A. Then $a \in \mathscr{K}(M)$ if and only if $Aa+\mathfrak{p} \neq A$ for all $\mathfrak{p} \in \operatorname{Ass} M$.*

PROOF. By the intersection theorem (cf. [3], (3.11)), $a \in \mathscr{K}(M)$ is equivalent to the condition that if $b-1 \in Aa$ for $b \in A$, then $b \notin \mathscr{Z}(M)$. It is also equivalent to the assertion that if $b \in \mathscr{Z}(M)$, then $b-1 \notin Aa$. Since $\mathscr{Z}(M)$ is the set-theoretic union of the associated prime ideals of M, it occurs if and only if $Aa+\mathfrak{p} \neq A$ for all $\mathfrak{p} \in \operatorname{Ass} M$.

COROLLARY 2.2. *Let A be a ring and let $\mathfrak{q}$ be a prime ideal of A. Let M be a non-zero A-module. If every associated prime ideal of M is contained in $\mathfrak{q}$, then $\mathfrak{q} \subset \mathscr{K}(M)$.*

COROLLARY 2.3. *Let A be a ring, M an A-module. Then $\mathscr{Z}(M) \subset \mathscr{K}(M)$ if and only if $\mathfrak{p}_i+\mathfrak{p}_j \neq A$ for all $\mathfrak{p}_i$ and $\mathfrak{p}_j$ in* Ass M.

REMARK 2.4. By prop. 2.1, we see well-known facts that if A is an integral domain or a local ring, then $\mathscr{K}(A)=A-\mathscr{U}(A)$. Furthermore if A is a semi-local ring, then its Jacobson radical is contained in $\mathscr{K}(A)$.

Let A be a ring, M an A-module. We denote by $\mathscr{W}(M)$ the subset of A consisting of those elements a of A which satisfy the following condition: for every prime ideal $\mathfrak{p}$ in Ass M, there exists a prime ideal $\mathfrak{q}$ in Ass M/aM such that $\mathfrak{p} \subset \mathfrak{q}$.

PROPOSITION 2.5. *Let A be a ring, M a non-zero A-module. Then $\mathscr{W}(M) \subset \mathscr{K}(M)$.*

PROOF. Let $a \in \mathscr{W}(M)$. For every $\mathfrak{p} \in \operatorname{Ass} M$, we can find a prime ideal $\mathfrak{q}$ in Ass M/aM with $\mathfrak{p} \subset \mathfrak{q}$. Since $a \in \mathfrak{q}$, $Aa+\mathfrak{p} \subset \mathfrak{q}$. Thus $Aa+\mathfrak{p} \neq A$, and prop. 2.1 implies $a \in \mathscr{K}(M)$.

COROLLARY 2.6. *Let A and M be as above. Then $\mathscr{W}(M) \cap \mathscr{R}(M)=\mathscr{K}(M) \cap \mathscr{R}(M)$.*

PROOF. Prop. 2.5 implies that $\mathscr{W}(M) \cap \mathscr{R}(M) \subset \mathscr{K}(M) \cap \mathscr{R}(M)$ and the other inclusion follows from ([2], (15, d)).

PROPOSITION 2.7. *Let A be a ring, M a non-zero A-module. Then the following conditions are equivalent:*

(i) $\mathscr{K}(M) \subset \mathscr{Z}(M)$.

(ii) *There exists a maximal ideal of A which belongs to* Ass M.

PROOF. (ii)$\Rightarrow$(i). Let $\mathfrak{m}$ be a maximal ideal of A in Ass M. Assume the contrary. Then we can find an element a of $\mathscr{K}(M)$ with $a \notin \mathscr{Z}(M)$. Whence $a \notin \mathfrak{m}$, and so $Aa+\mathfrak{m}=A$. It therefore follows from prop. 2.1 that $a \notin \mathscr{K}(M)$.

(i)$\Rightarrow$(ii). It is sufficient to prove that if any maximal ideal does not belong to

Ass M, then $\mathscr{X}(M) \not\subset \mathscr{Z}(M)$. First we assume that A is a semi-local ring. Let $\{\mathfrak{m}_1,\dots,\mathfrak{m}_t\}$ be the set of the maximal ideals of A. Since every $\mathfrak{m}_i$ does not belong to Ass M, there is an element a_i of $m_i \cap \mathscr{R}(M)$ for $1 \leqq i \leqq t$. Put $a = a_1 \cdots a_t$. Then $a \in \mathscr{X}(M)$ by prop. 2.1; in fact, for every $\mathfrak{p} \in \operatorname{Ass} M$ we find a maximal ideal $\mathfrak{m}_i$ with $\mathfrak{p} \subset \mathfrak{m}_i$, thus $Aa + \mathfrak{p} \subset Aa + \mathfrak{m}_i \subset \mathfrak{m}_i$, and hence $Aa + \mathfrak{p} \neq A$. On the other hand we see $a \in \mathscr{R}(M)$ because $a \notin \mathscr{Z}(M)$ and $\mathscr{X}(M) \subset \mathscr{Z}(M) \cup \mathscr{R}(M)$. We therefore obtain that $\mathscr{X}(M) \not\subset \mathscr{Z}(M)$.

We now procced to the general case. Let $\operatorname{Ass} M = \{\mathfrak{p}_1,\dots,\mathfrak{p}_u\}$. Since each $\mathfrak{p}_i$ is not a maximal ideal, we find a maximal ideal $\mathfrak{m}_i$ with $\mathfrak{p}_i \subset \mathfrak{m}_i$. Put $S = \cap (A - \mathfrak{m}_i)$, $1 \leqq i \leqq u$. Then S is a multiplicative subset of A and $S \subset \mathscr{R}(M) \cup \mathscr{U}(M)$, whence the natural mapping $M \to S^{-1}M$ is injective. Note that $\operatorname{Ass}_{S^{-1}A} S^{-1}M = \{\mathfrak{p}_1 S^{-1}A,\dots,\mathfrak{p}_u S^{-1}A\}$. It also follows from the definition of $\mathscr{X}(M)$ that if $a/1 \in \mathscr{X}_{S^{-1}A}(S^{-1}M)$, then $a \in \mathscr{X}(M)$. Now, since $S^{-1}A$ is a semi-local ring which satisfies the condition that any maximal ideal does not belong to $\operatorname{Ass}_{S^{-1}A} S^{-1}M$, the first arguments imply $\mathscr{X}_{S^{-1}A}(S^{-1}M) \not\subset \mathscr{Z}_{S^{-1}A}(S^{-1}M)$. Thus we may choose an element $a/1$ in $\mathscr{X}_{S^{-1}A}(S^{-1}M)$ which is not contained in $\mathscr{Z}_{S^{-1}A}(S^{-1}M)$. Hence we see that $a \in \mathscr{X}(M)$ and $a \notin \mathscr{Z}(M)$, which settles the assertion.

COROLLARY 2.8. *Let A be a ring, M a non-zero A-module. Then $\mathscr{X}(M) = \mathscr{Z}(M)$ if and only if there exists a maximal ideal $\mathfrak{m}$ in* Ass M *such that $\mathfrak{p} \subset \mathfrak{m}$ for all $\mathfrak{p} \in \operatorname{Ass} M$.*

PROOF. This is immediate from cor. 2.3 and prop. 2.7.

LEMMA 2.9. *Let A be a ring and let M be an A-module. Then $\mathscr{U}(M) = A - \cup\, \mathfrak{p}$, $\mathfrak{p} \in \operatorname{Supp} M$.*

PROOF. We may assume that M is a non-zero A-module. We shall show that any element a in $\mathscr{U}(M)$ does not belong to any prime ideal in Supp M. Assume on the contrary that there exists $\mathfrak{p} \in \operatorname{Supp} M$ such that $a \in \mathfrak{p}$. Then $M_\mathfrak{p} = aM_\mathfrak{p}$. By Nakayama's lemma we see that $M_\mathfrak{p} = 0$. But this is a contradiction.

Conversely let a be an element of A which does not belong to any prime ideal in Supp M. Then $a \notin \mathscr{Z}(M)$. We thus have an exact sequence

$$0 \longrightarrow M \xrightarrow{a} M \longrightarrow M/aM \longrightarrow 0.$$

Let $\mathfrak{p}$ be a prime ideal of A. Then we get an exact sequence

$$0 \longrightarrow M_\mathfrak{p} \xrightarrow{a} M_\mathfrak{p} \longrightarrow (M/aM)_\mathfrak{p} \longrightarrow 0.$$

If $\mathfrak{p} \notin \operatorname{Supp} M$, then $M_\mathfrak{p} = 0$. Thus $(M/aM)_\mathfrak{p} = 0$. If $\mathfrak{p} \in \operatorname{Supp} M$, then $a \notin \mathfrak{p}$ by hypothesis. It follows that $M_\mathfrak{p} = aM_\mathfrak{p}$, and this implies $(M/aM)_\mathfrak{p} = 0$. Whence $M/aM = 0$, so a belongs to $\mathscr{U}(M)$.

THEOREM 2.10. *Let A be a ring and let M be a non-zero A-module. Then the following assertions are equivalent*:

(i) $\mathscr{K}(M)=A-\mathscr{U}(M)$.

(ii) *For every prime ideal* $\mathfrak{p}$ *in* $\operatorname{Ass} M$ *and for every maximal ideal* $\mathfrak{m}$ *in* $\operatorname{Supp} M$, *we have* $\mathfrak{p}\subset\mathfrak{m}$.

(iii) $\mathscr{R}(M)\subset\mathscr{K}(M)$ *and* $\mathfrak{p}_1+\cdots+\mathfrak{p}_n\neq A$, *where* $\operatorname{Ass} M=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$.

PROOF. (i)$\Rightarrow$(ii). We assume the contrary. Then we can find a prime ideal $\mathfrak{p}\in\operatorname{Ass} M$ and a maximal ideal $\mathfrak{m}\in\operatorname{Supp} M$ such that $\mathfrak{p}\not\subset\mathfrak{m}$. We thus see that $\mathfrak{m}+\mathfrak{p}=A$, and hence there exist an element $a\in\mathfrak{m}$ and an element $p\in\mathfrak{p}$ with $a+p=1$. Thus $Aa+\mathfrak{p}=A$, consequently, in view of prop. 2.1, $a\notin\mathscr{K}(M)$. By hypothesis we obtain $a\in\mathscr{U}(M)$. It therefore follows from lemma 2.9 that $a\notin\mathfrak{m}$, a contradiction.

(ii)$\Rightarrow$(i). We have only to prove that $\mathscr{Z}(M)\cup\mathscr{R}(M)\subset\mathscr{K}(M)$. Let a be an element of $\mathscr{R}(M)\cup\mathscr{B}(M)$. Then, using lemma 2.9, there exists a maximal ideal $\mathfrak{m}$ in $\operatorname{Supp} M$ such that $a\in\mathfrak{m}$. Since $\mathfrak{m}$ contains all associated prime ideals of M, it follows from cor. 2.2 that $\mathfrak{m}\subset\mathscr{K}(M)$. In particular a belongs to $\mathscr{K}(M)$.

(i)$\Rightarrow$(iii). Since A-$\mathscr{U}(M)=\mathscr{Z}(M)\cup\mathscr{R}(M)$, it is clear that $\mathscr{R}(M)\subset\mathscr{K}(M)$. There is a maximal ideal $\mathfrak{m}$ in $\operatorname{Supp} M$, for M is a non-zero module. Thus, by (ii), $\mathfrak{p}_1+\cdots+\mathfrak{p}_n\subset\mathfrak{m}$, and so $\mathfrak{p}_1+\cdots+\mathfrak{p}_n\neq A$.

(iii)$\Rightarrow$(i). It is enough to prove that $\mathscr{Z}(M)\subset\mathscr{K}(M)$. But this follows from cor. 2.3, because $\mathfrak{p}_i+\mathfrak{p}_j\subset\mathfrak{p}_1+\cdots+\mathfrak{p}_n\neq A$ for all $\mathfrak{p}_i$ and $\mathfrak{p}_j$.

EXAMPLE 2.11. Let R be a ring and let $\operatorname{Ass} R=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$. Suppose $\mathfrak{p}_1+\cdots+\mathfrak{p}_n\neq R$ and put $S=1+\mathfrak{p}_1+\cdots+\mathfrak{p}_n$. Then S is a multiplicative subset of R and it does not contain 0. Put $A=S^{-1}R$. Then A satisfies the equivalent conditions of theorem 2.10 as an A-module.

LEMMA 2.12. *Let M be a non-zero module over a ring A. We let* $\operatorname{Ass} M=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$. *Suppose that* $\mathfrak{p}_1+\cdots+\mathfrak{p}_n=A$ *and* $\mathscr{R}(M)\neq\emptyset$. *Then* $\mathscr{R}(M)\not\subset\mathscr{K}(M)$.

PROOF. We may assume that any $\mathfrak{p}_i$ is not maximal by prop. 2.7. First we also assume that there exist $\mathfrak{p}_i$ and $\mathfrak{p}_j$ such that $\mathfrak{p}_i+\mathfrak{p}_j=A$. Without loss of generality, we may suppose that there is an integer k with $2\leqq k\leqq n$ which satisfies the following conditions: $\mathfrak{p}_1+\mathfrak{p}_2=A,\dots,\mathfrak{p}_1+\mathfrak{p}_k=A$, $\mathfrak{p}_1+\mathfrak{p}_{k+1}\neq A,\dots,\mathfrak{p}_1+\mathfrak{p}_n\neq A$. Thus $\mathfrak{p}_1+\mathfrak{p}_2\cdots\mathfrak{p}_k=A$. We can therefore find an element $p_1\in\mathfrak{p}_1$ and an element $p_2\in\mathfrak{p}_2\cdots\mathfrak{p}_k$ such that $p_1+p_2=1$. We can also find a maximal ideal $\mathfrak{m}$ with $\mathfrak{p}_2\cdots\mathfrak{p}_k\subset\mathfrak{m}$. Since each $\mathfrak{p}_i$ for $1\leqq i\leqq n$ is not a maximal ideal, there exists an element $x\in\mathfrak{m}$ which does not belong to any $\mathfrak{p}_i$. Then $y=p_1x+p_2\in\mathfrak{m}$, and this implies $y\notin\mathscr{U}(M)$, since $\mathfrak{m}\in\operatorname{Supp} M$.

We shall show that $y\in\mathscr{R}(M)$. If $y\in\mathfrak{p}_1$, then $p_2\in\mathfrak{p}_1$, whence $1=p_1+p_2\in\mathfrak{p}_1$, and this is impossible. If $y\in\mathfrak{p}_i$ for $2\leqq i\leqq k$, then $p_1x\in\mathfrak{p}_i$. Since $x\notin\mathfrak{p}_i$, we have

$p_1 \in \mathfrak{p}_i$. Thus we also get a contradiction that $1 = p_1 + p_2 \in \mathfrak{p}_i$. Finally we assume that $y \in \mathfrak{p}_i$ for $k+1 \leqq i \leqq n$. Then it follows from the relation $p_1(1-x)+y=1$ that $\mathfrak{p}_1+\mathfrak{p}_i=A$, and it shows a contradiction. Using those results we see that $y \in \mathscr{R}(M)$. However $y \notin \mathscr{K}(M)$. Indeed, if $\mathfrak{n}$ is a maximal ideal with $Ay+\mathfrak{p}_1 \subset \mathfrak{n}$, then $\mathfrak{p}_1 \subset \mathfrak{n}$, whence $p_1 \in \mathfrak{n}$. Therefore, since $y=p_1x+p_2$ and $y \in \mathfrak{n}$, we see that $p_2 \in \mathfrak{n}$, and so $1=p_1+p_2 \in \mathfrak{n}$. This contradiction shows that $Ay+p_1=A$. Consequently $y \notin \mathscr{K}(M)$ by prop. 2.1.

Next we assume that for all i and j with $1 \leqq i, j \leqq n$, $\mathfrak{p}_i+\mathfrak{p}_j \neq A$. Let t be the smallest integer among integers u which satisfy a condition that there is a relation $\mathfrak{p}_{i_1}+\mathfrak{p}_{i_2}+\cdots+\mathfrak{p}_{i_u}=A$. We may assume that $\mathfrak{p}_1+\mathfrak{p}_2+\cdots+\mathfrak{p}_t=A$. Then we find elements $p_i \in \mathfrak{p}_i$ with $p_1+p_2+\cdots+p_t=1$. Put $y=1-p_1$. Then $y \notin \mathscr{Z}(M)$, because if $y \in \mathscr{Z}(M)$, then there exists some $\mathfrak{p}_i$ such that $y \in \mathfrak{p}_i$, and hence $\mathfrak{p}_1+\mathfrak{p}_i=A$, which is contrary to our assumption. Since $Ay+\mathfrak{p}_1=A$, it follows from prop. 2.1 that $y \notin \mathscr{K}(M)$. To prove that $\mathscr{R}(M) \not\subset \mathscr{K}(M)$, it is sufficient to show that $y \notin \mathscr{U}(M)$. Assume on the contrary that $y \in \mathscr{U}(M)$. Then, in view of lemma 2.9, we find an element $c \in A$ and $d \in \operatorname{Ann} M$ such that $cy+d=1$. We thus verify the identity $cp_2+\cdots+cp_{t-1}+(cp_t+d)=1$. Since $d \in \mathfrak{p}_t$, we get that $\mathfrak{p}_2+\cdots+\mathfrak{p}_t=A$, which contradicts the minimal property of the integer t. Accordingly we see that $y \notin \mathscr{U}(M)$ and this completes the proof.

THEOREM 2.13. *Let M be a non-zero module over a ring A. Assume that $\mathscr{R}(M) \neq \emptyset$. Then $\mathscr{K}(M)=A-\mathscr{U}(M)$ if and only if $\mathscr{R}(M) \subset \mathscr{K}(M)$.*

PROOF. We have only to show that if $\mathscr{R}(M) \subset \mathscr{K}(M)$, then $\mathscr{K}(M)=A-\mathscr{U}(M)$. But this follows from theorem 2.10 and lemma 2.12.

3. Modules M with $\mathscr{K}(M)=A-\mathscr{U}(M)$

In this section we study a method of construction of A-modules M with $\mathscr{K}(M)=A-\mathscr{U}(M)$.

DEFINITION. Let A be a ring and let $\mathfrak{a}$ be an ideal of A. We denote by $S(\mathfrak{a})$ the set of those elements a such that $Aa+\mathfrak{a}=A$.

Let φ be the natural mapping $A \to A/\mathfrak{a}$. Then $S(\mathfrak{a})=\varphi^{-1}(\mathscr{U}(A/\mathfrak{a}))$. We see that $S(\mathfrak{a})$ is a saturated multiplicative subset of A and that $0 \in S(\mathfrak{a})$ if and only if $\mathfrak{a}=A$.

PROPOSITION 3.1. *Let A be a ring, and let M be an A-module. Let a be an element of A. Then $a \in \mathscr{K}(M)$ if and only if $a \notin S(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass} M$.*

PROOF. This is an immediate consequence of prop. 2.1.

LEMMA 3.2. *Let A be a ring, $\mathfrak{a}$ an ideal of A and M an A-module. If $\mathfrak{a} \supset \operatorname{Ann} M$, then $S(\mathfrak{a}) \supset \mathscr{U}(M)$.*

PROOF. Let a be an element of A with $a \notin S(\mathfrak{a})$. Then $Aa+\mathfrak{a} \neq A$, whence there is a maximal ideal $\mathfrak{m}$ with $Aa+\mathfrak{a} \subset \mathfrak{m}$. Therefore $\mathfrak{m} \in \operatorname{Supp} M$ and $a \in \mathfrak{m}$. By lemma 2.9, we have $a \notin \mathscr{U}(M)$.

PROPOSITION 3.3. *Let M be a non-zero module over a ring A. Then $\mathscr{K}(M)=A-\mathscr{U}(M)$ if and only if $S(\mathfrak{p})=\mathscr{U}(M)$ for all $\mathfrak{p} \in \operatorname{Ass} M$.*

PROOF. Assume that $\mathscr{K}(M)=A-\mathscr{U}(M)$. We shall show that for all $\mathfrak{p} \in \operatorname{Ass} M$, $S(\mathfrak{p})=\mathscr{U}(M)$. Assume the contrary. Then there exists some $\mathfrak{p} \in \operatorname{Ass} M$ with $S(\mathfrak{p}) \neq \mathscr{U}(M)$. It therefore follows from lemma 3.2 that $S(\mathfrak{p}) \supsetneqq \mathscr{U}(M)$, and hence we find an element $a \in S(\mathfrak{p})$ with $a \notin \mathscr{U}(M)$. By assumption, we have $a \in \mathscr{K}(M)$. We thus get a required contradiction by prop. 3.1.

Conversely we assume that $S(\mathfrak{p})=\mathscr{U}(M)$ for all $\mathfrak{p} \in \operatorname{Ass} M$. Let a be an element of A with $a \notin \mathscr{U}(M)$. Then $a \notin S(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass} M$, and so prop. 3.1 shows $a \in \mathscr{K}(M)$. We therefore obtain that $\mathscr{K}(M) \supset A-\mathscr{U}(M)$. The other inclusion is obvious.

LEMMA 3.4. *Let A be a ring and let $\mathfrak{a}$ be an ideal of A. Suppose that $\mathfrak{p}$ is a prime ideal of A. Then $\mathfrak{p} \cap S(\mathfrak{a})=\emptyset$ if and only if $\mathfrak{p}+\mathfrak{a} \neq A$.*

PROOF. Assume $\mathfrak{p} \cap S(\mathfrak{a}) \neq \emptyset$. Then we find an element p of $\mathfrak{p} \cap S(\mathfrak{a})$, whence $ap+b=1$ for suitable elements $a \in A$ and $b \in \mathfrak{a}$, and so $\mathfrak{p}+\mathfrak{a}=A$. We can easily prove the "only if" part in the same way.

PROPOSITION 3.5. *Let M be a non-zero module over a ring A. We let $\operatorname{Ass} M=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$ and put $S=S(\mathfrak{p}_1+\cdots+\mathfrak{p}_n)$. Then the following statements are equivalent:*

(i) $\mathfrak{p}_1+\cdots+\mathfrak{p}_n=A$.

(ii) $S \ni 0$.

(iii) $S^{-1}M=0$.

(iv) $S \cap \mathfrak{p}_i \neq \emptyset$ *for some* $\mathfrak{p}_i$.

(v) $S \cap \mathfrak{p}_i \neq \emptyset$ *for all* $\mathfrak{p}_i$.

PROOF. By lemma 3.4 and the definition of S we can easily prove this proposition.

COROLLARY 3.6. *Notation and assumptions being the same as in the previous proposition, if $\mathfrak{p}_1+\cdots+\mathfrak{p}_n \neq A$, then*

(i) $S \cap \mathscr{Z}(M)=\emptyset$.

(ii) *The natural mapping $M \rightarrow S^{-1}M$ is injective.*

(iii) $\operatorname{Ass}_{S^{-1}A} S^{-1}M=\{\mathfrak{p}_1S^{-1}A,\dots,\mathfrak{p}_nS^{-1}A\}$.

LEMMA 3.7. *Let M be a module over a ring A and let $\mathfrak{b}$ be an ideal of A with $\operatorname{Ann} M \subset \mathfrak{b}$. If $S(\mathfrak{b})=\mathscr{U}(A)$, then $S(\mathfrak{b})=\mathscr{U}(M)$.*

PROOF. We first show that every maximal ideal of A belongs to Supp M. Let $\mathfrak{m}$ be a maximal ideal of A. Assume that $\mathfrak{b} \not\subset \mathfrak{m}$. Then $\mathfrak{m}+\mathfrak{b}=A$, whence there exists an element $y \in \mathfrak{m}$ such that $Ay+\mathfrak{b}=A$. Thus this yields $y \in S(\mathfrak{b})$, and so $y \in \mathscr{U}(A)$. This contradiction shows $\mathfrak{b} \subset \mathfrak{m}$, and hence $\mathfrak{m} \in \operatorname{Supp} M$. Now, in view of lemma 2.9, we obtain that $\mathscr{U}(M)=\mathscr{U}(A)$, and so $S(\mathfrak{b})=\mathscr{U}(M)$.

LEMMA 3.8. *Let M be a non-zero module over a ring A. Let $S=S(\mathfrak{p}_1+\cdots+\mathfrak{p}_n)$, where $\mathfrak{p}_i$ runs through the set* Ass M. *Then $S_{S^{-1}A}(\mathfrak{p}_i S^{-1}A)=\mathscr{U}(S^{-1}A)$ for all $\mathfrak{p}_i \in \operatorname{Ass} M$.*

PROOF. We may assume that $\mathfrak{p}_1+\cdots+\mathfrak{p}_n \neq A$. It is sufficient to show that $S_{S^{-1}A}(\mathfrak{p}_i S^{-1}A) \subset \mathscr{U}(S^{-1}A)$ for all $\mathfrak{p}_i \in \operatorname{Ass} M$. Let a/s_1 be an element of $S_{S^{-1}A}(\mathfrak{p}_i S^{-1}A)$, where $a \in A$, $s_1 \in S$ and $\mathfrak{p}_i \in \operatorname{Ass} M$. Then we find elements $b \in A$, s_2, $s_3 \in S$ and $p_i \in \mathfrak{p}_i$ with $a/s_1 \cdot b/s_2+p_1/s_3=1$. Thes $abs_3s_4+p_is_1s_2s_4=s_1s_2s_3s_4$ for a suitable element s_4 of S. Since $s_1s_2s_3s_4$ belongs to S, there exist elements $c \in A$ and $q \in \mathfrak{p}_1+\cdots+\mathfrak{p}_n$ such that $cs_1s_2s_3s_4+q=1$. We therefore obtain $abcs_3s_4+cp_is_1s_2s_4=cs_1s_2s_3s_4=1-q$, whence $abcs_3s_4+(cp_is_1s_2s_4+q)=1$. Since $cp_is_1s_2s_4+q \in \mathfrak{p}_1+\cdots+\mathfrak{p}_n$, this relation yields $a \in S$. Thus we see that $a/s_1 \in \mathscr{U}(S^{-1}A)$, and we complete the proof.

PROPOSITION 3.9. *Let A be a ring. Let M a non-zero A-module with associated prime ideals $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$. Suppose that $\mathfrak{p}_1+\cdots+\mathfrak{p}_n \neq A$. Then $\mathscr{K}_{S^{-1}A}(S^{-1}M)=S^{-1}A-\mathscr{U}_{S^{-1}A}(S^{-1}M)$, where $S=S(\mathfrak{p}_1+\cdots+\mathfrak{p}_n)$.*

PROOF. By prop. 3.3 and cor. 3.6, it is enough to show that $S_{S^{-1}A}(\mathfrak{p}_i S^{-1}A)=\mathscr{U}_{S^{-1}A}(S^{-1}M)$ for all $\mathfrak{p}_i$. However the assertion follows from lemma 3.7 and lemma 3.8.

THEOREM 3.10. *Assumptions being the same as in prop.* 3.9, *the following conditions are equivalent*:

(i) $\mathscr{K}(M)=A-\mathscr{U}(M)$.

(ii) $S=\mathscr{U}(M)$.

(iii) *The natural mapping $M \to S^{-1}M$ is an isomorphism.*

PROOF. (i)$\Rightarrow$(ii). It is suffcient to prove $S \subset \mathscr{U}(M)$ by lemma 3.2. Let s be an element of S. Then $as+q=1$ for suitable elements $a \in A$ and $q \in \mathfrak{p}_1+\cdots+\mathfrak{p}_n$. Let $\mathfrak{m}$ be a maximal ideal in Supp M. Then $q \in \mathfrak{m}$ by theorem 2.10, and hence $s \notin \mathfrak{m}$. This implies $s \in \mathscr{U}(M)$ by lemma 2.9. Thus $S \subset \mathscr{U}(M)$.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i) follows from prop. 3.9. We complete the proof.

We continue with the assumptions of prop. 3.9. We set $T=A-\mathscr{Z}(M)$. Then it is clear that $S \subset T$, and hence the natural mapping $S^{-1}M \to T^{-1}M$ is injective. We denote by P the set of prime ideals of A which contain all $\mathfrak{p}_i$. Since $\mathfrak{p}_1+\cdots+$

$\mathfrak{p}_n \neq A$, we know that $P \neq \emptyset$. Let $\mathfrak{q}$ belong to P. It follows from lemma 3.4 that $S \subset A - \mathfrak{q} \subset T$, and thus we may assume that $S^{-1}M \subset M_{\mathfrak{q}} \subset T^{-1}M$. Consequently we have $S^{-1}M \subset \cap M_{\mathfrak{q}}$, $\mathfrak{q} \in P$.

PROPOSITION 3.11. *Let the assumptions be as above. Then* $S^{-1}M = \cap M_{\mathfrak{q}}$, *where* $\mathfrak{q}$ *ranges over the set* P.

PROOF. It will suffice to show that $S^{-1}M \supset \cap M_{\mathfrak{q}}$. By properties of the the localizations and cor. 3.6 we may assume that $M = S^{-1}M$. For an element $x \in \cap M_{\mathfrak{q}}$, we put $\mathfrak{b} = \{a \in A | ax \in M\}$. Then $\mathfrak{b}$ is an ideal of A with $\mathfrak{b} \supset \operatorname{Ann} M$. We want to show that $\mathfrak{b} = A$, which implies $x \in M$, and hence we get $M \supset \cap M_{\mathfrak{q}}$. Assume on the contrary that $\mathfrak{b} \neq A$. Then there exists a maximal ideal $\mathfrak{m}$ such that $\mathfrak{b} \subset \mathfrak{m}$. If $\mathfrak{m} \in P$, then $x \in M_{\mathfrak{m}}$. Whence we can write $x = m/t$ for suitable $m \in M$ and $t \notin \mathfrak{m}$. Accordingly $tx = m$, and so $t \in \mathfrak{b}$. This contradicts our assumption that $\mathfrak{b} \subset \mathfrak{m}$ and it yields $\mathfrak{m} \notin P$. We therefore find some $\mathfrak{p}_i$ with $\mathfrak{p}_i \not\subset \mathfrak{m}$. It is clear that $\mathfrak{p}_i + \mathfrak{m} = A$, and there are elements $p_i \in \mathfrak{p}_i$ and $a \in \mathfrak{m}$ with $p_i + a = 1$, which implies $a \in S$. But, since $M = S^{-1}M$, theorem 3.10 implies $a \in \mathscr{U}(M)$. We now get a required contradiction by lemma 2.9, because $a \in \mathfrak{m}$ and $\mathfrak{m} \in \operatorname{Supp} M$. This completes the proof.

4. Permutations of *M*-sequence

We consider permutations of M-sequences in this section. D. Taylor proved the following assertion in [4]: If A possesses an A-sequence of length 3, and if every permutation of an A-sequence is an A-sequence, then A is a local ring. Now we give some conditions which are equivalent to saying that $\{b, a\}$ is an M-sequence for every M-sequence $\{a, b\}$.

LEMMA 4.1. *Let A be a ring and let M be an A-module. If $\{a, b\}$ is an M-sequence, then* $0 :_M b \subset \cap a^n M$ $(n = 1, 2, \dots)$, *where* $0 :_M b = \{m \in M | bm = 0\}$.

PROOF. Let m be an element in $0 :_M b$. Then $bm = 0 = a0$. Since $\{a, b\}$ is an M-sequence, we find an element $m_1 \in M$ with $m = am_1$, and hence $abm_1 = 0$. Thus $bm_1 = 0$, for $a \notin \mathscr{Z}(M)$. Repeating this argument with m_1, we can write $m_1 = am_2$ for suitable $m_2 \in M$. Whence it implies $m = a^2 m_2$. It thus follows from these observations that $m \in \cap a^n M$.

COROLLARY 4.2. *Let M be a module over a ring A. If $\{a, b\}$ is an M-sequence with $a \in \mathscr{K}(M)$, then $\{b, a\}$ is an M-sequence.*

PROOF. Since $a \notin \mathscr{Z}(M/bM)$ ([1], Theorem 117), we have only to prove that $b \notin \mathscr{Z}(M)$. However the assertion follows from lemma 4.1.

COROLLARY 4.3. *Let M be a module over a ring A. Suppose that* $\mathscr{R}(M) \subset$

$\mathscr{K}(M)$. *Then* $\{b, a\}$ *is an* M*-sequence for every* M*-sequence* $\{a, b\}$.

PROPOSITION 4.4. *Let* M *be a module over a ring* A. *Let* a *be an element in* $\mathscr{R}(M)$. *Put* $N = \cap a^n M$ $(n=1, 2,\ldots)$ *and let* $\overline{M} = M/N$. *Then*:

(i) a *is an element in* $\mathscr{R}(\overline{M}) \cap \mathscr{K}(\overline{M})$.

(ii) *If* $\{a, b\}$ *is an* M*-sequence, then* $\{a, b\}$ *and* $\{b, a\}$ *are* $\overline{M}$*-sequences.*

(iii) *For a prime ideal* $\mathfrak{p}$ *of* A, $\mathfrak{p} \in \operatorname{Ass} \overline{M}$ *if and only if* $\mathfrak{p} \in \operatorname{Ass} M$ *and* $\mathfrak{p} \subset \mathfrak{q}$ *for some* $\mathfrak{q} \in \operatorname{Ass} M/aM$.

PROOF. (i) and (ii) are easily shown by elementary properties of M-sequences and cor. 4.2.

(iii) We first note that if $\mathfrak{p}$ is a prime ideal such that any associated prime ideal of M/aM does not contain $\mathfrak{p}$, then $\mathfrak{p} \notin \operatorname{Ass} \overline{M}$. Assume on the contrary that $\mathfrak{p} \in \operatorname{Ass} \overline{M}$. Then $\mathfrak{p} = \operatorname{Ann}(\overline{m})$ for suitable $m \in M$, where $\overline{m}$ denotes the image of m is $\overline{M}$. Furthermore we find an element $b \in \mathfrak{p}$ with $b \notin \mathscr{Z}(M/aM)$. Thus $b\overline{m} = 0$, whence $bm \in N$. In particular $bm \in a^n M$ for all positive integers n. By the fact that $\mathscr{Z}(M/a^n M) = \mathscr{Z}(M/aM)$ ([1], Ch. 3, Ex. 13), we have $b \notin \mathscr{Z}(M/a^n M)$. It thus implies $m \in a^n M$, and so $m \in N$, that is $\overline{m} = 0$. This is a required contradiction.

Now we are ready to prove (iii). We may only deal with a prime ideal $\mathfrak{p}$ which is contained in some $\mathfrak{q} \in \operatorname{Ass} M/aM$. Then $N_{\mathfrak{q}} = 0$, because $N_{\mathfrak{q}} \subset \cap_n (a/1)^n M_{\mathfrak{q}} = 0$. It therefore follows that $M_{\mathfrak{q}} = M_{\mathfrak{q}}/N_{\mathfrak{q}} = \overline{M}_{\mathfrak{q}}$. Since $\mathfrak{p} \in \operatorname{Ass} M$ if and only if $\mathfrak{p}A_{\mathfrak{q}} \in \operatorname{Ass}_{A_{\mathfrak{q}}} M_{\mathfrak{q}}$, we thus know that $\mathfrak{p} \in \operatorname{Ass} M$ if and only if $\mathfrak{p}A_{\mathfrak{q}} \in \operatorname{Ass}_{A_{\mathfrak{q}}} \overline{M}_{\mathfrak{q}}$, and it happens if and only if $\mathfrak{p} \in \operatorname{Ass} \overline{M}$.

LEMMA 4.5. *Let* M *be a module over a ring* A *and let* $\mathfrak{p}$ *be a prime ideal in* $\operatorname{Ass} M$. *Let* a *be an element of* $\mathscr{R}(M)$. *Then* $Aa + \mathfrak{p} \neq A$ *if and only if there exists an associated prime ideal* $\mathfrak{q}$ *of* M/aM *with* $\mathfrak{p} \subset \mathfrak{q}$.

PROOF. The "if" part is obvious. Suppose $Aa + \mathfrak{p} \neq A$; Then we find a maximal ideal $\mathfrak{m}$ such that $Aa + \mathfrak{p} \subset m$. Thus $\mathfrak{p}A_{\mathfrak{m}} \in \operatorname{Ass}_{A\mathfrak{m}} M_{\mathfrak{m}}$ and $a \in \mathfrak{m}A_{\mathfrak{m}}$. Since $a \in \mathscr{R}_{A\mathfrak{m}}(M_{\mathfrak{m}})$, it follows from cor. 2.6 that $a \in \mathscr{W}_{A\mathfrak{m}}(M_{\mathfrak{m}})$, that is to say, there is a prime ideal $\mathfrak{q}A_{\mathfrak{m}} \in \operatorname{Ass}_{A\mathfrak{m}} M_{\mathfrak{m}}/aM_{\mathfrak{m}}$ such that $\mathfrak{p}A_{\mathfrak{m}} \subset \mathfrak{q}A_{\mathfrak{m}}$. Thus we find a required prime ideal $\mathfrak{q} \in \operatorname{Ass} M/aM$ which contains $\mathfrak{p}$.

LEMMA 4.6. *Let* M *be a module over a ring* A. *Let* a *be an element of* $\mathscr{R}(M) - \mathscr{K}(M)$ *with* $\mathscr{R}(M/aM) \neq \emptyset$. *Then there exists an element* b *of* $\mathscr{Z}(M)$ *such that* $\{a, b\}$ *is an* M*-sequence.*

PROOF. By prop. 2.1, there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass} M$ with $Aa + \mathfrak{p} = A$, for $a \notin \mathscr{K}(M)$. It now follows from lemma 4.5 that $\mathfrak{p} \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in \operatorname{Ass} M/aM$. Since $\mathscr{R}(M/aM) \neq \emptyset$, we find a maximal ideal $\mathfrak{m}$ such that $Aa + \operatorname{Ann} M \subset \mathfrak{m}$ and $\mathfrak{m} \notin \operatorname{Ass} M/aM$. Then we see that $\mathfrak{m} \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in \operatorname{Ass} M/aM$. It follows from these results that $\mathfrak{m} \cap \mathfrak{p} \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in \operatorname{Ass} M/aM$, and hence there is an element $b \in \mathfrak{m} \cap \mathfrak{p}$ which is not contained in any associated prime ideal of M/aM. Since $\mathfrak{m} \in \operatorname{Supp} M$ and $(a, b)M \subset \mathfrak{m}M$, we see $(a, b)M \neq M$. Consequently $\{a, b\}$ is an M-sequence with $b \in \mathscr{Z}(M)$.

THEOREM 4.7. *Let A be a ring and M be an A-module. Then the following conditions are equivalent*:

(i) *For every M-sequence $\{a, b\}$ of length 2, $\{b, a\}$ is an M-sequence.*

(ii) *For every $a \in \mathscr{R}(M) - \mathscr{K}(M)$, $\mathscr{R}(M/aM) = \emptyset$.*

(iii) *For every $a \in \mathscr{R}(M) - \mathscr{K}(M)$ and every maximal ideal $\mathfrak{m}$ in* $\operatorname{Supp} M/aM$, $\operatorname{depth}_{A_\mathfrak{m}} M_\mathfrak{m} = 1$.

PROOF. (i)$\Rightarrow$(ii) is an immediate consequence of lemma 4.6. (ii)$\Rightarrow$(i) follows from cor. 4.2.

(ii)$\Leftrightarrow$(iii). Let a be an element of $\mathscr{R}(M)$. Then, since $\mathscr{R}(M/aM) = \emptyset$ means that every maximal ideal in $\operatorname{Supp} M/aM$ belongs to $\operatorname{Ass} M/aM$, we see that $\mathscr{R}(M/aM) = \emptyset$ if and only if for all maximal ideals $\mathfrak{m}$ in $\operatorname{Supp} M/aM$, $\operatorname{depth}_{A_\mathfrak{m}} M_\mathfrak{m} = 1$.

EXAMPLE 4.8. We consider a quotient ring $R = k[X, Y, Z]/(XY)$ of the polynomial ring over a field k and we write $R = k[x, y, z]$ as usual. Put $\mathfrak{n} = (x, y, z)$ and $\mathfrak{r} = (x-1, y)$. Then $\mathfrak{n}$ and $\mathfrak{r}$ are prime ideals of R. Let $A = S^{-1}R$, $\mathfrak{m} = \mathfrak{n}A$ and $\mathfrak{q} = \mathfrak{r}A$, where we put $S = (R - \mathfrak{n}) \cap (R - \mathfrak{r})$. Then A is a semi-local ring with its maximal ideals $\mathfrak{m}$ and $\mathfrak{q}$. Furthermore we see easily that $\operatorname{Ass} A = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, where $\mathfrak{p}_1 = Ax$ and $\mathfrak{p}_2 = Ay$. Since $\mathfrak{p}_1 \cup \mathfrak{p}_2 \subset \mathfrak{m}$, we know by cor 2.2 that $\mathfrak{m} \subset \mathscr{K}(A)$, and this implies $\mathscr{R}(A) - \mathscr{K}(A) \subset \mathfrak{q}$. On the other hand it follows from prop. 2.1 and the relation $A(x-1) + \mathfrak{p}_1 = A$ that $x - 1 \notin \mathscr{K}(A)$, and so $\mathscr{R}(A) - \mathscr{K}(A) \neq \emptyset$. We wish to show that A satisfies the equivalent conditions of theorem 4.7 as an A-module. This can be shown as follows. Let a be an element in $\mathscr{R}(A) - \mathscr{K}(A)$. Then $\mathfrak{q}$ is the only prime ideal which belongs to $\operatorname{Ass} A/Aa$, because $a \notin \mathfrak{m}$ and $ht\ \mathfrak{q} = 1$. We therefore conclude that $\mathscr{R}(A/Aa) = \emptyset$.

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