Hiroshima Math. J.
15 (1985), 393–410

On minimally thin and rarefied sets in R^p, $p \geq 2$

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(Received October 31, 1983)
(Revised December 20, 1984)

§ 1. Introduction

Let $D=\{x \in R^p: x_1>0\}$ where $x=(x_1,\dots, x_p)$ and $p \geq 2$ and let ∂D be the euclidean boundary of D. If u is subharmonic in D and $y \in \partial D$, we define $u(y)=\limsup u(x)$, $x \to y$, $x \in D$. If $u \leq 0$ on ∂D and if $\sup u(x)/x_1<\infty$, then it is known that

$$u(x)/x_1 \to \alpha, \quad x \to \infty, \quad x \in D \setminus E, \tag{1.1}$$

where the exceptional set E is minimally thin at infinity (cf. J. Lelong–Ferrand [8]). This result is best possible in the sense that the property of minimal thinness at infinity in D completely characterizes the exceptional set in question. If $p \geq 3$, it is also known that

$$(u(x)-\alpha x_1)/|x| \to 0, \quad x \to \infty, \quad x \in D \setminus F, \tag{1.2}$$

where the exceptional set F is rarefied at infinity in D (cf. Essén–Jackson [5b]).

In the present paper, we deduce precise descriptions of the geometrical properties of the exceptional sets E and F which will be new when $p=2$ and which will be improvements of the results of Essén and Jackson on problems (1.1) and (1.2) when $p \geq 3$. Our Theorems 1, 2 and 3 are best possible of their kind and contain the earlier of results of this type which are due to Ahlfors and Heins [1], Hayman [6] and Azarin [2]. (For details on earlier work, we refer the reader to the introduction in [5b]).

We shall say that a set $E \subset D$ has a covering $\{t_n, r_n, R_n\}$ if there exists a sequence of balls $\{B_n\}$ with centers in D such that $E \subset \cup B_n$ where r_n is the radius of B_n, R_n is the distance from the origin to the center of B_n and t_n is the distance from the center of B_n to ∂D.

It is known that the subharmonic function u can be uniquely decomposed as

$$u(x) = \alpha x_1 - G\mu(x) - P\mu_1(x),$$

where α is defined in (1.1), $G\mu$ is the Green potential of a mass distribution μ

1) Research supported by the Swedish National Science Research Council.
2) Support from NSERC Grant #A7322 is acknowledged.

on D and $P\mu_1$ is the Poisson integral of a mass distribution μ_1 on ∂D. Thus, it will be sufficient to find exceptional sets E and F such that

$$(G\mu(x)+P\mu_1(x))/x_1 \to 0, \quad x \to \infty, \quad x \in D \setminus E, \tag{1.1'}$$

$$(G\mu(x)+P\mu_1(x))/|x| \to 0, \quad x \to \infty, \quad x \in D \setminus F. \tag{1.2'}$$

As examples of our results, we mention that, for each $\beta > p-2$, there exists a covering $\{t_n, r_n, R_n\}$ of E such that

$$\sum (t_n/R_n)^p (r_n/t_n)^\beta < \infty. \tag{1.3}$$

Furthermore, for each $\beta > p-2$, there exists a covering $\{t_n, r_n, R_n\}$ of F such that

$$\sum (t_n/R_n)^{p-1} (r_n/t_n)^\beta < \infty. \tag{1.4}$$

In both cases, we have $0 < r_n \le t_n \sqrt{p}$ for all n.

The associated results in Essén and Jackson [5b] in the case $p \ge 3$ are that for each $\beta > p-2$, there exist coverings satisfying

$$\sum (t_n/R_n)^2 (r_n/R_n)^\beta < \infty, \tag{1.3'}$$

$$\sum (t_n/R_n)(r_n/R_n)^\beta < \infty \tag{1.4'}$$

for E and F, respectively. It is easy to check that the new results (1.3) and (1.4) are stronger.

REMARK. In the present paper, we study directly the sets where $G\mu(x)+P\mu_1(x)$ is large. An alternative approach following the main ideas in [5b] is also possible: the starting-point is to define the exceptional sets in terms of generalized Wiener conditions. This potential-theoretic approach has other interesting consequences which we shall pursue in a separate paper which will also give details on rarefied sets when $p=2$. We note that Theorems 1 and 2 below are inspired by the form these results have when $p=2$: this follows in a natural way from potential-theoretic considerations in the plane. It is a surprising fact that it carries over to higher dimensions in spite of the fact that the potential theory in higher dimensions is rather different from the potential theory in the plane.

§ 2. Notation

(i) As in §1, D denotes the half-space $\{x \in \mathbf{R}^p : x_1 > 0\}$, where $p \ge 2$.

(ii) By $B=(t, r, R)$, we mean a ball of radius r, centre $P=(t, x_2, \dots, x_p)$ where $t>0$ and $R=|P|$. We also introduce $\boldsymbol{H}$ to be the collection of all sets of the form $B \cap D$, where $0 < r \le t\sqrt{p}$. This means that if $Q \subset D \cup \partial D$ is a closed cube with sides parallel to the coordinate axis, then the ball B, whose centre is at the centre

of Q and whose diameter is that of Q, is such that $B \cap D \in \boldsymbol{H}$.
(iii) Let $S_n = \{x \in D \cup \partial D: 2^n \leq |x| < 2^{n+1}\}$.
(iv) Let $\phi_{p-2}(|x|) = |x|^{2-p}$, $p \geq 3$, and let $\phi_0(|x|) = -\log|x|$ be the fundamental kernels on $\boldsymbol{R}^p$, $p \geq 2$. We also introduce

$$F_0(r) = \begin{cases} \min\{(\log(1/r))^{-1}, 1\}, \ r > 0, \\ 0, \ r = 0. \end{cases}$$

(v) If $E \subset \boldsymbol{R}^p$, we let $E_n = E \cap S_n$.
(vi) If $x = (x_1, \ldots, x_p)$, then $\tilde{x} = (-x_1, x_2, \ldots, x_p)$ shall denote the reflection of x about the Euclidean boundary ∂D of D.
(vii) Let, for x and y in D, $G(x, y) = \phi_{p-2}(|x-y|) - \phi_{p-2}(|x-\tilde{y}|)$ be the Green kernel for D, and let $G\mu(x) = \int_{S(\mu)} G(x, y)\,d\mu(y)$ be the Green potential at x of the Radon measure μ whose support is $S(\mu)$. Let, for $x \in D$ and $y \in \partial D$, $P(x, y) = x_1|x-y|^{-p}$ be the Poisson kernel for D and let $P\mu_1(x) = \int_{\partial D} P(x, y)\,d\mu_1(y)$ be the Poisson integral of the Radon measure μ_1 with support contained in ∂D.
(viii) If $y \in \partial D$, we define the cone $\Gamma(y) = \{x \in D: 2x_1 > |x-y|\}$.
(ix) If f and g are positive real-valued functions on a set X, we shall say that f is comparable to g, and write $f \approx g$ iff there exist constants A, B, $0 < A \leq B$, such that $Ag \leq f \leq Bg$ on X.
(x) Coverings of the form $\{t_n, r_n, R_n\}$ were defined in the introduction. We shall say that a set $E \subset \boldsymbol{R}^p$ has a covering $\{r_n, R_n\}$ if there exists a sequence of balls $\{B_n\}$ in $\boldsymbol{R}^p$ such that $E \subset \cup B_n$, where r_n is the radius of B_n and R_n is the distance between the origin and the centre of B_n.

§ 3. The main results

Let $h: [0, \infty) \to [0, \infty)$ be a continuous non-decreasing function such that $h(0) = 0$.

THEOREM 1. *Let u be as in the introduction. Let h be as above and assume furthermore that*

$$\int_0^1 h(r) r^{1-p}\,dr < \infty. \tag{3.1}$$

Then the exceptional set E in (1.1) *can be covered by a sequence $\{B_n \cap D\}$ in $\boldsymbol{H}$ such*

$$\sum (t_n/R_n)^p h(r_n/t_n) < \infty, \tag{3.2}$$

where $B_n = (t_n, r_n, R_n)$, $n = 1, 2, \ldots$.

THEOREM 2. *Let u and h be as in Theorem 1. Then the exceptional set*

F in (1.2) *can be covered by a sequence* $\{B_n \cap D\}$ *in* $\boldsymbol{H}$ *such that*

$$\sum (t_n/R_n)^{p-1} h(r_n/t_n) < \infty, \tag{3.3}$$

where $B_n = (t_n, r_n, R_n)$, $n = 1, 2, \ldots$.

REMARK. If $h(r)r^{2-p}$ is non-decreasing on $[0, \sqrt{p}]$, then for $a = 0, 1$, we have

$$(t_n/R_n)^{1+a} h(r_n/R_n) \leq (t_n/R_n)^{p+a-1} h(r_n/t_n). \tag{3.4}$$

In Theorem 5.1 in [5b], Essén and Jackson proved that when $p \geq 3$, these exceptional sets have coverings $\{B_n \cap D\}$ in $\boldsymbol{H}$ such that

$$\sum (t_n/R_n)^{1+a} h(r_n/R_n) < \infty,$$

where we have $a = 1$ in the minimally thin case (1.1) and $a = 0$ in the rarefied case (1.2). Thus, assuming that (3.4) holds, we see that Theorems 1 and 2 give us a new result when $p = 2$ and an improvement of Theorem 5.1 in [5b], when $p \geq 3$.

REMARK. In Section 8, we give an example showing that our results will no longer be true if the integral in (3.1) is divergent.

We claim that the exponents p in (3.2) and $p-1$ in (3.3) are sharp. To see this, we note that if $p \geq 3$ if $\{B_n\}$ is a sequence of balls such that $E = \cup B_n$ and if $B_n \subset S_n$ for all n, the set E is minimally thin or rarefied at ∞ in D if and only if

$$\sum (t_n/R_n)^p (r_n/t_n)^{p-2} < \infty,$$

$$\sum (t_n/R_n)^{p-1} (r_n/t_n)^{p-2} < \infty,$$

(cf. Theorem 4.1 in [5b]). Here $B_n = (t_n, r_n, R_n)$, $n = 1, 2, \ldots$. For each $\varepsilon > 0$, there exists a function h satisfying (3.1) and a minimally thin set E which is such that if E is covered by $\cup B_n^*$, we have

$$\sum (t_n^*/R_n^*)^{p-\varepsilon} h(r_n^*/t_n^*) = \infty.$$

We choose $E = \cup B_n$ where $R_n = 2^n + 1$, $t_n/R_n = n^{-1/p}$ and $r_n/t_n = (\log n)^{-2/(p-2)}$, $n = 2, 3, \ldots$. We omit the details in the remaining cases.

COROLLARY 1. *Let* $p \geq 3$. *For each* $\beta > p-2$, *the exceptional set* E *in Theorem* 1 *has a covering* $\{B_n \cap D\}$ *in* $\boldsymbol{H}$ *such that* (1.3) *holds and the exceptional set* F *in Theorem* 2 *has a covering* $\{B_n \cap D\}$ *in* $\boldsymbol{H}$ *such that* (1.4) *holds.*

PROOF. We choose $h(r) = \min(r^\beta, 1)$, $r \geq 0$. Since (3.1) holds, we have coverings according to Theorems 1 and 2. Let $\sum'$ denote that we sum over those indices where $r_n/t_n \leq 1$ and let $\sum''$ denote the remaining indices. For the set E, since $r_n \leq t_n\sqrt{p}$, we have

$$\sum' (t_n/R_n)^p (r_n/t_n)^\beta = \sum' (t_n/R_n)^p h(r_n/t_n),$$
$$\sum'' (t_n/R_n)^p (r_n/t_n)^\beta \leq p^{\beta/2} \sum'' (t_n/R_n)^p h(r_n/t_n),$$

and the claim follows for the set E. The set F is treated in exactly the same way.

COROLLARY 2. *Let $p=2$. For each $\beta>1$, the exceptional sets E in Theorem 1 and F in Theorem 2 have coverings $\{B_n \cap D\}$ in $\boldsymbol{H}$ such that*

$$\sum (t_n/R_n)^2 F_0(r_n/t_n)^\beta < \infty,$$
$$\sum (t_n/R_n) F_0(r_n/t_n)^\beta < \infty,$$

respectively.

PROOF. We choose $h(r)=F_0(r)^\beta$ which satisfies (3.1) and argue in the same way as in the proof of Corollary 1.

We have also got a new proof of Azarin's result [2] which says that the set F in Theorem 2 has a covering such that $\sum (r_n/R_n)^{p-1}<\infty$ (just choose $h(r)=r^{p-1}$ in (3.3)!). We have also got a little more.

COROLLARY 3. *The exceptional set E in Theorem 1 has a covering $\{r_n, R_n\}$ such that $\sum (r_n/R_n)^p<\infty$.*

PROOF. Choose $h(r)=r^p$ in (3.2).

REMARK. These coverings are interesting only for the parts of the exceptional sets which are close to ∂D.

Theorem 2 is a direct consequence of the following somewhat more precise result.

THEOREM 3. *Let μ and μ_1 be non-negative measures on D and ∂D, respectively, which are such that $G\mu+P\mu_1$ is a superharmonic function in D. Let h be as in Theorem 1. Let*

$$F = \{x \in D\colon G\mu(x)+P\mu_1(x)>|x|\}. \tag{3.5}$$

Then there exist sets F_1 and F_2 such that $F\subset F_1 \cup F_2$, and F_1 and F_2 can be described as follows:

a) *There exists an open set $O\subset\partial D$ such that*

$$\int_O (1+|x|)^{1-p} dx < \infty, \tag{3.6}$$

and $F_1=D\setminus(\bigcup_{y\in\partial D\setminus O}\Gamma(y))$. F_1 can be covered by a union of p-dimensional balls $\{r_n, R_n\}$ with centres on ∂D such that

$$\sum (r_n/R_n)^{p-1} < \infty. \tag{3.7}$$

b) *$F_2 \subset D$ is an open set such that*

$$\int_{F_2} x_1^{-1}(|x|+1)^{1-p} dx < \infty, \tag{3.8}$$

and can be covered by balls $\{B_n \cap D\}$ in $\boldsymbol{H}$ such that

$$\sum (t_n/R_n)^{p-1} h(r_n/t_n) < \infty, \tag{3.9}$$

where $B_n = (t_n, r_n, R_n)$, $n=1, 2, \ldots$.

REMARK. In the proof of Theorem 1, we use a result of B. Dahlberg [4] in a form given by P. Sjögren [10] (also cf. Sjögren [11] for the notion of "convolution set" used in [10]): Let

$$E = \{x \in D\colon G\mu(x) + P\mu_1(x) > x_1\}, \tag{3.10}$$

where we use the notation of Theorem 3. Then we have

$$\int_E (1+|x|)^{-p} dx < \infty. \tag{3.11}$$

The corresponding property of the set F defined by (3.5) is (3.8). Sjögren gives an ingenious proof of Dahlberg's result and we use his ideas in the proof of Theorem 3.

§4. A potential-theoretic lemma

When deducing covering results, we can work either with balls or with dyadic cubes with sides parallel to the axis: the results are equivalent (cf. Section 5 in Essén and Jackson [5b]).

In this section, it is convenient to work with cubes. Let us say that a cube in $\boldsymbol{R}^p$ is half-open if it is of the form $\{x \in \boldsymbol{R}^p;\ a_i \le x < a_i + b,\ i=1, 2, \ldots, p\}$. Let $\boldsymbol{G}_n$ be a net of half-open cubes in D similar to those constructed in Carleson ([3], pp. 6–7) and let $\boldsymbol{G} = \cup_n \boldsymbol{G}_n$. We recall that all cubes have their sides parallel to the coordinate axis and that the length of a side of each cube in $\boldsymbol{G}_n$ is 2^{-n}. Furthermore, the cubes in $\boldsymbol{G}_n$ are obtained by dividing each side of every cube in $\boldsymbol{G}_{n-1}$ into halves so that every cube in $\boldsymbol{G}_{n-1}$ will be divided into 2^p equal subcubes. In addition, we arrange each net $\boldsymbol{G}_n$ so that the first coordinate of any vertex of each member of $\boldsymbol{G}_n$ is either 0 or of the form $m2^{-n}$ ($m \in \boldsymbol{N}$, $n \in \boldsymbol{N}$).

Let $h\colon[0, \infty) \to [0, \infty)$ be a non-decreasing continuous function such that $h(0)=0$. For each cube $Q \in \boldsymbol{G}$ we define the premeasure

$$\sigma_h^a(Q) = \sigma_h(Q) = t^a h(r/t),$$

where $2r$ is the side-length of Q and t is the distance from the centre of Q to ∂D and a is p or $p-1$. When it is clear from the context what value of a we are using, a will be suppressed.

Next, for each bounded subset E of $D \cup \partial D$, we define

$$L_h^a(E) = L_h(E) = \inf\left(\sum t_k^a h(r_k/t_k)\right),$$

where the infimum is taken over all coverings of E with cubes from $\boldsymbol{G}$. (For details, we refer to a similar discussion in Essén and Jackson [5b], Section 5.) Using the same argument as in the proof of Lemma 5.1 in [5b], we deduce

LEMMA 1. *Let h and a be as above and let F be a compact subset of $D \cap \{x \in \boldsymbol{R}^p: |x| < 8\}$. Then, there exists a mass distribution $\nu = \nu_a$ supported by F and constants C_1 and C_2 only depending on the dimension p such that*

$$L_h^a(F) \leq C_1 \int_F x_1 d\nu(x), \tag{4.1}$$

$$\int_Q x_1 d\nu(x) \leq C_2 \sigma_h^a(Q) \qquad \textit{for all} \quad Q \in \boldsymbol{G}. \tag{4.2}$$

§ 5. Proof of Theorem 1

Let us assume that, for each $\varepsilon > 0$, we can find a covering of the set $E_\varepsilon = \{x \in D: G\mu(x) + P\mu_1(x) > \varepsilon x_1\}$ such that (3.2) holds. Then it is easy to see that we can find a set E which has a covering satisfying (3.2) and for which (1.1′) holds, i.e., Theorem 1 is proved.

Thus, it is sufficient to study the covering problem for the set E_ε. Normalizing, we see that we can restrict ourselves to the study of the set $E = E_1$ defined by (3.10).

In the first part of the proof, we shall work with a collection $\{Q_k\}$ of disjoint half-open cubes (which we shall call Whitney cubes) with sides parallel to the axis which are such that if $Q \in \{Q_k\}$, we have

$$d(Q) = \text{dist.}\,(Q, \partial D) \geq 2\,\text{diam.}\,Q/\sqrt{p}, \tag{5.1}$$

(cf. e.g. Stein [12], p. 16). Furthermore, we have the following property: If $\tilde{Q}$ is the double of Q (i.e., $\tilde{Q} = \{x \in \boldsymbol{R}^p: x - x_Q = 2(y - x_Q)$ for some $y \in Q\}$, where x_Q is the centre of Q), $\tilde{Q} \subset D$ and there is a constant A only depending on the dimension p such that $\tilde{Q}$ meets at most A cubes in the collection $\{\tilde{Q}_k\}$. We also assume that

$$\text{diam.}\,\tilde{Q} \approx d(\tilde{Q}) \qquad \text{for} \quad Q \in \{Q_k\}. \tag{5.2}$$

We define,

$$\begin{cases} I(x) = \int_{\tilde{Q}_k} G(x, y)d\mu(y), \ x \in Q_k, \\ J(x) = \int_{D\setminus\tilde{Q}_k} G(x, y)d\mu(y) + P\mu_1(x), \ x \in Q_k, \end{cases} \qquad k = 1, 2, 3, \ldots$$

The functions $I(x)$ and $J(x)$ are defined everywhere in D and we put $U(x)=I(x)+J(x)$. Thus, if

$$E_k = \{x \in Q_k \colon |x| \geq 1, \ I(x) > x_1/2\},$$

$$F_k = \{x \in Q_k \colon |x| \geq 1, \ J(x) > x_1/2\},$$

we have $E \cap \{x \colon |x| \geq 1\} \subset \cup (E_k \cup F_k)$.

By Harnack's inequality (cf. e.g. Hayman–Kennedy [7], p. 35) we have for any k

$$J(x)/c_0 \leq J(y) \leq c_0 J(x), \quad x, y \in Q_k,$$

where c_0 depends only on the dimension p. It follows that if $F_k \neq \emptyset$, we have $J(x) \geq x_1/(2c_0) = c' x_1$, $x \in Q_k$. We obtain, with constants depending only on the dimension p,

$$\sum_{F_k \neq \emptyset} t_k^p (1+R_k)^{-p} \leq c'' \int_{\{J(x) \geq c' x_1\}} (1+|x|)^{-p} dx$$

$$\leq c'' \int_{\{U(x) > c' x_1\}} (1+|x|)^{-p} dx < \infty.$$

Here t_k is the distance from the centre of Q_k to ∂D and R_k is the distance of this centre from the origin. In the last step, we used the Dahlberg–Sjögren result quoted in (3.10) and (3.11). Since we have (5.2), we have obtained a covering of the set $\cup F_k$ which satisfies (3.2).

We now turn to $\cup E_k$. Let $E'_k = E_k/R_k$, $Q'_k = Q_k/R_k$ and $t'_k = t_k/R_k$. From (5.1), it is clear that $E'_k \subset \{x \in D \colon |x| < c(p)\}$. Here $c(p)$ is a constant which depends only on the dimension p. For a while, we shall consider a fixed k and we put $\tilde{Q}'_k = W$ and $t'_k = t$. Changing variables in $I(x)$, we see that

$$I(R_k x)/R_k = I'(x) = \int_W G(x, y) R_k^{1-p} d\mu(R_k y) = \int_W G(x, y) d\mu'(y) > x_1/2, \qquad x \in E'_k.$$

We claim that

$$L_h(E'_k) \leq \text{Const.} \int_W x_1 d\mu'(x) = \text{Const.}\, R_k^{-p} \int_{\tilde{Q}_k} x_1 d\mu(x), \tag{5.3}$$

where $L_h = L_h^p$ is the "Hausdorff measure" defined in Section 4 and the constant depends only on p and the value of the integral in (3.1).

To prove (5.3), it is sufficient to consider the case when F is a compact subset of the open set E'_k such that $I'(x) > x_1/2$ on F and deduce that

$$L_h(F) \leq \text{Const.} \int_W x_1 d\mu'(x). \tag{5.4}$$

According to Lemma 1, there exists a measure ν supported by F such that (4.1) holds. We obtain

$$\begin{aligned} L_h(F) &\leq C_1 \int_F x_1 d\nu(x) \leq 2C_1 \int_F \int_W G(x, y) d\mu'(y) d\nu(x) \\ &= 2C_1 \int_W G\nu(y) d\mu'(y). \end{aligned} \tag{5.5}$$

To estimate $G\nu(y)$ on W, we need the following fact:

$$\int_{|x-y|<s} x_1 d\nu(x) \leq C_3 t^p h(4s/t), \quad 0 < s \leq \text{diam. } W \leq t\sqrt{p}, \quad y \in W. \tag{5.6}$$

To prove (5.6), we let $l(W)$ be the sidelength of the cube W. The distance from the centre of W to ∂D is t. We know that (cf. (5.1))

$$l(W) \leq 4t/5, \quad d(W, \partial D) \leq 3t/5.$$

There are two cases to discuss:
a) $0 < s < t/20$. We cover $\{x : |x-y| < s\}$ by at most 2^p dyadic cubes from $\boldsymbol{G}$, all with side $2d$, $s \leq d < 2s$. For each such cube X, the distance from the centre of X to ∂D is at least $t/2$. Thus we have (cf. (4.2))

$$\int_X x_1 d\nu(x) \leq C_2 t^p h(2d/t) \leq C_2 t^p h(4s/t).$$

In this case, the left hand member of (5.6) is at most $C_2 2^p t^p h(4s/t)$.
b) $t/20 < s < t\sqrt{p}$. We cover $\{x : |x-y| < s\}$ by dyadic cubes from $\boldsymbol{G}$, all with side $2d$, where $t/20 < d < t$. For each such cube X, we have (cf. (4.2))

$$\int_X x_1 d\nu(x) \leq C_2 t^p h(2).$$

The number of such dyadic cubes has an upper bound only depending on p. We have proved (5.6). The constant C_3 depends only on p and on the function h. We note that we have worked with dyadic cubes from $\boldsymbol{G}$ and not with the Whitney cubes from the first part of the proof.

We define $H(s) = \log(5t/s)$ if $p = 2$ and $H(s) = s^{2-p}$, $p > 2$, $s > 0$. We have

$$G(x, y) \leq H(|x-y|), \quad y \in W, \quad x \in Q'_k.$$

Using these estimates, we obtain in a standard way that

$$Gv(y) \le \int_F H(|x-y|)dv(x) \le (2C_3/t)\int_0^{t\sqrt{p}} H(s)d_s(t^p h(4s/t))$$
$$\le C_4 y_1\left\{h(r\sqrt{p}) + \int_0^{4\sqrt{p}} s^{1-p}h(s)ds\right\} = C_5 y_1, \quad y \in W$$

where the constants C_i, $1 \le i \le 5$, depend only on the dimension p and the function h. Inserting this estimate into (5.5), we obtain (5.4) and thus also (5.3).

Since $u(x)$ is not identically infinite in D, we must have

$$B = \int_D x_1(1+|x|)^{-p} d\mu(x) < \infty.$$

Recalling the constant A giving an upper bound for the multiple coverings of the doubled cubes $\{\tilde{Q}_k\}$, we deduce from (5.3) that

$$\sum L_h(E'_k) \le \text{Const.} \sum \int_{\tilde{Q}_k} x_1(1+|x|)^{-p} d\mu(x) \le \text{Const.}\, AB < \infty,$$

where we sum (5.3) over all k such that $R_k > 1$. This gives the desired covering of the set $\cup E_k$ by (not necessarily dyadic) cubes (for a similar discussion, see p. 260 in [5b]). Dividing each of these cubes into 2^{Np} identical cubes and taking balls containing these smaller cubes we obtain the desired covering by sets from $\boldsymbol{H}$ if N is the smallest integer such that $2^N \ge 2\sqrt{p}$. Theorem 1 is now proved.

§ 6. Proof of Theorems 2 and 3

Arguing as in the beginning of the proof of Theorem 1, we see that it is sufficient to study the set F defined by (3.5). Since $G\mu + P\mu_1$ is not identically infinite, we have

$$\int_{\partial D}(1+|z|)^{-p} d\mu_1(z) + \int_D x_1(1+|x|)^{-p} d\mu(x) < \infty. \tag{6.1}$$

We have $F \subset \{x \in D\colon G\mu(x) > |x|/2\} \cup \{x \in D\colon P\mu_1(x) > |x|/2\}$. We first study $P\mu_1$. Let μ'_1 be the restriction of μ_1 to $\{z \in \partial D\colon |z| > L_1\}$ where L_1 is a large number to be chosen below. If $x \in S_n$ and $z \in \partial D \cap S_k$, where $|k-n| > 1$, we have

$$|x-z| \ge |x|/4, \quad k < n, \quad |x-z| \ge |z|/4, \quad k > n.$$

Let μ_k be the restriction of μ'_1 to S_k, $k \ge 2$. We now choose L_1 so large that

$$\sum\nolimits_{|k-n|>1} P\mu_k(x) \le 4^p x_1 \int_{\{|z|\ge 2\}} |z|^{-p} d\mu'_1(z) < |x|/10, \quad x \in S_n. \tag{6.2}$$

Let $\mu''_1 = \mu_1 - \mu'_1$. It is easy to see that there exists a number $L_2 > 16$ such that

$$P\mu''_1(x) = \int_{\partial D} x_1 |x-z|^{-p} d\mu''_1(z) \le |x|/10, \quad |x| \ge L_2. \tag{6.3}$$

From (6.2) and (6.3) we see that the set $\{x \in D\colon P\mu_1(x) > |x|/2 > L_2/2\}$ is contained in

$$\cup_n \{x \in S_n\colon (P\mu_{n-1} + P\mu_n + P\mu_{n+1})(x) > 3|x|/10,\ |x| \geq L_2 > 16\}.$$

Let us put $d\Lambda_n(z) = 2^{-np}(d\mu_{n-1} + d\mu_n + d\mu_{n+1})(2^n z)$. If $x = 2^n x'$ and $z = 2^n z'$, we see that the subset of S_n which we study, divided by 2^n, is contained in

$$\{x' \in D\colon P\Lambda_n(x') > 3/10,\ 1 \leq |x'| < 2\},$$

where $\operatorname{supp} \Lambda_n \subset \{z \in \partial D\colon 1/2 \leq |z| \leq 4\}$.

From (6.1), we see that $\sum \|\Lambda_n\| < \infty$, where $\|\cdot\|$ denotes the total mass of the measure. Let us for a while drop the n: s and the $'$: s. Thus Λ is a non-negative measure on $\{z \in \partial D\colon 1/2 \leq |z| \leq 4\}$. Consider the maximal function

$$N\Lambda(z) = \sup P\Lambda(x), \quad x \in \Gamma(z).$$

Let $G = \{z \in \partial D\colon N\Lambda(z) > 3/10\}$. G is an open set. Let $\Omega = \cup_{z \in \partial D \setminus G} \Gamma(z)$. To each $z \in \Omega$, there exists $z \in \partial D \setminus G$ such that $x \in \Gamma(z)$ and we have

$$P\Lambda(x) \leq N\Lambda(z) \leq 3/10.$$

We conclude that the set $\{x \in D\colon P\Lambda(x) > 3/10\}$ is contained in $D \setminus \Omega$.

It is well-known that there exists an absolute constant C such that

$$|G| \leq C\|\Lambda\|$$

where $|\cdot|$ denotes $(p-1)$-dimensional measure (cf. Theorem 1, p. 197 and Theorem 1, p. 5 in Stein [12]). We can cover G by $(p-1)$-dimensional Whitney balls with raddi $\{r_i\}$ in such a way that

$$\sum r_i^{p-1} \leq C(p)\|\Lambda\|.$$

This terminology means that we have $r_i \approx d(B_i, \partial G)$ for all i (cf. [12], p. 16). (By $C(p)$, we mean a constant which depends on p only. $C(p)$ may have different values in different formulas.) If $(x_1, z) \in D \setminus \Omega$ and $z \in G$ belongs to a Whitney ball of radius r, we have $d(z, \partial G) \leq C(p)r$ and thus $x_1 \leq \sqrt{3}C(p)r$. Hence the p-dimensional balls with centres at the centres of the Whitney balls and with radii $\{\sqrt{3}C(p)r_i\}$ will cover $D \setminus \Omega$. Returning to S_n and summing over n, we find an open set $O_1 = \cup (2^n G_n)$ which has a covering $\{r_i, R_i\}$ by $(p-1)$-dimensional balls such that

$$\sum (r_i/R_i)^{p-1} \leq C(p) \sum \|\Lambda_n\| < \infty. \tag{6.4}$$

Elementary calculations indicate that with the exception of finitely many indices the centres of the Whitney balls used in the covering of G will have distance at least 1/4 to the origin. In particular, (3.6) holds with O replaced by O_1. Furthermore, we see that the set $\{x\in D\colon P\mu_1(x)>|x|/2>L_2/2\}$ is contained in $\cup(D\setminus\Omega_n)$, where $D\setminus(\Omega_n/2^n)$ is the exceptional set for $P\Lambda_n$. (We note that the associated p-dimensional balls will cover $\cup(D\setminus\Omega_n)$ and that (6.4) holds for this covering.)

This concludes the first part of the proof.

To study the set $\{x\in D\colon G\mu(x)>|x|/2\}$, we use the Whitney cubes of Section 5 and the decomposition

$$\begin{cases} I(x)=\int_{\tilde{Q}_k} G(x,y)d\mu(y), & x\in Q_k,\\ J(x)=\int_{D\setminus\tilde{Q}_k} G(x,y)d\mu(y), & x\in Q_k,\end{cases} \qquad k=1,2,\ldots.$$

We have $G\mu(x)=I(x)+J(x)$. As usual, we see that the exceptional set for $G\mu$ is contained in the union of the two sets

$$\{x\in D\colon I(x)>|x|/4\} \quad\text{and}\quad \{x\in D\colon J(x)>|x|/4\}.$$

Let $H_k=\{x\in Q_k\colon I(x)>|x|/4\}$. If $|H_k|$ is the Lebesgue measure of H_k, we have

$$|H_k|\,|x|\le \text{Const.}\int_{\tilde{Q}_k} I(x)dx<\text{Const.}\,x_1^2\mu(\tilde{Q}_k),\qquad \text{for}\quad x\in Q_k.$$

In the last step, we used (5.6). If $F_2=(\cup H_k)\cap\{|x|\ge 1\}$, it follows from (6.1) that

$$\int_{F_2} x_1^{-1}(1+|x|)^{1-p}dx<\text{Const.}\int_D x_1(1+|x|)^{-p}d\mu(x)<\infty,$$

and we have proved (3.8).

To obtain the covering (3.9) fo F_2, we use the same argument as in the proof of (3.2) in Theorem 1. The only difference is that, when we apply Lemma 1, we use the premeasure $\sigma_h^{p-1}(Q)=t^{p-1}h(r/t)$ and the "Hausdorff measure" L_h^{p-1}. We omit the details.

It remains to study $\{x\in D\colon J(x)>|x|/4\}$. We shall use a trick of Sjögren which will reduce the problem to a study of a Poisson integral which we can handle by the first part of the proof. This idea of Sjögren can be found in his proof of Dahlberg's result (3.11) (cf. p. 280 in [10]).

Let $\varphi\colon D\to\partial D$ be a measurable mapping such that

$$|x-\varphi(x)|<2x_1,\quad x\in D, \tag{6.5}$$

$$|x|\approx|\varphi(x)|,\quad x\in D. \tag{6.6}$$

If $x_1<|x|/2$, we can let φ be orthogonal projection onto ∂D. If $x_1\ge|x|/2$, we can take $\varphi(x)=(0,x_2,\ldots,x_{p-1},x_p+x_1\operatorname{sgn}x_p)$. There are, of course, many other possibilities to choose φ.

Let $\mathscr{F}$ be the class of continuous functions on ∂D which is such that

$$\|f\| = \sup_{\partial D}(1+|z|^p)|f(z)| < \infty.$$

Consider the linear functional

$$L(f) = \int_D f(\varphi(y))y_1 d\mu(y), \quad f \in \mathscr{F}.$$

According to (6.6) and (6.1), we have

$$|L(f)| \leq \text{Const.}\ \|f\| \int_D y_1(1+|y|)^{-p} d\mu(y) < \infty.$$

We conclude in a standard way that there exists a measure v on ∂D such that

$$L(f) = \int_{\partial D} f(z) dv(z), \quad f \in \mathscr{F}, \tag{6.7}$$

$$\int_{\partial D} (1+|z|)^{-p} dv(z) < \infty.$$

We are going to compare $J(x)$ to the Poisson integral $Pv(x)$. If $c>0$ is given and $|x-y|>cx_1$, we have, according to (6.5), that

$$|x-\varphi(y)| \leq |x-y| + 2y_1 \leq 3|x-y| + 2x_1 \leq (3+2c^{-1})|x-y|.$$

According to the definition of the Whitney cubes, we can find $c>0$ such that

$$J(x) \leq \int_{|x-y|\geq cx_1} G(x, y) d\mu(y) \leq \text{Const.} \int_{|x-y|\geq cx_1} x_1 y_1 |x-y|^{-p} d\mu(y) \leq$$

$$\leq \text{Const.}\ x_1 \int_D y_1 |x-\varphi(y)|^{-p} d\mu(y) = \text{Const.}\ x_1 \int_{\partial D} |x-z|^{-p} dv(z).$$

In the last step, we applied the representation formula (6.7) to the function $f(z) = x_1|x-z|^{-p}$, $z \in D$, where $x \in D$ is fixed. Thus, we have proved that

$$J(x) \leq CPv(x), \quad x \in D, \tag{6.8}$$

where the constant C depends only on the dimension p. We have

$$\{x \in D\colon J(x) > |x|/4\} \subset \{x \in D\colon Pv(x) > |x|/4C\}.$$

Arguing as in the discussion of the set $\{x \in D\colon P\mu_1(x) > L_2/2\}$, we find an open set $O_2 \subset \partial D$ such that (3.6) holds with O replaced by O_2 and such that the set $\{x \in D\colon Pv(x) > |x|/4C\}$ is contained in $D \setminus (\bigcup_{z \in \partial D \setminus O_2} \Gamma(z))$ which has a covering $\{r_n, R_n\}$ such that (3.7) holds. Defining $O = O_1 \cup O_2$ and taking the union of the exceptional sets, we obtain Theorem 3.

§7. Examples

In this section, we give examples of potentials and associated minimally thin or rarefied sets (cf. (3.10) or (3.5)).

In the plane, we take a sequence of closed disks $B(z_n, r_n)=\{z: |z-z_n|\le r_n\}$, $n=1, 2,\dots$, which are such that $2r_n\le x_n$ and

$$\sum (x_n/|z_n|)^2(\log (x_n/r_n))^{-1} < \infty.$$

Here $z_n=x_n+iy_n$, $z_n\to\infty$. If $\mu_n=x_n(\log (x_n/r_n))^{-1}$, then

$$u(z) = \sum \mu_n \log |(z+\bar{z}_n)/(z-z_n)|$$

is a convergent Green potential in D and

$$u(z) \ge \mu_n \log |(z+\bar{z}_n)/r_n| \ge x_n, \quad z\in B(z_n, r_n).$$

Hence there exists a constant $c\in(0, 1)$ such that

$$u(z) \ge cx, \quad z\in \cup B(z_n, r_n),$$

and so $\cup B(z_n, r_n)$ is minimally thin at infinity in D.

In the rarefied case, we take a sequence of disks such that

$$\sum (x_n/|z_n|)(\log (x_n/r_n))^{-1} < \infty,$$

and use the same argument with $\mu_n=|z_n|(\log (x_n/r_n))^{-1}$.

When $p\ge 3$, we need the following estimate (cf. Essén and Jackson [5b], formula (2.3)):

$$G(x, y) \approx x_1 y_1 |x-y|^{2-p}|\tilde{x}-y|^{-2}, \quad x, y\in D. \tag{7.1}$$

In the minimally thin case, we take a sequence of closed balls

$$B(P_n, r_n) = \{x\in \boldsymbol{R}^p: |x-P_n|\le r_n\}, \quad n = 1, 2\dots$$

such that $2r_n\le t_n$, $R_n\to\infty$, and

$$\sum t_n^2 r_n^{p-2} R_n^{-p} < \infty. \tag{7.2}$$

If $\mu_n=t_n r_n^{p-2}$, then

$$u(x) = \sum \mu_n G(x, P_n)$$

is a convergent Green potential in D and

$$u(x) \ge \mu_n G(x, P_n) \ge cx_1, \quad x\in B(P_n, r_n),$$

and so $\cup B(P_n, r_n)$ is minimally thin at infinity in D.

In the rarefied case, (7.2) is replaced by

$$\sum t_n r_n^{p-2} R_n^{1-p} < \infty, \tag{7.3}$$

and we choose $\mu_n = R_n r_n^{p-2}$.

When $p \geq 3$, the results that a sequence of balls satisfying (7.2) or (7.3) is minimally thin or rarefied at infinity in D can be found in Theorems 4.3 or 1.1 in Essén and Jackson [5b]. There are also weak converse statements which illustrate the precision of Corollaries 1 and 2.

§8. A counterexample

It remains to show that condition (3.1) is best possible for Theorems 1 and 2. This is almost a consequence of a general result in Taylor (Theorem 3 in [13]). Before stating Taylor's result, we need some notation. If h: $[0, \infty) \to [0, \infty)$ is continuous and non-decreasing and such that $h(0) = 0$, we let

$$M_h(E) = \inf \sum h(r_j),$$

the infimum being taken over families of open balls which cover E.

If we only allow balls with radii at most r in this covering and then let $r \to 0+$, the limit of the infima will be the classical Hausdorff measure $\Lambda_h(E)$ (cf. [3, p. 6]). $\mathrm{Cap}(\cdot)$ will denote logarithmic capacity if $p = 2$ and newtonian capacity if $p > 2$. Taylor's result, specialized to our situation, can be stated as

THEOREM A. *Let h be as above. Suppose that*

$$\int_0^1 h(r) r^{1-p} dr = \infty. \tag{8.1}$$

If E is a bounded Borel set with $0 < \Lambda_h(E) < \infty$ and if at every point of E the lower spherical density $D(x, E)$ with respect to h is positive, then $\mathrm{Cap}(E) = 0$.

REMARK. $D(x, E) = \liminf_{r \to 0+} \Lambda_h (E \cap \{y\colon |y-x| < r\})/h(2r)$.

The point of Lemma 2 below is to construct a set of Cantor type which satisfies the requirements in Theorem A. Once we have this set, it is easy to find examples which show that condition (3.1) is sharp. For simplicity, we shall restrict ourselves to exceptional sets contained in a Stolz domain $K \subset D$.

THEOREM 4. *Let $h\colon [0, \infty) \to [0, \infty)$ be continuous and strictly increasing with $h(0) = 0$, and suppose that* (8.1) *holds and that*

$$h(r) r^{1-p} \textit{ is strictly decreasing in } (0, 1]. \tag{8.2}$$

Then, there is a polar set $E \subset K \subset D$ which has no covering in $\boldsymbol{H}$ satisfying (3.2) *or* (3.3).

Remark. A special example of this type is given in Remark 10 in [5a].

Since any polar set in D is both rarefied and minimally thin at infinity in D, this is the required example.

Lemma 2. *Let h be as in Theorem* 4. *Then there is a Cantor set F in $\mathbf{R}^{p-1}$ such that* $\operatorname{Cap}(F)=0$ *but* $M_h(F)>0$.

This result is certainly not new. For $p=2$, such a set was constructed by R. Nevanlinna [9, Theorem 6, p. 157] and the procedure is similar if $p>2$ (cf. [3, p. 35]). However, the details are not quite obvious and we give the proof of Lemma 2 in full.

First, note that (8.1) is equivalent to $\int_0^1 \varphi_0(r)dh(r)=\infty$, where φ_0 was defined in Section 2. Now, put $g(t)=h^{-1}(t^{p-1})$, $0\leq t^{p-1}\leq h(1)$, so that

$$\int_0^1 \varphi_0(r)dh(r) = (p-1)\int_0^{g^{-1}(1)} \varphi_0(g(t))t^{p-2}dt.$$

If $\varepsilon_n=g(2^{-n})$, $n\geq n_0$, where n_0 is the smallest integer such that $2^{-n_0(p-1)}\leq h(1)$, we deduce that

$$\textstyle\sum_{n=n_0}^{\infty} \varphi_0(\varepsilon_n)2^{-n(p-1)} = \infty. \tag{8.3}$$

Also, for $n\geq n_0$, $h(\varepsilon_n)=2^{p-1}h(\varepsilon_{n+1})$ and so, by (8.2),

$$\varepsilon_{n+1} < \varepsilon_n/2, \quad n \geq n_0. \tag{8.4}$$

We shall construct the Cantor set using the numbers $l_n=\varepsilon_{n_0+n}$, $n=0, 1, 2,\ldots$.

Let F_0 be a closed cube in $\mathbf{R}^{p-1}$ of sidelength l_0 and let F_1 be the subset of F_0 which consists of 2^{p-1} closed cubes of sidelength l_1 each having a vertex in common with F_0. Note that F_1 is a proper subset of F_0 by (8.4). We can continue in this way to construct a sequence of compact sets $F_0\supset F_1\supset F_2\supset\cdots$, where F_n consists of $2^{n(p-1)}$ closed cubes of sidelength l_n, each of which has a vertex in common with some cube in F_{n-1}. If $F=\bigcap_{n=0}^{\infty} F_n$, then it follows from (8.3) and [3, p. 31] that $\operatorname{Cap}(F)=0$.

The proof that $M_h(F)>0$ depends on the following easily verified geometric fact about the sets $\{F_n\}$: any ball of radius at most l_n can meet at most 5^p of the cubes in F_n.

Let $\{B_\nu\}$ be open balls which together cover F. Choose a finite subcover $B_1,\ldots, B_k$ with radii $r_1,\ldots, r_k$ and, whenever $r_j\leq l_0$, put $n_j=\max\{n\colon r\leq l_n\}$. If $r_j>l_0$, we put $n_j=0$. For $j=1,\ldots, k$, the ball B_j can then meet at most 5^p of the cubes in F_{n_j}. Also

$$h(r_j) > h(l_{n_j+1}) = 2^{-(n_0+n_j+1)(p-1)}, \quad j = 1, 2,\ldots, k. \tag{8.5}$$

If $N=\max\{n_j\colon j=1,\ldots, k\}$ and N_j denotes the number of cubes in F_N which meet B_j, then

$$N_j \leq 5^p 2^{(N-n_j)(p-1)}, \tag{8.6}$$

since each cube in E_{n_j} contains exactly $2^{(N-n_j)(p-1)}$ cubes in E_N. Also

$$\sum_{j=1}^k N_j \geq 2^{N(p-1)}, \tag{8.7}$$

since $\bigcup_{j=1}^k B_j$ covers F.

Combining (8.5), (8.6) and (8.7), we obtain

$$\sum_{j=1}^k h(r_j) > \sum_{j=1}^k 2^{-(n_0+n_j+1)(p-1)} = c(p, h)2^{-N(p-1)} \sum_{j=1}^k 2^{(N-n_j)(p-1)}$$
$$\geq c(p, h)2^{-N(p-1)} \sum_{j=1}^k N_j \geq c(p, h).$$

It follows that $M_h(F) \geq c(p, h) > 0$ and the proof of Lemma 2 is complete.

REMARK. We do not use Theorem A in the proof of Lemma 2. The reason is that Theorem A deals with Λ_h while Lemma 2 deals with M_h. It is shorter to use a result from [3] than to modify Theorem A to a statement involving M_h.

To construct the set E described in Theorem 4, we position a copy of F in the hyperplane $\{x_1=1\}$ with its centre at $(1, 0, \ldots, 0)$. Call this set E_0 and define $E_m = \{2^m x \colon x \in E_0\}$, $m=1, 2, \ldots$ and $E = \bigcup_{m=0}^\infty E_m$. Let $\{B_n\} = \{t_n, r_n, R_n\}$ be any covering of E which consists of sets from $\boldsymbol{H}$, and suppose that $B_n \cap E \neq \emptyset$, $n = 1, 2, \ldots$. It is clear from the definition of $\boldsymbol{H}$ that

$$t_n/R_n \geq c(p) > 0, \quad n = 1, 2, \ldots,$$

and so, if (3.2) or (3.3) holds, we deduce that

$$\sum_{n=1}^\infty h(r_n/t_n) < \infty.$$

In particular, $r_n/t_n \to 0$ as $n \to \infty$, which implies that all but a finite number of sets from the cover meet exactly one of the sets E_m. It is easy to check that

$$\sum_{B_n \cap E_m \neq \emptyset} h(r_n/t_n) \geq c(p)M_h(F), \quad m = 1, 2, \ldots,$$

and so we obtain a contradiction. Since E is a polar set, this proves Theorem 4.

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