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Noetherian property of symbolic Rees algebras

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In the course of giving a counter-example to a problem of Zariski, D. Rees [6] proved the following theorem: Let $\mathfrak{p}$ be a prime ideal of a two-dimensional noetherian normal local domain with $ht(\mathfrak{p})=1$. If the graded ring $\oplus_{n\geqq 0}\mathfrak{p}^{(n)}$ is noetherian, then $\mathfrak{p}^{(d)}$ is a principal ideal for some $d\geqq 1$.

The aim of this note is to give a generalization of this theorem, which is stated as follows:

THEOREM. *Let $\mathfrak{p}$ be a prime ideal of a noetherian normal Nagata local domain R. Assume that $\dim(R/\mathfrak{p})=1$ and $R_{\mathfrak{p}}$ is regular. Then the graded ring $\oplus_{n\geqq 0}\mathfrak{p}^{(n)}$ is noetherian if and only if $\ell(\mathfrak{p}^{(d)})=\dim(R)-1$ for some $d\geqq 1$. Here we denote by $\ell(I)$ the analytic spread of an ideal I. (Concerning Nagata domains, see [3].)*

Throughout this paper, let R be a commutative ring and let I be an ideal of R. We denote by $S=R-Z_R(R/I)$ the set of R/I-regular elements of R, and for an R-module M, we put $M_I=M_S$. If R is a noetherian domain, then we have $R_I=\bigcap_{\mathfrak{p}\in \mathrm{Ass}_R(R/I)}R_{\mathfrak{p}}$. For an integer $n\geqq 0$, we define the *n-th symbolic power* $I^{(n)}$ of I by $I^{(n)}=I^nR_I\cap R=\{x\in R;\ tx\in R$ for some R/I-regular element $t\in R\}$.

PROPOSITION 1. (1) *$Z_R(R/I^{(n)})\subset Z_R(R/I)$ for all $n\geqq 1$.*

(2) *$I^{(1)}=I$, $\mathrm{rad}(I^{(n)})=\mathrm{rad}(I)$, $I^{(m)}I^{(n)}\subset I^{(m+n)}$ and $I^{(mn)}\subset I^{(m)(n)}$ for all $m, n\geqq 1$.*

(3) *Assume that R is noetherian and $\mathrm{Ass}_R(R/I)=\mathrm{Min}_R(R/I)$. Then $\mathrm{Ass}_R(R/I^{(n)})=\mathrm{Min}_R(R/I)$ for all $n\geqq 1$. In particular, $Z_R(R/I^{(n)})=Z_R(R/I)$ for all $n\geqq 1$. Also, we have $I^{(mn)}=I^{(m)(n)}$ for all $m, n\geqq 1$. Here we denote by $\mathrm{Min}_R(R/I)$ the set of minimal prime ideals of I.*

PROOF. (1) Assume that $t\in R$ is R/I-regular and $tx\in I^{(n)}$ for some $x\in R$. Then we have $s(tx)\in I^n$ for some R/I-regular element $s\in R$. Hence st is R/I-regular and $(st)x\in I^n$. This implies that $x\in I^{(n)}$.

(2) We prove the inclusion $I^{(mn)}\subset I^{(m)(n)}$. Take an element x of $I^{(mn)}$. Then for some R/I-regular element $t\in R$, we have $tx\in I^{mn}\subset I^{(m)n}$. Since t is $R/I^{(m)}$-regular by (1), we have $x\in I^{(m)(n)}$.

(3) If $\mathfrak{p}\in \mathrm{Ass}_R(R/I^{(n)})$, then $\mathfrak{p}\subset Z_R(R/I^{(n)})\subset Z_R(R/I)$. Hence $I\subset\mathfrak{p}\subset\mathfrak{q}$ for some $\mathfrak{q}\in \mathrm{Ass}_R(R/I)=\mathrm{Min}_R(R/I)$. Therefore we have $\mathfrak{p}=\mathfrak{q}\in \mathrm{Min}_R(R/I)$.

Let x be an element of $I^{(m)(n)}$. Then for every $\mathfrak{p} \in \mathrm{Ass}_R(R/I^{(m)}) = \mathrm{Ass}_R(R/I)$, we have $x/1 \in I^{(m)n}R_{\mathfrak{p}} = (I^{(m)}R_{\mathfrak{p}})^n = (I^m R_{\mathfrak{p}})^n = I^{mn}R_{\mathfrak{p}}$. Therefore $x \in I^{(mn)}$.

Q. E. D.

We define the *symbolic Rees algebra* of I by $R^s(I) = \oplus_{n \geqq 0} I^{(n)}$. This ring can be identified with the graded subring $\oplus_{n \geqq 0} I^{(n)}X^n$ of $R[X]$, and we have $R^s(I) = R(I)_I \cap R[X]$ and $R^s(I)_I = R(IR_I)$, where $R(I) = \oplus_{n \geqq 0} I^n$.

PROPOSITION 2. *Assume that R is a noetherian normal domain.*

(1) *$R^s(I)$ is normal if and only if $R(IR_{\mathfrak{p}})$ is normal for all $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$.*

(2) *If $G(IR_{\mathfrak{p}})$ is reduced for all $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$, then $R^s(I)$ is normal. In particular, if I is a radical ideal which is generically a complete intersection, then $R^s(I)$ is normal. Here we denote by $G(I)$ the associated graded ring $\oplus_{n \geqq 0} I^n/I^{n+1}$ of I.*

(3) *Let $\mathfrak{p}$ be a prime ideal of R such that $R_{\mathfrak{p}}$ is regular. Then $R^s(\mathfrak{p})$ is normal.*

PROOF. (1) We have $R^s(I) = R(I)_I \cap R[X]$ and $R^s(I)_I = R(I)_I$. Hence $R^s(I)$ is normal $\Leftrightarrow R(I)_I$ is normal $\Leftrightarrow R(I)_{\mathfrak{p}}$ is normal for all $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$.

(2) follows from (1) and the following fact (cf. Barshay [1]): If $G(I)$ is reduced, then $R(I)$ is integrally closed in $R[X]$. Q. E. D.

PROPOSITION 3. *The following conditions are equivalent:*

(1) $R^s(I) = R(I)$, i.e., $I^{(n)} = I^n$ *for all* $n \geqq 0$.

(2) *$G(I)$ is a torsion-free R/I-module.*

Moreover if R is a locally quasi-unmixed noetherian ring, $\mathrm{Ass}_R(R/I) = \mathrm{Min}_R(R/I)$ and $R(I)$ is integrally closed in $R[X]$, then the above conditions are also equivalent to each of the following conditions:

(3) $\bar{A}^*(I) = \mathrm{Min}_R(R/I)$, *where* $\bar{A}^*(I) = \cup_{n \geqq 0} \mathrm{Ass}_R(R/\overline{I^n})$.

(4) *$\ell(IR_{\mathfrak{p}}) < ht(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of R such that $\mathfrak{p} \supset I$ and $\mathfrak{p} \notin \mathrm{Min}_R(R/I)$.*

(5) (*Assume that R is local and* $\dim(R/I) = 1$) $\ell(I) = ht(I)$.

PROOF. (1)$\Leftrightarrow R/I^n \to R/I^n \otimes_R R_I$ is injective for all $n \geqq 0 \Leftrightarrow I^n/I^{n+1} \to I^n/I^{n+1} \otimes_R R_I$ is injective for all $n \geqq 0 \Leftrightarrow G(I) \to G(I) \otimes_R R_I$ is injective$\Leftrightarrow$(2). (1)$\Leftrightarrow Z_R(R/I^n) \subset Z_R(R/I)$ for all $n \geqq 1 \Leftrightarrow \mathrm{Ass}_R(R/I^n) = \mathrm{Min}_R(R/I)$ for all $n \geqq 1 \Leftrightarrow$(3) (note that we have $\overline{I^n} = I^n$ by the assumption). For the equivalence of (3) and (4), see [4], [5]. (4)$\Rightarrow$(5) is clear. (5)$\Rightarrow$(4): Assume that $\mathfrak{p} \supset I$, $\mathfrak{p} \notin \mathrm{Min}_R(R/I)$, and take $\mathfrak{q} \in \mathrm{Min}_R(R/I)$ such that $\mathfrak{p} \supsetneqq \mathfrak{q} \supset I$. Then we have $\ell(IR_{\mathfrak{p}}) = \ell(I) = ht(\mathfrak{q}) < ht(\mathfrak{p})$. Q. E. D.

THEOREM 4. *Assume that R is a locally quasi-unmixed noetherian normal domain, $\mathrm{Ass}_R(R/I) = \mathrm{Min}_R(R/I)$, and $R^s(I)$ is normal. If $R^s(I)$ is noetherian,* *then for some* $d \geqq 1$, *we have* $\ell(I^{(d)}R_{\mathfrak{p}}) < ht(\mathfrak{p})$ *for all prime ideals* $\mathfrak{p}$ *of* R *such that* $\mathfrak{p} \supset I$ *and* $\mathfrak{p} \notin \mathrm{Min}_R(R/I)$. *Moreover the converse also holds if* R *is a Nagata domain. Note that if* R *is local and* $\dim(R/I) = 1$, *the above condition is equivalent to the condition* $\ell(I^{(d)}) = \dim(R) - 1$.

PROOF. If $R^s(I)$ is noetherian, then $R^s(I)^{(d)} = R(I^{(d)})$ for some d, where $R^s(I)^{(d)}$ denotes the d-th Veronesean subring of $R^s(I)$. The converse also holds if R is a Nagata domain (see Lemma 5 below). Now $R^s(I)^{(d)} = R(I^{(d)}) \Leftrightarrow I^{(dn)} = I^{(d)n}$ for all $n \geqq 0 \Leftrightarrow I^{(d)(n)} = I^{(d)n}$ for all $n \geqq 0$ (cf. Prop. 1, (3)) $\Leftrightarrow \ell(I^{(d)}R_{\mathfrak{p}}) < ht(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of R such that $\mathfrak{p} \supset I$ and $\mathfrak{p} \notin \mathrm{Min}_R(R/I)$ (cf. Prop. 3). Q.E.D.

LEMMA 5. *Let* $A = \oplus_{n \geqq 0} A_n$ *be a graded ring with* $A_0 = R$. *Assume that* R *is a Nagata domain,* A *is reduced and* $A^{(d)}$ *is noetherian for some* $d \geqq 1$. *Then* A *is also noetherian.*

PROOF. We may assume that A is an integral domain. In fact, since $A^{(d)}$ is noetherian, $\mathrm{Min}(A^{(d)})$ is a finite set, and it is easy to show that $\mathrm{Min}(A)$ is also a finite set. Put $\mathrm{Min}(A) = \{\mathfrak{P}_1, \dots, \mathfrak{P}_r\}$. Since $(A/\mathfrak{P}_i)^{(d)} \cong A^{(d)}/\mathfrak{P}_i^{(d)}$ is noetherian, $A/\mathfrak{P}_i$ is noetherian by the assumption. Therefore $A \subset \prod_{i=1}^r A/\mathfrak{P}_i$ is a finite extension and $\prod_{i=1}^r A/\mathfrak{P}_i$ is noetherian. This implies that A is noetherian.

Now let A be an integral domain and let $^*Q(A)$ be the "*graded quotient field*" of A, i.e., $^*Q(A) = \{a/b \in Q(A);\ a, b$ are homogeneous elements of $A\}$. Then it is well-known that $^*Q(A) = {}^*Q(A)_0[x, x^{-1}]$ for some $x = a/b$, and $^*Q(A)_0 = {}^*Q(A^{(d)})_0$. Put $B = A^{(d)}[a, b]$. Then we have $B \subset A \subset Q(B)$ and A is integral over B. Since R is a Nagata domain, A is finite over B. Therefore A is noetherian. Q.E.D.

Let $Q = Q(R)$ be the total quotient ring of R, and for an R-submodule J of Q, put $J^{-1} = (R : J)_Q$. For the ideal I, put $\tilde{I} = (I^{-1})^{-1}$. If R is a noetherian normal domain and I is a non-zero ideal of R, then $\tilde{I} = I$ (or equivalently, I is a reflexive R-module) if and only if $\mathrm{Ass}_R(R/I) \subset \mathrm{Ht}_1(R) = \{\mathfrak{p} \in \mathrm{Spec}(R);\ ht(\mathfrak{p}) = 1\}$. We call the graded ring $\tilde{R}(I) = \oplus_{n \geqq 0} \widetilde{I^n}$ the *divisorial Rees algebra* of I. If R is a noetherian normal domain, then the ring $\tilde{R}(I)$ is also a normal domain and it is easy to see $\tilde{R}(I) = R^s(\tilde{I})$.

COROLLARY 6. *Assume that* R *is a locally quasi-unmixed noetherian normal domain. If* $\tilde{R}(I)$ *is noetherian, then for some* $d \geqq 1$, *we have* $\ell(\widetilde{I^d}R_{\mathfrak{p}}) < ht(\mathfrak{p})$ *for all prime ideals* $\mathfrak{p}$ *of* R *such that* $\mathfrak{p} \supset I$ *and* $ht(\mathfrak{p}) \geqq 2$. *The converse also holds if* R *is a Nagata domain.*

COROLLARY 7. *Assume that* R *is a two-dimensional noetherian normal domain.*

(1) *If* $\tilde{R}(I)$ *is noetherian, then* $\widetilde{I^d}$ *is invertible for some* $d \geqq 1$. *The converse*

also holds if R is a Nagata domain.

(2) *Assume moreover that R is a Nagata local domain. Then the following conditions are equivalent:*

(a) *$\tilde{R}(I)$ is noetherian for every ideal I of R.*

(b) *$\oplus_{n\geq 0}\mathfrak{p}^{(n)}$ is noetherian for every $\mathfrak{p}\in \mathrm{Ht}_1(R)$.*

(c) *The divisor class group $Cl(R)$ of R is a torsion group.*

For the assertion (1) of Cor. 7, we need the following

LEMMA 8 (cf. Cowsik and Nori [2]). *Let R be a noetherian local ring which satisfies the Serre's condition (S_{n+1}). If $\ell(I)=ht(I)=n$ and I is generically a complete intersection, then I is generated by an R-regular sequence. In particular, if R is a noetherian normal local domain and $\ell(I)=1$, then I is a non-zero principal ideal.*

References

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