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The Fock representations of the Virasoro algebra and the Hirota equations of the modified KP hierarchies

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0. Introduction

The Virasoro algebra $\mathfrak{l}$ is the Lie algebra over the complex number field $\boldsymbol{C}$ of the form $\mathfrak{l}=\sum_{k\in\boldsymbol{Z}} \boldsymbol{C}l_k \oplus \boldsymbol{C}c$ with the bracket relations

$$[l_k, l_m] = (k-m)l_{k+m} + \frac{1}{12}(k^3-k)\delta_{k+m,0}c, \quad [c, \mathfrak{l}] = \{0\}.$$

This algebra is the one-dimensional central extension of the so-called Witt algebra. The Virasoro algebra was introduced by physicists in their string theory of the elementary particles (cf. [6]). Mathematicians started to develop a representation theory of this algebra very recently.

Let us make a survey of the contents of this paper.

We recall the definition and some properties of the Virasoro algebra and the highest weight modules over it in section 1. An $\mathfrak{l}$-module M is called a "highest weight module" if there exists a non-zero vector v_o (the highest weight vector) such that 1) $U(\mathfrak{l})v_o=M$, where $U(\mathfrak{l})$ is the universal enveloping algebra of $\mathfrak{l}$, 2) there exists $\lambda\in(\boldsymbol{C}l_0\oplus\boldsymbol{C}c)^*$ (the highest weight) such that $Hv_o=\lambda(H)v_o$ for all $H\in(\boldsymbol{C}l_0\oplus\boldsymbol{C}c)$, 3) $l_k v_o=0$ for all positive k. The study of such modules was started by V. Kac ([3, 4]). He obtained the determinant formula for the matrix of the vacuum expectation values, and gave the "formal character" of some irreducible highest weight $\mathfrak{l}$-modules (THEOREM 1.1).

In section 2 we treat another kind of representations of $\mathfrak{l}$, which is called the "Fock representations" (cf. [6], [12]). Let a_j ($j\in\boldsymbol{Z}$) be the operators, acting on some "Fock space", with the following commutation relations:

(0.1) $$[a_j, a_i] = j\delta_{i+j,0}.$$

Define the operators

(0.2) $$L_k = \frac{1}{2}\sum_{j\in\boldsymbol{Z}} : a_{-j}a_{j+k} :$$

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for $k \in \boldsymbol{Z}$. Here : : denotes the normal product. Then the mapping $l_k \to L_k$, $c \to$ identity, definies a representation of I. This is the original form which was considered by physicists. More explicitly, one can construct the Fock representation as follows(cf. [12]). Let $V = \boldsymbol{C}[x_1, x_2, x_3, \dots]$ and $a_j = \partial/\partial x_j$, $a_{-j} = jx_j$ for positive j, and $a_0 = -\mu/\sqrt{2}$ where $\mu \in \boldsymbol{R}$, so that the commutation relations (0.1) hold. The operators given by (0.2), with the normal product

$$:a_i a_j: = a_i a_j \quad \text{for} \quad i \leqslant j, \quad = a_j a_i \quad \text{for} \quad i > j,$$

define the Fock representation on $V = V_\mu$. The vector space V has a positive definite hermitian inner product

$$\langle f, g \rangle = f(\tilde{\partial})\bar{g}(x)|_{x=0}$$

where $\tilde{\partial} = (\partial/\partial x_1, \partial/2\partial x_2, \partial/3\partial x_3, \dots)$. One can check that $\langle L_k f, g \rangle = \langle f, L_{-k} g \rangle$, and that the space V is completely reducible. The Fock representation is irreducible if and only if $\mu \notin \boldsymbol{Z}$ (COROLLARY 2.2). We give an explicit formula of the highest weight vectors of the irreducible components in terms of the Schur polynomials (THEOREM 2.3), and obtain the irreducible decomposition of the Fock representations in the case that they are reducible (COROLLARY 2.4).

We discuss in section 3 the relation between the tensor product of two Fock representations and the Hirota equations of the so-called "KP hierarchy". The Kadomtsev-Petviashvili (KP) hierarchy was first introduced by M. Sato ([7, 8]). His colleagues studied the transformation groups of the solutions of the KP hierarchy and of some subhierarchies ([1, 2]).

Let p be an indeterminate and put

$$\eta(x, p) = \textstyle\sum_{j=1}^{\infty} p^j x_j .$$

We define the polynomials $p_k(x)$ by

$$e^{\eta(x,p)} = \textstyle\sum_{k=0}^{\infty} p_k(x) p^k .$$

The Hirota bilinear operator D is defined by

$$P(D) f(x) \cdot g(x) = P(\partial/\partial y) f(x+y) g(x-y)|_{y=0} .$$

The following system of differential equations with the unknown function $\tau = \tau(x_1, x_2, x_3, \dots)$ is called the Hirota form of the KP hierarchy:

$$\textstyle\sum_{j=0}^{\infty} p_j(-2y) p_{j+1}(\tilde{D}) \exp\left(\sum_{k=1}^{\infty} y_k D_k\right) \tau(x) \cdot \tau(x) = 0$$

for any $y = (y_1, y_2, y_3, \dots)$, where $\tilde{D} = (D_1, D_2/2, D_3/3, \dots)$. The first two nontrivial equations are

$$(D_1^4 - 4D_1 D_3 + 3D_2^2)\tau \cdot \tau = 0,$$

$$(D_1^3D_2+2D_2D_3-3D_1D_4)\tau\cdot\tau = 0.$$

Following [2] the modified KP hierarchies are defined in section 3. We show that if the Fock representation V_μ (0.2) of the Virasoro algebra is decomposed as $V_\mu=\Omega\oplus\Omega^\perp$, where Ω is the irreducible component containing 1, then the equations $P\left(\frac{1}{\sqrt{2}}\tilde{D}\right)\tau\cdot\tau'=0$ for $P(x)\in\Omega^\perp$ is one of the Hirota bilinear equations of the modified KP hierarchy, and vice versa (THEOREM 3.3).

After our announcement [10] of the contents of this paper had spread out, N. Wallach [11] gave an elegant proof of Theorem 2.3. In the present paper we give another proof of it.

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1. The Virasoro algebra and the highest weight modules

We first consider the Lie algebra consisting of the polynomial vector fields on the circle. Let $l_k=\sqrt{-1}\,e^{\sqrt{-1}k\theta}(d/d\theta)\,(k\in \boldsymbol{Z})$ be the vector fields on the circle. Then the bracket relations

$$[l_k, l_m] = (k-m)l_{k+m}$$

hold. The Lie algebra $\mathfrak{l}_o=\sum_{k\in \boldsymbol{Z}} \boldsymbol{C}l_k$ is called the Witt algebra. The Virasoro algebra $\mathfrak{l}$ is the one-dimensional central extension of $\mathfrak{l}_o$. Namely,

$$\mathfrak{l} = \textstyle\sum_{k\in \boldsymbol{Z}} \boldsymbol{C}l_k \oplus \boldsymbol{C}c$$

with the bracket relations

$$[l_k, l_m] = (k-m)l_{k+m} + \frac{1}{12}(k^3-k)\delta_{k+m,0}c, \quad [c, \mathfrak{l}] = \{0\}.$$

We put $\mathfrak{h}=\boldsymbol{C}l_0\oplus \boldsymbol{C}c$, which is the Cartan subalgebra of $\mathfrak{l}$, and $\mathfrak{n}^\pm=\sum_{k>0}\boldsymbol{C}l_{\pm k}$ so that $\mathfrak{l}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$. An $\mathfrak{l}$-module M is called a "highest weight module" if there exists a non-zero vector $v_o\in M$ such that 1) $U(\mathfrak{l})v_o=M$, where $U(\mathfrak{l})$ is the universal enveloping algebra of $\mathfrak{l}$, 2) there exists $\lambda\in\mathfrak{h}^*$ such that $Hv_o=\lambda(H)v_o$ for all $H\in\mathfrak{h}$, 3) $\mathfrak{n}^+v_o=\{0\}$. For such a module M, $\lambda\in\mathfrak{h}^*$ is called the "highest weight" and $v_o\in M$ the "highest weight vector". For an arbitrarily given $\lambda\in\mathfrak{h}^*$, there is a universal highest weight module $M(\lambda)$, the Verma module, which is defined by $M(\lambda)=U(\mathfrak{l})\otimes_{U(\mathfrak{h}\oplus\mathfrak{n}^+)}\boldsymbol{C}$, where the action of $\mathfrak{h}\oplus\mathfrak{n}^+$ on $\boldsymbol{C}$ is given by $(H+X)\cdot 1=\lambda(H)\cdot 1$ for $H\in\mathfrak{h}$ and $X\in\mathfrak{n}^+$. Any highest weight module with highest weight $\lambda\in\mathfrak{h}^*$ is a quotient of the Verma module. There is a unique irreducible $\mathfrak{l}$-module with highest weight λ, which is denoted by $L(\lambda)$. If $\lambda(l_0)=\xi\in\boldsymbol{C}$ and $\lambda(c)=\eta\in\boldsymbol{C}$, we write $\lambda=(\xi,\eta)$ and $M(\lambda)=M(\xi,\eta)$, $L(\lambda)=L(\xi,\eta)$.

Let $N(\lambda)$ be a highest weight module with the highest weight vector $v_o \in N(\lambda)$. For a non-negative integer m, we put

$$N(\lambda)_m = \textit{linear span of } \{l_{-\nu_1} l_{-\nu_2} \cdots l_{-\nu_k} v_o; \quad k \geqslant 0,\ \nu_i \geqslant 1,\ \nu_1 + \nu_2 + \cdots + \nu_k = m\},$$

which is a finite dimensional vector space. The generating function

$$\operatorname{ch} N(\lambda) = \textstyle\sum_{m=0}^{\infty} \dim N(\lambda)_m q^m$$

is called the "formal character" of $N(\lambda)$. For the Verma module $M(\lambda)$, it is easy to see that $\dim M(\lambda)_m = p(m)$, where $p(m) = \#\{$partition of m into positive integers$\}$. Hence $\operatorname{ch} M(\lambda) = \phi(q)^{-1}$, where $\phi(q) = \prod_{n=1}^{\infty}(1-q^n)$.

Our starting point is the following theorem of V. Kac.

THEOREM 1.1 ([3, 4]). 1) *The Verma module $M(\xi, 1)$ is irreducible if and only if $\xi \neq m^2/4$ for $m \in \boldsymbol{Z}$.*

2) $\operatorname{ch} L(m^2/4, 1) = \phi(q)^{-1}(1-q^{m+1})$ *for* $m = 0, 1, 2, \ldots$.

2. The Fock representations of the Virasoro algebra

We prepare the infinite dimensional vector space $V = \boldsymbol{C}[x_1, x_2, x_3, \ldots]$ of the polynomials of infinitely many variables. Let a_j be the operators on V defined by, for positive integer j, $a_j = \partial_j$ $(= \partial/\partial x_j)$, $a_{-j} = jx$, and $a_0 = -\mu/\sqrt{2}$ $(\mu \in \boldsymbol{R})$. Note that they have the commutation relations (0.1). Now, using a_j, define the operators

$$L_k^{(\mu)} = \frac{1}{2} \textstyle\sum_{j \in \boldsymbol{Z}} :a_{-j} a_{j+k}:$$

for $k \in \boldsymbol{Z}$. Here : : is the normal product:

$$:a_i a_j: = a_i a_j + j\delta_{i+j,0} Y_-(j) = \begin{cases} a_i a_j & (i \leqslant j) \\ a_j a_i & (i > j) \end{cases}$$

where $Y_-(j) = 0$ for $j \geqslant 0$, $= 1$ for $j < 0$.

We write down some $L_k^{(\mu)}$. For example,

$$L_0^{(\mu)} = \mu^2/4 + \textstyle\sum_{j=1}^{\infty} jx_j \partial_j,$$

$$L_1^{(\mu)} = -\frac{\mu}{\sqrt{2}} \partial_1 + \textstyle\sum_{j=1}^{\infty} jx_j \partial_{j+1},$$

$$L_2^{(\mu)} = \frac{1}{2} \partial_1^2 - \frac{\mu}{\sqrt{2}} \partial_2 + \textstyle\sum_{j=1}^{\infty} jx_j \partial_{j+2}.$$

One can easily see that $[L_k^{(\mu)}, a_j]=-ja_{k+j}$ and hence

$$[L_k^{(\mu)}, L_m^{(\mu)}] = (k-m)L_{k+m}^{(\mu)} + \frac{1}{12}(k^3-k)\delta_{k+m,0}\,.$$

Therefore the mapping π_μ: $l_k \to L_k^{(\mu)}$, $c \to$ identity, defines a representation of $\mathfrak{l}$ on V. Considered as the representation space of π_μ, the vector space V is denoted by V_μ. We call (π_μ, V_μ) the "Fock representation". This representation itself is not necessarily a highest weight representation. However it includes the highest weight module with highest weight vector $1 \in V_\mu$. Obviously the highest weight is $(\mu^2/4, 1)$. Therefore, if (π_μ, V_μ) is irreducible, then $V_\mu \cong L(\mu^2/4, 1)$.

Lemma 2.1. *If (π_μ, V_μ) is irreducible, then $V_\mu \cong M(\mu^2/4, 1)$.*

Proof. From the definition of the Verma module, there is a natural $\mathfrak{l}$-invariant surjective homomorphism

$$f\colon M(\mu^2/4, 1) \longrightarrow V_\mu;\ l_{-\nu_1}\cdots l_{-\nu_k}v_o \longrightarrow L_{\nu_1}^{(\mu)}\cdots L_{\nu_k}^{(\mu)}\cdot 1.$$

If we introduce the gradation on V_μ by $\deg x_j = j$, then we see that the homomorphism f preserves the gradation. The space of homogeneous polynomials of degree m is of $p(m)$-dimensional. This proves the lemma.

Corollary 2.2. *The Fock representation (π_μ, V_μ) is irreducible if and only if $\mu \notin \boldsymbol{Z}$.*

We introduce a positive definite hermitian inner product on V_μ by

$$\langle f, g\rangle = f(\tilde{\partial})\bar{g}(x)|_{x=0}$$

where $\tilde{\partial}=(\partial_1, \partial_2/2, \partial_3/3,\dots)$. One checks that

$$\langle a_j f, g\rangle = \langle f, a_{-j}g\rangle, \quad \langle L_k^{(\mu)}f, g\rangle = \langle f, L_{-k}^{(\mu)}g\rangle\,.$$

It follows that V_μ is completely reducible and is a direct sum of irreducible highest weight modules.

Let us recall the "Schur polynomial" of $x_1, x_2, x_3,\dots$ for a given Young diagram. Let $Y=(f_1, f_2,\dots,f_n)$ $(f_1 \geqslant f_2 \geqslant \cdots \geqslant f_n > 0)$ be a Young diagram of size N. For the set of non-negative integers $(\nu_1, \nu_2,\dots, \nu_N)$ with $\nu_1+2\nu_2+\cdots+N\nu_N=N$, let $\pi_Y(1^{\nu_1}, 2^{\nu_2},\dots, N^{\nu_N})$ be the value of the irreducible character of S_N, the symmetric permutation group of N letters, labeled by the Young diagram Y, and evaluated at the conjugacy class consisting of ν_1 cycles of size 1, ν_2 cycles of size 2, and so on. The Schur polynomial for Y is by definition

$$\chi_Y(x) = \sum \pi_Y(1^{\nu_1}, 2^{\nu_2},\dots, N^{\nu_N})\frac{x_1^{\nu_1}x_2^{\nu_2}\cdots x_N^{\nu_N}}{\nu_1!\nu_2!\cdots\nu_N!}\,.$$

It is a homogeneous polynomial of degree N ($\deg x_j = j$) with rational coefficients. We will denote by $p_r(x)$ the Schur polynomial $\chi_{(r)}(x)$. They have the simple generating function:

$$\sum_{r=0}^{\infty} p_r(x)p^r = e^{\eta(x,p)}, \tag{2.1}$$

where $\eta(x, p) = \sum_{j=1}^{\infty} p^j x_j$. It is known the following formula representing the Schur polynomial for any Young diagram $Y = (f_1, \dots, f_n)$:

$$\chi_Y(x) = \det (P_{f_i - i + j})_{1 < i, j < n} \tag{2.2}$$

where we have set $p_r(x) = 0$ for negative r.

We can describe the highest weight vectors of the irreducible components of the Fock representations in terms of the Schur polynomials.

THEOREM 2.3. *Fix a non-negative integer n. For a non-negative integer r, let $\Delta_{r,r+n} = \underbrace{(r, \dots, r)}_{r+n}$ be a rectangular Young diagram. Then*

$$L_k^{(n)} \chi_{\Delta_{r,r+n}}(\sqrt{2}x) = 0 \tag{2.3}$$

for all positive k.

PROOF. The proof of the case $n=0$ is given in [9; Prop. 6.4]. Here we give a reduction to the case $n=0$. We only have to show (2.3) for $k=1$ and $k=2$. Put $K_k = \sum_{j=1}^{\infty} j x_j \partial_{j+k}$ for positive k. By an easy calculation we have

$$K_k p_r(\sqrt{2}x) = (r-k) p_{r-k}(\sqrt{2}x). \tag{2.4}$$

Hence, putting $p'_r(x) = p_r(\sqrt{2}x)$,

$$(2.5)\quad K_1 \begin{vmatrix} p'_{r+n} & \cdots & p'_{2r+n-1} \\ & \cdots & \\ p'_{1+n} & \cdots & p'_{r+n} \end{vmatrix}$$

$$= \begin{vmatrix} (r+n-1)p'_{r+n-1} & \cdots & (2r+n-2)p'_{2r+n-2} \\ p'_{r+n-1} & \cdots & p'_{2r+n-2} \\ & \cdots & \\ p'_{1+n} & \cdots & p'_{r+n} \end{vmatrix} + \cdots + \begin{vmatrix} p'_{r+n} & \cdots & p'_{2r+n-1} \\ & \cdots & \\ p'_{2+n} & \cdots & p'_{r+n-1} \\ np'_n & \cdots & (r+n-1)p'_{r+n-1} \end{vmatrix}$$

$$= n \begin{vmatrix} p'_{r+n-1} & \cdots & p'_{2r+n-2} \\ p'_{r+n-1} & \cdots & p'_{2r+n-2} \\ & \cdots & \\ p'_{1+n} & \cdots & p'_{r+n} \end{vmatrix} + \cdots + n \begin{vmatrix} p'_{r+n} & \cdots & p'_{2r+n-1} \\ & \cdots & \\ p'_{1+n} & \cdots & p'_{r+n} \\ p'_{1+n} & \cdots & p'_{r+n} \end{vmatrix} + n \begin{vmatrix} p'_{r+n} & \cdots & p'_{2r+n-1} \\ & \cdots & \\ p'_{2+n} & \cdots & p'_{r+n+1} \\ p'_n & \cdots & p'_{r+n-1} \end{vmatrix} +$$

$$\begin{vmatrix} (r-1)p'_{r+n-1} & \cdots & (2r-2)p'_{2r+n-2} \\ p'_{r+n-1} & \cdots & p'_{2r+n-2} \\ & \cdots & \\ p'_{1+n} & \cdots & p'_{r+n} \end{vmatrix} + \cdots + \begin{vmatrix} p'_{r+n} & \cdots & p'_{2r+n-1} \\ & \cdots & \\ p'_{2+n} & \cdots & p'_{r+n+1} \\ 0 & \cdots & (r-1)p'_{r+n-1} \end{vmatrix}.$$

The first $(r-1)$ terms are equal to zero. The r-th term is equal to

$$\frac{n}{\sqrt{2}}\frac{\partial}{\partial x_1}\begin{vmatrix} p'_{r+n} & \cdots & p'_{2r+n-1} \\ & \cdots & \\ p'_{1+n} & \cdots & p'_{r+n} \end{vmatrix}.$$

We put $X_r = p'_{r+n}$. The right hand side of (2.5) is equal to

$$\frac{n}{\sqrt{2}}\frac{\partial}{\partial x_1}\begin{vmatrix} p'_{r+n} & \cdots & p'_{2r+n-1} \\ & \cdots & \\ p'_{1+n} & \cdots & p'_{r+n} \end{vmatrix} +$$

$$\begin{vmatrix} (r-1)X_{r-1} & \cdots & (2r-2)X_{2r-2} \\ X_{r-1} & \cdots & X_{2r-2} \\ & \cdots & \\ X_1 & \cdots & X_r \end{vmatrix} + \cdots + \begin{vmatrix} X_r & \cdots & X_{2r-1} \\ & \cdots & \\ X_2 & \cdots & X_{r+1} \\ 0 & \cdots & (r-1)X_{r-1} \end{vmatrix}.$$

On the other hand, from the case $n=0$, we have

$$0 = L_1^{(0)}\begin{vmatrix} p'_r & \cdots & p'_{2r-1} \\ & \cdots & \\ p'_1 & \cdots & p'_r \end{vmatrix}$$

$$= \begin{vmatrix} (r-1)p'_{r-1} & \cdots & (2r-2)p'_{2r-2} \\ p'_{r-1} & \cdots & p'_{2r-2} \\ & \cdots & \\ p'_1 & \cdots & p'_r \end{vmatrix} + \cdots + \begin{vmatrix} p'_r & \cdots & p'_{2r-1} \\ & \cdots & \\ p'_2 & \cdots & p'_{r+1} \\ 0 & \cdots & (r-1)p'_{r-1} \end{vmatrix}.$$

By the algebraic independence of p'_1, $p'_2, \ldots$, we obtain

$$\left(K_1 - \frac{n}{\sqrt{2}}\frac{\partial}{\partial x_1}\right)\chi_{\Delta_{r,r+n}}(\sqrt{2}x) = L_1^{(n)}\chi_{\Delta_{r,r+n}}(\sqrt{2}x) = 0.$$

Since the proof for $L_2^{(n)}$ is similar, we omit it here.

REMARK. If we modify the definition of a_j so that $a'_j = \sqrt{2}\,\partial_j$, $a_{-j}' = (1/\sqrt{2})jx_j$, and $a'_0 = -n/\sqrt{2}$, then, defining $L_k^{(n)\prime} = \frac{1}{2}\sum_{j\in Z} : a'_{-j}a'_{j+k} :$, we have $L_k^{(n)\prime}\chi_{\Delta_{r,r+n}}(x) = 0$ for all positive k.

Applying $\sum_{j=1}^{\infty} jx_j\partial_j$ to a homogeneous polynomial, we get the homogeneous degree as the eigenvalue. Hence

$$\begin{aligned} L_0^{(n)}\chi_{\Delta_{r,r+n}}(\sqrt{2}x) &= (\tfrac{1}{4}n^2+r(r+n))\,\chi_{\Delta_{r,r+n}}(\sqrt{2}x) \\ &= \tfrac{1}{4}(n+2r)^2\,\chi_{\Delta_{r,r+n}}(\sqrt{2}x). \end{aligned}$$

This equation shows that the Schur polynomial $\chi_{\Delta_{r,r+n}}(\sqrt{2}x)$ is the highest weight vector of highest weight $\frac{1}{4}(n+2r)^2$. Therefore, for a non-negative integer n, $V_n \supset \oplus_{r=0}^{\infty} L(\frac{1}{4}(n+2r)^2, 1)$. We compute the sum of the formal characters of the right-hand side:

$$\begin{aligned} &\sum_{r=0}^{\infty} \operatorname{ch} L(\tfrac{1}{4}(n+2r)^2, 1)q^{r(r+n)} \\ &= \sum_{r=0}^{\infty} \frac{1}{\phi(q)}(1-q^{n+2r+1})q^{r(r+n)} = \frac{1}{\phi(q)}. \end{aligned}$$

COROLLARY 2.4. *For a non-negative integer n, the Fock representation (π_n, V_n) is decomposed as*

$$V_n \cong \oplus_{r=0}^{\infty} L(\tfrac{1}{4}(n+2r)^2, 1).$$

3. The Virasoro algebra and the modified KP hierarchies

In this section we discuss the relation between the Virasoro algebra and the modified KP (Kadomtsev-Petviashvili) hierarchies. The main tool is the Fock representation of the Lie algebra $\mathfrak{gl}(\infty)$, which was introduced by Date, Jimbo, Kashiwara and Miwa (cf. [1, 2]).

Fix a non-negative integer n, and put $V(n)=\boldsymbol{C}[x_1, x_2, x_3, \ldots]$. Let p and q be the indeterminates. Set

$$\eta(x, p) = \sum_{j=1}^{\infty} p^j x_j, \quad \eta(\tilde{\partial}, p^{-1}) = \sum_{j=1}^{\infty} j^{-1}p^{-j}\frac{\partial}{\partial x_j},$$

and

(3.1) $\quad \Gamma(p, q) = \exp(\eta(x, p) - \eta(x, q))\exp(-(\eta(\tilde{\partial}, p^{-1}) - \eta(\tilde{\partial}, q^{-1})))$.

The "n-th modified vertex operator" for the KP hierarchy is defined as follows:

$$X^{(n)}(p, q) = \frac{q}{p-q}\left(\left(\frac{p}{q}\right)^n \Gamma(p, q) - 1\right) = \sum_{i,j\in\boldsymbol{Z}} X_{ij}^{(n)}\, p^i q^{-j}. \tag{3.2}$$

Here we denote

$$\frac{q}{p-q} = -\sum_{k=0}^{\infty}\left(\frac{p}{q}\right)^k.$$

We define $Y_+(i)=1-Y_-(i)$.

It is proved in [1, 2] that (3.2) defines a representation of the Lie algebra $\mathfrak{gl}(\infty)$ on $V(n)$. That is, the commutation relations

$$[X_{ij}^{(n)}, X_{kl}^{(n)}] = \delta_{jk}X_{il}^{(n)} - \delta_{il}X_{kj}^{(n)} + \delta_{jk}\delta_{il}(Y_+(j) - Y_+(i)) \tag{3.3}$$

hold. This representation is called the Fock representation (the vertex representation) of $\mathfrak{gl}(\infty)$.

We define the Heisenberg Lie algebra $\mathfrak{s}$ by

$$\mathfrak{s} = \sum_{k\in \boldsymbol{Z}, k\neq 0} \boldsymbol{C}h_k \oplus \boldsymbol{C}c$$

with the bracket relations

$$[h_k, h_m] = k\delta_{k+m,0}c, \ [c, \mathfrak{s}] = \{0\}.$$

LEMMA 3.1. *Put* $H_k^{(n)} = \sum_{j\in \boldsymbol{Z}} X_{j,j+k}^{(n)}$ for $k \in \boldsymbol{Z}$. *Then*

$$H_k^{(n)} = \begin{cases} \partial_k & (k > 0) \\ 0 & (k = 0) \\ -kx_{-k} & (k < 0). \end{cases}$$

Namely, the mapping $h_k \to H_k^{(n)}$, $c \to$ *identity, defines a representation of* $\mathfrak{s}$.

PROOF. By an easy calculation, we obtain

$$\begin{aligned} X^{(n)}(p, q)\Big|_{q=p} &= q\frac{\partial\Gamma(p, q)}{\partial q}\Big|_{q=p} = p\Big(\frac{\partial\eta(x, p)}{\partial p} - \frac{\partial\eta(\tilde{\partial}, p^{-1})}{\partial p}\Big) \\ &= \sum_{j=1}^{\infty} jx_j p^j + \sum_{j=1}^{\infty} \partial_j p^{-j}. \end{aligned}$$

On the other hand, we have

$$X^{(n)}(p, q)|_{q=p} = \sum_{i,j\in \boldsymbol{Z}} X^{(n)} p^{i-j} = \sum_{k\in \boldsymbol{Z}} (\sum_{i\in \boldsymbol{Z}} X_{i,i+k}^{(n)})p^{-k}.$$

The lemma is proved by comparing the coefficients of p^{-k}.

From this lemma, we deduce that the Fock representation on $V(n)$ is irreducible.

We take the tensor product of the Fock representations of $\mathfrak{gl}(\infty)$:

$$\begin{aligned} V(0) \otimes V(n) &= \boldsymbol{C}[x_1^{(1)}, x_2^{(1)}, \dots] \otimes \boldsymbol{C}[x_1^{(2)}, x_2^{(2)}, \dots] \\ &= \boldsymbol{C}[x_1, x_2, \dots; y_1, y_2, \dots] \end{aligned}$$

where we have set

$$x_j = (x_j^{(1)} + x_j^{(2)})/\sqrt{2}, \ y_j = (x_j^{(1)} - x_j^{(2)})/\sqrt{2}.$$

This space is decomposed as $V(n)_{high} \oplus V(n)_{low}$, where $V(n)_{high}$ is the irreducible component containing $1 \otimes 1$, and $V(n)_{low}$ is the orthogonal complement of $V(n)_{high}$ with respect to the inner product $\langle \, , \rangle$, which is defined naturally on $V(0) \otimes V(n)$. We set $\tilde{\Omega}(n) = V(n)_{high} \cap \boldsymbol{C}[y_1, y_2, \ldots]$ and $\tilde{\Omega}(n)^{\perp} = V(n)_{low} \cap \boldsymbol{C}[y_1, y_2, \ldots]$. An element of the latter space is called a "Hirota polynomial".

The Heisenberg subalgebra $\mathfrak{s}$ acts on $V(0) \otimes V(n)$ as $h_j \to \partial/\partial x_j^{(1)} + \partial/\partial x_j^{(2)} = \sqrt{2}\, \partial/\partial x_j$ and $h_{-j} \to j x_j^{(1)} + j x_j^{(2)} = \sqrt{2}\, j x_j$ for positive j. We denote by $\Gamma^{(1)}(p, q)$ (resp. $\Gamma^{(2)}(p, q)$) the operator which is obtained from $\Gamma(p, q)$ (3.1) with x_j replaced by $x_j^{(1)}$ (resp. $x_j^{(2)}$). The homogeneous parts of $\Gamma^{(1)}(p, q) + \Gamma^{(2)}(p, q)$ act on the space $V(0) \otimes V(n)$.

Any element $f_0(x^{(1)}) \otimes f_n(x^{(2)}) \in V(n)_{high}$ satisfies the orthogonal relations described by the system of differential equations

$$\langle P, f_0(x^{(1)}) \otimes f_n(x^{(2)}) \rangle = 0$$

for all $P(y) \in \tilde{\Omega}(n)^{\perp}$. These equations are equivalent to

$$P(\tilde{\partial}_y/\sqrt{2})\, f_0(x+y) f_n(x-y) = 0 \tag{3.4}$$

for all $P(y) \in \tilde{\Omega}(n)^{\perp}$.

Now we introduce the Hirota bilinear operators by

$$P(D) f(x) \cdot g(x) = P\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots\right) f(x+y) g(x-y)|_{y_1 = y_2 = \cdots = 0}.$$

Put $\tilde{D} = (D_1, D_2/2, D_3/3, \ldots)$. The orthogonal relations (3.4) are written simply as $P(\tilde{D}/\sqrt{2}) f_0 \cdot f_n = 0$ for all $P(y) \in \tilde{\Omega}(n)^{\perp}$. This system of non-linear differential equations is called the "n-th modified KP hierarchy". The generating function representation of the n-th modified KP hierarchy is known as

$$\textstyle\sum_{j=0}^{\infty} p_j(-2z) p_{j+n+1}(\tilde{D}) \exp\left(\sum_{k=1}^{\infty} z_k D_k\right) f_0 \cdot f_n = 0 \tag{3.5}$$

for any $z = (z_1, z_2, z_3, \ldots)$. The first two non-trivial equations of the (0-th modified) KP hierarchy are

$$(D_1^4 - 4D_1 D_3 + 3D_2^2) f_0 \cdot f_0 = 0, \tag{3.6}$$

$$(D_1^3 D_2 + 2D_2 D_3 - 3D_1 D_4) f_0 \cdot f_0 = 0. \tag{3.7}$$

If we put $u = (\partial/\partial x_1)^2 \log f_0$, and $x = x_1$, $y = x_2$, $t = x_3$, then we see that the equation (3.6) takes the form

$$\frac{3}{4} u_{yy} = \left(u_t - 3uu_x - \frac{1}{4} u_{xxx}\right)_x$$

which is the original KP equation.

Denote by $\tilde{\Omega}(n)_m$ (resp $\tilde{\Omega}(n)_m^\perp$) the subspace of $\tilde{\Omega}(n)$ (resp. $\tilde{\Omega}(n)^\perp$) consisting of the polynomials of degree m, so that $\tilde{\Omega}(n)=\oplus_{m=0}^{\infty}\tilde{\Omega}(n)_m$ (resp. $\tilde{\Omega}(n)^\perp=\oplus_{m=1}^{\infty}\tilde{\Omega}(n)_m^\perp$). The following proposition is important to the theory of universal Grassmann manifold ([7, 8]).

PROPOSITION 3.2 ([1]). $\dim \tilde{\Omega}(n)_m^\perp = p(m-n-1)$.

By using this proposition, we have

$$\sum_{m=0}^{\infty} \dim \tilde{\Omega}(n)_m q^m = \frac{1}{\phi(q)}(1-q^{n+1}).$$

This is exactly the formal character of $L(n^2/4, 1)$ for the Virasoro algebra.

Now let us return to the Virasoro algebra. As we have seen in the previous section, the Fock representation (π_n, V_n) is decomposed as $V_n=\Omega_n\oplus\Omega_n^\perp$, where Ω_n is the subrepresentation isomorphic to $L(n^2/4, 1)$, and $\Omega_n^\perp$ is the orthogonal complement of Ω_n with respect to $\langle\,,\,\rangle$. The proof of the following theorem for the case $n=0$ was obtained jointly with V. Kac and K. Ueno.

THEOREM 3.3. *For any $P(x)\in\Omega_n^\perp$, the equation $P(\tilde{D}/\sqrt{2})f_0\cdot f_n=0$ is one of the Hirota bilinear equations of the n-th modified KP hierarchy. Conversely, any Hirota bilinear equation of the n-th modified KP hierarchy corresponds to some $P(x)\in\Omega_n^\perp$.*

PROOF. Define the operator

$$\begin{aligned} Z^{(n)}(p, q) = \frac{1}{(p-q)^2}\Big\{&\exp\Big(-\frac{1}{\sqrt{2}}(\eta(x, p)-\eta(x, q))\Big)\\ &\Big(\Big(\frac{p}{q}\Big)^{-n/2}\Gamma^{(1)}(p, q)+\Big(\frac{p}{q}\Big)^{n/2}\Gamma^{(2)}(p, q)\Big)\\ &\exp\Big(\frac{1}{\sqrt{2}}(\eta(\tilde{\partial}_x, p^{-1})-\eta(\tilde{\partial}_x, q^{-1}))\Big)-2\Big\}. \end{aligned}$$

We can see that the homogeneous parts of $Z^{(n)}(p, q)$ act on $V(0)\otimes V(n)$. By an easy calculation such as

$$\begin{aligned} &\exp\Big(-\frac{1}{\sqrt{2}}\eta(x, p)\Big)\exp\Big(\eta(x^{(1)}, p)\Big)\\ &=\exp\Big(-\frac{1}{\sqrt{2}}\eta(x, p)\Big)\exp\Big(\frac{1}{\sqrt{2}}(\eta(x, p)+\eta(y, p))\Big)=\exp\Big(\frac{1}{\sqrt{2}}\eta(y, p)\Big), \end{aligned}$$

we see that

$$Z^{(n)}(p, q) = \frac{1}{(p-q)^2} \left\{ \left(\frac{p}{q}\right)^{-n/2} \exp\left(\frac{1}{\sqrt{2}}(\eta(y, p) - \eta(y, q))\right) \right.$$
$$\exp\left(-\frac{1}{\sqrt{2}}(\eta(\tilde{\partial}_y, p^{-1}) - \eta(\tilde{\partial}_y, q^{-1}))\right) + \left(\frac{p}{q}\right)^{n/2}$$
$$\left. \exp\left(-\frac{1}{\sqrt{2}}(\eta(y, p) - \eta(y, q))\right) \exp\left(\frac{1}{\sqrt{2}}(\eta(\tilde{\partial}_y, p^{-1}) - \eta(\tilde{\partial}_y, q^{-1}))\right) - 2 \right\}.$$

Hence the homogeneous parts of $Z^{(n)}(p, q)$ act both on $\tilde{\Omega}(n)$ and $\tilde{\Omega}(n)^{\perp}$. Put

$$e_1 = \exp\left(\frac{1}{\sqrt{2}}(\eta(y, p) - \eta(y, q))\right),$$
$$e_2 = \exp\left(-\frac{1}{\sqrt{2}}(\eta(\tilde{\partial}_y, p^{-1}) - \eta(\tilde{\partial}_y, q^{-1}))\right),$$
$$e_3 = \exp\left(-\frac{1}{\sqrt{2}}(\eta(y, p) - \eta(y, q))\right),$$
$$e_4 = \exp\left(\frac{1}{\sqrt{2}}(\eta(\tilde{\partial}_y, p^{-1}) - \eta(\tilde{\partial}_y, q^{-1}))\right),$$

and

$$\Gamma_1^{(n)} = \left(\frac{p}{q}\right)^{-n/2} e_1 e_2, \quad \Gamma_2^{(n)} = \left(\frac{p}{q}\right)^{n/2} e_3 e_4,$$

so that

$$Z^{(n)}(p, q) = \frac{1}{(p-q)^2}(\Gamma_1^{(n)} + \Gamma_2^{(n)} - 2).$$

We differentiate $\Gamma_1^{(n)}$ with respect to q. Then

$$\frac{\partial \Gamma_1^{(n)}}{\partial q} = \left(\frac{n}{2}\right)\left(\frac{p}{q}\right)^{-n/2} q^{-1} e_1 e_2 - \frac{1}{\sqrt{2}}\left(\frac{p}{q}\right)^{-n/2} \frac{\partial \eta(y, q)}{\partial q} e_1 e_2$$
$$+ \frac{1}{\sqrt{2}}\left(\frac{p}{q}\right)^{-n/2} e_1 e_2 \frac{\partial \eta(\tilde{\partial}_y, q^{-1})}{\partial q},$$

and

$$\frac{\partial^2 \Gamma_1^{(n)}}{\partial q^2}$$
$$= \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)\left(\frac{p}{q}\right)^{-n/2} q^{-2} e_1 e_2 - \frac{1}{\sqrt{2}}\left(\frac{n}{2}\right)\left(\frac{p}{q}\right)^{-n/2} q^{-1} \frac{\partial \eta(y, q)}{\partial q} e_1 e_2$$
$$+ \frac{1}{\sqrt{2}}\left(\frac{n}{2}\right)\left(\frac{p}{q}\right)^{-n/2} q^{-1} e_1 e_2 \frac{\partial \eta(\tilde{\partial}_y, q^{-1})}{\partial q} - \frac{1}{\sqrt{2}}\left(\frac{n}{2}\right)\left(\frac{p}{q}\right)^{-n/2} q^{-1} \frac{\partial \eta(y, q)}{\partial q} e_1 e_2$$

$$-\frac{1}{\sqrt{2}}\left(\frac{p}{q}\right)^{-n/2}\frac{\partial^2\eta(y,q)}{\partial q^2}e_1e_2+\frac{1}{2}\left(\frac{p}{q}\right)^{-n/2}\left(\frac{\partial\eta(y,q)}{\partial q}\right)^2 e_1e_2$$
$$-\frac{1}{2}\left(\frac{p}{q}\right)^{-n/2}\frac{\partial\eta(y,q)}{\partial q}e_1e_2\frac{\partial\eta(\tilde{\partial}_y,q^{-1})}{\partial q}+\frac{1}{\sqrt{2}}\left(\frac{n}{2}\right)\left(\frac{p}{q}\right)^{-n/2}q^{-1}e_1e_2\frac{\partial\eta(\tilde{\partial}_y,q^{-1})}{\partial q}$$
$$-\frac{1}{2}\left(\frac{p}{q}\right)^{-n/2}\frac{\partial\eta(y,q)}{\partial q}e_1e_2\frac{\partial\eta(\tilde{\partial}_y,q^{-1})}{\partial q}+\frac{1}{2}\left(\frac{p}{q}\right)^{-n/2}e_1e_2\left(\frac{\partial\eta(\tilde{\partial}_y,q^{-1})}{\partial q}\right)^2$$
$$+\frac{1}{\sqrt{2}}\left(\frac{p}{q}\right)^{-n/2}e_1e_2\frac{\partial^2\eta(\tilde{\partial}_y,q^{-1})}{\partial q^2}.$$

Taking the limit $q\to p$, we have

$$\left.\frac{\partial^2\Gamma_1^{(n)}}{\partial q^2}\right|_{q=p}=\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)p^{-2}-\frac{n}{\sqrt{2}}p^{-1}\frac{\partial\eta(y,p)}{\partial p}+\frac{n}{\sqrt{2}}p^{-1}\frac{\partial\eta(\tilde{\partial}_y,p^{-1})}{\partial p}$$
$$-\frac{1}{2}\frac{\partial^2\eta(y,p)}{\partial p^2}+\frac{1}{2}\frac{\partial^2\eta(\tilde{\partial}_y,p^{-1})}{\partial p^2}-\frac{\partial\eta(y,p)}{\partial p}\frac{\partial\eta(\tilde{\partial}_y,p^{-1})}{\partial p}$$
$$+\frac{1}{2}\left(\frac{\partial\eta(y,p)}{\partial p}\right)^2+\frac{1}{2}\left(\frac{\partial\eta(\tilde{\partial}_y,p^{-1})}{\partial p}\right)^2.$$

By a similar calculation, we obtain

$$\left.\frac{\partial^2\Gamma_2^{(n)}}{\partial q^2}\right|_{q=p}=\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)p^{-2}-\frac{n}{\sqrt{2}}p^{-1}\frac{\partial\eta(y,p)}{\partial p}+\frac{n}{\sqrt{2}}p^{-1}\frac{\partial\eta(\tilde{\partial}_y,p^{-1})}{\partial p}$$
$$+\frac{1}{\sqrt{2}}\frac{\partial^2\eta(y,p)}{\partial p^2}-\frac{1}{\sqrt{2}}\frac{\partial^2\eta(\tilde{\partial}_y,p^{-1})}{\partial p^2}-\frac{\partial\eta(y,p)}{\partial p}\frac{\partial\eta(\tilde{\partial}_y,p^{-1})}{\partial p}$$
$$+\frac{1}{2}\left(\frac{\partial\eta(y,p)}{\partial p}\right)^2+\frac{1}{2}\left(\frac{\partial\eta(\tilde{\partial}_y,p^{-1})}{\partial p}\right)^2.$$

Hence

$$Z^{(n)}(p,p)=\frac{1}{2}\frac{\partial^2}{\partial q^2}(\Gamma_1^{(n)}+\Gamma_2^{(n)}-2)|_{q=p}$$
$$=\frac{1}{4}n^2p^{-2}-\frac{n}{\sqrt{2}}p^{-1}\frac{\partial\eta(y,p)}{\partial p}+\frac{n}{\sqrt{2}}p^{-1}\frac{\partial\eta(\tilde{\partial}_y,p^{-1})}{\partial p}$$
$$-\frac{\partial\eta(y,p)}{\partial p}\frac{\partial\eta(\tilde{\partial}_y,p^{-1})}{\partial p}+\frac{1}{2}\left(\frac{\partial\eta(y,p)}{\partial p}\right)^2+\frac{1}{2}\left(\frac{\partial\eta(\tilde{\partial}_y,p^{-1})}{\partial p}\right)^2.$$

If we set $Z^{(n)}(p,q)=\sum_{k\in Z}L_k^{(n)\prime}p^{-(k+2)}$, then we see that the operators $L_k^{(n)\prime}$ are nothing but the Virasoro operators

$$L_k^{(n)\prime}=\frac{1}{2}\sum_{j\in Z}:b_{-j}^{(n)}b_{j+k}^{(n)}:$$

where $b_j^{(n)}=\partial/\partial y_j$, $b_{-j}^{(n)}=jy_j$ for $j\geqslant 1$, and $b_0^{(n)}=-n/\sqrt{2}$. This proves that the spaces $\tilde{\Omega}(n)$ and $\tilde{\Omega}(n)^{\perp}$ are invariant under the Fock representation $\pi_0 \otimes \pi_n$ of the Virasoro algebra.

Using Proposition 3.2, we see that $\tilde{\Omega}(n)=\{P(y);\ P(x)\in\Omega_n\}$. Thus the theorem is proved.

We give some examples. The polynomial $\chi_{\Delta_{2,2}}(\sqrt{2}x)$ is one of the highest weight vectors of $\Omega_0^{\perp}$. One has

$$(3.8)\qquad \chi_{\Delta_{2,2}}(\sqrt{2}x)|_{x=\tilde{D}/\sqrt{2}} = \frac{1}{12}(D_1^4 - 4D_1D_3 + 3D_2^2).$$

Next, take $L_{-1}^{(0)}\chi_{\Delta_{2,2}}(\sqrt{2}x)$ of $\Omega_0^{\perp}$. Then

$$(3.9)\qquad L_{-1}^{(0)}\chi_{\Delta_{2,2}}(\sqrt{2}x)|_{x=\tilde{D}/\sqrt{2}} = \frac{1}{3}(D_1^3D_2 + 2D_2D_3 - 3D_1D_4).$$

We see that (3.8) and (3.9) are proportional to the operators in (3.6) and (3.7) respectively. The polynomial $p_2(\sqrt{2}x)$ is one of the highest weight vectors of $\Omega_1^{\perp}$. One has

$$(3.10)\qquad p_2(\sqrt{2}x)|_{x=\tilde{D}/\sqrt{2}} = \frac{1}{2}(D_1^2 + D_2).$$

Take $L_{-1}^{(1)}p_2(\sqrt{2}x)$ of $\Omega_1^{\perp}$. Then

$$(3.11)\qquad L_{-1}^{(1)}p_2(\sqrt{2}x)|_{x=\tilde{D}/\sqrt{2}} = -\frac{1}{4}(D_1^3 - 3D_1D_2 - 4D_3).$$

We see that (3.10) is proportional to the constant term of (3.5) ($n=1$), and (3.11) to the coefficient of z_1. The equations corresponding to them are the "Miura transformation" and the original modified KP equation respectively (cf. [2]).

References

[1] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, Transformation groups for soliton equations —Euclidean Lie algebras and reduction of the KP hierarchy—, Publ. RIMS. **18** (1982), 1077–1110.

[2] M. Jimbo and T. Miwa, Solitons and infinite dimensional Lie algebras, ibid. **19** (1983), 943–1001.

[3] V. G. Kac, Contravariant form for infinite dimensional Lie algebras and superalgebras, Lecture Notes in Phys. **94**, Springer, 1978, 441–445.

[4] ———, Highest weight representations of infinite dimensional Lie algebras, Proc. of ICM., Helsinki, 1978, 299–304.

[5] ———, Infinite Dimensional Lie Algebras, Birkhäuser, 1983.

[6] S. Mandelstam, Dual resonance models, Phys. Rep. **13** (1974), 259–353.

[7] M. Sato, Soliton equations as dynamical systems on an infinite dimensional Grassmann manifold, RIMS-Kokyuroku **439** (1981), 30–46.

[8] M. Sato and Y. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifold, in "Nonlinear Partial Differential Equations in Applied Science", H. Fujita et al. ed., Kinokuniya/North-Holland, 1982, 259–271.

[9] G. Segal, Unitary representations of some infinite dimensional groups, Commun. Math. Phys. **80** (1981), 301–342.

[10] M. Wakimoto and H. Yamada, Irreducible decompositions of Fock representations of the Virasoro algebra, Lett. Math. Phys. **7** (1983), 513–516.

[11] N. R. Wallach, Classical invariant theory and the Virasoro algebra, in "Vertex operators in Mathematics and Physics", J. Lepowsky et al. ed., Springer, 1985, 475–482.

[12] V. G. Kac, Some problems on infinite dimensional Lie algebras and their representations, Lecture Notes in Math. **933**, Springer, 1982, 117–126.

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