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On the regularity of solutions of a degenerate parabolic Bellman equation

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1. Introduction

The control of diffusion processes leads to the parabolic Bellman equation of the type:

$$(1.1)\qquad \sup\{u_t + Lu^\alpha - f^\alpha; \alpha\in\Lambda\} = 0 \quad \text{in} \quad Q_T, \\ u = 0 \quad \text{on} \quad \partial_p Q_T,$$

where $Q_T = \Omega\times(0, T)$ for a smooth bounded domain $\Omega\subset \boldsymbol{R}^n$ and $T\in(0, \infty)$, $\partial_p Q_T$ denotes the parabolic boundary of Q_T, Λ denotes a set of indices, and each L^α is a second order elliptic operator of the form:

$$L^\alpha u = -a^\alpha_{ij}(x, t)\partial^2 u/\partial x_i\partial x_j + b^\alpha_i(x, t)\partial u/\partial x_i + c^\alpha(x, t)u.$$

Here and in the sequel we use the summation convention.

In case that L^α are uniformly elliptic operators and Λ is a finite set, L. C. Evans and S. Lenhart [2] have shown that there exists a unique function $u\in W^{2,1}_\infty(Q_T)\cap C^{\lambda,\lambda/2}(Q_T)$, for some $\lambda>0$, solving (1.1).

In this paper we investigate the following problem

$$(1.2)\qquad u_t + \max\{Lu - f, du - g\} = 0 \quad \text{a.e.} \quad \text{in } Q_T, \\ u = 0 \quad \text{on} \quad \partial_p Q_T,$$

where f, g and d are given functions, and L is a second order uniformly elliptic operator. We may regard (1.2) as a special degenerate case of (1.1), that is, a couple of a nondegenerate operator, L, and a special degenerate one, d.

The plan of this paper is as follows:
Section 2 is devoted to state and prove our main results. The proofs are done via elliptic regularization and penalization (see (2.10) below). The necessary a priori estimates of solutions to the corresponding approximate problems are obtained in Section 3. In Appendix we deal with the existence and regularity of the approximate problems.

The time independent case of (1.1) has been studied by N. V. Krylov [6] and P. L. Lions [7]. The time independent equation of (1.2) is called the obstacle problem in variational inequalities and has been investigated by many authors; see e.g. R. Jensen [4], D. Kinderlehere and G. Stampacchia [5] and their references.

Throughout this paper the letter C stands for various positive constants depending only on known quantities. We use the notation: for a function u and $k=1, 2, 3$

$$Du = (\partial u/\partial x_1,\dots, \partial u/\partial x_n)$$
$$|D^k u| = \{\textstyle\sum_{\alpha_1+\cdots+\alpha_n=k} |\partial^k u/\partial_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}|^2\}^{1/2}$$
$$|D^k u_t| = \{\textstyle\sum_{\alpha_1+\cdots+\alpha_n=k} |\partial^{k+1} u/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}\partial t|^2\}^{1/2}.$$

2. Main results

Consider the second order elliptic operator

$$Lu = - a_{ij}(x, t)\partial^2 u/\partial x_i\partial x_j + b_i(x, t)\partial u/\partial x_i + c(x, t)u.$$

We make the following assumptions on the coefficients of L:

$$a_{ij}(x, t) = a_{ji}(x, t) \quad \text{for all} \quad (x, t)\in Q_T \quad \text{and} \quad 1\le i, j\le n \tag{2.1}$$

$$a_{ij}(x, t)\xi_i\xi_j \ge \theta|\xi|^2 \quad \text{in} \quad Q_T, \tag{2.2}$$

for some $\theta>0$ and all $\xi=(\xi_1,\dots, \xi_n)\in \boldsymbol{R}^n$, and

$$a_{ij},\ b_i,\ c\in C^2(\overline{Q_T}) \qquad \text{for} \quad 1\le i, j\le n. \tag{2.3}$$

We also assume that

$$d, f, g\in C^2(\overline{Q_T}). \tag{2.4}$$

The boundary condition in (1.2) yields the following compatibility condition:

$$g(x, t)\ge 0 \qquad \text{for} \quad (x, t)\in\partial\Omega\times(0, T). \tag{2.5}$$

THEOREM 1. *Assume that the conditions* (2.1)−(2.5) *hold, and that there exist functions* $\{w^\varepsilon;\ \varepsilon\ge 0\}\subset C^2(\overline{Q_T})$ *satisfying*

$$\partial w^\varepsilon/\partial t + \max\{-\varepsilon\partial^2 w^\varepsilon/\partial t^2 + Lw^\varepsilon - f,\ dw^\varepsilon - g\}\le 0 \ \text{in} \ Q_T, \tag{2.6}$$
$$w^0 = w^\varepsilon = 0 \ \textit{on} \ \partial_p Q_T, \ \text{and} \ w^0\le w^\varepsilon \ \textit{in} \ Q_T \ \textit{for each} \ \varepsilon>0.$$

Then the problem (1.2) *has a unique solution* $u\in W^{1,\infty}_{loc}([0, T); L^\infty(\Omega))\cap L^\infty_{loc}([0, T); W^{1,\infty}(\Omega)\cap W^{2,\infty}_{loc}(\Omega))$.

THEOREM 2. *In addition to the assumptions of Theorem* 1, *we also assume* *that*

$$a_{ij} \in C^2([0, T]; C^{3,\alpha}(\partial\Omega)) \tag{2.7}$$

for some $\alpha > 0$ *and* $1 \le i, j \le n$. *Then the solution of* (1.2) *belongs to* $L^\infty_{loc}([0, T); W^{2,\infty}(\Omega))$.

REMARK 1. Without loss of generality we can assume that $\lambda_0 = \max\{\lambda \in \boldsymbol{R};\, c(x, t) \ge \lambda$ and $d(x, t) \ge \lambda$ for all $(x, t) \in Q_T\}$ is a large number. Indeed, apply the transformation $v(x, t) = u(x, t)\exp(-2\lambda_0 t)$.

REMARK 2. If $f, g \ge 0$ in Q_T, for all $\varepsilon \ge 0$ one can take $w^\varepsilon = 0$ in (2.6).

PROOF OF THEOREM 1. We begin by constructing an auxiliary function which will be needed to show $W^{2,\infty}$-regularity of the solution near $t = 0$. We define $h(x, t)$ by

$$h(x, t) = \int_0^t \{g(x, s) - d(x, s)w^0(x, s) + s\}ds. \tag{2.8}$$

Let ψ be a smooth nondecreasing function on $\boldsymbol{R}$ such that

$$\begin{aligned} &\psi(r) = 0 \quad \text{for} \quad r \le 0,\ \psi(r) = r - 1 \quad \text{for} \quad r \ge 2, \\ &\psi' > 0 \quad \text{for} \quad r > 0 \quad \text{and} \quad \psi'' \ge 0 \quad \text{in} \quad \boldsymbol{R}. \end{aligned} \tag{2.9}$$

For each $\delta \in (0, 1)$, let us define γ_δ, $\beta_\delta \in C^\infty(\boldsymbol{R})$ by $\gamma_\delta(r) = \psi(r/\delta)$ and $\beta_\delta(r) = \psi((r-\delta)/\delta)$ in $\boldsymbol{R}$, respectively.

We approximate solutions of (1.2) by those of the following elliptic regularization and penalization:

$$\begin{aligned} L^\varepsilon u + \beta_\delta(u_t + du - g) + \gamma_\sigma(u - h) = f \quad &\text{in} \quad Q_T, \\ u = w^\varepsilon \quad &\text{on} \quad \partial Q_T, \end{aligned} \tag{2.10}$$

where L^ε denote $-\varepsilon\partial^2/\partial t^2 + \partial/\partial t + L$. We also consider the semilinear equation:

$$\begin{aligned} L^\varepsilon v + \gamma_\sigma(v - h) = f \quad &\text{in} \quad Q_T, \\ v = w^\varepsilon \quad &\text{on} \quad \partial Q_T. \end{aligned} \tag{2.11}$$

Now we state the existence of solutions of (2.10) and (2.11).

PROPOSITION 1. *For each* ε, δ *and* $\sigma \in (0, 1)$, *problems* (2.10) *and* (2.11) *admit solutions* $u^{\varepsilon,\delta,\sigma}$ *and* $v^{\varepsilon,\sigma}$, *respectively, which belong to* $\overline{W}^{2,p}_{loc}(Q_T)$ *for all* $p \in (1, \infty)$. *Here we denote that* $\overline{W}^{2,p}_{loc}(Q_T) = \cap\{W^{2,p}(Q^\alpha_T);\, \alpha > 0\}$, *where* Q^α_T *is a smooth subdomain of* Q_T *containing* $\Omega \times (\alpha, T - \alpha)$ *and* $\{(x, t) \in Q_T;\, 0 < t \le T - \alpha$ *and* $\mathrm{dis}(x, \partial\Omega) > \alpha\}$. *Furthermore there exists a constant C, independent of* ε, *such that*

$$\|v^{\varepsilon,\sigma}\|_{L^\infty(Q_T)} \leq C. \tag{2.12}$$

We will investigate the convergence of $\{v^{\varepsilon,\sigma}; \sigma \in (0.1)\}$ to a solution of an obstacle problem (see (2.13) below).

PROPOSITION 2. *For each $\varepsilon \in (0, 1)$, $\{v^{\varepsilon,\sigma}; \sigma \in (0, 1)\}$ converges in $\overline{W}^{2;p}_{loc}(Q_T)$, as $\sigma \to 0$, to the unique solution v^ε of*

$$\begin{aligned} \max\{L^\varepsilon v^\varepsilon - f, v^\varepsilon - h\} &= 0 \quad \text{in} \quad Q_T, \\ v^\varepsilon &= w^\varepsilon \quad \text{on} \quad \partial Q_T. \end{aligned} \tag{2.13}$$

The above two propositions will be proved in Appendix. The difficulty in proving them is that Q_T has a corner.

In order to prove Theorem 1, we need the following a priori estimtes of solutions to the problem (2.10).

LEMMA 1. *For suffciently small $\alpha > 0$, there is a constant $C > 0$ so that*

$$\|u^{\varepsilon,\delta,\sigma}\|_{L^\infty([0,T);L^\infty(\Omega))} \leq C, \tag{2.14}$$

$$\|u^{\varepsilon,\delta,\sigma}\|_{L^\infty([0,T-\alpha);W^{1,\infty}(\Omega))} \leq C, \tag{2.15}$$

$$\|u^{\varepsilon,\delta,\sigma}\|_{W^{1,\infty}([0,T-\alpha);L^\infty(\Omega))} \leq C, \tag{2.16}$$

and for each open set U with $\overline{U} \subset \Omega$,

$$\|u^{\varepsilon,\delta,\sigma}\|_{L^\infty([0,T-\alpha);W^{2,\infty}(U))} \leq C. \tag{2.17}$$

In view of the bounds obtained in the above lemma, there exist sequences $\varepsilon(k) \to 0$, $\delta(k) \to 0$ and $\sigma(k) \to 0$, as $k \to \infty$, and a function $u \in L^\infty_{loc}([0, T); W^{1,\infty}(\Omega) \cap W^{2;\infty}_{loc}(\Omega)) \cap W^{1;\infty}_{loc}([0, T); L^\infty(\Omega)) \cap L^\infty(Q_T)$ such that

$$u^{\varepsilon(k),\delta(k),\sigma(k)} \longrightarrow u, \quad \text{as} \quad k \longrightarrow \infty, \tag{2.18}$$

weakly star in $L^\infty_{loc}([0, T); W^{1,\infty}(\Omega) \cap W^{2;\infty}_{loc}(\Omega)) \cap W^{1;\infty}_{loc}([0, T); L^\infty(\Omega))$ and strongly in $L^p_{loc}([0, T); C^1(\Omega))$, for all $p \in (1, \infty)$. (2.18) and Mazur's lemma (see e.g. p. 120 in [9]) yield

$$u_t + du \leq g \quad \text{and} \quad u \leq h \quad \text{in} \quad Q_T. \tag{2.19}$$

By the comparison theorem and (2.6) we have

$$w^0(x, t) \leq u^{\varepsilon,\delta,\sigma}(x, t) \quad \text{in} \quad Q_T.$$

Thus (2.18) implies

$$w^0(x, t) \leq u(x, t) \quad \text{in} \quad Q_T.$$

By the first inequality of (2.19) we have

$$u(x, t) \le \int_0^t \{g(x, s) - d(x, s)w^0(x, s)\}ds.$$

Hence the definition of h (see (2.8)) implies

$$u < h \quad \text{in} \quad Q_T. \tag{2.20}$$

On the other hand, (2.9), (2.10) and (2.18) yield

$$u_t + Lu \le f \quad \text{a.e.} \quad \text{in} \quad Q_T. \tag{2.21}$$

Now we will show that u satisfies (1.2). For notational simplicity we write u^ε, β and γ instead of $u^{\varepsilon(k),\delta(k),\sigma(k)}$, $\beta_{\delta(k)}$ and $\gamma_{\sigma(k)}$, respectively. We also omit the argument $u_t + du - g$ of β and its derivatives and $u - h$ of γ and its derivatives. In addition, we write simply u_{ij}, u_i, u_{it}, $a_{ij,t}$, $b_{i,k}$ etc. instead of $\partial^2 u/\partial x_i \partial x_j$, $\partial u/\partial x_i$, $\partial^2 u/\partial x_i \partial t$, $\partial a_{ij}/\partial t$, $\partial b_i/\partial x_k$ etc. For any nonnegative function $\xi \in C_0^\infty(0, T)$, we have

$$\begin{aligned}
0 &= \int_0^T \int_\Omega \{L^\varepsilon u^\varepsilon + \beta + \gamma - f\}\{u_t^\varepsilon + du^\varepsilon - g\}\xi dxdt \\
&= \int_0^T \int_\Omega [\varepsilon\{(u_t^\varepsilon)^2 \xi_t/2 + (du^\varepsilon - g)_t u_t^\varepsilon \xi + (du^\varepsilon - g)u_t^\varepsilon \xi_t\} \\
&\quad + \{a_{ij} u_i^\varepsilon (u_t^\varepsilon + du^\varepsilon - g)_j \xi\} \\
&\quad + \{b_i \partial u^\varepsilon/\partial x_i + cu^\varepsilon - f\}\{u_t^\varepsilon + du^\varepsilon - g\}\xi \\
&\quad + \{\beta + \gamma\}\{u_t^\varepsilon + du^\varepsilon - g\}\xi]dxdt.
\end{aligned} \tag{2.22}$$

The first term of the right hand side of (2.22) converges to 0 as $\varepsilon \to 0$. Indeed, (2.14) and (2.16) yield

$$\int_0^T \int_\Omega \{(u_t^\varepsilon)^2 \xi_t/2 + (du^\varepsilon - g)_t u_t^\varepsilon \xi + (du^\varepsilon - g)u_t^\varepsilon \xi_t\}dxdt \le C,$$

where C is a constant independent of ε. The last term of the right hand side of (2.22) converges to a nonnegative number as $\varepsilon \to 0$. Indeed, (2.18) and (2.20) yield that $\gamma \to 0$ pointwise in $\Omega \times [0, T)$ as $\varepsilon \to 0$. Therefore $\int_0^T \int_\Omega \gamma(u_t^\varepsilon + du^\varepsilon - g)\xi dxdt$ converges to 0 as $\varepsilon \to 0$, by the bounded convergence theorem. Clearly the other part of the last term converges to a nonnegative number by the monotonicity of β. We further calculate that

$$\int_0^T \int_\Omega a_{ij} u_i^\varepsilon u_{j,t}^\varepsilon \xi dxdt = -\int_0^T \int_\Omega (a_{ij,t} u_i^\varepsilon u_j^\varepsilon \xi + a_{ij} u_i^\varepsilon u_j^\varepsilon \xi_t)/2 dxdt, \tag{2.23}$$

$$\int_0^T \int_\Omega (u_t^\varepsilon)^2 \xi \, dxdt \ge \int_0^T \int_\Omega (2u_t^\varepsilon u_t - u_t^2)\xi \, dxdt.$$

We combine (2.22) and (2.23) to obtain

$$\int_0^T \int_\Omega (u_t + Lu - f)(u_t + du - g)\xi dxdt \leq 0 . \tag{2.24}$$

In view of (2.19), (2.21) and (2.24) we see that u solves (1.2).

Now we prove the uniqueness of the solution of (1.2). Let u and v be two solutions of (1.2). For any nonnegative nonincreasing function $\xi \in C_0^\infty([0, T))$, $z = u - v$ satisfies

$$\int_0^T \int_\Omega (z + Lz)(z_t + dz)\xi dxdt \leq 0 .$$

Noting Remark 1, we easily obtain

$$\begin{aligned} &\int_0^T \int_\Omega (z_t^2 + \lambda^2 z^2)\xi dxdt \\ &\quad \leq \int_0^T \int_\Omega \{(C - \lambda\theta)|Dz|^2 + z_t^2/2 + \lambda^2 z^2/2\}\xi dxdt . \end{aligned}$$

Hence we get $z = 0$ in Q_T. Q.E.D.

PROOF OF THEOREM 2. We need the following lemma:

LEMMA 2. *For sufficiently small* $\alpha > 0$, *there exists a constant* C, *independent of* ε, δ, $\sigma \in (0, 1)$, *such that*

$$\|u^{\varepsilon,\delta,\sigma}\|_{L^\infty([0,T-\alpha);W^{2,\infty}(\Omega))} \leq C . \tag{2.25}$$

The above lemma implies that there exist sequences $\varepsilon(k) \to 0$, $\delta(k) \to 0$ and $\sigma(k) \to 0$, as $k \to \infty$, and a function $u \in L_{loc}^\infty([0, T);\ W^{2,\infty}(\Omega)) \cap W_{loc}^{1,\infty}([0, T);\ L^\infty(\Omega))$ such that $u^{\varepsilon(k),\delta(k),\sigma(k)} \to u$ weakly star in $L_{loc}^\infty([0, T);\ W^{2,\infty}(\Omega))$. The remaining assertions of Theorem 2 are obtained in the same way as in the proof of Theorem 1. Q.E.D.

J. I. Diaz [1] has studied a simpler equation than that considered here and has established the existence and the asymptotic behavior of solutions. We have independently obtained a similar result on the asymptotic behavior of solutions to (1.2). Our result as well as regularizing effects will be discussed in the forthcoming paper.

3. Proof of Lemma 1 and 2

PROOF OF LEMMA 1. By Proposition 1, $u^{\varepsilon,\delta,\sigma} \in W_{loc}^{2,p}(Q_T)$ and $v^{\varepsilon,\sigma} \in W_{loc}^{2,p}(Q_T)$ for all $p \in (1, \infty)$. The standard comparison theorem and (2.6) thus yield

$$w^0 \le u^{\varepsilon,\delta,\sigma} \le v^{\varepsilon,\sigma} \quad \text{in} \quad Q_T. \tag{3.1}$$

Hence (3.1) and (2.12) yield (2.14).

Since $\partial\Omega$ is smooth, the uniform exterior sphere condition holds, i.e. there exists a positive number ρ such that for each $x_0 \in \partial\Omega$ we can choose $\hat{x} \in \boldsymbol{R}^n \setminus \bar{\Omega}$ satisfying $\{z \in \boldsymbol{R}^n;\ |z-\hat{x}| \le \rho\} \cap \bar{\Omega} = \{x_0\}$. Let μ, $\lambda > 0$ be numbers to be chosen later and consider the barrier:

$$w(x) = \lambda\{\exp(-\mu\rho^2) - \exp(-\mu|x-\hat{x}|^2)\}.$$

We can take λ, $\mu > 0$ so large that the inequality

$$L^\varepsilon w(x) \ge \|f\|_{L^\infty(Q_T)}$$

holds in Q_T. By the comparison theorem and (2.6) we have

$$w^0(x,t) \le u^{\varepsilon,\delta,\sigma}(x,t) \le w(x) \quad \text{in} \quad Q_T.$$

Thus we find that there exists a constant C, independent of x_0, such that

$$|Du^{\varepsilon,\delta,\sigma}(x_0,t)| \le C. \tag{3.2}$$

Similary, using the function

$$\bar{w}(t) = \lambda\{\exp(-\mu^2) - \exp(-|t+\mu|^2)\},$$

where, λ, $\mu > 0$ are large numbers, we obtain

$$w^0(x,t) \le u^{\varepsilon,\delta,\sigma}(x,t) \le \bar{w}(t) \quad \text{in} \quad Q_T.$$

We thus find that there exists a constant C, independent of $x \in \Omega$, such that

$$|u_t^{\varepsilon,\delta,\sigma}(x,0)| \le C. \tag{3.3}$$

We next prove estimates (2.15) and (2.16) for solutions of (2.10). The proof is similar to that of Lemma 4.2 in [3]. For each $T' \in (0,T)$ we choose a non-negative nonincreasing function $\xi \in C^\infty(\boldsymbol{R})$ such that

$$\xi(t) = 1 \quad \text{for } t \in [0,T'] \quad \text{and } \operatorname{supp}\xi \subset (-\infty, T). \tag{3.4}$$

For simplicity we suppress the superscripts ε, δ and σ of $u^{\varepsilon,\delta,\sigma}$ in the following calculations. We set

$$V = \xi^4(|Du|^2 + u_t^2) + \mu\xi^2(u+\mu)^2,$$

where μ is a positive number to be chosen later. Here and in the sequel we regard u_t and Du as one-sided derivatives on $\partial_p Q_T$. Let (x_0, t_0) be a point in $\overline{Q_T}$ such that $V(x_0,t_0) = \sup\{V(x,t);\ (x,t) \in Q_T\}$. By (3.2), (3.3) and (2.14) we can assume that $(x_0, t_0) \in Q_T$; then the maximum principle applied to V gives

$$\begin{aligned} 0 &\le (L^\varepsilon - c)V \\ &\le -16\varepsilon\xi^3\xi_t(u_k u_{kt} + u_t u_{tt}) - 4\varepsilon\xi^3\xi_{tt}(|Du|^2 + u^2) \\ &\quad - 8\varepsilon\mu\xi\xi_t(u+\mu)u_t - 2\varepsilon\mu\xi\xi_{tt}(u+\mu)^2 - 2\varepsilon\mu\xi^2 u_t^2 \\ &\quad - 2\varepsilon\xi^4(|Du_t|^2 + u_{tt}^2) - 2\xi^4\theta(|D^2u|^2 + |Du_t|^2) \\ &\quad - 2\theta\xi^2\mu|Du|^2 + 2\xi^4(u_k L^\varepsilon u_k + u_t L^\varepsilon u_t) \\ &\qquad + 2\xi^2\mu(u+\mu)L^\varepsilon(u+\mu) - 2cV, \quad \text{at} \quad (x_0, t_0). \end{aligned}$$

Here we use the monotonicity of ξ. Since $V_t(x_0, t_0) = 0$, we have

$$\xi^2(u_k u_{kt} + u_t u_{tt}) \ge -\mu(u+\mu)u_t \quad \text{at} \quad (x_0, t_0). \tag{3.5}$$

Differentiating (2.10), we have

$$u_k L^\varepsilon u_k + u_t L^\varepsilon u_t = u_k \bar{D}_k^2 u + u_t \bar{D}_t^2 u - u_k(\beta+\gamma)_k - u_t(\beta+\gamma)_t + u_k f_k + u_t f_t, \tag{3.6}$$

where $\bar{D}_k^2 u = \sum_{|\alpha|\le 2} \sigma_k^\alpha D^\alpha u$, σ_k^α bounded and $\bar{D}_t^2 u = \sum_{|\alpha|\le 2} \sigma_t^\alpha D^\alpha u$, σ_t^α bounded. Let μ be a large number depending only on known qufintities. Then using (3.5) and (3.6) we have

$$\begin{aligned} 2cV &\le C + \beta'\{-2\xi^4 d(|Du|^2 + u_t^2) + C\xi^4(|Du|^2 + u_t^2) + C\} \\ &\quad + \gamma'\{-2\xi^4(|Du|^2 + u_t^2) + C\} \quad \text{at} \quad (x_0, t_0). \end{aligned}$$

By virtue of Remark 1 we have $V(x_0, t_0) \le C$. Hence there exists a positive constant C independent of ε, δ, $\sigma \in (0, 1)$ such that

$$\|\xi^2 Du\|_{L^\infty(Q_T)} + \|\xi^2 u_t\|_{L^\infty(Q_T)} \le C. \tag{3.7}$$

This yields (2.15) and (2.16).

Next we will prove (2.17) in a similar way to the proof of the Theorem in [8]. For any open set U such that $\bar{U} \subset \Omega$, we choose a function $\bar{\zeta} \in C_0^\infty(\Omega)$ satisfying $\bar{\zeta} = 1$ in U and $\bar{\zeta} \ge 0$ in Ω. We will derive a bound of $M = M(U) = \sup\{\bar{\zeta}^4\xi^4(|D^2u| + |\varepsilon u_{tt}|)\}$, where ξ is a nonnegative nonincreasing smooth function satisfying (3.4). Putting $\zeta = \bar{\zeta}\xi$, we set

$$V = \zeta^8(|D^2u|^2 + \varepsilon^2 u_{tt}^2) + \mu M\zeta^4(\beta+\gamma) + \mu\zeta^4(|Du|^2 + u_t^2).$$

Let (x_0, t_0) be a point at which V attains its maximum in $\bar{Q}_T$. First assume $(x_0, t_0) \in \partial Q_T$; then we can suppose that $x_0 \in \operatorname{supp} \bar{\zeta}$ and $t_0 = 0$ by the choice of ζ. On the other hand (2.6), (2.8) and (2.13) yield

$$\partial v^\varepsilon / \partial t(x, 0) \le g(x, 0) \qquad \text{for} \quad x \in \Omega.$$

Proposition 2 and the standard imbedding theorem yield that $\partial v^{\varepsilon,\sigma}/\partial t(x, 0)$ converges to $\partial v^{\varepsilon}/\partial t(x, 0)$ uniformly on $\operatorname{supp}\zeta$, as $\sigma\to 0$. Hence for any $\varepsilon\in(0, 1)$ there exists a function $\delta(\varepsilon, \sigma)$ such that $\delta(\varepsilon, \sigma)\to 0$, as $\sigma\to 0$, and

$$\partial v^{\varepsilon,\sigma}/\partial t(x, 0) \le g(x, 0) + \delta(\varepsilon, \sigma)$$

for any $x\in\operatorname{supp}\zeta$. The above inequality and (3.1) imply

$$\partial u^{\varepsilon,\delta,\sigma}/\partial t(x, 0) \le g(x, 0) + \delta(\varepsilon, \sigma) \quad (\delta = \delta(\varepsilon, \sigma)). \tag{3.8}$$

Note that $\delta(\varepsilon, \sigma)$ depends on $\operatorname{supp}\zeta$. This is the reason why we can only obtain the estimate (2.17) instead of (2.25). For notational simplicity we write δ instead of $\delta(\varepsilon, \sigma)$. By (3.8) and the boundary condition in (2.10), we have

$$\begin{aligned} (3.9)\qquad & -\varepsilon u_{tt} + u_t = f,\ \partial^2 u/\partial x_i\partial x_j = \partial u/\partial x_i = u = 0 \quad \text{for} \quad 1\le i, j\le n,\\ & \zeta^4(\beta+\gamma) = 0, \quad \text{on} \quad \operatorname{supp}\zeta\times\{0\}. \end{aligned}$$

Therefore in the case that $(x_0, t_0)\in\partial Q_T$, we see that there exists a constant C such that $M\le C$.

In the case that $(x_0, t_0)\in Q_T$, we apply the maximum principle to V. At (x_0, t_0) we have

$$0\le -\varepsilon V_{tt} - a_{k\ell}V_{k\ell} + V_t + b_kV_k.$$

A simple calculation using (2.2) yields

$$\begin{aligned} (3.10)\qquad 0\le & -2\zeta^8\{\varepsilon(|D^2u_t|^2+\varepsilon^2u_{ttt}^2) + \theta(|D^3u|^2+\varepsilon^2|Du_{tt}|^2)\}\\ & + 2\zeta^8\{(u_{ij}L^\varepsilon u_{ij}+\varepsilon^2u_{tt}L^\varepsilon u_{tt}) - c(|D^2u|^2+\varepsilon^2u_{tt}^2)\}\\ & - 2\mu\zeta^4\{\varepsilon(|Du_t|^2+u_{tt}^2) + \theta(|D^2u|^2+|Du_t|^2)\}\\ & + 2\mu\zeta^4\{(u_kL^\varepsilon u_k+u_tL^\varepsilon u_t) - c(|Du|^2+u_t^2)\}\\ & + \mu M\zeta^4\{\beta'(L^\varepsilon-c)(u_t+du-g) + \gamma'(L^\varepsilon-c)(u-h)\}\\ & - \mu M\zeta^4[\varepsilon\{\beta''(u_t+du-g)_t^2 + \gamma''(u-h)_t^2\}\\ & \quad + \theta\{\beta''|D(u_t+du-g)|^2 + \gamma''|D(u-h)|^2\}]\\ & - 8\zeta^7\{\varepsilon\zeta_{tt}(|D^2u|^2+\varepsilon^2u_{tt}^2) + \varepsilon\zeta_t(|D^2u|^2+\varepsilon^2u_{tt}^2)_t\\ & \quad + a_{k\ell}\zeta_{k\ell}(|D^2u|^2+\varepsilon^2u_{tt}^2) + \varepsilon\zeta_t(|D^2u|^2+\varepsilon^2u_{tt}^2)_\ell\\ & \quad + b_k\zeta_k(|D^2u|^2+\varepsilon^2u_{tt}^2)\}\\ & - 4\mu M\zeta^3\{\varepsilon\zeta_{tt}(\beta+\gamma) + \varepsilon\zeta_t(\beta+\gamma)_t + a_{k\ell}\zeta_{k\ell}(\beta+\gamma)\\ & \quad + a_{k\ell}\zeta_k(\beta+\gamma)_\ell + b_k\zeta_k(\beta+\gamma)\}\\ & - 4\mu\zeta^3\{\varepsilon\zeta_{tt}(|D_u|^2+u_t^2) + \varepsilon\zeta_t(|Du|^2+u_t^2)_t\\ & \quad + a_{k\ell}\zeta_{k\ell}(|Du|^2+u_t^2) + a_{k\ell}\zeta_k(|Du|^2+u_t^2)_\ell - b_k\zeta_k(|Du|^2+u_t^2)\}. \end{aligned}$$

Using Young's inequality and Remark 1, and letting μ sufficiently large, we have

$$
\begin{aligned}
(3.11)\qquad & \zeta^8\{\varepsilon(|D^2u_t|^2+\varepsilon^2u_{ttt}^2) + \theta(|D^3u|^2+\varepsilon^2|Du_{tt}|^2)\} \\
& + \mu\zeta^4\{\varepsilon(|Du_t|^2+u_{tt}^2) + \theta(|D^2u|^2+|Du_t|^2)\} \\
& + 2\lambda\{\zeta^8(|D^2u|^2+\varepsilon^2u_{tt}^2) + \mu\zeta^4(|Du|^2+u_t^2)\} \\
& \quad \le 2\zeta^8(u_{ij}L^\varepsilon u_{ij} + \varepsilon^2u_{tt}L^\varepsilon u_{tt}) \\
& \qquad + 2\mu\zeta^4(u_kL^\varepsilon u_k+u_tL^\varepsilon u_t) \\
& \qquad + \mu M\zeta^4\{\beta'(L^\varepsilon-c)(u_t+du-g) + \gamma'(L^\varepsilon-c)(u-h)\} \\
& \qquad - \mu M\zeta^4[\varepsilon\{\beta''(u_t+du-g)_t^2 + \gamma''(u-h)_t^2\} \\
& \qquad\quad + \theta\{\beta''|D(u_t+du-g)|^2 + \gamma''|D(u-h)|^2\}] \\
& \qquad - 4\mu M\zeta^3\{\varepsilon\zeta_{tt}(\beta+\gamma) + \varepsilon\zeta_t(\beta+\gamma)_t + a_{k\ell}\zeta_{k\ell}(\beta+\gamma) \\
& \qquad\quad + a_{k\ell}\zeta_k(\beta+\gamma)_\ell + b_k\zeta_k(\beta+\gamma)\}.
\end{aligned}
$$

Since $V_t/\zeta(x_0, t_0) = V_k/\zeta(x_0, t_0) = 0$, we have

$$
\begin{aligned}
(3.12)\qquad \mu M\zeta^3(\beta+\gamma)_t = & - \zeta^7(|D^2u|^2+\varepsilon^2u_{tt}^2)_t \\
& - 8\zeta^6\zeta_t(|D^2u|^2+\varepsilon^2u_{tt}^2) - 4\mu M\zeta^2\zeta_t(\beta+\gamma) \\
& - \mu\zeta^3(|Du|^2+u_t^2)_t - 4\mu\zeta^2\zeta_t(|Du|^2+u_t^2), \\
\mu M\zeta^3(\beta+\gamma)_k = & - \zeta^7(|D^2u|^2+\varepsilon^2u_{tt}^2)_k \\
& - 8\zeta^6\zeta_k(|D^2u|^2+\varepsilon^2u_{tt}^2) - 4\mu M\zeta^2\zeta_k(\beta+\gamma) \\
& - \mu\zeta^3(|Du|^2+u_t^2)_k - 4\mu\zeta^2\zeta_k(|Du|^2+u_t^2).
\end{aligned}
$$

By (3.12) and the convexity of β and γ, the last term of the right hand side of (3.11) is estimated as

$$
\begin{aligned}
& - 4\mu M\zeta^3\{\varepsilon\zeta_{tt}(\beta+\gamma) + \varepsilon\zeta_t(\beta+\gamma)_t + a_{k\ell}\zeta_{k\ell}(\beta+\gamma) \\
& \quad + a_{k\ell}\zeta_k(\beta+\gamma)_\ell + b_k\zeta_k(\beta+\gamma)\} \\
& \le \beta'\{C\mu\zeta^3M|(u_t+du-g)|\} + \gamma'\{C\mu\zeta^3M|(u-h)|\} \\
& \quad - 4\varepsilon\zeta_t\{- \zeta^7(|D^2u|^2+\varepsilon^2u_{tt}^2)_t - 8\zeta^6\zeta_t(|D^2u|^2+\varepsilon^2u_{tt}^2) \\
& \qquad - 4\mu M\zeta^2\zeta_t(\beta+\gamma) - \mu\zeta^3(|Du|^2+u_t^2)_t \\
& \qquad - 4\mu\zeta^2\zeta_t(|Du|^2+u_t^2)\} \\
& - 4a_{k\ell}\zeta_k\{- \zeta^7(|D^2u|^2+\varepsilon^2u_{tt}^2)_\ell - 8\zeta^6\zeta_\ell(|D^2u|^2+\varepsilon^2u_{tt}^2) \\
& \qquad - 4\mu M\zeta^2\zeta_\ell(\beta+\gamma) - \mu\zeta^3(|Du|^2+u_t^2)_\ell \\
& \qquad - 4\mu\zeta^2\zeta_\ell(|Du|^2+u_t^2)\}.
\end{aligned}
$$

Applying Young's inequality to the right hand side of above yields

$$
\begin{aligned}
(3.13)\qquad & -4\mu M\zeta^3\{\varepsilon\zeta_{tt}(\beta+\gamma)+\varepsilon\zeta_t(\beta+\gamma)_t+a_{k\ell}\zeta_{k\ell}(\beta+\gamma)\\
&\quad +a_{k\ell}\zeta_k(\beta+\gamma)_\ell+b_k\zeta_k(\beta+\gamma)\}\\
&\le \beta'\{C\mu\zeta^3M|(u_t+du-g)|\}+\gamma'\{C\mu\zeta^3M|(u-h)|\}\\
&\quad +\zeta^8\{\varepsilon(|D^2u_t|^2+\varepsilon^2u_{ttt}^2)+\theta(|D^3u|^2+\varepsilon^2|Du_{tt}|^2)\}/2\\
&\quad +C\zeta^6\{\varepsilon(|D^2u|^2+\varepsilon^2u_{tt}^2)+(|D^2u|^2+\varepsilon^2u_{tt}^2)\}\\
&\quad +C\mu M\zeta^2\{\beta'|(u_t+du-g)|+\gamma'|(u-h)|\}\\
&\quad +\mu\zeta^4\{\varepsilon(|Du_t|^2+|u_{tt}|^2)+\theta(|D^2u|^2+|Du_t|^2)\}/2\\
&\quad +C\mu\zeta^2(|Du|^2+u_t^2).
\end{aligned}
$$

In the fourth term of the right hand side of (3.13) we have used $\beta(x)\le\beta'|x|$ and $\gamma(x)\le\gamma'|x|$ which follow from the convexity of β and γ. On the other hand, differentiating (2.10) gives

$$
\begin{aligned}
(3.14)\qquad L^\varepsilon u_{ij} &= \textstyle\sum_{|\alpha|\le 3} e^\alpha(i,j)D^\alpha u-(\beta+\gamma)_{ij}+f_{gj},\\
L^\varepsilon u_{tt} &= \textstyle\sum_{|\tau|\le 1,|\alpha|\le 2} e^{\tau,\alpha}(t)D^\alpha\partial^\tau u/\partial t^\tau-(\beta+\tau)_{tt}+f_{tt},\\
L^\varepsilon u_t &= \textstyle\sum_{|\alpha|\le 2} e^\alpha(t)D^\alpha u-(\beta+\gamma)_t+f_t,\\
L^\varepsilon u_k &= \textstyle\sum_{|\alpha|\le 2} e^\alpha(k)D^\alpha u-(\beta+\gamma)_k+f_k,
\end{aligned}
$$

where $e^\alpha(i,j)$, $e^{\tau,\alpha}(t)$, $e^\alpha(t)$ and $e^\alpha(k)$ are bounded functions. Using (3.7), (3.14) and Young's inequality, we estimate the first two terms of the right hand side of (3.11) as

$$
\begin{aligned}
(3.15)\qquad & 2\zeta^8(u_{ij}L^\varepsilon u_{ij}+\varepsilon^2u_{tt}L^\varepsilon u_{tt})+2\mu\zeta^4(u_kL^\varepsilon u_k+u_tL^\varepsilon u_t)\\
&\le \zeta^8(\theta|D^3u|^2+\varepsilon|D^2u_t|^2)/2+\mu\zeta^4\theta|D^2u|^2/2\\
&\quad +C\zeta^4\varepsilon u_{tt}^2+C\zeta^4|D^2u|^2+C\zeta^4|Du_t|^2+C\\
&\quad -2\zeta^8u_{ij}(\beta+\gamma)_{ij}-2\varepsilon^2\zeta^8u_{tt}(\beta+\gamma)_{tt}\\
&\quad -2\mu\zeta^4u_k(\beta+\gamma)_k-2\mu\zeta^4u_t(\beta+\gamma)_t,
\end{aligned}
$$

where C is a constant independent of μ. Substituting (3.13) and (3.15) into (3.11) yields

$$
\begin{aligned}
(3.16)\qquad & 2\lambda\{\zeta^8(|D^2u|^2+\varepsilon^2u_{tt}^2)+\mu\zeta^4(|Du|^2+u_t^2)\}\\
&+\zeta^4\varepsilon(M\mu-C\varepsilon|u_{tt}|)\{\beta''(u_t+du-g)_t^2+\gamma''(u-h_t^2)\}\\
&+\zeta^4\theta(M\mu-C|D^2u|)\{\beta''|D(u_t+du-g)|^2+\gamma''|D(u-h)|^2\}\\
&\le C(1+\mu)
\end{aligned}
$$

$$
\begin{aligned}
&+ \beta'\{- 2\zeta^8 u_{ij}(u_t + du - g)_{ij} - 2\varepsilon^2\zeta^8 u_{tt}(u_t + du - g)_{tt} \\
&\qquad - 2\mu\zeta^4 u_k(u_t + du - g)_k - 2\mu\zeta^4 u_t(u_t + du - g)_t \\
&\qquad + CM\mu + M\zeta^4(L^\varepsilon - c)(u_t + du - g)\} \\
&+ \gamma'\{ - 2\zeta^8 u_{ij}(u - h)_{ij} - 2\varepsilon^2\zeta^8 u_{tt}(u - h)_{tt} \\
&\qquad - 2\mu\zeta^4 u_k(u - h)_k - 2\mu\zeta^4 u_t(u - h)_t \\
&\qquad + CM\mu + \mu M\zeta^4(L^\varepsilon - c)(u - h)\}.
\end{aligned}
$$

We can assume that $(M\mu - C\varepsilon|u_{tt}|)$ and $(M\mu - C|D^2u|)$ are nonnegative. Hence the second and the third terms of the left hand side of (3.16) are nonnegative.

If the third term of the right hand side of (3.16) is positive, then we obtain

$$\zeta^8(|D^2u|^2 + \varepsilon^2 u_{tt}^2)(x_0, t_0) \le C\mu(M+1).$$

Therefore,

$$
\begin{aligned}
M^2 &\le 2\sup\{V(x, t); (x, t) \in Q_T\} = 2V(x_0, t_0) \\
&\le 2C\mu(M+1) + 2\mu M(C + CM)^{1/2} + C.
\end{aligned}
$$

Thus we have

$$M \le C(\mu). \tag{3.17}$$

We can therefore assume that the third term on the right of (3.16) is nonpositive.

If the second term of the right hand side of (3.16) is negative, then we have (3.17) similarly. Hence we can assume that

$$
\begin{aligned}
&2\zeta^8 u_{ij}(u_t + du - g)_{ij} + 2\varepsilon^2\zeta^8 u_{tt}(u_t + du - g)_{tt} \\
&\quad + 2\mu\zeta^4 u_k(u_t + du - g)_k + 2\mu\zeta^4 u_t(u_t + du - g)_t \\
&\quad \le CM\mu + \mu M\zeta^4 L^\varepsilon(u_t + du - g).
\end{aligned} \tag{3.18}
$$

Since $V_t(x_0, t_0) = 0$, we have

$$
\begin{aligned}
&2\zeta^8(u_{ij}u_{ijt} + \varepsilon^2 u_{tt}u_{ttt}) + 2\mu\zeta^4(u_i u_{it} + u_t u_{tt}) \\
&\quad + 2\mu M\zeta^4(\beta + \gamma)_t \ge 0.
\end{aligned}
$$

Inserting the above inequality into (3.18) we have

$$\lambda\zeta^8\{(|D^2u|^2 + \varepsilon^2 u_{tt}^2) + \mu\zeta^4(|D|^2 + u_t^2)\} \le C\mu(M+1),$$

where λ is a constant in Remark 1. Hence similarly we obtain (3.17). Since μ is a large constant depending only on known quantities, we have proved the estimates (2.17). Q.E.D.

PROOF OF LEMMA 2. For any $x_0 \in \partial\Omega$, we may assume that $x_0=0$ and $\Omega \cap B_1 = B_1^+$ by a smooth change of variables. Here we use the notation:

$$B_r = \{x \in \boldsymbol{R}^n;\ |x| < r\},\quad B_r^+ = \{x \in B_r;\ x_n > 0\}$$
$$\text{and}\quad S_r = \{x \in B_r;\ x_n = 0\},\quad \text{for}\quad r > 0.$$

We may further assume that the coefficients of L satisfy

$$a_{in}(x, t) = 0 \quad \text{for} \quad x \in S_{1/2} \quad \text{and} \quad 1 \le i \le n-1. \tag{3.19}$$

Indeed, choose a smooth domain such that $\bar{B}_{1/2}^+ \subset \bar{\Omega}_0 \subset \bar{B}_1^+$ and functions $\phi_k \in C^2([0, T];\ C^{3,\alpha}(\partial\Omega_0))$ $(1 \le k \le n-1)$ satisfying $\phi_k = -a_{kn}/a_{nn}$ in $S_{1/2} \times [0, T]$. Let $T^k(x, t)\,(1 \le k \le n-1)$ be the unique solution of the problem:

$$-\triangle T^k + T^k = 0 \quad \text{in} \quad \Omega_0 \times [0, T];\ \partial T^k/\partial\nu = \phi_k \quad \text{on} \quad \partial\Omega_0,$$

and set

$$Y_k(x, t) = \begin{cases} x_k + T^k(x, t) - T^k(x', 0, t) & x \in B_{1/2}^+,\ 1 \le k \le n-1 \\ x_n & x \in B_{1/2}^+,\ k = n. \end{cases}$$

By a standard theory, we see that $Y(x, t) = (Y_1(x, t), \dots, Y_n(x, t)) \in C^2([0, T];$ $C^{4,\alpha}(\bar{B}_{1/2}^+))$. It is easily checked that $a_{in}Y_{k,i} = 0$ in $S_{1/2} \times [0, T]$, $1 \le k \le n-1$. Making the change of variables: $x \to Y$, we arrive at the situation (3.19).

To prove Lemma 2 we must choose $\delta(\varepsilon, \sigma)$ independently of supp $\bar{\zeta}$ (see (3.8)). In view of the argument used to derive (3.8) we need the following fact:

PROPOSITION 3. *As* $\sigma \to 0$, $\{\partial v^{\varepsilon,\sigma}/\partial t(x, 0);\ \sigma \in (0, 1)\}$ *converges to* $\partial v^{\varepsilon}/\partial t(x, 0)$ *uniformly in* Ω.

The above proposition will be proved in Section 4. From compatibility condition (2.5), (3.19) and Proposition 3, we have

$$-a_{nn}u_{nn} + b_n u_n = f \qquad \text{on} \quad S_{1/2} \times [0, T]. \tag{3.20}$$

We let $\bar{u}_{nn} = u_{nn} - b_n u_n/a_{nn} + f/a_{nn}$ and let $\bar{u}_{ij} = u_{ij}$ except for $i = j = n$.

Choose a nonnegative function $\bar{\zeta} \in C_0^\infty(B_1)$ such that $\bar{\zeta}(0) = 1$ and $\bar{\zeta}_n = 0$ on S_1. Let ξ be a smooth nonnegative function satisfying (3.4) and define $\zeta = \bar{\zeta}\xi$. Set

$$V = \zeta^8(\bar{u}_{ij}^2 + \varepsilon^2 u_{tt}^2) + \zeta\mu^4 M(\beta + \gamma) + \mu\zeta^4(|Du|^2 + u_t^2).$$

In view of (3.20) we have

$$\begin{aligned} V_n &= \zeta^8(\bar{u}_{ij}^2)_n + \mu\zeta(|^4 Du|^2 + u_t^2)_n \\ &= 2\zeta^8 \textstyle\sum_{i=1}^{n-1} u_{in}\{(b_n u_n - f)/a_{nn}\}_i + 2\mu\zeta^4 u_n\{(b_n - f)/a_{nn}\} \\ &\ge -CV \qquad \text{on} \quad S_1 \times [0, T]. \end{aligned} \tag{3.21}$$

If $(\tilde{x}, \tilde{t})$ is a point attaining $\sup\{\exp(Ax_n)V(x, t); (x, t) \in Q_T\}$, then (3.21) implies that $\tilde{x} \in \Omega$, for some $A>0$. Therefore we apply the maximum principle to $\exp(Ax_n)V$ to get

$$0 \leq \{-A^2 a_{nn} V - 2\textstyle\sum_{i=1}^{n-1} a_{in} V_i + (L^\varepsilon - c)V\} \exp(A\tilde{x}_n).$$

Since $V_i = 0$ at $(\tilde{x}, \tilde{t})$ for $1 \leq i \leq n-1$, we have

$$0 \leq (L^\varepsilon - c)V + CV. \tag{3.22}$$

The first term of the right hand side of (3.22) is calculated in the same way as in (3.10)–(3.16). As in the proof of (3.17), (3.16) implies

$$\lambda\zeta^8(|D^2u|^2 + \varepsilon^2 u_{tt}^2) \leq C + C\mu + C(M^2 + \mu M).$$

In view of Remark 1, this inequality implies (3.17). Q.E.D.

4. Appendix

PROOF OF PROPOSITION 1. For small $\alpha>0$ there are unique functions $u = u^{\varepsilon,\delta,\sigma,\alpha}$ and $v = v^{\varepsilon,\sigma,\alpha}$ solving

$$L^\varepsilon u + \beta_\delta(u_t + du - g) + \gamma_\sigma(u - h) = f \quad \text{in} \quad Q_T^\alpha$$
$$u = w^\varepsilon \quad \text{on} \quad \partial Q_T^\alpha,$$
$$L^\varepsilon v + \gamma_\sigma(v - h) = f \quad \text{in} \quad Q_T^\alpha$$
$$v = w^\varepsilon \quad \text{on} \quad \partial Q_T^\alpha,$$

respectively. For notational simplicity we write u^α and v^α instead of $u^{\varepsilon,\delta,\sigma,\alpha}$ and $v^{\varepsilon,\sigma,\alpha}$, respectively.

The comparison theorem yields that

$$w^0 \leq w^\varepsilon \leq u^\alpha \leq v^\alpha \quad \text{in} \quad Q_T^\alpha, \tag{4.1}$$

We put

$$V^\alpha = \begin{cases} v^\alpha & \text{in} \quad \bar{Q}_T^\alpha \\ w^\varepsilon & \text{in} \quad \bar{Q}_T \backslash \bar{Q}_T^\alpha, \end{cases}$$

$$U^\alpha = \begin{cases} u^\alpha & \text{in} \quad \bar{Q}_T^\alpha \\ w^\varepsilon & \text{in} \quad \bar{Q}_T \backslash \bar{Q}_T^\alpha. \end{cases}$$

Let (x_0, t_0) be a point in $\bar{Q}_T$ such that $V^\alpha(x_0, t_0) - \mu t_0 = \sup\{V^\alpha(x, t) - \mu t; (x, t) \in Q_T\}$, where μ is a positive constant to be chosen later. If we suppose that $(x_0, t_0) \in Q_T^\alpha$, then we have

$$0 \le L^\varepsilon V^\alpha - cV^\alpha - \mu \le f - cV^\alpha - \mu .$$

Since we can assume that $\mu > \|f\|_{L^\infty(Q_T)}\|$ and $V^\alpha > 0$, the above inequality yields that $(x_0, t_0) \in \bar{Q}_T \backslash \bar{Q}_T^\alpha$. Therefore we have

(4.2) $$v^\alpha \le C \quad \text{in} \quad Q_T^\alpha,$$

where C is a constant independent of ε, δ, σ and α. Thus there exists a constant C, independent of α, such that

$$|\gamma(v-h)| \le C \quad \text{in} \quad Q_T^\alpha .$$

Hence by the standard diagonal argument we can show the existence and its regularity of solutions of (2.11) in Proposition 1. Also we have (2.12) by (4.1) and (4.2).

Since $\partial\Omega$ is smooth, there exists a positive number ρ such that for each $(x_0, t_0) \in \partial Q_T^\alpha$ we can choose $(\tilde{x}, \tilde{t}) \in \boldsymbol{R}^{n+1} \backslash \bar{Q}_T^\alpha$ satisfying $\{(z, t) \in \boldsymbol{R}^{n+1};\ |(z, t) - (\tilde{x}, \tilde{t})| \le \rho\} \cap \bar{Q}_T^\alpha = \{(x_0, t_0)\}$. Let λ, $\mu > 0$ be numbers to be chosen later and consider the barrier:

$$\begin{aligned} w(x, t) = \lambda\{&\exp(-\mu\rho^2) - \exp(-\mu|t-\tilde{t}|^2 - \mu|x-\tilde{x}|^2)\} \\ &+ w^\varepsilon(x, t). \end{aligned}$$

We can take λ, $\mu > 0$ so large that the inequality

$$L^\varepsilon w \ge |f| + |L^\varepsilon w^\varepsilon|$$

holds in Q_T. By the comparison theorem and (4.1), we have

$$w^\varepsilon(x, t) \le u^{\varepsilon,\delta,\sigma,\alpha} \le w(x, t) \quad \text{in} \quad Q_T^\alpha .$$

In the same way as in (3.2) and (3.3) we find that there exists a constant $C > 0$ independent of (x_0, t_0) and α, such that

(4.3) $$\begin{aligned} &|Du^{\varepsilon,\delta,\sigma,\alpha}(x_0, t_0)| \le C, \\ &|\partial u^{\varepsilon,\delta,\sigma,\alpha}/\partial t(x_0, t_0)| \le C . \end{aligned}$$

Set $V = |Du^\alpha|^2 + (u_t^\alpha)^2 + \mu(u^\alpha - \mu)^2$, where μ is a positive number to be chosen later. Similarly to the argument used in deducing (3.7), we have

$$|Du^\alpha| + |u_t^\alpha| \le C \quad \text{in} \quad Q_T^\alpha .$$

Therefore there exists a constant C, independent of α, satisfying

$$|\beta(u_t^\alpha + du^\alpha - f) + \gamma(u^\alpha - h)| \le C .$$

Thus we can show the existence and regularity of solutions of (2.10) in Proposition 1. Q.E.D.

PROOF OF PROPOSITION 2. In order to prove Proposition 2, we establish the following a priori estimates of solutions to the problem (2.11). For each $\varepsilon \in (0, 1)$ there exists a constant $\hat{\alpha}>0$ such that for all $\alpha \in (0, \hat{\alpha})$

$$|\gamma(v^{\varepsilon,\sigma,\alpha} - h)| \leq C, \tag{4.4}$$

in $Q_T^{\hat{\alpha}}$, where C is a constant independent of σ and α.

Let ξ be a smooth function such that $\xi=1$ in $Q_T^{\hat{\alpha}}$ and $\xi=0$ near $\{(x, t) \in \partial Q_T^{\hat{\alpha}/2};$ $\mathrm{dis}(x, \partial\Omega)<\hat{\alpha}/2$ and $t \in [0, \hat{\alpha}/2]\} \cup \{(x, T); x \in \Omega\}$. Set $V=\xi^2\gamma$, and let (x_0, t_0) be a point of $\bar{Q}_T^{\alpha}$ satisfying $V(x_0, t_0)=\sup\{V(x, t); (x, t) \in Q_T^{\alpha}\}$. By the compatibility condition and our choice of ξ, we can assume that $(x_0, t_0) \in Q_T^{\alpha}$. The maximum principle yields

$$\begin{aligned} 0 &\leq (L^\varepsilon - c)V \\ &= \xi^2\gamma'(L^\varepsilon - c)(v-h) - \xi^2\gamma''\{\theta|D(v-h)|^2 + \varepsilon(v-h)_t^2\} \\ &\quad - 2\varepsilon\xi\xi_t\gamma_t - \varepsilon(\xi^2)_{tt} - 2a_{ij}\xi\xi_i\gamma_j - a_{ij}(\xi^2)_{ij}\gamma \\ &\quad - 2\xi\xi_t\gamma - 2\xi b_k\xi_k\gamma. \end{aligned}$$

Since $V_k/\xi(x_0, t_0) = V_t/\xi(x_0, t_0) = 0$, we have

$$\xi\gamma_t = -2\xi_t\gamma, \quad \text{and} \quad \xi_k\gamma = -2\xi_k\gamma \quad \text{at} \quad (x_0, t_0).$$

By making use of (4.5) and the convexity of γ, we have

$$\begin{aligned} 0 &\leq \gamma'\{\xi^2(L^\varepsilon - c)(v-h) + C|(v-h)|\} \\ &= \gamma'\{-\xi^2(\gamma + f - cv - L^\varepsilon h) + C|(v-h)|\}. \end{aligned}$$

Since we can assume that $V(x_0, t_0)>0$, the third assumption of (2.9) implies

$$\xi^2\gamma \leq -\xi^2(f - cv - L^\varepsilon h) + C|(v-h)|.$$

This implies (4.4). Q.E.D.

PROOF OF PROPOSITION 3. It suffices to prove that there exists a constant C, independent of σ, such that

$$\|\partial^2 v^{\varepsilon,\sigma}/\partial t^2\|_{L^\infty(\Omega\times[0,T/2])} \leq C. \tag{4.6}$$

Indeed, simple calculation yields that

$$\begin{aligned} & v_t^\varepsilon(x, 0) - v_t^{\varepsilon,\sigma}(x, 0) \\ &= \int_0^1 \{v_{tt}^{\varepsilon,\sigma}(x, \theta t) - v_{tt}^\varepsilon(x, \theta t)\}d\theta \times t \\ &\quad + v_t^\varepsilon(x, t) - v_t^{\varepsilon,\sigma}(x, t). \end{aligned}$$

If we let $t=\tau/4C$, for any small $\tau>0$, (4.6) implies

$$\left|\int_0^1 \{v_{t\varepsilon}^{\varepsilon,\sigma}(x, \theta t) - v_{tt}^{\varepsilon}(x, \theta t)\} d\theta \times t\right| < \tau/2 .$$

From Proposition 2 and the Sobolev imbedding theorem, it follows that there exists a positive constant σ_0, independent of $x \in \Omega$, such that

$$|v_t^{\varepsilon}(x, \tau/4C) - v_t^{\varepsilon,\sigma}(x, \tau/4C)| < \tau/2$$

for all $\sigma \in (0, \sigma_0)$. Therefore the above considerations imply Proposition 3. The estimates (4.6) are easily obtained by the same method as those for obtaining the estimates in Lemma 1 and 2. We only note that $\gamma_\sigma(v^{\varepsilon,\sigma}-h)=0$ on $\partial_p Q_T$. The same calculations as in the proof of Lemmas 1 and 2 imply that the same estimate as (2.25) holds for $v^{\varepsilon,\sigma}$. Q.E.D.

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References

[1] J. I. Diaz, On a fully nonlinear parabolic equation and the asymptotic behavior of its solutions, J. Math. Anal. Appl. **95** (1983), 144–168.
[2] L. C. Evans and S. Lenhart, The parabolic Bellman equation, Nonlinear Anal., Theo. Meth. Appl. **5** (1981), 756–773.
[3] H. Ishii and S. Koike, Boundary regularity and uniqueness for an elliptic equation with gradient constraint, Comm. in P. D. E. **8** (1983), 317–346.
[4] R. Jensen, Boundary regularity for variational inequalities, Indiana Univ. Math. J. **29** (1980), 495–504.
[5] D. Kinderlehrer and G. Stampacchia, Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.
[6] N. V. Krylov, Controlled Diffusion Processes, Springer-Verlag, New York-Heidelberg-Berlin, 1980.
[7] P. L. Lions, Résolution analytique des problèmes de Bellman-Dirichlet, Acta Math. **146** (1981), 151–166.
[8] M. Wiegner, The $C^{1,1}$-character of solutions of second order elliptic equations with gradient constraint, Comm. in P. D. E. **6** (1981), 361–371.
[9] K. Yosida, Functional Analysis, Springer-Verlag, Berlin-Heidelberg-New York, 1978.

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