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Extremal problems with respect to ideal boundary components of an infinite network

Dedicated to Professor Kôtaro Oikawa on his 60th birthday

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Introduction

We introduce a notion of ideal boundary components of an infinite network as a discrete analogue of that in the theory of Riemann surfaces. This notion gives a fine information on the ideal boundary of the infinite network. Given an ideal boundary component α of N and a finite set A of nodes, the extremal length $EL_p(A, \alpha)$ and the extremal width $EW_p(A, \alpha)$ of N of order p relative to A and α will be studied in Section 2 and Section 4. A discrete analogue of the continuity lemma due to Marden and Rodin [3] plays an important role in our study. It will be shown that a generalized inverse relation $[EL_p(A, \alpha)]^{1/p}[EW_p(A, \alpha)]^{1/q}=1$ $(1/p+1/q=1, p>1)$ holds in the present case.

§1. Ideal boundary components

Let X be a countable set of nodes, Y be a countable set of arcs, K be the node-arc incidence function and r be a strictly positive real function on Y. We assume that the graph $\{X, Y, K\}$ is connected, locally finite and has no self-loop. The quartet $N=\{X, Y, K, r\}$ is called an infinite network. For notation and terminology, we mainly follow [2] and [4].

For each $a\in X$ and $y\in Y$, let us put

$$Y(a)=\{y\in Y;\ K(a, y)\neq 0\},$$
$$e(y)=\{x\in X;\ K(x, y)\neq 0\},$$
$$X(a)=\bigcup\{e(y);\ y\in Y(a)\}.$$

We say that a subset A of X is connected if, for every $x, x'\in A$, there exists a path P from x to x' such that $C_X(P)\subset A$. A node $a\in A$ is called an interior node of A if $X(a)\subset A$, i.e., every neighboring node of a is contained in A. Denote by $i(A)$ the set of all interior nodes of A. We put $b(A)=A-i(A)$ and call it the boundary of A.

For two subnetworks $N'=\langle X', Y'\rangle$ and $N''=\langle X'', Y''\rangle$ of N, we write $N'\leqslant N''$ if N' is a subnetwork of N'' and $X'\subset i(X'')$. An infinite subnetwork $N^*=\langle X^*, Y^*\rangle$ of N is called an end of N if the following conditions are fulfilled:

(1.1) $b(X^*)$ is a finite connected set,

(1.2) $Y^* = \{y \in Y;\ e(y) \subset X^*\}$,

(1.3) $X - X^*$ is connected.

Denote by $ed(N)$ the set of all ends of N.

A sequence $\{N_n^*\}$ $(N_n^* = \langle X_n^*, Y_n^* \rangle)$ of ends is called a determining sequence of an ideal boundary component if the following conditions are fulfilled:

(1.4) $N_n^* \geqslant N_{n+1}^*$,

(1.5) $\bigcap_{n=1}^{\infty} X_n^* = \phi$.

We say that two determining sequences $\{N_n^*\}$ and $\{\bar{N}_n^*\}$ are equivalent if for each N_n^* there exists $\bar{N}_m^*$ such that $\bar{N}_m^* \leqslant N_n^*$ and if for each $\bar{N}_n^*$ there exists N_m^* such that $N_m^* \leqslant \bar{N}_n^*$. Each equivalence class is called an ideal boundary component of N. Denote by $ibc(N)$ the totality of ideal boundary components.

For an end $N^* = \langle X^*, Y^* \rangle$ of N and a nonempty finite subset A of X, denote by $P_{A,\infty}^*(N^*)$ the set of all $P \in P_{A,\infty}$ (the set of all paths from A to the ideal boundary ∞ of N) such that $C_X(P) - X^*$ is a finite set (possibly, the empty set). Let $\alpha \in ibc(N)$ and $\{N_n^*\}$ be its determining sequence. Then $P_{A,\infty}^*(N_{n+1}^*) \subset P_{A,\infty}^*(N_n^*)$. Let us put

(1.6) $$P_{A,\alpha} = \bigcap_{n=1}^{\infty} P_{A,\infty}^*(N_n^*)$$

and call its element a path from A to α. Clearly this definition does not depend on the choice of the determining sequence of α. We may say that $\alpha \in ibc(N)$ is an ideal boundary of an end N^* if $P_{A,\infty}^*(N^*)$ contains $P_{A,\alpha}$ for a nonempty finite set A.

Let Γ be a family of paths. The extremal length $\lambda_p(\Gamma)$ of Γ of order p $(1 < p < \infty)$ is defined by

$$\lambda_p(\Gamma)^{-1} = \inf\{H_p(W);\ W \in E_p(\Gamma)\},$$

where $H_p(w) = \sum_{y \in Y} r(y)|w(y)|^p$ and $E_p(\Gamma)$ is the set of all $W \in L^+(Y)$ such that $H_p(W) < \infty$ and

$$\sum_P r(y)W(y) = \sum_{y \in C_{Y(P)}} r(y)W(y) \geqq 1$$

for all $P \in \Gamma$. We also use notation $EL_p(A, \alpha)$ for $\lambda_p(P_{A,\alpha})$. It is called the extremal length of order p of N relative to A and α. Since $E_p(P_{A,\alpha}) \neq \phi$ for a finite set A, we always have $EL_p(A, \alpha) > 0$.

We say that a property holds for p-almost every path of Γ if it does for the members of Γ except for those belonging to a subfamily with infinite extremal length of order p.

For $u \in L(X)$ and $P \in P_\infty = \bigcup\{P_{\{x\},\infty};\ x \in X\}$, denote by $u(P)$ the limit of $u(x)$ as x tends to the ideal boundary ∞ of N along P if it exists. It is proved in [2] that $u(P)$ exists for p-almost every $P \in P_\infty$ if u is a Dirichlet function of order p, i.e., $u \in D^{(p)}(N)$

$=\{u\in L(X);\ D_p(u)<\infty\}$, where

$$D_p(u)=H_p(du) \quad \text{and} \quad du(y)=-r(y)^{-1}\textstyle\sum_{x\in X}K(x,y)u(x).$$

We write $u(\alpha)=t$ for $\alpha\in ibc(N)$ and $t\in R$ if $u(P)$ exists and is equal to t for p-almost every $P\in P_\alpha=\bigcup\{P_{\{x\},\alpha};\ x\in X\}$.

We prepare some lemmas. By [2; Theorem 2.3], we have

LEMMA 1.1. *If $W\in L^+(Y)$ and $H_p(W)<\infty$, then $\sum_P r(y)W(y)=\infty$ for p-almost every $P\in P_\infty$.*

For later use, we introduce an operation on the set of paths. Let P' be a path from a to b with $C_X(P')=\{x'_0, x'_1,\cdots, x'_n\}$ $(x'_0=a, x'_n=b)$, $C_Y(P')=\{y'_1,\cdots, y'_n\}$ and let P'' be a path from b to c with $C_X(P'')=\{x''_0, x''_1,\cdots, x''_m\}$ $(x''_0=b, x''_m=c)$, $C_Y(P'')=\{y''_1,\cdots, y''_m\}$. Put $v=\max\{k;\ x''_k\in C_X(P')\}$ and let $x''_v=x'_q$. We define two ordered set $X_0=\{x_k;\ 0\leqq k\leqq m+q-v\}$ and $Y_0=\{y_k;\ 1\leqq k\leqq m+q-v\}$ by

$$x_0=x'_0,\ x_k=x'_k \quad \text{and} \quad y_k=y'_k \quad \text{if} \quad 1\leqq k\leqq q,$$
$$x_k=x''_{k-q+v} \quad \text{and} \quad y_k=y''_{k-q+v} \quad \text{if} \quad q+1\leqq k\leqq m+q-v.$$

Let p' and p'' be the path indexes of P' and P'' respectively and define $p\in L(Y)$ by

$$p(y)=p'(y) \quad \text{if} \quad y\in Y_0\cap C_Y(P'),$$
$$p(y)=p''(y) \quad \text{if} \quad y\in Y_0\cap C_Y(P'')-C_Y(P'),$$
$$p(y)=0 \quad \text{if} \quad y\notin Y_0.$$

Then the triple $\{X_0, Y_0, p\}$ defines a path P from a to c. We call P the path generated by P' and P'' and denote it by $P'+P''$. In the case where P'' is a path from b to the ideal boundary ∞, we can define $P'+P''$ similarly.

LEMMA 1.2. *Let A_1 and A_2 be nonempty finite subsets of X and $\alpha\in ibc(N)$. Then $\lambda_p(P_{A_1,\alpha})=\infty$ if and only if $\lambda_p(P_{A_2,\alpha})=\infty$.*

PROOF. Assume that $\lambda_p(P_{A_1,\alpha})=\infty$. Then there exists $W\in L^+(Y)$ such that $H_p(W)<\infty$ and $\sum_P r(y)W(y)=\infty$ for every $P\in P_{A_1,\alpha}$ by Lemma 2.3 in [2]. Let $P\in P_{A_2,\alpha}$. If $C_X(P)\cap A_1\neq\phi$, then P contains a subpath $P'\in P_{A_1,\alpha}$, so that $\sum_P r(y)W(y)\geqq\sum_{P'}r(y)W(y)=\infty$. If $C_X(P)\cap A_1=\phi$, then there exists a path P_0 from A_1 to A_2 such that $P''=P_0+P\in P_{A_1,\alpha}$, so that

$$\textstyle\sum_P r(y)W(y)=\sum_{P''}r(y)W(y)-\sum_{P_0}r(y)W(y)=\infty,$$

since $C_Y(P_0)$ is a finite set. Therefore $\sum_P r(y)W(y)=\infty$ for every $P\in P_{A_2,\alpha}$, and hence $\lambda_p(P_{A_2,\alpha})=\infty$ by Lemma 2.3 in [2].

As a discrete analogue of the fundamental lemma due to Marden and Rodin [3], we have

LEMMA 1.3. *Assume that* $W_n \in L^+(Y)$ *and* $H_p(W_n) \to 0$ *as* $n \to \infty$. *Then there exists a subsequence* $\{W_{n_k}\}$ *of* $\{W_n\}$ *such that for p-almost every* $P \in P_\infty$

$$\lim_{k \to \infty} \sum_P r(y) W_{n_k}(y) = 0.$$

PROOF. Choose a subsequence $\{W_{n_k}\}$ such that $H_p(W_{n_k}) < 2^{-2kp}$. Set $\Gamma_k = \{P \in P_\infty; \sum_P r(y) W_{n_k}(y) > 2^{-k}\}$, $\Gamma'_k = \bigcup_{\ell=k}^{\infty} \Gamma_\ell$ and $\Gamma' = \bigcap_{k=1}^{\infty} \Gamma'_k$. Since $2^k W_{n_k} \in E_p(\Gamma_k)$ for each k, we have by Lemma 2.2 in [2]

$$\lambda_p(\Gamma')^{-1} \leqq \lambda_p(\Gamma'_k)^{-1} \leqq \sum_{\ell=k}^{\infty} \lambda_p(\Gamma_\ell)^{-1} \leqq \sum_{\ell=k}^{\infty} H_p(2^\ell W_{n_\ell}) \leqq \sum_{\ell=k}^{\infty} 2^{-\ell p} \to 0$$

as $k \to \infty$. Hence $\lambda_p(\Gamma') = \infty$. If $\limsup_{k \to \infty} \sum_P r(y) W_{n_k}(y) > 0$ for some $P \in P_\infty$, then $P \in \Gamma'_k$ for all k and therefore $P \in \Gamma'$.

In order to assure the existence of a limit function of a sequence of functions on Y or X, we need the following type of Clarkson's inequality (cf. [1], [5]):

LEMMA 1.4. *For* w, $w' \in L_p(Y; r) = \{w \in L(Y); H_p(w) < \infty\}$, *the following inequalities hold*:

(1.7) $$H_p(w+w') + H_p(w-w') \leqq 2^{p-1}[H_p(w) + H_p(w')] \text{ in case } 2 \leqq p;$$

(1.8) $$[H_p(w+w')]^{1/(p-1)} + [H_p(w-w')]^{1/(p-1)} \leqq 2[H_p(w) + H_p(w')]^{1/(p-1)} \text{ in case } 1 < p \leqq 2.$$

§2. Extremum problems related to $\alpha \in ibc(N)$

Let $\alpha \in ibc(N)$, $c \in L^+(Y)$ and A be a nonempty finite subset of X. Consider the following linear programming problems related to α:

(2.1) Find $N(P_{A,\alpha}; c) = \inf\{\sum_P c(y); P \in P_{A,\alpha}\}$;

(2.2) Find $N^*(A, \alpha; c)$

$$= \sup\{[\inf_{x \in A} u(x)] - [\sup_{P \in \Gamma_{A,\alpha;c}} u(P)]; u \in S^*\},$$

where S^* is the set of all $u \in L(X)$ satisfying $|\sum_{x \in X} K(x, y) u(x)| \leqq c(y)$ on Y and $\Gamma_{A,\alpha;c} = \{P \in P_{A,\alpha}; \sum_P c(y) < \infty\}$. We remark that $u(P)$ exists for any $u \in S^*$ and $P \in \Gamma_{A,\alpha;c}$.

We have the following duality theorem:

THEOREM 2.1. *If* $\Gamma_{A,\alpha;c} \neq \phi$, *then* $N(P_{A,\alpha}; c) = N^*(A, \alpha; c)$ *holds and problem* (2.2) *has an optimal solution.*

PROOF. Let $u \in S^*$ and $P \in \Gamma_{A,\alpha;c}$ with $C_X(P) = \{x_n; n \geqq 0\}$ $(x_0 \in A)$ and $C_Y(P) = \{y_n; n \geqq 1\}$. Then we have

$$\sum_P c(y) \geqq \sum_{k=1}^{n+1} c(y_k) \geqq \sum_{k=0}^{n} |u(x_{k+1}) - u(x_k)| \geqq u(x_0) - u(x_{n+1}).$$

Letting $n\to\infty$, we have $\sum_P c(y)\geqq u(x_0)-u(P)$ and hence

$$\sum_P c(y)\geqq \inf_{x\in A} u(x)-\sup_{P\in\Gamma_{A,\alpha;c}} u(P).$$

Thus the inequality $N(P_{A,\alpha};\, c)\geqq N^*(A,\, \alpha;\, c)$ holds.

Next we define $\hat{u}\in L(X)$ by

$$\hat{u}(x)=\inf\{\textstyle\sum_P c(y);\ P\in P_{\{x\},\alpha}\}$$

for $x\in X$. By the assumption of the theorem, $\hat{u}(x)<\infty$. To prove that $\hat{u}\in S^*$, let $\bar{y}\in Y$ with $e(\bar{y})=\{x_1,\, x_2\}$. Let $P\in P_{\{x_1\},\alpha}$ be arbitrarily given. In case $\bar{y}\in C_Y(P)$, there exists a subpath P' of P such that $P'\in P_{\{x_2\},\alpha}$. Then $\hat{u}(x_2)\leqq\sum_{P'}c(y)\leqq\sum_P c(y)+c(\bar{y})$. In case $\bar{y}\notin C_Y(P)$, let $P''\in P_{\{x_2\},\alpha}$ be the path generated by $\{\bar{y}\}$ and P. Then $\hat{u}(x_2)\leqq\sum_{P''}c(y)=\sum_P c(y)+c(\bar{y})$. Thus we have $\hat{u}(x_2)\leqq\sum_P c(y)+c(\bar{y})$ for any $P\in P_{\{x_1\},\alpha}$, and hence $\hat{u}(x_2)\leqq\hat{u}(x_1)+c(\bar{y})$. By interchanging the role of x_1 and x_2, we have $\hat{u}(x_1)\leqq\hat{u}(x_2)+c(\bar{y})$ and hence $|\sum_{x\in X}K(x,\, \bar{y})\hat{u}(x)|\leqq c(\bar{y})$.

Let $P\in\Gamma_{A,\alpha;c}$ with $C_X(P)=\{x_n;\, n\geqq 0\}$ $(x_0\in A)$ and denote by P_n the subpath of P from x_n to α. Then we have $\hat{u}(x_n)\leqq\sum_{P_n}c(y)\to 0$ as $n\to\infty$, so that $\hat{u}(P)=0$. Therefore $\sup_{P\in\Gamma_{A,\alpha;c}}\hat{u}(P)=0$ and $N(P_{A,\alpha};\, c)=\inf_{x\in A}\hat{u}(x)\leqq N^*(A,\, \alpha;\, c)$. Note that $\hat{u}$ is an optimal solution of problem (2.2). This completes the proof.

As a dual quantity of $EL_p(A,\, \alpha)=\lambda_p(P_{A,\alpha})$, let us consider the following value of an extremum problem:

(2.3) Find $d_p(A,\, \alpha)=\inf\{D_p(u);\ u=1$ on $A,\ u(\alpha)=0\}$.

Note that $d_p(A,\, \alpha)<\infty$, since A is a finite set. We have

THEOREM 2.2. $d_p(A,\, \alpha)=\lambda_p(P_{A,\alpha})^{-1}$.

PROOF. In case $\lambda_p(P_{A,\alpha})=\infty$, we have $d_p(A,\alpha)=0$, since $u=1$ is an admissible function for problem (2.3). We consider the case where $\lambda_p(P_{A,\alpha})<\infty$. To prove the inequality $\lambda_p(P_{A,\alpha})^{-1}\leqq d_p(A,\, \alpha)$, let $u\in D^{(p)}(N)$ satisfy $u=1$ on A and $u(\alpha)=0$. Put $W(y)=|du(y)|$. Then $W\in L^+(Y)$ and $H_p(W)=D_p(u)$. Set $\Gamma(\alpha)=\{P\in P_{A,\alpha};\ u(P)=0\}$. Then we see easily that $\sum_P r(y)\,W(y)\geqq 1-u(P)=1$ for all $P\in\Gamma(\alpha)$, so that $W\in E_p(\Gamma(\alpha))$. Since $\lambda_p(P_{A,\alpha}-\Gamma(\alpha))=\infty$, we have by Lemma 2.2 in [2]

$$\lambda_p(P_{A,\alpha})^{-1}=\lambda_p(\Gamma(\alpha))^{-1}\leqq H_p(W)=D_p(u).$$

Thus $\lambda_p(P_{A,\alpha})^{-1}\leqq d_p(A,\alpha)$. To prove the converse inequality, let $W\in E_p(P_{A,\alpha})$. Then $\sum_P r(y)\,W(y)<\infty$ for p-almost every $P\in P_{A,\alpha}$ by Lemma 1.1. On account of Theorem 2.1, we can find $u\in L(X)$ such that $u(x)\geqq 1$ on A, $u(\alpha)=0$ and $|\sum_{x\in X}K(x,\, y)u(x)|\leqq r(y)\,W(y)$ on Y. Define $v\in L(X)$ by $v(x)=\min(u(x),\, 1)$. Then $v(x)=1$ on A, $v(\alpha)=0$ and $|dv(y)|\leqq|du(y)|\leqq W(y)$, so that $d_p(A,\, \alpha)\leqq D_p(v)\leqq H_p(W)$. Therefore $d_p(A,\, \alpha)\leqq\lambda_p(P_{A,\alpha})^{-1}$.

As for the existence of an optimal solution of problem (2.3), we have

THEOREM 2.3. *There exists a unique optimal solution of problem* (2.3).

PROOF. Let $\{u_n\}$ be a sequence in $D^{(p)}(N)$ such that $u_n=1$ on A, $u_n(\alpha)=0$ and $D_p(u_n)\to d_p(A, \alpha)$ as $n\to\infty$. Since $(u_n+u_m)/2$ is an admissible function, we see by Clarkson's inequality that $D_p(u_n-u_m)\to 0$ as $n, m\to\infty$ (cf. [5]). Since $D^{(p)}(N)$ is a Banach space with the norm $\|u\|_p=[D_p(u)+|u(b)|^p]^{1/p}$ $(b\in X)$, there exists $\hat{u}\in D^{(p)}(N)$ such that $\|u_n-\hat{u}\|_p\to 0$ as $n\to\infty$. It follows that $\hat{u}=1$ on A and $d_p(A, \alpha) =D_p(\hat{u})$. To prove $\hat{u}(\alpha)=0$, put $W_n(y)=|du_n(y)-d\hat{u}(y)|$. Then $H_p(W_n)=D_p(u_n-\hat{u}) \to 0$ as $n\to\infty$. Set $\Gamma'(\alpha)=\{P\in P_{A,\alpha};\ \hat{u}(P)$ exists and $u_n(P)=0$ for all $n\}$. Then $\lambda_p(P_{A,\alpha} -\Gamma'(\alpha))=\infty$. By means of Lemma 1.3, we can find a subfamily $\Gamma''(\alpha)$ of $\Gamma'(\alpha)$ and a subsequence $\{W_{n_k}\}$ of $\{W_n\}$ such that $\lim_{k\to\infty}\sum_P r(y)W_{n_k}(y)=0$ for every $P\in\Gamma''(\alpha)$ and $\lambda_p(\Gamma'(\alpha)-\Gamma''(\alpha))=\infty$. Denoting by $p(y)$ the path index of P, we have the relations

$$\textstyle\sum_P r(y)p(y)du_n(y)=1 \quad \text{and} \quad \sum_P r(y)p(y)d\hat{u}(y)=1-\hat{u}(P)$$

for every $P\in\Gamma'(\alpha)$, so we see that $\hat{u}(P)=0$ for every $P\in\Gamma''(\alpha)$. Since $\lambda_p(P_{A,\alpha}-\Gamma''(\alpha)) =\infty$, we have $\hat{u}(\alpha)=0$, and hence $\hat{u}$ is an optimal solution of problem (2.3). The uniqueness of the optimal solution follows from Clarkson's inequality.

Let $\{N_n^*\}$ $(N_n^*=\langle X_n^*, Y_n^*\rangle)$ be a determining sequence of $\alpha\in ibc(N)$ such that $A\cap X_1^*=\phi$. Denote by P_{A,X_n^*} the set of all paths from A to X_n^*. Let us study the relation between $\lambda_p(P_{A,\alpha})$ and the extremal length $\lambda_p(P_{A,X_n^*})$ of order p of N relative to A and X_n^*.

We begin with

LEMMA 2.1. *Let* $c\in L^+(Y)$ *and set* $t=N(P_{A,\alpha}; c)$ *and* $t_n=N(P_{A,X_n^*}; c) =\inf\{\sum_P c(y);\ P\in P_{A,X_n^*}\}$. *Then* $t_n\leqq t_{n+1}\leqq t$ *and* $t_n\to t$ *as* $n\to\infty$.

PROOF. Since each path of P_{A,X_{n+1}^*} (resp. $P_{A,\alpha}$) contains a path of P_{A,X_n^*} (resp. P_{A,X_{n+1}^*}), we have $t_n\leqq t_{n+1}\leqq t$. Suppose that $\lim_{n\to\infty}t_n=t_0<t$. Let ε be a positive number such that $\varepsilon<t-t_0$. For each n there exists $P_n\in P_{A,X_n^*}$ such that $\sum_{P_n}c(y)<t_n +\varepsilon/4^n$. Since $\{t_n\}$ is monotone, by taking a subsequence if necessary, we may assume that $t_0-t_n<1/2^n$. Since A is a finite set, we may also assume that all elements of $\{P_n\}$ have the same node $a\in A$. Let $C_X(P_n)=\{x_i^{(n)};\ 0\leqq i\leqq q_n\}$ $(x_0^{(n)}=a,\ x_{q_n}^{(n)}\in b(X_n^*))$. For every n and k with $n>k$, let $\nu(k, n)=\max\{i;\ x_i^{(n)}\in b(X_k^*)\}$. Then $x_i^{(n)}\in X_k^*$ for all i with $\nu(k, n)\leqq i\leqq q_n$. We call $x_{\nu(k,n)}^{(n)}$ $(n>k)$ the last exit node of P_n from $X-X_k^*$. Since $b(X_1^*)$ is a finite set, we can select a subsequence $\{P_{n_i^{(1)}}\}$ of $\{P_n\}$, all elements of which have the same last exit node z_1 from $X-X_1^*$. Put $n_1^{(1)}=n_1$. Similarly we can select a subsequence $\{P_{n_i^{(2)}}\}$ of $\{P_{n_i^{(1)}}\}$, all of whose elements have the same last exit node z_2 from $X-X_2^*$. Let n_2 be the first number of $\{n_i^{(2)}\}$such that $n_i^{(2)}>n_1$. By induction we obtain for each k a subsequence $\{P_{n_i^{(k)}}\}$ of the preceding one, all of whose elements have the same last exit node z_k from $X-X_k^*$, and the number n_k. We consider the sequence of paths $\{P_{n_k}\}$ and denote it by $\{\tilde{P}_k\}$. Let k_0 be a number such as $\sum_{n=k_0}^{\infty} 1/2^n < \varepsilon/2$. We shall construct a path $P^* \in P_{\{a\},\infty}$. For each $k \geqq 2$, let P'_k be the subpath of $\tilde{P}_k$ such that P'_k is a path from z_{k-1} to z_k and let P''_k be the subpath of $\tilde{P}_k$ such that P''_k is a path from a to z_{k-1}. We define a set $\{P_k^*; k \geqq k_0\}$ of paths by $P_{k_0}^* = P''_{k_0} + P'_{k_0}$ (the path generated by P''_{k_0} and P'_{k_0}) and $P_{k+1}^* = P_k^* + P'_{k+1}$ for $k \geqq k_0$. We see that for each $k \geqq k_0$, the restriction of P_m^* to the subnetwork $N - N_k^* = \langle X - X_k^*, Y - Y_k^* \rangle$ is identical for all $m \geqq k+1$. Thus we can define an infinite path $P^* \in P_{\{a\},\infty}$ by the condition that the restriction of P^* to $N - N_k^*$ is equal to P_{k+1}^* for every $k \geqq k_0$. Then $P^* \in P_{A,\alpha}$. Since P''_k contains a path belonging to P_{A,X_{k-1}^*}, $\sum_{P''_k} c(y) \geqq t_{k-1}$, so that

$$\textstyle\sum_{P'_k} c(y) \leqq \sum_{\tilde{P}_k} c(y) - \sum_{P''_k} c(y) < t_{n_k} + \varepsilon/4^k - t_{k-1} < \varepsilon/4^k + 1/2^{k-1}.$$

We have

$$\begin{aligned} \textstyle\sum_{P_k^*} c(y) &\leqq \textstyle\sum_{P_{k_0}^*} c(y) + \sum_{i=k_0+1}^{k} \sum_{P'_i} c(y) \\ &< t_{n_{k_0}} + \varepsilon/4^{k_0} + \textstyle\sum_{i=k_0+1}^{k} (\varepsilon/4^i + 1/2^{i-1}) \\ &< t_0 + \varepsilon \end{aligned}$$

for all $k \geqq k_0 + 1$, so that $\sum_{P^*} c(y) \leqq t_0 + \varepsilon < t$. This is a contradiction. Therefore $t_n \to t$ as $n \to \infty$.

Now we have a discrete analogue of the continuity lemma due to Marden and Rodin [3]:

THEOREM 2.4. $\lim_{n\to\infty} \lambda_p(P_{A,X_n^*}) = \lambda_p(P_{A,\alpha})$.

PROOF. Since $E_p(P_{A,\alpha}) \supset E_p(P_{A,X_{n+1}^*}) \supset E_p(P_{A,X_n^*})$, we have $\lambda_p(P_{A,\alpha}) \geqq \lambda_p(P_{A,X_{n+1}^*}) \geqq \lambda_p(P_{A,X_n^*})$. Put $s = \lim_{n\to\infty} \lambda_p(P_{A,X_n^*})$. Then $0 < s \leqq \lambda_p(P_{A,\alpha})$. To prove the converse inequality, let $W \in E_p(P_{A,\alpha})$ and put $c(y) = r(y)\,W(y)$. Then, by Lemma 2.1, we have $t_n = N(P_{A,X_n^*}; c) \to t = N(P_{A,\alpha}; c)$ as $n \to \infty$. We note that $t \geqq 1$ since $W \in E_p(P_{A,\alpha})$. For any ε with $0 < \varepsilon < 1$, there exists n_0 such that $t_n > 1 - \varepsilon > 0$ for all $n \geqq n_0$. Then $W/(1-\varepsilon)$ belongs to $E_p(P_{A,X_n^*})$ and

$$1/s \leqq \lambda_p(P_{A,X_n^*})^{-1} \leqq H_p(W/(1-\varepsilon)) = H_p(W)/(1-\varepsilon)^p$$

for all $n \geqq n_0$. Since ε is arbitrary, we have $1/s \leqq H_p(W)$, so that $1/s \leqq \lambda_p(P_{A,\alpha})^{-1}$. This completes the proof.

§3. Flow problems

For a node $x \in X$, a subset B of X and $w \in L(Y)$, let us put

$$I(w; x) = \textstyle\sum_{y\in Y} K(x, y) w(y),$$

$$I(w;\ B) = \sum_{x\in B} I(w;\ x) \quad \text{if} \quad \sum_{x\in B} |I(w;\ x)| < \infty.$$

Let A be a nonempty finite subset of X, $\alpha \in ibc(N)$ and $\{N_n^*\}$ ($N_n^* = \langle X_n^*, Y_n^* \rangle$) be a determining sequence of α. Denote by $F(A, X_n^*)$ the set of all flows w from A to X_n^*, i.e., the set of $w \in L(Y)$ satisfying the conditions: $I(w; x) = 0$ for all $x \in X - A - X_n^*$ and $I(w; A) + I(w; X_n^*) = 0$. Note that $F(A, X_{n+1}^*) \subset F(A; X_n^*)$. Let $L_0(Y)$ be the set of all $w \in L(Y)$ with finite support and $F_q(A, X_n^*)$ be the closure of $F(A, X_n^*) \cap L_0(Y)$ in the Banach space $L^q(Y; r)$ with the norm $[H_q(w)]^{1/q}$. Here q is a positive number such that $1 < q < \infty$.

We say that $w \in L(Y)$ is a flow of order q from A to α if there exists a sequence $\{w_n\}$ of flows such that $w_n \in F_q(A, X_n^*)$ and $H_q(w_n - w) \to 0$ as $n \to \infty$. Denote by $F_q(A, \alpha)$ the set of all flows of order q from A to α. Let us consider the following extremum problems related to flows:

(3.1) Find $d_q^*(A, X_n^*) = \inf\{H_q(w);\ w \in F_q(A, X_n^*),\ I(w;\ A) = -1\}$;

(3.2) Find $d_q^*(A, \alpha) = \inf\{H_q(w);\ w \in F_q(A, \alpha),\ I(w;\ A) = -1\}$.

We have

THEOREM 3.1. $\lim_{n\to\infty} d_q^*(A, X_n^*) = d_q^*(A, \alpha)$.

PROOF. Since $F_q(A, X_{n+1}^*) \subset F_q(A, X_n^*)$, $d_q^*(A, X_n^*) \leqq d_q^*(A, X_{n+1}^*)$. Let $w \in F_q(A, \alpha)$ such that $I(w;\ A) = -1$. Then there exists a sequence $\{w_n\}$ of flows such that $w_n \in F_q(A, X_n^*)$ and $H_q(w_n - w) \to 0$ as $n \to \infty$. Since $w_n(y) \to w(y)$ as $n \to \infty$ for each $y \in Y$, $I(w_n;\ A) \to I(w;\ A) = -1$ as $n \to \infty$. We have

$$d_q^*(A, X_n^*) \leqq H_q(w_n/I(w_n;\ A)) = H_q(w_n)/|I(w_n;\ A)|^q$$

for large n, so that $\lim_{n\to\infty} d_q^*(A, X_n^*) \leqq H_q(w)$. Therefore $\lim_{n\to\infty} d_q^*(A, X_n^*) \leqq d_q^*(A, \alpha)$. To prove the converse inequality, we may assume that $\lim_{n\to\infty} d_q^*(A, X_n^*) < \infty$. For each n, there exists an optimal solution $\bar{w}_n$ of problem (3.1), i.e., $\bar{w}_n \in F_q(A, X_n^*)$ such that $I(w_n;\ A) = -1$ and $d_q^*(A, X_n^*) = H_q(\bar{w}_n)$. By a standard argument and Lemma 1.4, we can verify that $H_q(\bar{w}_n - \bar{w}_m) \to 0$ as $n, m \to \infty$. Since $L_q(Y;\ r)$ is a Banach space, there exists $\bar{w} \in L_q(Y; r)$ such that $H_q(\bar{w}_n - \bar{w}) \to 0$ as $n \to \infty$. Therefore $\bar{w} \in F_q(A, \alpha)$. Since $\bar{w}_n(y) \to \bar{w}(y)$ as $n \to \infty$ for each $y \in Y$, we have $I(\bar{w}; A) = -1$. Hence $\lim_{n\to\infty} d_q^*(A, X_n^*) = \lim_{n\to\infty} H_q(\bar{w}_n) = H_q(\bar{w}) \geqq d_q^*(A, \alpha)$. This completes the proof.

In connection with problem (3.1), we considered the following problem in [4]:

(3.3) Find $d_p(A, X_n^*) = \inf\{D_p(u);\ u = 1 \text{ on } A,\ u = 0 \text{ on } X_n^*\}$.

By [4; Theorems 2.1 and 5.1] we have

$$d_p(A, X_n^*) = \lambda_p(P_{A,X_n^*})^{-1}$$

and the reciprocal relation

$$[d_p(A, X_n^*)]^{1/p}[d_q^*(A, X_n^*)]^{1/q} = 1 \quad \text{if} \quad 1/p + 1/q = 1.$$

On account of Theorems 2.2, 2.4 and 3.1, we obtain the following reciprocal relation:

THEOREM 3.2. *If $d_p(A, \alpha) > 0$ and $1/p + 1/q = 1$, then*

$$[d_p(A, \alpha)]^{1/p}[d_q^*(A, \alpha)]^{1/q} = 1.$$

§4. Extremal width of N relative to A and α

Let B_1 and B_2 be mutually disjoint nonempty subsets of X. Denote by $B_1 \ominus B_2$ the set of all $y \in Y$ which connects B_1 and B_2, i.e., $e(y) \cap B_1 \neq \phi$ and $e(y) \cap B_2 \neq \phi$. Let Q_{B_1,B_2} be the set of all cuts between B_1 and B_2, namely $Q \in Q_{B_1,B_2}$ if there exist mutually disjoint subsets $Q(B_1)$ and $Q(B_2)$ such that $Q(B_i) \supset B_i$ $(i=1, 2)$, $X = Q(B_1) \cup Q(B_2)$ and $Q = Q(B_1) \ominus Q(B_2)$.

In general, we say that a nonempty subset Q of Y is a cut of N if there exists a subset X' of X such that $Q = X' \ominus (X - X')$. The pair of X' and $X - X'$ is uniquely determined by Q.

Let A be a finite nonempty subset of X, $\alpha \in ibc(N)$ and $\{N_n^*\}$ ($N_n^* = \langle X_n^*, Y_n^* \rangle$) be a determining sequence of α such that $A \cap X_1^* = \phi$. Then $Q_{A,X_n^*} \subset Q_{A,X_{n+1}^*}$. Let us put

$$Q_{A,\alpha} = \cup_{n=1}^{\infty} Q_{A,X_n^*} \tag{4.1}$$

and call an element of $Q_{A,\alpha}$ a cut between A and α. Note that the definition of $Q_{A,\alpha}$ does not depend on the choice of the determining sequence of α.

For a set Λ of cuts, let us define the extremal width $\mu_q(\Lambda)$ of Λ of order q by

$$\mu_q(\Lambda)^{-1} = \inf\{H_q(W);\ W \in E_q^*(\Lambda)\}, \tag{4.2}$$

where $E_q^*(\Lambda)$ is the set of all $W \in L^+(Y)$ such that $H_q(W) < \infty$ and $\sum_Q W(y) \geqq 1$ for all $Q \in \Lambda$. Here we put $\sum_Q W(y) = \sum_{y \in Q} W(y)$. The following properties of the extremal width can be proved analogously to the case of the extremal length (cf. [2]):

LEMMA 4.1. *Let Λ_1 and Λ_2 be sets of cuts. If $\Lambda_1 \subset \Lambda_2$, then $\mu_q(\Lambda_1) \geqq \mu_q(\Lambda_2)$.*

LEMMA 4.2. *Let $\{\Lambda_n; n = 1, 2, \cdots\}$ be a family of cuts in N. Then $\sum_{n=1}^{\infty} \mu_q(\Lambda_n)^{-1} \geqq \mu_q(\cup_{n=1}^{\infty} \Lambda_n)^{-1}$.*

We say that a property holds for q-almost every cut of Λ if it does for the members of Λ except for those belonging to a subfamily with infinite extremal width of order q.

Similarly to Lemma 1.3, we can prove

LEMMA 4.3. *Let Λ be a set of cuts and assume that $W_n \in L^+(Y)$ and $H_q(W_n) \to 0$*

as $n \to \infty$. Then there exists a subsequence $\{n\}$ such that for q-almost every $Q \in \Lambda$, $\lim_{n\to\infty} \sum_Q W_n(y) = 0$.

We call $EW_p(A, X_n^*) = \mu_q(Q_{A,X_n^*})$ (resp. $EW_p(A, \alpha) = \mu_q(Q_{A,\alpha})$) the extremal width of N of order p relative to A and X_n^* (resp. A and α.), where $1/p + 1/q = 1$. We have

THEOREM 4.1. *$\lim_{n\to\infty} \mu_q(Q_{A, X_n^*}) = \mu_q(Q_{A,\alpha})$, i.e.,*

$$\lim_{n\to\infty} EW_p(A, X_n^*) = EW_p(A, \alpha).$$

PROOF. Since $Q_{A,X_n^*} \subset Q_{A,X_{n+1}^*} \subset Q_{A,\alpha}$, we have by Lemma 4.1 $\mu_q(Q_{A,X_n^*}) \geqq \mu_q(Q_{A,X_{n+1}^*}) \geqq \mu_q(Q_{A,\alpha})$, so that $\lim_{n\to\infty} \mu_q(Q_{A,X_n^*}) = s \geqq \mu_q(Q_{A,\alpha})$. To prove the converse inequality, we may assume that $\mu_q(Q_{A,\alpha}) < \infty$ and $s > 0$. By [4; Theorem 4.1] (note that the definition of $E_q^*(\Lambda)$ in [4] is different from the present one), we have $d_q^*(A, X_n^*) = \mu_q(Q_{A,X_n^*})^{-1}$ for each n. There exists $w_n \in F_q(A, X_n^*)$ such that $I(w_n; A) = -1$ and $H_q(w_n) = d_q^*(A, X_n^*)$. By the proof of Theorem 3.1, there exists $\bar{w} \in F_q(A, \alpha)$ such that $I(\bar{w}; A) = -1$, $1/s = d_q^*(A, \alpha) = H_q(\bar{w})$ and $H_q(w_n - \bar{w}) \to 0$ as $n \to \infty$. For each n, choose $w_n' \in F(A, X_n^*) \cap L_0(Y)$ such that $H_q(w_n - w_n') < 1/n$. Then $H_q(\bar{w} - w_n') \to 0$ as $n \to \infty$. Since $I(w_n'; A) \to I(\bar{w}; A) = -1$ as $n \to \infty$, we may assume that $I(w_n'; A) \neq 0$. Put $\bar{w}_n = -w_n'/I(w_n'; A)$. Then $\bar{w}_n \in F(A, X_n^*) \cap L_0(Y)$ and $I(\bar{w}_n; A) = -1$. Let $Q \in Q_{A,\alpha}$. Then there exists n_0 such that $Q \in Q_{A,X_n^*}$ for all $n \geqq n_0$. Define $u \in L(X)$ by $u = 1$ on $Q(A)$ and $u = 0$ on $Q(X_{n_0}^*)$. For every $n \geqq n_0$, we have

$$\begin{aligned} -1 = I(\bar{w}_n; A) &= \sum_{x\in A} \sum_{y\in Y} K(x, y) \bar{w}_n(y) \\ &= \sum_{x\in X} u(x) \sum_{y\in Y} K(x, y) \bar{w}_n(y) \\ &= \sum_{y\in Y} \bar{w}_n(y) \sum_{x\in X} K(x, y) u(x), \end{aligned}$$

so that

$$1 \leqq \sum_{y\in Y} |\bar{w}_n(y)| |\sum_{x\in X} K(x, y) u(x)| = \sum_Q |\bar{w}_n(y)|.$$

Let us put $W_n(y) = ||\bar{w}(y)| - |\bar{w}_n(y)||$ for every $y \in Y$. Then $H_q(W_n) \leqq H_q(\bar{w} - \bar{w}_n) \to 0$ as $n \to \infty$. By Lemma 4.3, there exist a subset Λ of $Q_{A,\alpha}$ and a subsequence $\{W_{n_k}\}$ of $\{W_n\}$ such that $\mu_q(Q_{A,\alpha} - \Lambda) = \infty$ and $\lim_{k\to\infty} \sum_Q W_{n_k}(y) = 0$ for all $Q \in \Lambda$. We have

$$1 - \sum_Q |\bar{w}(y)| \leqq \sum_Q [|\bar{w}_{n_k}(y)| - |\bar{w}(y)|] \leqq \sum_Q W_{n_k}(y),$$

so that $1 \leqq \sum_Q |\bar{w}(y)|$ for all $Q \in \Lambda$. Thus $|\bar{w}| \in E_q^*(\Lambda)$ and $\mu_q(\Lambda)^{-1} \leqq H_q(\bar{w}) = d_q^*(A, \alpha) = 1/s$. By Lemma 4.2, we have $\mu_q(\Lambda) = \mu_q(Q_{A,\alpha})$, and hence $\mu_q(Q_{A,\alpha})^{-1} \leqq 1/s$. This completes the proof.

The relation $[\lambda_p(P_{A,X_n^*})]^{1/p} [\mu_q(Q_{A,X_n^*})]^{1/q} = 1$ being known by [4; Theorem 5.2], we have

COROLLARY 4.1. $[EL_p(A, \alpha)]^{1/p} [EW_p(A, \alpha)]^{1/q} = 1$.

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