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# A note on Picard principle for rotationally invariant density

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A nonnegative locally Hölder continuous function $P$ on the punctured closed unit disk $0 < |z| \leq 1$ will be referred to as a *density* on $\Omega: 0 < |z| < 1$. For a density $P$ on $\Omega$ we consider the Martin compactification $\Omega_P^*$ of $\Omega$ with respect to the equation

$$(1) \qquad L_P u \equiv \Delta u - Pu = 0 \qquad (\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2)$$

on $\Omega$. We say that the Picard principle is valid for $P$ if the set of Martin minimal boundary points over the origin $z = 0$ consists of a single point. In the case that $P$ is a *rotationally invariant* density on $\Omega$, i.e., a density $P$ satisfying $P(z) = P(|z|)$ $(z \in \Omega)$, the Martin compactification $\Omega_P^*$ is characterized completely by Nakai [3] in terms of what he calls the singularity index $\alpha(P)$ of $P$ at $z = 0$:

$$\Omega_P^* \simeq \{\alpha(P) \leq |z| \leq 1\} ;$$

in particular, the Picard principle is valid for $P$ if and only if $\alpha(P) = 0$.

Take two sequences $\{a_n\}$ and $\{b_n\}$ $(n = 1, 2, \cdots)$ in the interval $(0, 1)$ satisfying $b_{n+1} < a_n < b_n$ with $\{a_n\}$ tending to zero as $n \to \infty$. We consider a sequence of annuli:

$$A_n \equiv \{z \in \boldsymbol{C}: a_n \leq |z| \leq b_n\}, \qquad A = \bigcup_{n=1}^{\infty} A_n ,$$

in $\Omega$ and set

$$P_n \equiv (2\pi)^{-1} \iint_{A_n} P(z)\, dx\, dy + 1 .$$

The purpose of this note is to show the following

THEOREM. *Let $P$ be a rotationally invariant density. If sequences $\{a_n\}$, $\{b_n\}$, and $\{P_n\}$ satisfy the condition*

$$(2) \qquad \sum_{n=1}^{\infty} \frac{\{\log (b_n/a_n)\}^2}{1 + P_n \log (b_n/a_n)} = +\infty ,$$

*then the Picard principle is valid for $P$ at $z = 0$.*

COROLLARY 1. *If sequences $\{a_n\}$ and $\{b_n\}$ satisfy the conditions*

$$\sum_{n=1}^{\infty} \{\log (b_n/a_n)\}^2 = +\infty \tag{3}$$

*and*

$$\sup_n \iint_{A_n} P(z)\, dx\, dy \equiv \sup_n 2\pi \int_{a_n}^{b_n} P(r)r\, dr < +\infty\,, \tag{4}$$

*then the Picard principle is valid for P.*

COROLLARY 2. *If sequences $\{a_n\}$ and $\{b_n\}$ satisfy the condition* (3) *and if*

$$P(|z|) = O(|z|^{-2}) \qquad (|z| \to 0) \qquad on\ A\,, \tag{5}$$

*then the Picard principle is valid for P* (*cf.* [1]).

**1.** To prove the theorem, we recall the *P-unit criterion* for rotationally invariant densities $P(z) \equiv P(r)$ ($|z| \equiv r$), (cf. [4], [2]). Change the variable $r \in (0, 1]$ to $t \in [0, \infty)$ by $r = e^{-t}$. The function $Q(t)$ associated with $P(r)$ is the function on $[0, \infty)$ defined by $Q(t) = e^{-2t}P(e^{-t})$. The Riccati component $a_Q$ of $Q$ is the unique nonnegative solution of the equation

$$-\frac{da(t)}{dt} + a(t)^2 = Q(t) \tag{6}$$

on $[0, \infty)$. It is known ([4], [2]) that the Picard principle is valid for $P$ if and only if

$$\int_0^{\infty} \frac{dt}{a_Q(t) + 1} = +\infty\,. \tag{7}$$

**2.** To apply this criterion we need the following

LEMMA. *Let $a_Q$ be the Riccati component of $Q$. Then*

$$a_Q(\beta) < (\alpha - \beta)^{-1} + \int_{\beta}^{\alpha} Q(t)\, dt \qquad (0 \le \beta < \alpha) \tag{8}$$

*and*

$$a_Q(x) < 2(\alpha - \beta)^{-1} + \int_{\beta}^{\alpha} Q(t)\, dt \qquad (\beta \le x \le (\alpha + \beta)/2)\,. \tag{9}$$

PROOF. Letting $\beta = x$ in (8) and then letting $x$ lie in $[\beta, (\alpha + \beta)/2]$ for $0 \le \beta \le \alpha$, we easily deduce (9). Thus we have only to prove (8).

Integrating both sides of the equality

$$-\frac{da_Q(t)}{dt} + a_Q(t)^2 = Q(t)$$

over $[\beta, x]$ $(\beta \leq x \leq \alpha)$, we have

$$p + \int_{\beta}^{x} a_Q(t)^2 \, dt \leq a_Q(x) , \tag{10}$$

where $p = a_Q(\beta) - \int_{\beta}^{\alpha} Q(t) \, dt$. If $p \leq 0$, then (8) is evident. Assuming $p > 0$, we shall show that

$$\sum_{k=0}^{n} p^{k+1}(x - \beta)^k \leq a_Q(x) \qquad \text{for } x \in [\beta, \alpha] , \qquad n = 0, 1, \cdots . \tag{11}$$

Since $a_Q(\alpha) < \infty$, it follows that $p(\alpha - \beta) < 1$, which implies (8).

To prove (11), let $f_n(x)$ denote the left-hand side of (11). By (10), $f_0(x) = p \leq a_Q(x)$ for $x \in [\beta, \alpha]$. Assume $f_n(x) \leq a_Q(x)$ for $x \in [\beta, \alpha]$. Then

$$\sum_{k=0}^{n} (k + 1)p^{k+2}(t - \beta)^k \leq f_n(t)^2 \leq a_Q(t)^2 \qquad \text{for } t \in [\beta, \alpha] .$$

Integrating both sides over $[\beta, x]$ and using (10), we obtain

$$p + \sum_{k=0}^{n} p^{k+2}(x - \beta)^{k+1} \leq p + \int_{\beta}^{x} a_Q(t)^2 \, dt \leq a_Q(x) \qquad \text{for } x \in [\beta, \alpha] ,$$

i.e., $f_{n+1}(x) \leq a_Q(x)$. Therefore, by induction, (11) is valid for all $n$.

3. Proof of the theorem. We set $\alpha_n = -\log a_n$ and $\beta_n = -\log b_n$. We estimate from below the integrand of the left-hand side of (7).

By the inequality (9) in the lemma we have

$$\{a_Q(t) + 1\}^{-1} \geq 2^{-1}(\alpha_n - \beta_n)\left\{1 + (\alpha_n - \beta_n)\left(\int_{\beta_n}^{\alpha_n} Q(t) \, dt + 1\right)\right\}^{-1}$$

on each interval $I_n \equiv [\beta_n, (\alpha_n + \beta_n)/2]$. Integrating both sides of the above inequality on $I_n$ with respect to $t$ and adding resulting inequalities with respect to $n$, we obtain

$$\int_{0}^{\infty} \{a_Q(t) + 1\}^{-1} \, dt \geq 4^{-1} \sum_{n=1}^{\infty} \{\log (b_n/a_n)\}^2 \{1 + P_n \log (b_n/a_n)\}^{-1}$$

since $\log (b_n/a_n) = \alpha_n - \beta_n$ and $P_n = \int_{\beta_n}^{\alpha_n} Q(t) \, dt + 1$. Hence condition (2) in the theorem implies condition (7) in the $P$-unit criterion.

The proof of the theorem is herewith complete.

4. Proof of the corollaries. Assume that (4) in Corollary 1 is valid. In the case that

$$\limsup_{n \to \infty} \log (b_n/a_n) \leq 1 , \tag{12}$$

it is easy to see that (2) is valid if and only if (3) is valid. If (12) is not valid, then there exist infinitly many $n$ such that $\log(b_n/a_n) > 1$, and hence, in this case, (2) is valid. Next, assume that (5) in Corollary 2 is valid. Observe that

$$P_n \leq c \log(b_n/a_n) + 1$$

for a positive constant $c$. As in the proof of Corollary 1, we can show that if (3) and (5) are both valid, then (2) is valid.

REMARK. In the following example, (4) in Corollary 1 is not valid, while (2) in the theorem is valid: Take a constant $\lambda$ in (0, 1) and set $a_n = \lambda^{2n+1}$ and $b_n = \lambda^{2n}$, and set $P(r) = |\log r| r^{-2}$ on $A_n$.

## References

[1] M. Kawamura, On a conjecture of Nakai on Picard principle, J. Math. Soc. Japan, **31** (1979), 359–371.

[2] M. Kawamura and M. Nakai, A test of Picard principle for rotation free desities II, J. Math. Soc. Japan, **28** (1976), 323–342.

[3] M. Nakai, Martin boundary over an isolated singularity of rotation free density, J. Math. Soc. Japan, **26** (1974), 483–504.

[4] M. Nakai, A test of Picard principle for rotation free densities, J. Math. Soc. Japan, **27** (1975), 412–431.

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