

Instability of singularly perturbed Neumann layer solutions in reaction-diffusion systems

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Abstract. Instability of mono-Neumann layer solutions to reaction-diffusion systems is proved by using the SLEP method. Mono-Neumann layers are singularly perturbed solutions of boundary layer type which are close to the stable constant state except in a neighborhood of a boundary point and satisfy the Neumann boundary conditions. We also show the dimension of the associated unstable manifold and the asymptotic behavior of the unstable eigenvalue when one of the diffusion coefficients tends to zero.

1. Introduction

For PDE systems of dissipative type such as reaction-diffusion systems and the Navier-Stokes equations, it has been anticipated that the associated global attractors are finite dimensional. Especially the recent progress of the study of inertial manifolds guarantees that this is the case for several typical equations (see, for example, [2]). Still so, in order to understand the precise dynamics on it, it is quite important to know the number of unstable solutions, their profiles, and the dimension of the unstable manifolds. We shall study such problems in its most simplest case for the following reaction-diffusion system:

$$(1.1a) \quad \begin{cases} u_t = \varepsilon^2 u_{xx} + f(u, v), \\ v_t = Dv_{xx} + g(u, v), \end{cases} \quad (t, x) \in (0, \infty) \times I,$$

with Neumann boundary conditions

$$(1.1b) \quad u_x = 0 = v_x, \quad (t, x) \in (0, \infty) \times \partial I.$$

The associated stationary problem is given by

$$(1.2a) \quad \begin{cases} \varepsilon^2 u_{xx} + f(u, v) = 0, \\ Dv_{xx} + g(u, v) = 0, \end{cases} \quad x \in I,$$

with

$$(1.2b) \quad u_x = 0 = v_x, \quad x \in \partial I,$$

where I denotes the interval $(0, 1)$, ε and D are positive diffusion coefficients.

We assume that the nullcline of f is S-shaped, $g = 0$ intersects once with $f = 0$ at $\bar{U} = (\bar{u}, \bar{v})$ like Figure 1, and ε is sufficiently small. Note that \bar{U} is a stable constant solution of (1.1). See the end of this section for detailed assumptions.

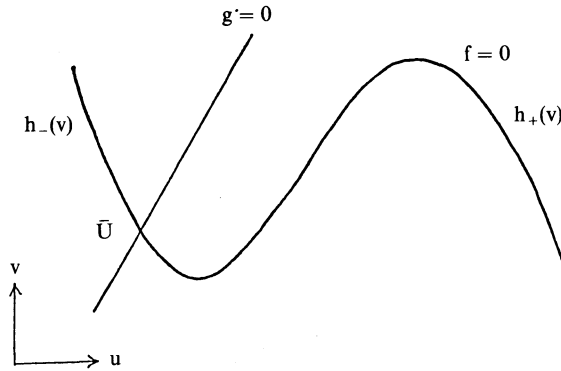


Figure 1: Functional forms of f and g .

The system (1.2) models a variety of phenomena such as chemical reaction, solidification, population dynamics, and so on (see, for example, [3], [12], [14] and references therein). It is known that (1.2) displays a variety of solution with layers, what is called the singularly perturbed solutions, as in Figure 2 when ε is sufficiently small. They are divided into two classes according to whether or not they have boundary layers (i.e., sharp transition at the boundary). Solutions with boundary layers like Figure 2(b)(c)(d) were constructed by [5], and they are called the *Neumann layer solutions*. Loosely speaking, those boundary layers satisfy the Neumann boundary conditions at both ends (This explains why the name “Neumann layer” is given to those layers). Under Dirichlet boundary conditions, boundary layers usually appear in order to fill the gap between the outer solutions and the boundary conditions. However boundary layers in Figure 2 are not of this type, in fact the solution without boundary layers (see Figure 2(a)) already satisfy the boundary conditions (1.1b). In this sense Neumann layers are essentially different from the usual ones.

Solutions which have only internal layers (see Figure 2(a)) were constructed by [4], [13], [8] and they are proved to be stable (see [16], [17], [20]). The important observation suggested in [6] and [15] is that Neumann layer solutions play the role of *separators* of these stable inner layer or constant solutions. Namely they play the similar role as that of the separatrix in ODE system. Note that for the scalar PDE case the existence and stability of such intermediate solutions has been discussed even for higher dimensional case (see, for example, [1], [10], [9]). As for the system (1.2), the situation seems to be much more complicated than the scalar case. One of the main reason for this

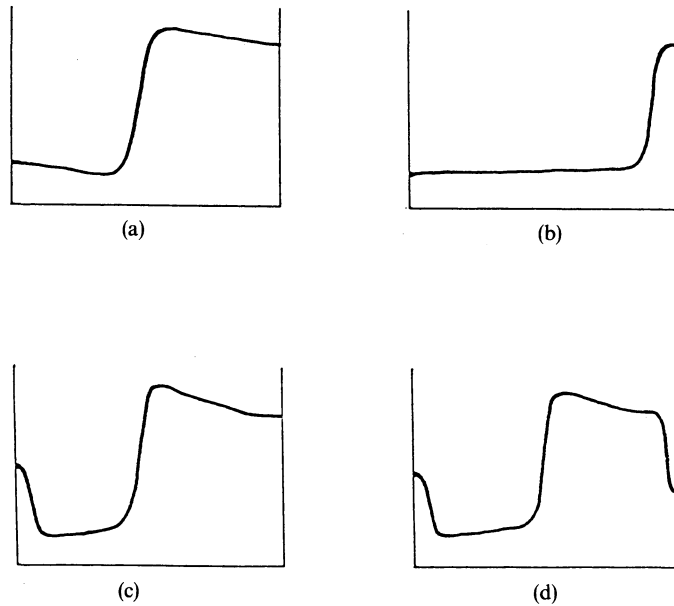


Figure 2: Profiles of the u -component of singularly perturbed solutions

- (a) Internal layer solution
- (b) Mono-Neumann layer solution
- (c) Neumann layer at the left end and the internal layer
- (d) Neumann layers at both ends and the internal layer.

is that many stable stationary solutions coexist for suitably fixed parameters (see [6], [17]).

Now we restate the problems more concretely:

- (1) How are the stability properties of Neumann layer solutions?
- (2) If they are unstable, what are the dimensions of unstable manifolds?
Especially how do they relate to the number of layers?
- (3) What are the destinations of unstable manifolds?

These are fundamental to understand the global dynamics of (1.2). Note that there is a gap between (1) and (2). More precise analysis is needed to know the dimension of the unstable manifold.

Here we consider the most simplest case (see Figure 2(b)), i.e., the Neumann layer solution which is close to the constant state \bar{U} except in a neighborhood of $x = 0$ or 1 . We call it the mono-Neumann layer solution. In this case Figure 3 answers the above questions numerically: a monotone initial data bigger (resp. smaller) than the mono-Neumann layer solution evolves (resp. decays) to the internal layer solution (resp. the constant solution \bar{U}).

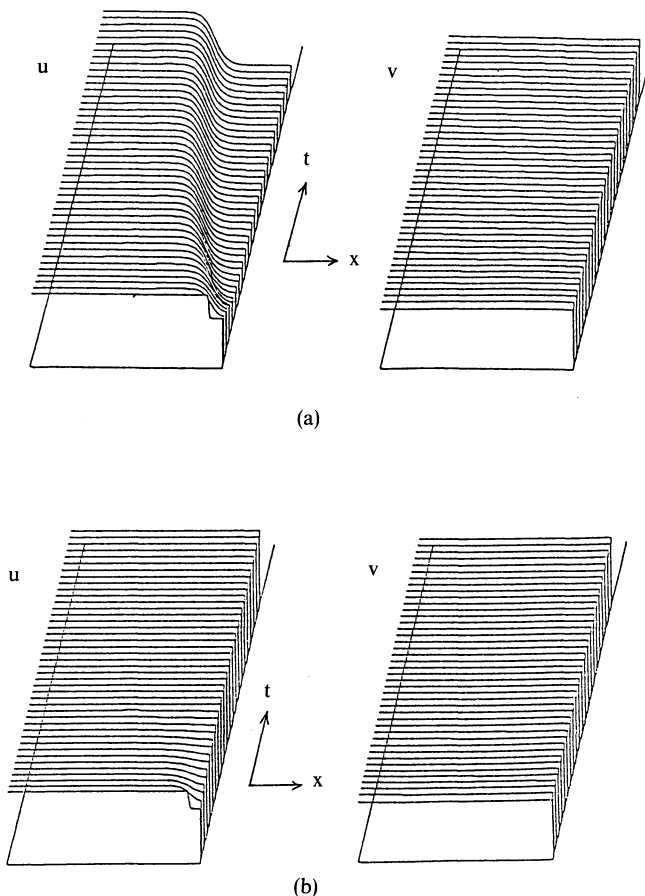


Figure 3: Destinations of the unstable manifold of mono-Neumann layer solution
 (a) From Neumann layer to the internal layer
 (b) From Neumann layer to the constant state \bar{U} .

Namely this suggests that the dimension of the unstable manifold is equal to one, and the destinations of it are the internal layer solution and the constant state \bar{U} , respectively (see Figure 4).

As the first step to solve the above problems rigorously, we intend to prove in this paper the instability of mono-Neumann layer solution by studying the spectral distribution of the linearized problem, and that the dimension of the unstable manifold is equal to one. This gives us a detailed proof for the corresponding results of [15].

Our goal is as follows.

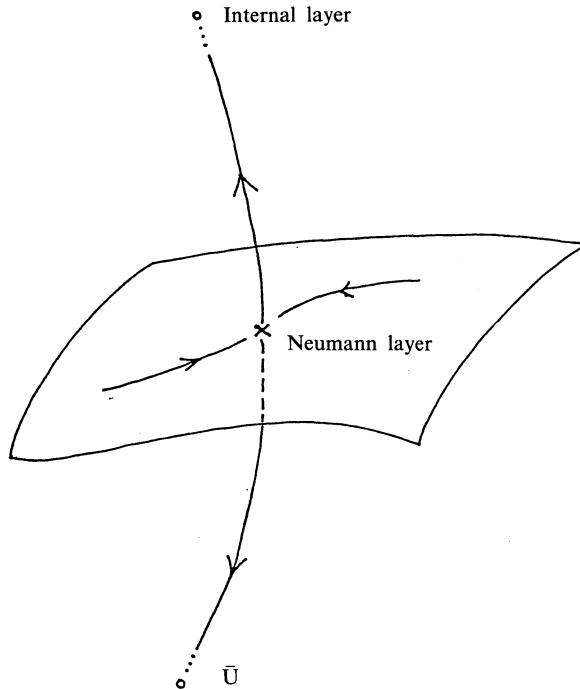


Figure 4: Behavior of the unstable manifold of mono-Neumann layer solution.

MAIN THEOREM. *The eigenvalue problem linearized at a mono-Neumann layer solution U^ε has a unique real simple positive eigenvalue $\lambda_c(\varepsilon)$ for small ε which behaves like*

$$(1.3) \quad \lambda_c(\varepsilon) = \zeta_0^* + \varepsilon\tau(\varepsilon)$$

where ζ_0^* is a positive constant and $\tau(\varepsilon)$ is continuous up to $\varepsilon = 0$. The rest of the spectrum has strictly negative real parts for small ε .

The SLEP method of [16] also works to prove this result. However, in order to know the asymptotic order (1.3), i.e., $\lambda_c(\varepsilon) - \zeta_0^* = \varepsilon\tau(\varepsilon)$, we need to construct the approximate solutions more accurately than [5]. In fact it turns out that the approximation up to order ε is sufficient for our purpose (see Appendix A). Also note that in order to obtain the asymptotic behavior of the principal eigenvalue of the singular Sturm-Liouville operator (see Lemma 2.3), which is indispensable to show (1.3), we can not apply the same technique as in [16] to it, since the spatial derivative of the stretched Neumann layer solution does not converge to the principal eigenfunction of the limiting stretched

singular Sturm-Liouville problem as $\varepsilon \downarrow 0$. We shall show it in Appendix B with the aid of the approximate solutions of order ε .

We close this section with the list of the assumptions for f and g and the notation.

ASSUMPTIONS

- (A-1) f and g are smooth functions of u and v defined on some open set O in \mathbb{R}^2 .
 (A-2) The nullcline of f is sigmoidal and consists of three continuous curves $u = h_-(v)$, $h_0(v)$ and $h_+(v)$ defined on the intervals I_- , I_0 and I_+ , respectively. Let $\min I_- = v_-$ and $\max I_+ = v_+$, then the inequality $h_-(v) < h_0(v) < h_+(v)$ holds for $v \in I^* = (v_-, v_+)$, and $h_+(v)$ ($h_-(v)$) coincides with $h_0(v)$ at only one point $v = v_+$ (v_-), respectively. Moreover, the unique equilibrium point $\bar{U} = (\bar{u}, \bar{v})$ satisfies $\bar{u} = h_-(\bar{v})$.
 (A-3) It holds that $J(\bar{v}) > 0$, where

$$J(v) = \int_{h_-(v)}^{h_+(v)} f(s, v) ds.$$

- (A-4) $f_u|_{\bar{U}} < 0$ and $g_v|_{\bar{U}} < 0$.
 (A-5) $\det(\partial(f, g)/\partial(u, v))|_{\bar{U}} > 0$ (see Figure 1).

REMARK 1.1. Note that additional assumptions are necessary besides the above in order to guarantee the existence and stability of internal layer solutions. See [16] for the details.

We use the following notation with p being a nonnegative integer and α any nonnegative constant:

$C^p(I)$ = the space of p -times continuously differentiable functions on I with the usual norm,

$C_\alpha^p(I)$ = the space of p -times continuously differentiable functions on I with the norm

$$\|u\|_{C_\alpha^p} = \sum_{k=0}^p \max \left| \left(\alpha \frac{d}{dx} \right)^k u(x) \right|,$$

$$C_{\alpha,0}^p(I) = \{ u \in C_\alpha^p(I) \mid u_x(0) = 0, u(1) = 0 \},$$

$$C_{\alpha,1}^p(I) = \{ u \in C_\alpha^p(I) \mid u(0) = 0, u_x(1) = 0 \},$$

$C_{c.u.}^p(\mathbf{R}_+)$ = the compact uniform convergence in C^p -sense in \mathbf{R}_+ , namely, the uniform convergence on any compact subset of \mathbf{R}_+ in C^p -sense,

$H^p(I)$ = the usual Sobolev space,

$H_N^p(I)$ = the space of closure of $\{\cos(n\pi x)\}_{n=0}^{+\infty}$ in $H^p(I)$,

$\mathcal{L}(X, Y)$ = the space of bounded linear operators from X into Y with the usual norm.

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2. Existence theorem and preliminaries

We first show the existence theorem of mono-Neumann layer solution of (1.2).

THEOREM 2.1. *There is a positive constant ε_0 such that (1.2) has an ε -family of the solutions $U^\varepsilon(x) = (u(x, \varepsilon), v(x, \varepsilon)) \in C^2_\varepsilon(I) \times C^2(I)$ for $\varepsilon \in (0, \varepsilon_0)$. Moreover they satisfy*

$$\lim_{\varepsilon \downarrow 0} u(x, \varepsilon) = \bar{u} \quad \text{uniformly on } [k, 1]$$

and

$$\lim_{\varepsilon \downarrow 0} v(x, \varepsilon) = \bar{v} \quad \text{uniformly on } I,$$

for any $k > 0$ (see Figure 5).

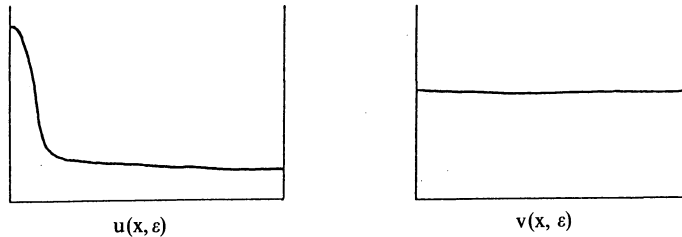


Figure 5: Profile of mono-Neumann layer solution.

PROOF. See Appendix A.

It is convenient to introduce the stretched variable $s = x/\varepsilon$ to see the internal structure of U^ε near $x = 0$. We have

LEMMA 2.2. (*Behavior of the stretched solution*). *Let \tilde{U}^ε be the stretched solution corresponding to U^ε , i.e. $\tilde{U}^\varepsilon(s) \equiv U^\varepsilon(\varepsilon s)$, and let \tilde{u}^* be the unique monotone solution of*

$$\begin{cases} \tilde{u}_{ss} + f(\tilde{u}, \bar{v}) = 0, & s \in \mathbf{R}_+, \\ \tilde{u}_s(0) = 0, & \tilde{u}(+\infty) = \bar{u}. \end{cases}$$

Then it holds that

$$\lim_{\varepsilon \downarrow 0} \tilde{U}^\varepsilon = \tilde{U}^* \quad \text{in } C^2_{c.u.}(\mathbf{R}_+)\text{-sense,}$$

where $\tilde{U}^* = (\tilde{u}^*, \bar{v})$.

PROOF. This is easily seen from the construction of the solution U^ε (see Appendix A).

Let us consider the following Sturm-Liouville problem at U^ε ,

$$(2.1) \quad \begin{cases} L^\varepsilon \phi \equiv (\varepsilon^2(d^2/dx^2) + f_u^\varepsilon)\phi = \zeta \phi & \text{in } I, \\ \phi_x = 0 & \text{on } \partial I. \end{cases}$$

where $f_u^\varepsilon = f_u(u(x, \varepsilon), v(x, \varepsilon))$. Let $\{\phi_n^\varepsilon\}_{n \geq 0}$ be the complete orthonormal set (CONS) in $L^2(I)$ -sense and $\{\zeta_n^\varepsilon\}_{n \geq 0}$ the associated eigenvalues of (2.1), which are all real and simple. It is convenient to define the stretched Sturm-Liouville problem for (2.1):

$$(2.2) \quad \begin{cases} \tilde{L}^\varepsilon \hat{\phi} \equiv ((d^2/ds^2) + \tilde{f}_u^\varepsilon)\hat{\phi} = \zeta \hat{\phi} & \text{in } \tilde{I}, \\ \hat{\phi}_s = 0 & \text{on } \partial \tilde{I}, \end{cases}$$

where \tilde{f}_u^ε is the stretched potential of f_u^ε and \tilde{I} is the stretched interval $(0, 1/\varepsilon)$. Similarly, let $\{\hat{\phi}_n^\varepsilon\}_{n \geq 0}$ be the CONS in $L^2(I)$ -sense and $\{\zeta_n^\varepsilon\}_{n \geq 0}$ the associated real simple eigenvalues of (2.2). Note that the set of eigenvalues $\{\zeta_n^\varepsilon\}_{n \geq 0}$ remain the same after stretching. On the other hand, we need $\sqrt{\varepsilon}$ -factor for the eigenfunctions $\hat{\phi}_n^\varepsilon = \sqrt{\varepsilon} \phi_n^\varepsilon$.

Next, we introduce the Sturm-Liouville problem on \mathbf{R}_+ which is obtained by taking a limit of $\varepsilon \downarrow 0$ in (2.2):

$$(2.3) \quad \begin{cases} \tilde{L}^* \hat{\phi} \equiv ((d^2/ds^2) + \tilde{f}_u^*)\hat{\phi} = \zeta \hat{\phi} & \text{in } \mathbf{R}_+, \\ \hat{\phi} \in L^2(\mathbf{R}_+), \end{cases}$$

where $\tilde{f}_u^* = f_u(\tilde{u}^*, \bar{v})$.

The main aim of this section is to show the spectral behavior of (2.1). More precisely, the principal eigenvalue ζ_0^ε of (2.1) has a positive limit ζ_0^* as $\varepsilon \downarrow 0$, which becomes the principal eigenvalue of (2.3), and the rest of its spectrum is strictly negative for small ε . Moreover, $(\zeta_0^\varepsilon - \zeta_0^*)/\varepsilon$ has a definite limit as $\varepsilon \downarrow 0$ which can be explicitly expressed in terms of the approximations up to order ε . We see in Lemma 2.4 that the principal eigenfunction $\hat{\phi}_0^\varepsilon$ also converges to that of (2.3) in an appropriate sense.

LEMMA 2.3. (Spectral behavior of L^ε). (a) Let $\{\zeta_n^\varepsilon\}_{n \geq 0}$ be the complete set of eigenvalues of (2.1). Then we have

$$\begin{aligned} \zeta_0^\varepsilon &> 0 > -\Delta^* > \zeta_1^\varepsilon > \cdots > \zeta_n^\varepsilon > \cdots, \\ \zeta_0^\varepsilon &= \zeta_0^* + \varepsilon \hat{\zeta}_0^\varepsilon, \end{aligned}$$

where ζ_0^* , Δ^* are positive constants independent of ε , and $\hat{\zeta}_0^\varepsilon$ is a continuous function of ε up to $\varepsilon = 0$ with

$$\hat{\zeta}_0^* = \lim_{\varepsilon \downarrow 0} \hat{\zeta}_0^\varepsilon$$

(see (44) in Appendix B for the detailed expression of $\hat{\zeta}_0^*$ in terms of $O(\varepsilon)$ -approximation).

(b) The principal eigenfunction ϕ_0^ε satisfies the following properties

$$(i) \quad \hat{\phi}_0^\varepsilon(s) \leq \hat{c} \exp(-\gamma s), \quad s \in \tilde{I},$$

where \hat{c} and γ are positive constants independent of ε .

(ii) It holds that

$$\int_I \phi_0^\varepsilon(x) dx = L(\varepsilon) \sqrt{\varepsilon},$$

where $L(\varepsilon)$ is a positive continuous function of ε up to $\varepsilon = 0$ with

$$L^* = \lim_{\varepsilon \downarrow 0} L(\varepsilon) = \int_{\mathbf{R}_+} \hat{\phi}_0^*(s) ds > 0$$

and $\hat{\phi}_0^*$ is the principal eigenfunction of (2.3) to ζ_0^* .

PROOF. (a) We only show that the principal eigenvalue ζ_0^ε is strictly positive and the remaining spectrum is bounded away from zero. The detailed behavior of ζ_0^ε and its limiting formula is proved in Appendix B. First, we extend the x -interval of the linearized problem from $(0, 1)$ to $(-1, 1)$ in an even way and impose the Neumann boundary conditions on both ends. Note that any eigenvalue and the associated eigenfunction of (2.1) becomes that of the extended problem after folding over. Especially, they have common principal eigenvalue. We distinguish the extended problem and its solutions by adding the subscript e like $(2.1)_e$ and $(\phi_0^\varepsilon)_e$. The key idea lies in the behavior of the second eigenvalue of the extended problem. More precisely, we first note that $d\tilde{u}_e^*/ds$ satisfies $(2.3)_e$ with $\zeta = 0$, which has the unique zero at $s = 0$ (nodal one) and decays exponentially as $|s| \rightarrow \infty$ (and hence belongs to $L^2(\mathbf{R})$). Namely the second eigenvalue ζ_1^* of $(2.3)_e$ with the eigenfunction $d\tilde{u}_e^*/ds$ of nodal one is equal to zero. A direct consequence of this is that the principal eigenvalue ζ_0^* of $(2.3)_e$ is strictly positive and the spectrum (including continuous spectrum) except ζ_0^* and ζ_1^* lies strictly in the negative real axis. The associated principal eigenfunction $(\hat{\phi}_0^*)_e$ of $(2.3)_e$ becomes an even function because of the even symmetry of \tilde{f}_u^* and the simplicity of the principal eigenvalue. Hence the half of $(\hat{\phi}_0^*)_e$ becomes the principal eigenfunction of (2.3) with the same eigenvalue ζ_0^* . In view of these observations and using similar arguments used in the proof of Lemma 1.3 of [16], we can verify without difficulty that the principal eigenvalue ζ_0^ε of (2.1), which is a continuous function of ε , converges to ζ_0^* as

$\varepsilon \downarrow 0$, and that the next eigenvalue ζ_1^ε , the associated eigenfunction of which has two nodal points in the extended interval, is strictly bounded away from zero.

(b) The result (i) is a direct consequence of the potential form f_u^ε , and the proof of (ii) is quite similar to that of Corollary 1.3 of [16], so we leave the details to the reader.

LEMMA 2.4. *It holds that*

$$\lim_{\varepsilon \downarrow 0} \hat{\phi}_0^\varepsilon (= \sqrt{\varepsilon} \tilde{\phi}_0^\varepsilon) = \hat{\phi}_0^* \quad \text{in } C_{c.u.}^2(\mathbf{R}_+)\text{-sense,}$$

where $\hat{\phi}_0^\varepsilon$ (resp. $\hat{\phi}_0^*$) is the L^2 -normalized principal eigenfunction of (2.2) (resp. (2.3)).

PROOF. Using a similar argument as in the proof of Lemma 1.3 of [16], any sequence of $\{\hat{\phi}_0^\varepsilon\}_{\varepsilon > 0}$ has a convergent subsequence $\{\hat{\phi}_0^{\varepsilon_n}\}_{n \geq 1}$ in $C_{c.u.}^2(\mathbf{R}_+)\text{-sense}$ and its limit satisfies (2.3) with $\zeta = \zeta_0^*$. Since ζ_0^* is simple, the limit does not depend on the choice of the sequence, and hence the conclusion follows.

3. Instability of Neumann layer solutions

Let us solve the linearized eigenvalue problem of (1.2) around $U^\varepsilon(x)$:

$$(3.1) \quad \mathcal{L}^\varepsilon \begin{pmatrix} w \\ z \end{pmatrix} \equiv \begin{pmatrix} L^\varepsilon & f_v^\varepsilon \\ g_u^\varepsilon & M^\varepsilon \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \lambda \begin{pmatrix} w \\ z \end{pmatrix},$$

where $M^\varepsilon \equiv D(d^2/dx^2) + g_v^\varepsilon$.

In view of Lemma 2.3, the first equation of (3.1) can be solved as

$$(3.2) \quad w = (L^\varepsilon - \lambda)^{-1}(-f_v^\varepsilon z) = \sum_{n=0}^{\infty} \frac{\langle -f_v^\varepsilon z, \phi_n^\varepsilon \rangle}{\zeta_n^\varepsilon - \lambda} \phi_n^\varepsilon \\ = \frac{\langle -f_v^\varepsilon z, \phi_0^\varepsilon \rangle}{\zeta_0^\varepsilon - \lambda} \phi_0^\varepsilon + (L^\varepsilon - \lambda)^\dagger(-f_v^\varepsilon z).$$

Here we introduce the reduced resolvent $(L^\varepsilon - \lambda)^\dagger$ defined by

$$(3.3) \quad (L^\varepsilon - \lambda)^\dagger(\cdot) \equiv \sum_{n \geq 1} \frac{\langle \cdot, \phi_n^\varepsilon \rangle}{\zeta_n^\varepsilon - \lambda} \phi_n^\varepsilon.$$

Substituting (3.2) into the second equation of (3.1), we have

$$(3.4) \quad Dz_{xx} + \frac{\langle -f_v^\varepsilon z, \phi_0^\varepsilon \rangle}{\zeta_0^\varepsilon - \lambda} g_u^\varepsilon \phi_0^\varepsilon + g_u^\varepsilon (L^\varepsilon - \lambda)^\dagger(-f_v^\varepsilon z) + (g_v^\varepsilon - \lambda)z = 0.$$

We shall show the existence of real positive eigenvalue λ^ε which converges to ζ_0^* (the limiting value of the principal eigenvalue of L^ε) as $\varepsilon \downarrow 0$, namely

$$(3.5a) \quad \lambda^\varepsilon = \zeta_0^* + \tau(\varepsilon),$$

where τ is a continuous function of ε with $\lim_{\varepsilon \downarrow 0} \tau(\varepsilon) = 0$. The asymptotic order of τ as $\varepsilon \downarrow 0$ is not *a priori* known, however it turns out later (see Lemma 3.5) that τ is at least of $O(\varepsilon)$, i.e. τ can be written in the following form

$$(3.5b) \quad \tau(\varepsilon) = \varepsilon \hat{\tau}(\varepsilon),$$

where $\hat{\tau}$ is a bounded continuous function of ε up to $\varepsilon = 0$. Let us proceed further under the assumption (3.5b) and defer its justification till Lemma 3.5. Although λ^ε and ζ_0^ε have the same limiting value ζ_0^* as $\varepsilon \downarrow 0$, λ^ε is not equal to ζ_0^ε for small positive ε . In fact it holds that

LEMMA 3.1.

$$\zeta_0^\varepsilon \notin \sigma(\mathcal{L}^\varepsilon) \quad \text{for small positive } \varepsilon.$$

PROOF. See Appendix 2 in [16].

When λ^ε belongs to $\sigma(\mathcal{L}^\varepsilon)$, this lemma guarantees that $\zeta_0^\varepsilon - \lambda^\varepsilon \neq 0$ for small positive ε . It follows from Lemma 2.3(a) and (3.5) that the second term of (3.4) becomes

$$(3.6) \quad \frac{\langle -f_v^\varepsilon z, \phi_0^\varepsilon \rangle}{\zeta_0^\varepsilon - \lambda^\varepsilon} g_u^\varepsilon \phi_0^\varepsilon = \frac{\langle -f_v^\varepsilon z, \phi_0^\varepsilon \rangle}{\varepsilon(\zeta_0^\varepsilon - \hat{\tau})} g_u^\varepsilon \phi_0^\varepsilon = \frac{\langle z, -f_v^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon} \rangle}{\zeta_0^\varepsilon - \hat{\tau}} g_u^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}}.$$

Hence (3.4) becomes

$$(3.7) \quad Dz_{xx} + \frac{\langle z, -f_v^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon} \rangle}{\zeta_0^\varepsilon - \hat{\tau}} g_u^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}} + g_u^\varepsilon (L^\varepsilon - \lambda^\varepsilon)^\dagger (-f_v^\varepsilon z) + (g_v^\varepsilon - \lambda^\varepsilon)z = 0.$$

The second term of (3.7) is called the *critical part*, since it behaves in a singular way as $\varepsilon \downarrow 0$ in the sense of Lemma 3.3 below. The rest of (3.7) is called the *noncritical part*. To proceed further, we need the following three lemmas. The first two lemmas concern about the asymptotic characterization of the second and third terms of (3.7). The third one shows the existence of inverse operator of the noncritical part of (3.7). We leave the proof of them to the reader, since they are obvious modifications of those of Lemmas 2.3, 2.4 and 3.1 in [16].

LEMMA 3.2. Let $F(u, v)$ be a smooth function of u and v . Then it holds that

$$\lim_{\varepsilon \downarrow 0} (L^\varepsilon - \lambda)^\dagger (F^\varepsilon h) = F^* h / (f_u^* - \lambda) \quad \text{strongly in } L^2\text{-sense,}$$

for any function $h \in L^2(I) \cap L^\infty(I)$ and $\lambda \in C_\mu$, where $F^\varepsilon \equiv F(u(x, \varepsilon), v(x, \varepsilon))$, $F^* \equiv F(\bar{u}, \bar{v})$, $f_u^* = f_u(\bar{u}, \bar{v})$ (see Theorem 2.1), $\mu = \min \{A^*, -f_u^*\}$ and $C_\mu = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda > -\mu\}$. Moreover, if h belongs to $H^1(I)$, the above convergence is also uniform on a bounded set in $H^1(I)$.

LEMMA 3.3.

- (i) $\lim_{\varepsilon \downarrow 0} (-f_v^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon}) = c_1^* \delta_0$ in $(H_N^1(I))^\#$ -sense,
(ii) $\lim_{\varepsilon \downarrow 0} (g_u^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon}) = c_2^* \delta_0$ in $(H_N^1(I))^\#$ -sense,

where δ_0 is the Dirac's δ -function at $x = 0$, $(H_N^1(I))^\#$ means the dual space of $H_N^1(I)$ and c_i^* ($i = 1, 2$) are positive constants defined by

$$c_1^* = - \int_0^{+\infty} \tilde{f}_v^* \hat{\phi}_0^* ds,$$

$$c_2^* = \int_0^{+\infty} \tilde{g}_u^* \hat{\phi}_0^* ds.$$

Here $\tilde{f}_v^* = f_v(\tilde{u}^*(s), \bar{v})$, $\tilde{g}_u^* = g_u(\tilde{u}^*(s), \bar{v})$.

LEMMA 3.4. The differential operator $T^{\varepsilon, \lambda}: H_N^1(I) \rightarrow (H_N^1(I))^\#$ defined by $T^{\varepsilon, \lambda} z \equiv -D(d^2/dx^2)z - g_u^\varepsilon(L^\varepsilon - \lambda)^\dagger(-f_v^\varepsilon z) - (g_v^\varepsilon - \lambda)z$ has a uniform bounded inverse $K^{\varepsilon, \lambda}: (H_N^1(I))^\# \rightarrow H_N^1(I)$ for $0 \leq \varepsilon < \varepsilon_0$ and $\lambda \in \mathbf{C}_\mu$ with ε_0 being an appropriate positive constant. $K^{\varepsilon, \lambda}$ depends continuously (resp. analytically) on ε (resp. λ) in operator norm sense.

Let us solve (3.7) in $H_N^1(I)$. Applying $K^{\varepsilon, \lambda^\varepsilon}$ to (3.7), we have $z = \langle z, -f_v^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon} \rangle K^{\varepsilon, \lambda^\varepsilon} \{g_u^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon}\} / (\zeta_0^\varepsilon - \hat{\tau})$. Hence z is a constant multiple of $K^{\varepsilon, \lambda^\varepsilon} \{g_u^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon}\}$, i.e.

$$z = \alpha K^{\varepsilon, \lambda^\varepsilon} \{g_u^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon}\}.$$

Substituting this into (3.7), we see that a nontrivial z satisfying (3.7) exists if and only if τ satisfies the following equation

$$(3.8) \quad \mathcal{F}(\hat{\tau}, \varepsilon) \equiv \hat{\zeta}_0^\varepsilon - \hat{\tau} - \left\langle K^{\varepsilon, \lambda^\varepsilon} \left\{ g_u^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}} \right\}, -f_v^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}} \right\rangle = 0,$$

where $\lambda^\varepsilon = \zeta_0^* + \varepsilon \hat{\tau}$ (see (3.5)). It follows directly from Lemmas 3.1–3.4 that \mathcal{F} is continuous with respect to ε up to $\varepsilon = 0$, and holomorphic with respect to $\hat{\tau} \in \mathcal{K}$, where \mathcal{K} is an arbitrary compact subset of \mathbf{C} . Hence \mathcal{F} is well-defined at $\varepsilon = 0$ with the limiting value

$$(3.9) \quad \mathcal{F}(\hat{\tau}, 0) = \hat{\zeta}_0^* - \hat{\tau} - c_1^* c_2^* \langle K^{0, \zeta_0^*} \{\delta_0\}, \delta_0 \rangle.$$

Now we are ready to apply the implicit function theorem to (3.8) at $(\hat{\tau}, \varepsilon) = (\hat{\tau}^*, 0)$. Here $\hat{\tau}^*$ is defined by

$$(3.10) \quad \hat{\tau}^* \equiv \hat{\zeta}_0^* - c_1^* c_2^* \langle K^{0, \zeta_0^*} \{\delta_0\}, \delta_0 \rangle.$$

In fact, it is clear from (3.9) that $\mathcal{F}(\hat{t}^*, 0) = 0$, and it holds that

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial \hat{t}}(\hat{t}^*, 0) &= -1 - \varepsilon \frac{d}{d\lambda^\varepsilon} \left[\left\langle K^{\varepsilon, \lambda^\varepsilon} \left\{ g_u^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}} \right\}, -f_v^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}} \right\rangle \right] \Big|_{\varepsilon=0} \\ &= -1. \end{aligned}$$

Thus, using the implicit function theorem, we see that there exists a unique continuous function $\hat{t} = \hat{t}^u(\varepsilon)$ with $\hat{t}^u(0) = \hat{t}^*$ for small positive ε satisfying

$$\mathcal{F}(\hat{t}^u(\varepsilon), \varepsilon) = 0.$$

Note that \hat{t}^u is real-valued, since \mathcal{F} is real-valued when \hat{t} is real. The simplicity of this unstable eigenvalue can be verified in an analogous way as [16]. So we leave it to the reader.

Now let us return to the justification of asymptotic order (3.5b).

LEMMA 3.5. *Suppose that there exists an eigenvalue $\lambda = \lambda^\varepsilon$ of (3.1) which approaches ζ_0^* as $\varepsilon \downarrow 0$, then it must have the asymptotic form (3.5b).*

PROOF. We prove this by contradiction. Suppose $|\tau(\varepsilon)|$ tends to zero strictly slower than ε . Then we can find a sequence ε_n for $n > 1$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ such that $\hat{t}(\varepsilon_n)$ ($\equiv \tau(\varepsilon_n)/\varepsilon_n$) is a solution of (3.8) for $n > 1$ with $\lim_{n \rightarrow \infty} |\hat{t}(\varepsilon_n)| = \infty$. However this is not possible since we see from Lemmas 3.3 and 3.4 that the rest of \mathcal{F} , that is, $\hat{\zeta}_0^\varepsilon - \langle K^{\varepsilon, \kappa^\varepsilon} \{ g_u^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon} \}, -f_v^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon} \rangle$, remains bounded as $\varepsilon (= \varepsilon_n) \downarrow 0$.

We conclude that

THEOREM 3.6. *There exists a positive constant ε_0 such that the linearized eigenvalue problem (3.1) has a unique real simple positive eigenvalue $\lambda = \lambda^u(\varepsilon)$ for $0 \leq \varepsilon < \varepsilon_0$, which tends to ζ_0^* as $\varepsilon \downarrow 0$ where $\zeta_0^* = \lim_{\varepsilon \downarrow 0} \zeta_0^\varepsilon$ (see Lemma 2.3). Moreover, it has the asymptotic form*

$$\lambda^u(\varepsilon) = \zeta_0^* + \varepsilon \hat{t}^u(\varepsilon),$$

where $\hat{t}^u(\varepsilon)$ is a real continuous function of ε for $0 \leq \varepsilon < \varepsilon_0$ with $\hat{t}^u(0) = \hat{t}^*$ being given by (3.10).

So far we only focus on the eigenvalue converging to ζ_0^* as $\varepsilon \downarrow 0$ (i.e., the unstable eigenvalue), however we can show much stronger result if we reconsider the above discussions. Namely we can prove the following

PROPOSITION 3.7. *Any eigenvalue $\lambda = \lambda^\varepsilon$ of (3.1) which stays in the region C_μ for any small ε must converge to ζ_0^* when $\varepsilon \downarrow 0$.*

PROOF. We prove this by contradiction. Suppose that $\lambda = \lambda^\varepsilon$ remains in C_μ and away from ζ_0^* uniformly for small ε . In view of (3.7), we see that there exists a nontrivial z^ε satisfying

$$(3.11) \quad Dz_{xx}^\varepsilon + \frac{\langle z^\varepsilon, -f_v^\varepsilon \phi_0^\varepsilon \rangle}{\zeta_0^\varepsilon - \lambda^\varepsilon} g_u^\varepsilon \phi_0^\varepsilon + g_u^\varepsilon (L^\varepsilon - \lambda^\varepsilon)^\dagger (-f_v^\varepsilon z^\varepsilon) + (g_v^\varepsilon - \lambda^\varepsilon) z^\varepsilon = 0.$$

Using the same procedure as before (see (3.8) and (3.9)), this is equivalent to say that λ^ε satisfies this equation

$$(3.12) \quad \mathcal{F}(\lambda^\varepsilon, \varepsilon) = \zeta_0^\varepsilon - \lambda^\varepsilon - \varepsilon \left\langle K^{\varepsilon, \lambda^\varepsilon} \left\{ g_u^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}} \right\}, -f_v^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}} \right\rangle = 0,$$

In view of Lemmas 2.3, 3.3 and 3.4, we see that λ^ε must remain bounded, since both ζ_0^ε and $\left\langle K^{\varepsilon, \lambda^\varepsilon} \left\{ g_u^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}} \right\}, -f_v^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}} \right\rangle$ are bounded for small ε . Hence λ^ε converges to some value λ^* in C_μ different from ζ_0^* (if necessary, we take a subsequence of it).

Thus we have in the limit of $\varepsilon \downarrow 0$

$$\mathcal{F}(\lambda^*, 0) = \zeta_0^* - \lambda^* = 0,$$

which is a contradiction since $\lambda^* \neq \zeta_0^*$.

Combining Proposition 3.7 and Theorem 3.6, we have

THEOREM 3.8. *The unstable eigenvalue $\lambda = \lambda^u(\varepsilon)$ is a unique eigenvalue of (3.1) in C_μ , and hence the rest of the spectrum has strictly negative real parts uniformly for small ε .*

This apparently shows that the mono-Neumann layer solution is unstable and the dimension of the unstable manifold of it is equal to one. Also it is clear that Main Theorem in Section 1 is contained in the statements of Theorems 3.6 and 3.8.

4. Concluding remarks

(a) *Instability of spike solutions.* Folding over even times the mono-Neumann layer solution and normalizing the length of the interval to 1, we obtain the solutions of (1.2) with sharp peaks as in Figure 6. We call these the *spike solutions*. All these spike solutions are *unstable*, since, by folding over the unstable eigenfunction for mono-Neumann layer, we see that the resulting one automatically becomes an eigenfunction of the linearized problem with keeping the same unstable eigenvalue.

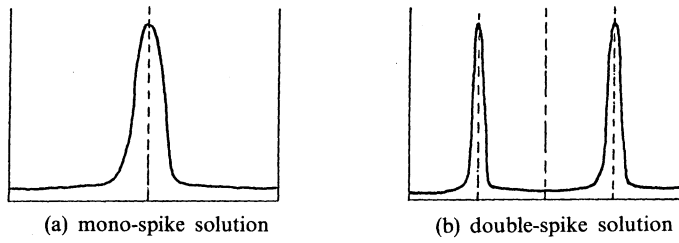
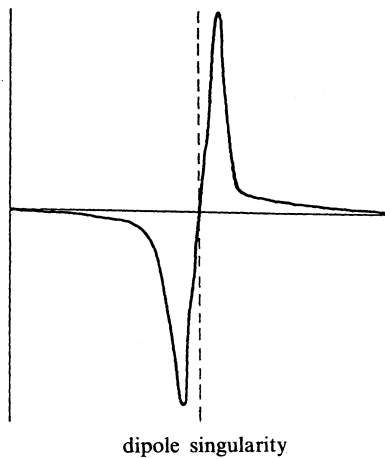


Figure 6: Profiles of spike solutions.

(b) *Singularity of dipole type.* For simplicity we only consider the mono-spike solution like Figure 6(a), which is obtained by even extension of mono-Neumann layer. As we saw in Lemma 3.3, the scaled principal eigenfunction of L^ε was characterized as a convergent sequence to Dirac's δ -function when $\varepsilon \downarrow 0$. An interesting phenomenon for the spike solution is that a new type of singularity appears for the second eigenfunction of L^ε as $\varepsilon \downarrow 0$, namely the *dipole* singularity. Loosely speaking, this can be observed by differentiating the u -component of the spike solution with respect to x , although it does not satisfy the Neumann boundary conditions, but Dirichlet ones. See Figure 7. One can prove that the appropriately scaled second eigenfunction of L^ε at the mono-spike solution, which has nodal one, is close to Figure 7 and converges to a constant multiple of the derivative of Dirac's δ -function at $x = 1/2$ (i.e., dipole) as $\varepsilon \downarrow 0$. The detailed discussions will be reported elsewhere.

Figure 7: Profile of the derivative of the u -component of mono-spike solution.

(c) *Instability of general Neumann layer solutions.* Our approach and the instability result seem to be valid for more general Neumann layer solutions with internal transition layers like Figure 2(c)(d). We shall discuss more about this in a future paper.

(d) *The destination of the unstable manifold for the shadow system.* When the second diffusion coefficient D goes to infinity, we have the following limiting system, what is called the shadow system (see [14] and [16])

$$(4.1) \quad \begin{cases} u_t = \varepsilon^2 u_{xx} + f(u, \xi), & x \in I, \\ \xi_t = \int_I g(u, \xi) dx \\ u_x = 0, & x \in \partial I, \end{cases}$$

where $v = \xi$ is a constant function with respect to x . This could be regarded to be an intermediate system between the full system (1.2) and the scalar reaction-diffusion equation. In fact, it can be proved that (4.1) has both mono-Neumann layer solution and mono-internal transition layer solution which are unstable and stable, respectively (see [6]), and at the same time, (4.1) has a *Lyapounov function* when g is a linear function of u and ξ . Using these properties, we can determine the destinations of the one-dimensional unstable manifold of mono-Neumann layer solution, namely the stable internal transition layer solution and the stable constant state \bar{U} like Figure 3. More precise discussions will appear in [18].

(e) *Stability and instability of standing pulse solutions.* When the interval I becomes infinite, Ermentrout, Hastings and Troy [3] showed the existence of two different standing pulse solutions of (1.2a) with boundary conditions $\lim_{|x| \rightarrow \infty} (u, v)(x) = \bar{U}$ by using a shooting method. Singular perturbation method also works to obtain similar solutions, moreover the SLEP method clarifies the stability properties of them. Namely the large pulse solution is stable and the small pulse solution, which is an extension of mono-spike solution to the whole line, is unstable. The proofs of these results can be obtained by the combination of those of [15], [16], [19] and this paper. Note that, because of translation invariance, zero is always a known critical eigenvalue for this case. Hence the analysis of asymptotic behavior of critical eigenvalues becomes slightly easier than the finite interval case. Finally the instability of the small pulse solution is also obtained by Mimura and Ikeda [11] independently.

Appendix A (Construction of the mono-Neumann layer solution up to $O(\varepsilon)$).

We shall prove Theorem 2.1 which is a finer version of Fujii and Hosono

[5] showing the existence of the Neumann layer solution by using the approximate solution of $O(1)$ with respect to ε .

First, we shall construct the solutions (u_{\pm}, v_{\pm}) of the following two problems:

$$(1a) \quad \begin{cases} \varepsilon^2(u_-)_{xx} + f(u_-, v_-) = 0, \\ D(v_-)_{xx} + g(u_-, v_-) = 0, \end{cases} \quad x \in (0, \varepsilon\kappa),$$

with the boundary conditions

$$(1b) \quad \begin{cases} (u_-)_x(0) = 0, & u_-(\varepsilon\kappa) = h_0(\bar{v}), \\ (v_-)_x(0) = 0, & v_-(\varepsilon\kappa) = \alpha, \end{cases}$$

and

$$(2a) \quad \begin{cases} \varepsilon^2(u_+)_{xx} + f(u_+, v_+) = 0, \\ D(v_+)_{xx} + g(u_+, v_+) = 0, \end{cases} \quad x \in (\varepsilon\kappa, 1),$$

with the boundary conditions

$$(2b) \quad \begin{cases} (u_+)_x(1) = 0, & u_+(\varepsilon\kappa) = h_0(\bar{v}), \\ (v_+)_x(1) = 0, & v_+(\varepsilon\kappa) = \alpha. \end{cases}$$

Next, we obtain α and κ to be the functions of ε such that u_{\pm} and v_{\pm} satisfy

$$\begin{cases} (u_-)_x(\varepsilon\kappa) = (u_+)_x(\varepsilon\kappa) \\ (v_-)_x(\varepsilon\kappa) = (v_+)_x(\varepsilon\kappa), \end{cases}$$

that is, they are classical solutions of (1.2).

A.1. Construction of solutions on $(0, \varepsilon\kappa)$.

Using the stretched variable $\xi = x/\varepsilon\kappa$, (1) can be converted to the problem on $I = (0, 1)$ as

$$(3a) \quad \begin{cases} (u_-)_{\xi\xi} + \kappa^2 f(u_-, v_-) = 0, \\ D(v_-)_{\xi\xi} + \varepsilon^2 \kappa^2 g(u_-, v_-) = 0, \end{cases} \quad \xi \in I$$

with

$$(3b) \quad \begin{cases} (u_-)_{\xi}(0) = 0, & u_-(1) = h_0(\bar{v}), \\ (v_-)_{\xi}(0) = 0, & v_-(1) = \alpha. \end{cases}$$

Let $\kappa = \kappa_0 + \varepsilon\kappa_1$ and $\alpha = \alpha_0 + \varepsilon\alpha_1$, where $\kappa_0 (>0)$, κ_1 , α_0 and α_1 will be determined later. We seek the solution of (3) in the following form:

$$(4) \quad \begin{cases} u_-(\xi) = U_{0-}(\xi) + \varepsilon U_{1-}(\xi) + \varepsilon p_-(\xi) \\ v_-(\xi) = V_{0-}(\xi) + \varepsilon V_{1-}(\xi) + \varepsilon^2 V_{2-}(\xi) + \varepsilon^2 q_-(\xi), \end{cases}$$

where p_- and q_- are remainder terms. Substituting (4) into (3), we have

$$(5a) \quad \begin{cases} (U_{0-})_{\xi\xi} + \varepsilon(U_{1-})_{\xi\xi} + \varepsilon(p_-)_{\xi\xi} \\ \quad + (\kappa_0 + \varepsilon\kappa_1)^2 f(U_{0-} + \varepsilon U_{1-} + \varepsilon p_-, V_{0-} + \varepsilon V_{1-} + \varepsilon^2 V_{2-} + \varepsilon^2 q_-) = 0, \\ D(V_{0-})_{\xi\xi} + D\varepsilon(V_{1-})_{\xi\xi} + D\varepsilon^2(V_{2-})_{\xi\xi} + D\varepsilon^2(q_-)_{\xi\xi} \\ \quad + \varepsilon^2(\kappa_0 + \varepsilon\kappa_1)^2 g(U_{0-} + \varepsilon U_{1-} + \varepsilon p_-, V_{0-} + \varepsilon V_{1-} + \varepsilon^2 V_{2-} + \varepsilon^2 q_-) = 0, \end{cases} \quad \xi \in I,$$

with

$$(5b) \quad \begin{cases} (U_{0-})_\xi(0) + \varepsilon(U_{1-})_\xi(0) + \varepsilon(p_-)_\xi(0) = 0, \\ U_{0-}(1) + \varepsilon U_{1-}(1) + \varepsilon p_-(1) = h_0(\bar{v}), \\ (V_{0-})_\xi(0) + \varepsilon(V_{1-})_\xi(0) + \varepsilon^2(V_{2-})_\xi(0) + \varepsilon^2(q_-)_\xi(0) = 0, \\ V_{0-}(1) + \varepsilon V_{1-}(1) + \varepsilon^2 V_{2-}(1) + \varepsilon^2 q_-(1) = \alpha_0 + \varepsilon\alpha_1. \end{cases}$$

Equating like power of ε^0 , we have $V_{0-}(\xi) \equiv \alpha_0$, and (5) is reduced to the scalar problem of U_{0-} :

$$(6) \quad \begin{cases} (U_{0-})_{\xi\xi} + \kappa_0^2 f(U_{0-}, \alpha_0) = 0, \\ (U_{0-})_\xi(0) = 0, \quad U_{0-}(1) = h_0(\bar{v}), \end{cases} \quad \xi \in I.$$

By using a phase plane analysis and assumption (A-3), we have the following two lemmas.

LEMMA 1. *Let $\alpha_0 = \bar{v}$. Then, for some positive constant κ_0^* , there exists a unique monotone decreasing solution $U_{0-}(\xi)$ defined on \mathbf{R}_+ such that $U_{0-}(\xi)$ satisfies (6) and $U_{0-}(+\infty) = \bar{u}$, that is, $U_{0-}(\xi) = \tilde{u}^*(\kappa_0^* \xi)$ (see Lemma 2.2).*

LEMMA 2. *There is a positive constant c_1 such that for any $\alpha_0 \in (\bar{v} - c_1, \bar{v} + c_1)$ and $\kappa_0 \in (\kappa_0^* - c_1, \kappa_0^* + c_1)$, there exists a unique monotone decreasing solution $U_{0-}(\xi, \alpha_0, \kappa_0)$ of (6).*

Using Lemma 2 and equating like power of ε in (5), we have $V_{1-}(\xi) \equiv \alpha_1$ and the following problem for U_{1-} :

$$(7) \quad \begin{cases} (U_{1-})_{\xi\xi} + \kappa_0^2 \{f_u(U_{0-}, \alpha_0)U_{1-} + f_v(U_{0-}, \alpha_0)\alpha_1\} \\ \quad + 2\kappa_0\kappa_1 f(U_{0-}, \alpha_0) = 0, \\ (U_{1-})_\xi(0) = 0, \quad U_{1-}(1) = 0, \end{cases} \quad \xi \in I.$$

Since it follows from (6) that $\phi_-(\xi) \equiv (U_{0-})_\xi(\xi, \alpha_0, \kappa_0) < 0$ is a nontrivial solution of the differential operator acting on U_{1-} , we can construct the Green function $G(\xi, \eta)$ by using it, namely, setting $\psi(\xi) \equiv \phi_-(\xi) \int_1^\xi d\eta / \phi_-(\eta)^2$ and $\bar{\phi}(\xi) \equiv \phi_-(\xi) - \phi'_-(0)\psi(\xi)/\psi'(0)$, we have the Green function defined by

$$G(\xi, \eta) \equiv \begin{cases} \psi(\xi)\bar{\phi}(\eta) & \text{for } 0 \leq \eta < \xi \\ \bar{\phi}(\xi)\psi(\eta) & \text{for } \xi \leq \eta \leq 1. \end{cases}$$

Therefore, a solution $U_{1-}(\xi, \Delta)$ of (7) is given by

$$(8) \quad U_{1-}(\xi, \Delta) = - \int_0^1 G(\xi, \eta) \{ \kappa_0^2 \alpha_1 f_v(U_{0-}, \alpha_0) + 2\kappa_0 \kappa_1 f(U_{0-}, \alpha_0) \} d\eta,$$

where $\Delta = (\alpha_0, \alpha_1, \kappa_0, \kappa_1)$. It follows from (8) that

LEMMA 3. For any $\Delta \in \Gamma_1 \equiv \{ \Delta \in \mathbf{R}^4 | \alpha_0 \in (\bar{v} - c_1, \bar{v} + c_1), \kappa_0 \in (\kappa_0^* - c_1, \kappa_0^* + c_1) \text{ and } \alpha_1, \kappa_1 \in (-c_1, c_1) \}$, there exists a unique solution $U_{1-}(\xi, \Delta)$ of (7).

Equating like power of ε^2 in (5), we have the following problem of V_{2-} :

$$\begin{cases} D(V_{2-})_{\xi\xi} + \kappa_0^2 g(U_{0-}, \alpha_0) = 0, & \xi \in I. \\ (V_{2-})_\xi(0) = 0, \quad V_{2-}(1) = 0, \end{cases}$$

The solution V_{2-} is given by

$$V_{2-}(\xi, \alpha_0, \kappa_0) = \frac{\kappa_0^2}{D} \int_\xi^1 \int_0^\xi g(U_{0-}, \alpha_0) dt d\xi.$$

From (5b) and lemmas 2, 3, p_- and q_- satisfy the boundary conditions

$$\begin{cases} (p_-)_\xi(0) = 0, & p_-(1) = 0, \\ (q_-)_\xi(0) = 0, & q_-(1) = 0. \end{cases}$$

Therefore, we look for the solution (p_-, q_-) which belongs to $X = C_{1,0}^2(I) \times C_{1,0}^2(I)$. Let $t = (p_-, q_-)$ and $Y = C^0(I) \times C^0(I)$. Dividing the left hand side of (5a) by ε and putting it to $T(t, \varepsilon, \Delta)$, we find that T is the operator from X to Y and continuously differentiable of t for (ε, Δ) . Analogously as in Lemma 9 of [12], we have the following lemma.

LEMMA 4. There exists a positive constant ε_1 such that the following estimates hold for $(\varepsilon, \Delta) \in (0, \varepsilon_1) \times \Gamma_1$:

- (i) $\|T(0, \varepsilon, \Delta)\|_Y \leq K_0 \varepsilon,$
- (ii) $\|T_t(0, \varepsilon, \Delta)^{-1}\|_{\mathcal{L}(Y, X)} \leq K_1,$
- (iii) $\|T_t(t_1, \varepsilon, \Delta) - T_t(t_2, \varepsilon, \Delta)\|_{\mathcal{L}(X, Y)} \leq K_2 \|t_1 - t_2\|_X,$

where K_i ($i = 0, 1, 2$) are positive constants independent of (ε, Δ) and T_t is the Fréchet derivative of T with respect to t .

Applying the generalized implicit function theorem in [4] to $T = 0$, we have

THEOREM 5. *There exist solutions $t(\varepsilon, \Delta) = (p_-(\varepsilon, \Delta), q_-(\varepsilon, \Delta))$ of $T = 0$ for $(\varepsilon, \Delta) \in (0, \varepsilon_1) \times \Gamma_1$ such that $t(\varepsilon, \Delta)$ depends continuously on (ε, Δ) in X -topology, and satisfies $\lim_{\varepsilon \downarrow 0} \|t(\varepsilon, \Delta)\|_X = 0$ uniformly in $(0, \varepsilon_1) \times \Gamma_1$.*

Thus, we obtain solutions of (1) of the form

$$(9) \quad \begin{cases} u_-(x, \varepsilon, \Delta) = U_{0-}(x/\varepsilon\kappa, \alpha_0, \kappa_0) + \varepsilon U_{1-}(x/\varepsilon\kappa, \Delta) + \varepsilon p_-(x/\varepsilon\kappa, \varepsilon, \Delta) \\ v_-(x, \varepsilon, \Delta) = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2 V_{2-}(x/\varepsilon\kappa, \alpha_0, \kappa_0) + \varepsilon^2 q_-(x/\varepsilon\kappa, \varepsilon, \Delta). \end{cases}$$

A.2. Construction of solutions on $(\varepsilon\kappa, 1)$.

By using the transformation $y = (x - \varepsilon\kappa)/(1 - \varepsilon\kappa)$, (2) can be written as

$$(10a) \quad \begin{cases} \varepsilon^2(u_+)_{yy} + (1 - \varepsilon\kappa)^2 f(u_+, v_+) = 0, \\ D(v_+)_{yy} + (1 - \varepsilon\kappa)^2 g(u_+, v_+) = 0, \end{cases} \quad y \in I,$$

with

$$(10b) \quad \begin{cases} (u_+)_y(1) = 0, & u_+(0) = h_0(\bar{v}), \\ (v_+)_y(1) = 0, & v_+(0) = \alpha. \end{cases}$$

We first construct outer approximations of (10) in the following form:

$$\begin{cases} u_+(y) = u_{0+}(y) + \varepsilon u_{1+}(y) \\ v_+(y) = v_{0+}(y) + \varepsilon v_{1+}(y). \end{cases}$$

Substituting this into (10), we have

$$(11a) \quad \begin{cases} \varepsilon^2(u_{0+})_{yy} + \varepsilon^3(u_{1+})_{yy} + (1 - \varepsilon\kappa)^2 f(u_{0+} + \varepsilon u_{1+}, v_{0+} + \varepsilon v_{1+}) = 0, \\ D(v_{0+})_{yy} + D\varepsilon(v_{1+})_{yy} + (1 - \varepsilon\kappa)^2 g(u_{0+} + \varepsilon u_{1+}, v_{0+} + \varepsilon v_{1+}) = 0, \end{cases} \quad y \in I,$$

with

$$(11b) \quad \begin{cases} (u_{0+})_y(1) + \varepsilon(u_{1+})_y(1) = 0, & u_{0+}(0) + \varepsilon u_{1+}(0) = h_0(\bar{v}), \\ (v_{0+})_y(1) + \varepsilon(v_{1+})_y(1) = 0, & v_{0+}(0) + \varepsilon v_{1+}(0) = \alpha_0 + \varepsilon\alpha_1. \end{cases}$$

Equating like power of ε^0 , we have $f(u_{0+}, v_{0+}) = 0$. Letting $u_{0+} = h_-(v_{0+})$, v_{0+} must satisfy

$$\begin{cases} D(v_{0+})_{yy} + G_-(v_{0+}) = 0, & y \in I, \\ (v_{0+})_y(1) = 0, & v_{0+}(0) = \alpha_0, \end{cases}$$

where $G_-(v) = g(h_-(v), v)$. It turns out later that when ε tends to zero, v_{0+} must satisfy the Neumann boundary condition at $x = 0$ from the matching condition of v . Hence in view of the above equations and the monotonicity of $G_-(v)$, we see that the only solution, which meets the matching condition, is \bar{v} , that is, $\alpha_0 = \bar{v}$.

Equating like power of ε in (11), we have $f_u^* u_{1+} + f_v^* v_{1+} = 2\kappa_0 f(\bar{u}, \bar{v})$, that is, $f_u^* u_{1+} + f_v^* v_{1+} = 0$, where $f_u^* = f_u(\bar{u}, \bar{v})$ and $f_v^* = f_v(\bar{u}, \bar{v})$. Therefore, v_{1+} must satisfy

$$(12) \quad \begin{cases} D(v_{1+})_{yy} + \left(g_v^* - \frac{f_v^* g_u^*}{f_u^*}\right) v_{1+} = 0, \\ (v_{1+})_y(1) = 0, \quad v_{1+}(0) = \alpha_1, \end{cases} \quad y \in I,$$

where $g_u^* = g_u(\bar{u}, \bar{v})$ and $g_v^* = g_v(\bar{u}, \bar{v})$. Let $a^* = g_v^* - f_v^* g_u^* / f_u^* (< 0, \text{ see assumption (A-5)(b)})$ and $\sigma_{\pm} = \pm \sqrt{-a^*/D}$. Then the solution $v_{1+}(y, \alpha_1)$ of (12) and u_{1+} are obtained by

$$(13) \quad \begin{cases} v_{1+}(y, \alpha_1) = C_1 e^{\sigma_+(1-y)} + C_2 e^{\sigma_-(1-y)}, \\ u_{1+}(y, \alpha_1) = -\frac{f_v^*}{f_u^*} v_{1+}(y, \alpha_1), \end{cases}$$

where $C_1 = \alpha_1 \sigma_- / (\sigma_- e^{\sigma_+} - \sigma_+ e^{\sigma_-})$ and $C_2 = -\sigma_+ C_1 / \sigma_-$.

Next, we construct inner approximations of (10). By using $\eta = y(1 - \varepsilon\kappa)/\varepsilon\kappa$, (10) can be rewritten as

$$(14a) \quad \begin{cases} (u_+)_{\eta\eta} + \kappa^2 f(u_+, v_+) = 0, \\ D(v_+)_{\eta\eta} + \varepsilon^2 \kappa^2 g(u_+, v_+) = 0, \end{cases}$$

with

$$(14b) \quad u_+(0) = h_0(\bar{v}), \quad v_+(0) = \alpha.$$

We seek approximations of the form

$$(15) \quad \begin{cases} u_+(\eta) = \bar{u} + \varepsilon u_{1+}(\eta) + U_{0+}(\eta) + \varepsilon U_{1+}(\eta), \\ v_+(\eta) = \bar{v} + \varepsilon v_{1+}(\eta) + \varepsilon^2 V_{2+}(\eta) + \varepsilon^3 V_{3+}(\eta). \end{cases}$$

Substituting (15) into (14), we have

$$(16a) \quad \begin{cases} \varepsilon^3 \kappa^2 (u_{1+})_{yy} + (1 - \varepsilon\kappa)^2 [(U_{0+})_{\eta\eta} + \varepsilon (U_{1+})_{\eta\eta}] \\ \quad + \kappa^2 (1 - \varepsilon\kappa)^2 f(u_+, v_+) = 0, \\ D\varepsilon \kappa^2 (v_{1+})_{yy} + D\varepsilon^2 (1 - \varepsilon\kappa)^2 [(V_{2+})_{\eta\eta} + \varepsilon (V_{3+})_{\eta\eta}] \\ \quad + (1 - \varepsilon\kappa)^2 \kappa^2 g(u_+, v_+) = 0, \end{cases}$$

with

$$(16b) \quad \begin{cases} \bar{u} + \varepsilon u_{1+}(0) + U_{0+}(0) + \varepsilon U_{1+}(0) = h_0(\bar{v}), \\ \bar{v} + \varepsilon v_{1+}(0) + \varepsilon^2 V_{2+}(0) + \varepsilon^3 V_{3+}(0) = \alpha. \end{cases}$$

Equating like power of ε^0 , we have the equation of U_{0+} as follows:

$$(17) \quad \begin{cases} (U_{0+})_{\eta\eta} + \kappa_0^2 f(\bar{u} + U_{0+}, \bar{v}) = 0, \\ \bar{u} + U_{0+}(0) = h_0(\bar{v}). \end{cases}$$

It follows from Lemma 1 that there exists a unique monotone decreasing solution $U_{0+}(\eta, \kappa_0)$ of (17) with the boundary condition $U_{0+}(+\infty) = 0$ for any positive constant κ_0 .

Using this result and equating like power of ε in (16), we have

$$(18) \quad \begin{cases} (U_{1+})_{\eta\eta} + \kappa_0^2 f_u(\bar{u} + U_{0+}, \bar{v}) U_{1+} = F, \\ U_{1+}(0) = -u_{1+}(0), \end{cases}$$

where $F = -\kappa_0^2 \{f_u(\bar{u} + U_{0+}, \bar{v}) u_{1+}(0) + f_v(\bar{u} + U_{0+}, \bar{v}) v_{1+}(0)\} - 2\kappa_0 \kappa_1 f(\bar{u} + U_{0+}, \bar{v})$. Letting $(U_{0+})_{\eta}(\eta, \kappa_0) = \phi_+(\eta)$, we obtain a solution $U_{1+}(\eta, \alpha_1, \kappa_0, \kappa_1)$ of (18) with the boundary condition $U_{1+}(+\infty) = 0$ by

$$(19) \quad U_{1+}(\eta, \alpha_1, \kappa_0, \kappa_1) = -\phi_+(\eta) \int_0^\eta \int_\sigma^{+\infty} \frac{\phi_+(t)}{\phi_+(\sigma)^2} F(t) dt d\sigma - u_{1+}(0) \frac{\phi_+(\eta)}{\phi_+(0)}.$$

Next, equating like powers of ε^2 and ε^3 in (16), we have the problem of V_{2+} and V_{3+} as follows:

$$\begin{cases} D(V_{2+})_{\eta\eta} + \kappa_0^2 g(\bar{u} + U_{0+}, \bar{v}) = 0, \\ V_{2+}(0) = 0, \\ \begin{cases} D\kappa_0^2(v_{1+})_{yy}(0) + D(V_{3+})_{\eta\eta} + \kappa_0^2 \{g_u(\bar{u} + U_{0+}, \bar{v})(u_{1+}(0) + U_{1+}) \\ \quad + g_v(\bar{u} + U_{0+}, \bar{v})v_{1+}(0)\} + 2\kappa_0 \kappa_1 g(\bar{u} + U_{0+}, \bar{v}) = 0, \\ V_{3+}(0) = 0. \end{cases} \end{cases}$$

The solutions V_{2+} and V_{3+} bounded on \mathbf{R}_+ are given by

$$\begin{cases} V_{2+}(\eta, \kappa_0) = \bar{V}_2(\eta, \kappa_0) - \bar{V}_2(0, \kappa_0), \\ \bar{V}_2(\eta, \kappa_0) = -\frac{\kappa_0^2}{D} \int_\eta^{+\infty} \int_\sigma^{+\infty} g(\bar{u} + U_{0+}, \bar{v}) dt d\sigma, \end{cases}$$

and

$$\left\{ \begin{aligned} V_{3+}(\eta, \alpha_1, \kappa_0, \kappa_1) &= \bar{V}_3(\eta, \alpha_1, \kappa_0, \kappa_1) - \bar{V}_3(0, \alpha_1, \kappa_0, \kappa_1), \\ \bar{V}_{3+}(\eta, \alpha_1, \kappa_0, \kappa_1) &= -\frac{1}{D} \int_{\eta}^{+\infty} \int_{\sigma}^{+\infty} \{ -D\kappa_0^2(v_{1+})_{tt}(0) - 2\kappa_0\kappa_1 g(\bar{u} + U_{0+}, \bar{v}) \\ &\quad - \kappa_0^2 [g_u(\bar{u} + U_{0+}, \bar{v})(u_1(0) + U_{1+}) \\ &\quad + g_v(\bar{u} + U_{0+}, \bar{v})v_1(0)] \} dt d\sigma. \end{aligned} \right.$$

We seek the solution of (10) in the following form:

$$(20) \quad \begin{cases} u_+ = \bar{u} + \varepsilon u_{1+} + (U_{0+} + \varepsilon U_{1+})\theta + \varepsilon p_+ + \varepsilon h'_-(\bar{v})q_+, \\ v_+ = \bar{v} + \varepsilon v_{1+} + \varepsilon^2(V_{2+} + \varepsilon V_{3+})\theta + \varepsilon q_+, \end{cases}$$

where $\theta(y)$ is a C^∞ -cutoff function defined by

$$\theta(y) = \begin{cases} 1 & \text{for } y \in [0, 1/4], \\ 0 & \text{for } y \in [1/2, 1]. \end{cases}$$

Substituting (20) into (10) and dividing it by ε , we have for $t \equiv (p_+, q_+)$

$$\left\{ \begin{aligned} P(t, \varepsilon, \alpha_0, \kappa_0, \kappa_1) &\equiv \left\{ \varepsilon^2(u_{1+})_{yy} + \varepsilon^2\theta_{yy}(U_{0+} + \varepsilon U_{1+}) \right. \\ &\quad + 2\theta_y \frac{\varepsilon}{\kappa} (1 - \varepsilon\kappa)[(U_{0+})_\eta + \varepsilon(U_{1+})_\eta] \\ &\quad + \frac{\theta}{\kappa^2} (1 - \varepsilon\kappa)^2[(U_{0+})_{\eta\eta} + \varepsilon(U_{1+})_{\eta\eta}] \\ &\quad + \varepsilon^2(p_+)_{yy} + \varepsilon^3 h'_-(\bar{v})(q_+)_{yy} \\ &\quad \left. + (1 - \varepsilon\kappa)^2 f(u_+, v_+) \right\} / \varepsilon = 0, \\ Q(t, \varepsilon, \alpha_1, \kappa_0, \kappa_1) &\equiv \left\{ D\varepsilon(v_{1+})_{yy} + D\varepsilon^2\theta_{yy}(V_{2+} + \varepsilon V_{3+}) \right. \\ &\quad + 2D\theta_y \frac{\varepsilon}{\kappa} (1 - \varepsilon\kappa)[(V_{2+})_\eta + \varepsilon(V_{3+})_\eta] \\ &\quad + \frac{D\theta}{\kappa^2} (1 - \varepsilon\kappa)^2[(V_{2+})_{\eta\eta} + \varepsilon(V_{3+})_{\eta\eta}] + D\varepsilon(q_+)_{yy} \\ &\quad \left. + (1 - \varepsilon\kappa)^2 g(u_+, v_+) \right\} / \varepsilon = 0. \end{aligned} \right.$$

Then, the boundary conditions of (p_+, q_+) are given by

$$\begin{cases} (p_+)_y(1) = 0, & p_+(0) = 0, \\ (q_+)_y(1) = 0, & q_+(0) = 0. \end{cases}$$

Letting $T(t, \varepsilon, \pi) \equiv (P, Q)$, $\pi \equiv (\alpha_1, \kappa_0, \kappa_1)$ and $X^\varepsilon = C_{\varepsilon,1}^2(I) \times C_{1,1}^2(I)$, we find that T is the operator from X^ε to Y and differentiable of t for (ε, π) . Analogously as in Lemma 4.3 of [8] and Lemma 4.3 of [13], we have the following lemma.

LEMMA 6. *There exist positive constants c_2 and ε_2 such that for any $\pi \in \Gamma_2 \equiv \{\pi \in \mathbf{R}^3 | \kappa_0 \in (\kappa_0^* - c_2, \kappa_0^* + c_2)\}$ and $\alpha_1, \kappa_1 \in (-c_2, c_2)\}$ and $\varepsilon \in (0, \varepsilon_2)$, the following estimates hold:*

- (i) $\|T(0, \varepsilon, \pi)\|_Y \leq K_3 \varepsilon,$
- (ii) $\|T_t(0, \varepsilon, \pi)^{-1}\|_{\mathcal{L}(Y, X^\varepsilon)} \leq K_4,$
- (iii) *for any $t_1, t_2 \in X^\varepsilon$,*

$$\|T_t(t_1, \varepsilon, \pi) - T_t(t_2, \varepsilon, \pi)\|_{\mathcal{L}(X^\varepsilon, Y)} \leq K_5 \|t_1 - t_2\|_{X^\varepsilon},$$

where K_i ($i = 3, 4, 5$) are positive constants independent of (ε, π) .

Applying the generalized implicit function theorem to $T = 0$, we have

THEOREM 7. *There exist solutions $t(\varepsilon, \pi) = (p_+(\varepsilon, \pi), q_+(\varepsilon, \pi))$ of $T = 0$ for $(\varepsilon, \pi) \in (0, \varepsilon_2) \times \Gamma_2$ such that $t(\varepsilon, \pi)$ depends continuously on (ε, π) in X^ε -topology, and satisfies $\lim_{\varepsilon \downarrow 0} \|t(\varepsilon, \pi)\|_{X^\varepsilon} = 0$ uniformly in $(0, \varepsilon_2) \times \Gamma_2$.*

Thus, we obtain solutions of (10) of the form

$$(21) \quad \begin{cases} u_+(x, \varepsilon, \pi) = \bar{u} + \varepsilon u_{1+}((x - \varepsilon\kappa)/(1 - \varepsilon\kappa), \alpha_1) \\ \quad + \theta[U_{0+}(x/\varepsilon\kappa - 1, \kappa_0) + \varepsilon U_{1+}(x/\varepsilon\kappa - 1, \pi)] \\ \quad + \varepsilon p_+((x - \varepsilon\kappa)/(1 - \varepsilon\kappa), \varepsilon, \pi) \\ \quad + \varepsilon h'_-(\bar{v})q_+((x - \varepsilon\kappa)/(1 - \varepsilon\kappa), \varepsilon, \pi), \\ v_+(x, \varepsilon, \pi) = \bar{v} + \varepsilon v_{1+}((x - \varepsilon\kappa)/(1 - \varepsilon\kappa), \alpha_1) \\ \quad + \varepsilon^2 \theta[V_{2+}(x/\varepsilon\kappa - 1, \kappa_0) + \varepsilon V_{3+}(x/\varepsilon\kappa - 1, \pi)] \\ \quad + \varepsilon q_+((x - \varepsilon\kappa)/(1 - \varepsilon\kappa), \varepsilon, \pi). \end{cases}$$

A.3. Construction of solutions on I .

We seek solutions on the whole interval I . Putting $\alpha_0 = \bar{v}$ and $\kappa_0 = \kappa_0^*$, we see from (9) and (21) that u_\pm and v_\pm constructed in the above subsections

satisfy the C^1 -matching conditions up to $O(1)$ as $\varepsilon \downarrow 0$. Taking α_1 and κ_1 appropriately, we match the derivatives up to $O(\varepsilon)$ as follows:

$$(22) \quad \left\{ \begin{array}{l} \Phi(\varepsilon, \alpha_1, \kappa_1) = \left[\frac{d}{dx} u_-(x, \varepsilon, \bar{v}, \alpha_1, \kappa_0^*, \kappa_1) \right. \\ \quad \left. - \frac{d}{dx} u_+(x, \varepsilon, \alpha_1, \kappa_0^*, \kappa_1) \right] \Big|_{x=\varepsilon\kappa} = 0, \\ \Psi(\varepsilon, \alpha_1, \kappa_1) = \frac{1}{\varepsilon} \left[\frac{d}{dx} v_-(x, \varepsilon, \bar{v}, \alpha_1, \kappa_0^*, \kappa_1) \right. \\ \quad \left. - \frac{d}{dx} v_+(x, \varepsilon, \alpha_1, \kappa_0^*, \kappa_1) \right] \Big|_{x=\varepsilon\kappa} = 0, \end{array} \right.$$

where u_{\pm} and v_{\pm} are given in (9) and (21). By Theorems 5 and 7, it holds that Φ and Ψ are uniformly continuous in $(\varepsilon, \alpha_1, \kappa_1)$, that is, they are extended continuously to $\varepsilon = 0$. Setting $\varepsilon = 0$ in (22) and using (8), (13), (19), we have

$$\Phi(0, \alpha_1, \kappa_1) = \frac{1}{\kappa_0^*} \left[\frac{d}{d\xi} U_{1-}(1, \bar{v}, \alpha_1, \kappa_0^*, \kappa_1) - \frac{d}{d\xi} U_{1+}(0, \alpha_1, \kappa_0^*, \kappa_1) \right],$$

$$\begin{aligned} \Psi(0, \alpha_1, \kappa_1) &= -\frac{d}{dy} v_{1+}(0, \alpha_1) - \frac{\kappa_0^*}{D} \int_0^{+\infty} g(U_{0-}, \bar{v}) d\xi \\ &= \alpha_1 \sigma_- \sigma_+ \left(\frac{e^{\sigma_+} - e^{\sigma_-}}{\sigma_+ e^{\sigma_-} - \sigma_- e^{\sigma_+}} \right) - \frac{\kappa_0^*}{D} \int_0^{+\infty} g(U_{0-}, \bar{v}) d\xi. \end{aligned}$$

Putting $\alpha_1^* = \frac{\kappa_0^*}{D} \int_0^{+\infty} g(U_{0-}, \bar{v}) d\xi / (\sigma_+ e^{\sigma_-} - \sigma_- e^{\sigma_+}) / \sigma_- \sigma_+ (e^{\sigma_+} - e^{\sigma_-})$, we have $\Psi(0, \alpha_1^*, \kappa_1) = 0$ and $(\partial/\partial\alpha_1)\Psi(0, \alpha_1, \kappa_1)|_{\alpha_1=\alpha_1^*} \neq 0$. Next, we seek κ_1 such that $\Phi(0, \alpha_1^*, \kappa_1) = 0$. From (8) and (19), it follows that

$$\begin{aligned} \Phi(0, \alpha_1^*, \kappa_1) &= \frac{-\kappa_0^*}{\phi_-(1)} \left\{ \alpha_1^* \kappa_0^* \left[\int_0^1 \psi(\xi) f_v(U_{0-}, \bar{v}) d\xi \right. \right. \\ &\quad \left. \left. + \int_1^{+\infty} \phi(\eta) f_v(U_{0+}, \bar{v}) d\eta \right] + 2\kappa_1 \int_0^1 (\psi - \phi) f(U_{0-}, \bar{v}) d\xi \right\}. \end{aligned}$$

We find that $\Phi(0, \alpha_1^*, \kappa_1^*) = 0$, where $\kappa_1^* = -\alpha_1^* \kappa_0^* [\int_0^1 \psi(\xi) f_v(U_{0-}, \bar{v}) d\xi + \int_1^{+\infty} \phi(\eta) f_v(u_{0-}, \bar{v}) d\eta] / 2 \int_0^1 (\psi - \phi) f(U_{0-}, \bar{v}) d\xi$. Applying the implicit function theorem to $\Phi = \Psi = 0$, we can show that there is a positive constant ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$, there exist two functions $\alpha_1(\varepsilon)$ and $\kappa_1(\varepsilon)$ satisfying $\Phi(\varepsilon, \alpha_1(\varepsilon), \kappa_1(\varepsilon)) = \Psi(\varepsilon, \alpha_1(\varepsilon), \kappa_1(\varepsilon)) = 0$, $\lim_{\varepsilon \downarrow 0} \alpha_1(\varepsilon) = \alpha_1^*$ and $\lim_{\varepsilon \downarrow 0} \kappa_1(\varepsilon) = \kappa_1^*$.

Therefore, $U^\varepsilon(x) \equiv (u(x, \varepsilon), v(x, \varepsilon))$ defined by

$$u(x, \varepsilon) = \begin{cases} u_-(x, \varepsilon, \bar{v}, \alpha_1(\varepsilon), \kappa_0^*, \kappa_1(\varepsilon)) & \text{for } x \in (0, \varepsilon\kappa(\varepsilon)), \\ u_+(x, \varepsilon, \bar{v}, \alpha_1(\varepsilon), \kappa_0^*, \kappa_1(\varepsilon)) & \text{for } x \in (\varepsilon\kappa(\varepsilon), 1), \end{cases}$$

and

$$v(x, \varepsilon) = \begin{cases} v_-(x, \varepsilon, \bar{v}, \alpha_1(\varepsilon), \kappa_0^*, \kappa_1(\varepsilon)) & \text{for } x \in (0, \varepsilon\kappa(\varepsilon)), \\ v_+(x, \varepsilon, \bar{v}, \alpha_1(\varepsilon), \kappa_0^*, \kappa_1(\varepsilon)) & \text{for } x \in (\varepsilon\kappa(\varepsilon), 1), \end{cases}$$

becomes an ε -family of solutions to (1.2). This completes the proof of Theorem 2.1.

Appendix B (Asymptotic behavior of the principal eigenvalue and its eigenfunction).

We shall prove the remaining part of Lemma 2.3 by constructing the principal eigenvalue ζ and its eigenfunction w such that

$$(23a) \quad \varepsilon^2 \frac{d^2}{dx^2} w + (f_u^\varepsilon - \zeta)w = 0, \quad x \in I,$$

with

$$(23b) \quad w_x(0) = 0, \quad w_x(1) = 0, \quad w(\varepsilon\kappa(\varepsilon)) = 1,$$

where $\kappa(\varepsilon) = \kappa_0^* + \kappa_1(\varepsilon)$.

In order to solve (23), we shall construct the solutions w_\pm of the following problems for any ζ belonging to some real interval:

$$(24a) \quad \varepsilon^2 \frac{d^2}{dx^2} w_- + (f_u^\varepsilon - \zeta)w_- = 0, \quad x \in (0, \varepsilon\kappa(\varepsilon))$$

with

$$(24b) \quad (w_-)_x(0) = 0, \quad w_-(\varepsilon\kappa(\varepsilon)) = 1,$$

and

$$(25a) \quad \varepsilon^2 \frac{d^2}{dx^2} w_+ + (f_u^\varepsilon - \zeta)w_+ = 0, \quad x \in (\varepsilon\kappa(\varepsilon), 1)$$

with

$$(25b) \quad (w_+)_x(1) = 0, \quad w_+(\varepsilon\kappa(\varepsilon)) = 1.$$

Next, we determine ζ to be a function of ε such that w_\pm satisfy

$$(w_-)_x(\varepsilon\kappa(\varepsilon)) = (w_+)_x(\varepsilon\kappa(\varepsilon)).$$

B.1. Construction of solutions on $(0, \varepsilon\kappa(\varepsilon))$.

By using the transformation $\xi = x/\varepsilon\kappa(\varepsilon)$, (24) can be written as

$$(26) \quad \begin{cases} (w_-)_{\xi\xi} + \kappa(\varepsilon)^2(\tilde{f}_u^\varepsilon - \zeta)w_- = 0, & \xi \in I, \\ (w_-)_\xi(0) = 0, & w_-(1) = 1. \end{cases}$$

We seek the solution of (26) in the following form:

$$(27) \quad \begin{cases} w_-(\xi) = W_{0-}(\xi) + \varepsilon W_{1-}(\xi) + \varepsilon r_-(\xi), \\ \zeta = \zeta_0 + \varepsilon \hat{\zeta}_0, \end{cases}$$

where r_- is a remainder term. Substituting (27) into (26), we have

$$(28) \quad \begin{cases} (W_{0-})_{\xi\xi} + \varepsilon(W_{1-})_{\xi\xi} + \varepsilon(r_-)_{\xi\xi} \\ \quad + \kappa(\varepsilon)^2[\tilde{f}_u^\varepsilon - (\zeta_0 + \varepsilon \hat{\zeta}_0)](W_{0-} + \varepsilon W_{1-} + \varepsilon r_-) = 0, & \xi \in I, \\ (W_{0-})_\xi(0) + \varepsilon(W_{1-})_\xi(0) + \varepsilon(r_-)_\xi(0) = 0, \\ W_{0-}(1) + \varepsilon W_{1-}(1) + \varepsilon r_-(1) = 1. \end{cases}$$

Equating like power of ε^0 , we have the following problem of W_{0-} and ζ_0 :

$$(29) \quad \begin{cases} (W_{0-})_{\xi\xi} + (\kappa_0^*)^2(\tilde{f}_u^0 - \zeta_0)W_{0-} = 0, & \xi \in I, \\ (W_{0-})_\xi(0) = 0, & W_{0-}(1) = 1. \end{cases}$$

LEMMA 8. *There exists a unique positive constant ζ_0^* with which (29) has a unique bounded solution $W_0(\xi)$ on \mathbf{R}_+ .*

PROOF. (29) is the linearized eigenvalue problem of the following problem:

$$\begin{cases} (U_0)_{\xi\xi} + f(U_0, \bar{v}) = 0, & \xi \in \mathbf{R}_+, \\ (U_0)_\xi(0) = 0, & U_0(+\infty) = \bar{u}, \end{cases}$$

where the solution $U_0(\xi)$ is given in Lemma 1. Therefore, (29) has a real positive eigenvalue and the corresponding eigenfunction has a definite sign (for example, see [7]). Without loss of generality, we can assume that $W_0(\xi) > 0$ for $\xi \in \mathbf{R}_+$, $W_{0-}(\xi) = W_0(\xi)/W_0(1)$ is a solution of (29) with $\zeta_0 = \zeta_0^*$.

REMARK 1. Since we showed in the proof of Lemma 2.3 that (23a) has a unique positive eigenvalue, the eigenpair of (27) corresponding to ζ_0^* must be the principal one.

Equating like power of ε in (28), we have the following problem of $\hat{\zeta}_0$ and

W_{1-} :

$$(30) \quad \begin{cases} (W_{1-})_{\xi\xi} + (\kappa_0^*)^2(\tilde{f}_u^0 - \zeta_0^*)W_{1-} = F_- + \zeta_0(\kappa_0^*)^2W_{0-}, & \xi \in I, \\ (W_{1-})_{\xi}(0) = 0, & W_{1-}(1) = 0, \end{cases}$$

where $F_- = -[(\kappa_0^*)^2\tilde{f}_{uu}^0U_{1-} + (\kappa_0^*)^2\tilde{f}_{uu}^0\alpha_1^* + 2\kappa_0^*\kappa_1^*(\tilde{f}_u^0 - \zeta_0^*)]W_{0-}$.

LEMMA 9. For any $\zeta_0 \in \mathbf{R}$, (30) has a solution $W_{1-}(\xi, \zeta_0)$.

PROOF. Set $\phi(\xi) \equiv W_{0-}(\xi) \int_1^\xi d\eta / W_{0-}(\eta)^2$. (30) has the Green function $G(\xi, \eta)$ defined by

$$G(\xi, \eta) \equiv \begin{cases} \phi(\xi)W_{0-}(\eta) & \text{for } 0 \leq \eta < \xi \\ W_{0-}(\xi)\phi(\eta) & \text{for } \xi \leq \eta < 1. \end{cases}$$

Therefore, a solution $W_{1-}(\xi, \zeta_0)$ of (30) is given by

$$(31) \quad W_{1-}(\xi, \zeta_0) = \int_0^1 G(\xi, \eta)F_-(\eta) d\eta + \zeta_0(\kappa_0^*)^2 \int_0^1 G(\xi, \eta)W_{0-}(\eta) d\eta.$$

From (28) and Lemmas 8, 9, it follows that r_- satisfies the boundary conditions $(r_-)_{\xi}(0) = 0$, $r_-(1) = 0$. Dividing the left hand side of the equation in (28) by ε and putting it to $T(r_-, \varepsilon, \zeta_0)$, we find that T is the operator from $C_{1,0}^2(I)$ to $C^0(I)$ and continuously differentiable of r_- for ε . Then, we have the following lemma for T .

LEMMA 10. For any bounded interval B in \mathbf{R} , there exists a positive constant ε_3 such that the following estimates hold for $(\varepsilon, \zeta_0) \in (0, \varepsilon_3) \times B$:

$$(i) \quad \|T(0, \varepsilon, \zeta_0)\|_{C^0(I)} \leq K_6\varepsilon,$$

$$(ii) \quad \|T_r(0, \varepsilon, \zeta_0)^{-1}\|_{\mathcal{L}(C^0(I), C_{1,0}^2(I))} \leq K_7,$$

$$(iii) \quad \|T_r(r_1, \varepsilon, \zeta_0) - T_r(r_2, \varepsilon, \zeta_0)\|_{\mathcal{L}(C_{1,0}^2(I), C^0(I))} \leq K_8\|r_1 - r_2\|_{C_{1,0}^2(I)},$$

where K_i ($i = 6, 7, 8$) are positive constants independent of (ε, ζ_0) .

PROOF. Since (i) and (iii) are obvious, we only consider (ii). Noting that $T_r(0, \varepsilon, \zeta_0) = d^2/d\xi^2 + (\kappa_0^*)^2(\tilde{f}_u^0 - \zeta_0^*) + O(\varepsilon)$ as $\varepsilon \downarrow 0$ and using the Green function in Lemma 9, we can prove (ii).

Applying the generalized implicit function theorem to $T = 0$, we have

THEOREM 11. There exist solutions $r_-(\varepsilon, \zeta_0)$ of $T = 0$ for $(\varepsilon, \zeta_0) \in (0, \varepsilon_3) \times B$ such that $r_-(\varepsilon, \zeta_0)$ depends continuously on (ε, ζ_0) with respect to the topology of $C_{1,0}^2(I)$, and satisfies $\lim_{\varepsilon \downarrow 0} \|r_-(\varepsilon, \zeta_0)\|_{C_{1,0}^2(I)} = 0$ uniformly in $(\varepsilon, \zeta_0) \in (0, \varepsilon_3) \times B$.

Thus, we obtain solutions of (24) of the form

$$(32) \quad \begin{cases} w_-(x, \varepsilon, \hat{\zeta}_0) = W_{0-}(x/\varepsilon\kappa(\varepsilon)) + \varepsilon W_{1-}(x/\varepsilon\kappa(\varepsilon), \hat{\zeta}_0) + \varepsilon r_-(x/\varepsilon\kappa(\varepsilon), \varepsilon, \hat{\zeta}_0), \\ \zeta(\varepsilon, \hat{\zeta}_0) = \zeta_0^* + \varepsilon \hat{\zeta}_0. \end{cases}$$

B.2. Construction of solutions on $(\varepsilon\kappa(\varepsilon), 1)$.

By using the transformation $y = (x - \varepsilon\kappa(\varepsilon))/(1 - \varepsilon\kappa(\varepsilon))$, (25) can be written as

$$(33) \quad \begin{cases} \varepsilon^2 \frac{d^2}{dy^2} w_+ + (1 - \varepsilon\kappa(\varepsilon))^2 (f_u^\varepsilon - (\zeta_0^* + \varepsilon \hat{\zeta}_0)) w_+ = 0, & y \in I, \\ (w_+)_y(1) = 0, & w_+(0) = 1. \end{cases}$$

We first construct outer approximations of (33) in the following form:

$$(34) \quad w_+(y) = w_{0+}(y) + \varepsilon w_{1+}(y).$$

Substituting (34) into (33), we have

$$(35) \quad \begin{cases} \varepsilon^2 \frac{d^2}{dy^2} (w_{0+} + \varepsilon w_{1+}) + (1 - \varepsilon\kappa(\varepsilon))^2 (f_u^\varepsilon - (\zeta_0^* + \varepsilon \hat{\zeta}_0)) (w_{0+} + \varepsilon w_{1+}) = 0, & y \in I, \\ (w_{0+})_y(1) + (\varepsilon w_{1+})_y(1) = 0, \\ w_{0+}(0) + w_{1+}(0) = 1. \end{cases}$$

Equating like power of ε^0 , we have $(f_u^0 - \zeta_0^*) w_{0+} = 0$. Because $f_u^0 - \zeta_0^* \neq 0$ for $y \in (0, 1]$, it holds that $w_{0+} = 0$. Using this result and equating like power of ε in (35), we have $(f_u^0 - \zeta_0^*) w_{1+} = 0$, that is, $w_{1+} = 0$.

Next, we consider inner approximations of the form

$$(36) \quad w_+(\eta) = W_{0+}(\eta) + \varepsilon W_{1+}(\eta)$$

by using $\eta = x/(\varepsilon\kappa(\varepsilon)) - 1$. Substituting (36) into (33), we have

$$(37) \quad \begin{cases} (W_{0+} + \varepsilon W_{1+})_{\eta\eta} + \kappa(\varepsilon)^2 (\tilde{f}_u^\varepsilon - (\zeta_0^* + \varepsilon \hat{\zeta}_0)) (W_{0+} + \varepsilon W_{1+}) = 0, \\ W_{0+}(0) + \varepsilon W_{1+}(0) = 1. \end{cases}$$

Equating like power of ε^0 , we have the equation of W_{0+} as follows:

$$(38) \quad \begin{cases} (W_{0+})_{\eta\eta} + (\kappa_0^*)^2 (\tilde{f}_u^0 - \zeta_0^*) W_{0+} = 0, \\ W_{0+}(0) = 1. \end{cases}$$

Then, it follows from Lemma 8 that the bounded solution of (38) is given by $W_{0+}(\eta) = W_0(\eta + 1)/W_0(1)$.

Using this result and equating like power of ε in (37), we have

$$(39) \quad \begin{cases} (W_{1+})_{\eta\eta} + (\kappa_0^*)^2(\tilde{f}_u^0 - \zeta_0^*)W_{1+} - \hat{\zeta}_0(\kappa_0^*)^2W_{0+} + F_+ = 0, \\ W_{1+}(0) = 0, \end{cases}$$

where $F_+ = 2\kappa_0^*\kappa_1^*(\tilde{f}_u^0 - \zeta_0^*)W_{0+} + (\kappa_0^*)^2[\tilde{f}_{uu}^0(u_{1+}(0, 0) + U_{1+}\theta) + \tilde{f}_{uv}^0v_{1+}(0, 0)]W_{0+}$. By using W_{0+} , a bounded solution $W_{1+}(\eta, \hat{\zeta}_0)$ of (39) is given by the form

$$(40) \quad W_{1+}(\eta, \hat{\zeta}_0) = W_{0+}(\eta) \int_0^\eta \int_\sigma^{+\infty} \frac{W_{0+}(t)}{W_{0+}(\sigma)^2} (F_+(t) - (\kappa_0^*)^2\hat{\zeta}_0 W_{0+}(t)) dt d\sigma.$$

We seek the solution of (33) in the following form:

$$(41) \quad w_+ = (W_{0+} + \varepsilon W_{1+})\theta + \varepsilon r_+,$$

where r_+ is a remainder term and θ is the cutoff function. Substituting (41) into (33) and dividing it by ε , we have

$$\begin{cases} T(r_+, \varepsilon, \hat{\zeta}_0) \equiv \{ [(W_{0+})_{\eta\eta} + \varepsilon(W_{1+})_{\eta\eta}]\theta + \varepsilon^2\theta_{yy}(W_{0+} + \varepsilon W_{1+}) \\ \quad + 2\varepsilon\theta_y[(W_{0+})_\eta + \varepsilon(W_{1+})_\eta] + \varepsilon^3(r_+)_{yy} + (1 - \varepsilon\kappa(\varepsilon))^2 \\ \quad \times [f_u^\varepsilon - (\zeta_0^* + \varepsilon\hat{\zeta}_0)](W_{0+} + \varepsilon W_{1+})\theta + \varepsilon r_+ \} / \varepsilon = 0. \end{cases}$$

The boundary conditions of r_+ are given by $(r_+)_y(1) = 0$ and $r_+(0) = 0$. Therefore, we find that the operator T from $C_{\varepsilon,1}^2(I)$ to $C^0(I)$ is continuously differentiable of r_+ for $(\varepsilon, \hat{\zeta}_0)$. Then we have the following lemma for T .

LEMMA 12. *There exists a positive constant ε_4 such that for $(\varepsilon, \hat{\zeta}_0) \in (0, \varepsilon_4) \times B$, the following estimates hold:*

$$(i) \quad \|T(0, \varepsilon, \hat{\zeta}_0)\|_{C^0(I)} \leq K_9\varepsilon,$$

$$(ii) \quad \|T_r(0, \varepsilon, \hat{\zeta}_0)^{-1}\|_{\mathcal{L}(C^0(I), C_{\varepsilon,1}^2(I))} \leq K_{10},$$

$$(iii) \quad \|T_r(r_1, \varepsilon, \hat{\zeta}_0) - T_r(r_2, \varepsilon, \hat{\zeta}_0)\|_{\mathcal{L}(C_{\varepsilon,1}^2(I), C^0(I))} \leq K_{11}\|r_1 - r_2\|_{C_{\varepsilon,1}^2(I)},$$

where K_i ($i = 9, 10, 11$) are positive constants independent of $(\varepsilon, \hat{\zeta}_0)$.

PROOF. Since (i) and (iii) are obvious, we only consider (ii). Note that $T_r(0, \varepsilon, \hat{\zeta}_0) = \varepsilon^2 d^2/dy^2 + f_u^0 - \zeta_0^* + O(\varepsilon)$ as $\varepsilon \downarrow 0$. Analogously as in Lemma 3.2 in [4], we can prove (ii).

Applying the generalized implicit function theorem to $T = 0$, we have

THEOREM 13. *There exist solutions $r_+(\varepsilon, \hat{\zeta}_0)$ of $T = 0$ for $(\varepsilon, \hat{\zeta}_0) \in (0, \varepsilon_4) \times B$ with respect to the topology of $C_{\varepsilon,1}^2(I)$, and it holds that $\lim_{\varepsilon \downarrow 0} \|r_+(\varepsilon, \hat{\zeta}_0)\|_{C_{\varepsilon,1}^2(I)} = 0$ uniformly in $(0, \varepsilon_4) \times B$.*

Thus, we obtain solutions of (25) of the form

$$(42) \quad w_+(x, \varepsilon, \hat{\zeta}_0) = [W_{0+}((x - \varepsilon\kappa(\varepsilon))/\varepsilon\kappa(\varepsilon)) \\ + \varepsilon W_{1+}((x - \varepsilon\kappa(\varepsilon))/\varepsilon\kappa(\varepsilon), \hat{\zeta}_0)]\theta((x - \varepsilon\kappa(\varepsilon))/(1 - \varepsilon\kappa(\varepsilon)) \\ + \varepsilon r_+((x - \varepsilon\kappa(\varepsilon))/(1 - \varepsilon\kappa(\varepsilon)), \varepsilon, \hat{\zeta}_0)$$

B.3. Construction of solutions on I .

We seek solutions of (23) on I . From (32) and (42), it follows that w_{\pm} satisfy the C^1 -matching condition up to $O(1)$ as $\varepsilon \downarrow 0$. So, we consider that up to $O(\varepsilon)$ as follows:

$$(43) \quad \Phi(\varepsilon, \hat{\zeta}_0) = \kappa(\varepsilon) \left[\frac{d}{dx} w_-(x, \varepsilon, \hat{\zeta}_0) - \frac{d}{dx} w_+(x, \varepsilon, \hat{\zeta}_0) \right] \Big|_{x=\varepsilon\kappa(\varepsilon)}.$$

By Theorems 11 and 13, it holds that Φ is uniformly continuous in ε and $\hat{\zeta}_0$, that is, it is extended continuously to $\varepsilon = 0$. Setting $\varepsilon = 0$ in (43) and using (31), (32), (40) and (42), we have

$$\begin{aligned} \Phi(0, \hat{\zeta}_0) &= (W_{1-})_{\xi}(1, \hat{\zeta}_0) - (W_{1+})_{\eta}(0, \hat{\zeta}_0) \\ &\quad + \hat{\zeta}_0(\kappa_0^*)^2 \int_0^1 (W_0)^2(\xi) d\xi - \frac{2\kappa_1^*}{\kappa_0^*} \int_0^{+\infty} [(W_0)_{\eta}(\eta)]^2 d\eta \\ &\quad - (\kappa_0^*)^2 \left\{ \int_0^1 [\tilde{f}_{uu}^0 U_{1-} + \tilde{f}_{uv}^0 \alpha_1^*] (W_0)^2(\xi) d\xi \right. \\ &\quad \left. + \int_1^{+\infty} [\tilde{f}_{uu}^0 (u_1(0) + U_{1-}) + \tilde{f}_{uv}^0 \alpha_1^*] (W_0)^2(\eta) d\eta \right\}. \end{aligned}$$

Putting

$$(44) \quad \hat{\zeta}_0^* = \left[2\kappa_1^* \int_0^{+\infty} ((W_0)_{\eta}(\eta))^2 d\eta + (\kappa_0^*)^3 \left\{ \int_0^1 [\tilde{f}_{uu}^0 U_{1-} + \tilde{f}_{uv}^0 \alpha_1^*] (W_0)^2(\xi) d\xi \right. \right. \\ \left. \left. + \int_1^{+\infty} [\tilde{f}_{uu}^0 (u_1(0) + U_{1-}) + \tilde{f}_{uv}^0 \alpha_1^*] (W_0)^2(\eta) d\eta \right\} \right] / (\kappa_0^*)^3 \int_0^{+\infty} (W_0(\xi))^2 d\xi,$$

we have $\Phi(0, \hat{\zeta}_0^*) = 0$. Noting that $(\partial/\partial \hat{\zeta}_0)\Phi(0, \hat{\zeta}_0^*) = (\kappa_0^*)^2 \int_1^{+\infty} (W_0)^2(\xi) d\xi > 0$ and applying the implicit function theorem to $\Phi(\varepsilon, \hat{\zeta}_0) = 0$, we obtain that there is a positive constant ε_5 such that for any $\varepsilon \in (0, \varepsilon_5)$, there exists a function $\hat{\zeta}_0(\varepsilon)$ satisfying $\Phi(\varepsilon, \hat{\zeta}_0(\varepsilon)) = 0$ and $\lim_{\varepsilon \downarrow 0} \hat{\zeta}_0(\varepsilon) = \hat{\zeta}_0^*$. Therefore, $(w(x, \varepsilon), \hat{\zeta}_0(\varepsilon))$ defined by

$$w(x, \varepsilon) = \begin{cases} w_-(x, \varepsilon, \hat{\zeta}_0(\varepsilon)) & \text{for } x \in (0, \varepsilon\kappa(\varepsilon)) \\ w_+(x, \varepsilon, \hat{\zeta}_0(\varepsilon)) & \text{for } x \in (\varepsilon\kappa(\varepsilon), 1) \end{cases}$$

and

$$\zeta(\varepsilon) = \zeta_0^* + \varepsilon \hat{\zeta}_0^*(\varepsilon),$$

becomes an ε -family of solutions to (23) with $\zeta = \zeta(\varepsilon)$. This completes the proof of Lemma 2.3.

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