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Fibred Riemannian spaces with quasi Sasakian structure

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Introduction

Recently Y. Tashiro and the present author [35] have studied fibred Riemannian spaces with almost Hermitian or almost contact metric structure, and given its applications to tangent bundles of Riemannian spaces. In [23] we have also studied fibred Riemannian spaces with vanishing contact Bochner curvature tensor and constructed an example of such spaces which is not a Sasakian space form.

As the first step, it is natural to consider fibred Riemannian spaces with invariant fibres normal to the structure vector. Such a space does not admit nearly Sasakian or contact structure but a quasi Sasakian or cosymplectic structure. This is a motivation for our study of fibred Riemannian spaces with quasi Sasakian or cosymplectic structure.

The notion of quasi Sasakian structure on an almost contact metric manifold was first introduced by D. E. Blair [2] and its properties have been studied by himself, J. C. Gonzalez and D. Chinea [16], S. Kanemaki [19, 20], J. A. Oubina [28] and S. Tanno [33]. It is known that a quasi Sasakian manifold with $d\eta = 0$ or $2\Phi = d\eta$ is cosymplectic or Sasakian, respectively, and there is no quasi Sasakian structure of even rank [2].

An almost contact metric manifold, the structure tensor ϕ of which is Killing, is called a nearly cosymplectic manifold, which was introduced by D. E. Blair [3]. A five-dimensional sphere S^5 admits a nearly cosymplectic structure but not a cosymplectic structure. Besides, M. Capursi [7], D. Chinea and C. Gonzalez [8, 9], I. Goldberg and K. Yano [15], Z. Olszak [26] and J. Oubina [28] have introduced new classes of almost contact metric structures as generalizations of cosymplectic structure. Essential examples of the various structures are given in the papers cited above.

On the other hand, cosymplectic space forms were studied by S. S. Eum [13], S. Kanemaki [20] and G. D. Ludden [24]. S. S. Eum [13] defined the cosymplectic Bochner curvature tensor and it vanishes in a cosymplectic space form. K. Yano [36, 37] considered complex conformal or contact conformal connection to give sufficient conditions in order that the Bochner or contact Bochner curvature tensor vanishes. Similar studies were made by T. Kashiwada

[21] and T. Sakaguchi [30] in the Kaehlerian case and by M. Seino [32] in the Sasakian case.

As the second step, it is natural to ask conditions in order that a cosymplectic manifold has the vanishing cosymplectic Bochner curvature tensor and to explore relations of a cosymplectic conformal connection and the cosymplectic Bochner curvature tensor.

In Chapter I, we shall give preliminaries of basic definitions and formulae of the fibred Riemannian space and study relations among various almost contact structures.

In Chapter II, we shall deal with nearly Sasakian, quasi Sasakian, Sasakian or cosymplectic structures in fibred almost contact spaces with invariant fibres tangent to the structure vector. As its application, we shall give examples of fibred almost contact spaces with quasi Sasakian, Sasakian and cosymplectic structures. Moreover, we shall characterize the cosymplectic space form.

In Chapter III, we shall treat the case when each fibre is invariant and normal to the structure vector and deal with quasi Sasakian or various cosymplectic structures induced in the total space.

A cosymplectic or closely cosymplectic manifold is locally the product of an almost complex manifold with a 1-dimensional Euclidean space, because the structure vector is parallel. At the first of Chapter IV, we shall give a discussion from this view point. We shall also study necessary and sufficient conditions for the cosymplectic Bochner curvature tensor to vanish by means of $\tilde{\phi}$-basis or a cosymplectic conformal connection.

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Chapter I. Preliminaries

§1. Fibred Riemannian spaces

Let $\{\tilde{M}, M, \tilde{g}, \pi\}$ be a fibred Riemannian space, that is, $\tilde{M}$ an m-dimensional total space with projectable Riemannian metric $\tilde{g}$, M an n-dimensional base space, and $\pi: \tilde{M} \to M$ a projection with maximal rank n. The fibre passing through a point $\tilde{P} \in \tilde{M}$ is denoted by $\overline{M}(\tilde{P})$ or generally $\overline{M}$, which is a p-dimensional submanifold of $\tilde{M}$, where $p = m - n$.

Manifolds, geometric objects and mappings we deal with are supposed to be of C^∞ class and manifolds to be connected. Throughout this paper the ranges of indices are as follows;

$$A, B, C, D, E : 1, 2, \ldots, m,$$

$$h, i, j, k, l : 1, 2, \ldots, m,$$

$$a, b, c, d, e : 1, 2, \ldots, n,$$

$$\alpha, \beta, \gamma, \delta, \varepsilon : n+1, \ldots, n+p = m,$$

unless otherwise is stated.

If we take coordinate neighborhoods $(\tilde{U}, z^h)$ in $\tilde{M}$ and (U, x^a) in M such that $\pi(\tilde{U}) = U$, then the projection π is expressed by equations

$$x^a = x^a(z^h) \tag{1.1}$$

with Jacobian $(\partial x^a/\partial z^i)$ of maximum rank n. There is a local coordinate system y^α in $\overline{M} \cap \tilde{U} \neq \varnothing$, (x^a, y^α) form a coordinate system in $\tilde{U}$ and each fibre $\overline{M}(\tilde{P})$ at $\tilde{P}$ in $\overline{M} \cap \tilde{U}$ is parametrized as

$$z^h = z^h(x^a, y^\alpha).$$

Then we can choose a local frame (E_a, C_α) and its dual frame (E^a, C^α) in $\tilde{U}$, where the components of E^a and C_α are given by

$$E_i{}^a = \partial x^a/\partial z^i \qquad \text{and} \qquad C^h{}_\alpha = \partial z^h/\partial y^\alpha. \tag{1.2}$$

The vector fields E_a span the horizontal distribution and C_α the tangent space of each fibre. The metric tensor g in the base space M is given by

$$g_{cb} = \tilde{g}(E_c, E_b), \tag{1.3}$$

and the induced metric tensor $\bar{g}$ in each fibre $\overline{M}$ by

$$\bar{g}_{\gamma\beta} = \tilde{g}(C_\gamma, C_\beta). \tag{1.4}$$

We write (E_B) for the frame (E_b, C_β) in all, if necessary. Let $h_{\gamma\beta}{}^a$ be components of the second fundamental tensor with respect to the normal vector E_a and $L = (L_{cb}{}^\alpha)$ the normal connection of each fibre $\overline{M}$. Then we have

$$h_{\gamma\beta}{}^a = h_{\beta\gamma}{}^a \qquad \text{and} \qquad L_{cb}{}^\alpha + L_{bc}{}^\alpha = 0. \tag{1.5}$$

Denoting by $\tilde{\nabla}$ the Riemannian connection of the total space $\tilde{M}$, we have the following equations [18, 22, 23, 35]:

$$\begin{aligned}
&\tilde{\nabla}_j E^h{}_b = \Gamma^a_{cb} E_j{}^c E^h{}_a - L_{cb}{}^\alpha E_j{}^c C^h{}_\alpha + L_b{}^a{}_\gamma C_j{}^\gamma E^h{}_a - h_\gamma{}^\alpha{}_b C_j{}^\gamma C^h{}_\alpha, \\
&\tilde{\nabla}_j C^h{}_\beta = L_c{}^a{}_\beta E_j{}^c E^h{}_a - (h_\beta{}^\alpha{}_c - P_{c\beta}{}^\alpha) E_j{}^c C^h{}_\alpha + h_{\gamma\beta}{}^a C_j{}^\gamma E^h{}_a + \bar{\Gamma}^\alpha_{\gamma\beta} C_j{}^\gamma C^h{}_\alpha, \\
&\tilde{\nabla}_j E_i{}^a = -\Gamma^a_{cb} E_j{}^c E_i{}^b - L_c{}^a{}_\beta (E_j{}^c C_i{}^\beta + C_j{}^\beta E_i{}^c) - h_{\gamma\beta}{}^a C_j{}^\gamma C_i{}^\beta, \\
&\tilde{\nabla}_j C_i{}^\alpha = L_{cb}{}^\alpha E_j{}^c E_i{}^b + (h_\beta{}^\alpha{}_c - P_{c\beta}{}^\alpha) E_j{}^c C_i{}^\beta + h_\gamma{}^\alpha{}_b C_j{}^\gamma E_i{}^b - \bar{\Gamma}^\alpha_{\gamma\beta} C_j{}^\gamma C_i{}^\beta,
\end{aligned} \tag{1.6}$$

where Γ_{cb}^a are connection coefficients of the projection $\nabla = p\tilde{\nabla}$ in M, $\bar{\Gamma}_{\gamma\beta}^\alpha$ those of the induced connection $\bar{\nabla}$ in $\bar{M}$,

$$L_c{}^a{}_\beta = L_{cb}{}^\alpha g^{ba}\bar{g}_{\alpha\beta}\,, \qquad h_\gamma{}^\alpha{}_b = h_{\gamma\beta}{}^a\bar{g}^{\beta\alpha}g_{ba}\,,$$

and $P_{c\beta}{}^\alpha$ are local functions in $\tilde{U}$ defined by

$$[E_b, C_\beta] = P_{b\beta}{}^\alpha C_\alpha\,.$$

From (1.6, 1), we see that $[E_c, E_b] = -2L_{cb}{}^\alpha C_\alpha$, and so the horizontal distribution is integrable if and only if the structure tensor L vanishes identically.

Let γ be a curve through a point P in the base space M and X be the tangent vector field of γ. There is a unique curve $\tilde{\gamma}$ through a point $\tilde{P} \in \pi^{-1}(P)$ such that the tangent vector field is the lift X^L. The curve $\tilde{\gamma}$ is called the *horizontal lift* of γ passing through $\tilde{P}$. If a curve γ joins points P and Q in M, then the horizontal lifts of γ through all points of the fibre $\bar{M}(P)$ define a fibre mapping $\Phi_\gamma : \bar{M}(P) \to \bar{M}(Q)$, called the *horizontal mapping covering* γ.

If the horizontal mapping covering any curve in M is an isometry of fibres, then $\{\tilde{M}, M, \tilde{g}, \pi\}$ is called a fibred Riemannian space with *isometric fibres.* A necessary and sufficient condition for $\tilde{M}$ to have isometric fibres is

$$(\mathscr{L}_{X^L}\tilde{g}^V)^V = 0$$

for any vector field X in M, or, equivalently $h_{\gamma\beta}{}^a = 0$. Here and hereafter A^H and A^V indicate the horizontal and vertical parts of A respectively.

If the horizontal mapping covering any curve in M is conformal mapping of fibres, then $\{\tilde{M}, M, \tilde{g}, \pi\}$ is called a fibred Riemannian space with *conformal fibres.* A condition for $\tilde{M}$ to have conformal fibres is $h_{\gamma\beta}{}^a = \bar{g}_{\gamma\beta}A^a$, where $A = A^a E_a$ is the mean curvature vector along each fibre in $\tilde{M}$. The following lemma is well known [18].

LEMMA 1.1. *If the components $L = (L_{cb}{}^\alpha)$ and $h = (h_{\gamma\beta}{}^a)$ vanish identically in a fibred Riemannian space, then the fibred space is locally the Riemannian product of the base space and a fibre.*

The curvature tensor of a fibred Riemannian space $\tilde{M}$ is defined by

$$\tilde{K}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z} \tag{1.7}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ in $\tilde{M}$. We put

$$\tilde{K}(E_D, E_C)E_B = \tilde{K}_{DCB}{}^A E_A = \tilde{K}_{DCB}{}^a E_a + \tilde{K}_{DCB}{}^\alpha C_\alpha\,, \tag{1.8}$$

then $\tilde{K}_{DCB}{}^A$ are components of the curvature tensor with respect to the basis (E_B). Denoting by $\tilde{K}_{kji}{}^h$ components of the curvature tensor in $(\tilde{U}, z^h)$, we have the relations

$$\tilde{K}_{DCB}{}^{A} = \tilde{K}_{kji}{}^{h}E^{k}{}_{D}E^{j}{}_{C}E^{i}{}_{B}E_{h}{}^{A} . \tag{1.9}$$

Substituting (1.6) into the definition (1.7) of the curvature tensor, we have the structure equations of a fibred Riemannian space as follows [1, 11, 22, 23, 27, 29, 35]:

$$\tilde{K}_{dcb}{}^{a} = K_{dcb}{}^{a} - L_{d}{}^{a}{}_{\varepsilon}L_{cb}{}^{\varepsilon} + L_{c}{}^{a}{}_{\varepsilon}L_{db}{}^{\varepsilon} + 2L_{dc}{}^{\varepsilon}L_{b}{}^{a}{}_{\varepsilon} , \tag{1.10}$$

$$\tilde{K}_{dcb}{}^{\alpha} = -{}^{*}\nabla_{d}L_{cb}{}^{\alpha} + {}^{*}\nabla_{c}L_{db}{}^{\alpha} - 2L_{dc}{}^{\varepsilon}h_{\varepsilon}{}^{\alpha}{}_{b} , \tag{1.11}$$

$$\begin{aligned}\tilde{K}_{dc\beta}{}^{\alpha} = {}& {}^{*}\nabla_{c}h_{\beta}{}^{\alpha}{}_{d} - {}^{*}\nabla_{d}h_{\beta}{}^{\alpha}{}_{c} + 2{}^{**}\nabla_{\beta}L_{dc}{}^{\alpha} + L_{de}{}^{\alpha}L_{c}{}^{e}{}_{\beta} \\ & - L_{ce}{}^{\alpha}L_{d}{}^{e}{}_{\beta} - h_{\varepsilon}{}^{\alpha}{}_{d}h_{\beta}{}^{\varepsilon}{}_{c} + h_{\varepsilon}{}^{\alpha}{}_{c}h_{\beta}{}^{\varepsilon}{}_{d} ,\end{aligned} \tag{1.12}$$

$$\tilde{K}_{d\gamma b}{}^{a} = {}^{*}\nabla_{d}L_{b}{}^{a}{}_{\gamma} - L_{d}{}^{a}{}_{\varepsilon}h_{\gamma}{}^{\varepsilon}{}_{b} + L_{db}{}^{\varepsilon}h_{\gamma\varepsilon}{}^{a} - L_{b}{}^{a}{}_{\varepsilon}h_{\gamma}{}^{\varepsilon}{}_{d} , \tag{1.13}$$

$$\tilde{K}_{d\gamma b}{}^{\alpha} = -{}^{*}\nabla_{d}h_{\gamma}{}^{\alpha}{}_{b} + {}^{**}\nabla_{\gamma}L_{db}{}^{\alpha} + L_{d}{}^{e}{}_{\gamma}L_{eb}{}^{\alpha} + h_{\gamma}{}^{\varepsilon}{}_{d}h_{\varepsilon}{}^{\alpha}{}_{b} , \tag{1.14}$$

$$\tilde{K}_{\delta\gamma b}{}^{a} = L_{\delta\gamma b}{}^{a} + h_{\delta}{}^{\varepsilon}{}_{b}h_{\gamma\varepsilon}{}^{a} - h_{\gamma}{}^{\varepsilon}{}_{b}h_{\delta\varepsilon}{}^{a} , \tag{1.15}$$

$$\tilde{K}_{\delta\gamma\beta}{}^{a} = {}^{**}\nabla_{\delta}h_{\gamma\beta}{}^{a} - {}^{**}\nabla_{\gamma}h_{\delta\beta}{}^{a} , \tag{1.16}$$

$$\tilde{K}_{\delta\gamma\beta}{}^{\alpha} = \bar{K}_{\delta\gamma\beta}{}^{\alpha} + h_{\delta\beta}{}^{e}h_{\gamma}{}^{\alpha}{}_{e} - h_{\gamma\beta}{}^{e}h_{\delta}{}^{\alpha}{}_{e} , \tag{1.17}$$

where we have put

$$K_{dcb}{}^{a} = \partial_{d}\Gamma_{cb}^{a} - \partial_{c}\Gamma_{db}^{a} + \Gamma_{de}^{a}\Gamma_{cb}^{e} - \Gamma_{ce}^{a}\Gamma_{db}^{e} , \tag{1.18}$$

$${}^{*}\nabla_{d}L_{cb}{}^{\alpha} = \partial_{d}L_{cb}{}^{\alpha} - \Gamma_{dc}^{e}L_{eb}{}^{\alpha} - \Gamma_{db}^{e}L_{ce}{}^{\alpha} + Q_{d\varepsilon}{}^{\alpha}L_{cb}{}^{\varepsilon} , \tag{1.19}$$

$${}^{*}\nabla_{d}L_{c}{}^{a}{}_{\beta} = \partial_{d}L_{c}{}^{a}{}_{\beta} + \Gamma_{de}^{a}L_{c}{}^{e}{}_{\beta} - \Gamma_{dc}^{e}L_{e}{}^{a}{}_{\beta} - Q_{d\beta}{}^{\varepsilon}L_{c}{}^{a}{}_{\varepsilon} , \tag{1.20}$$

$${}^{*}\nabla_{d}h_{\gamma\beta}{}^{a} = \partial_{d}h_{\gamma\beta}{}^{a} + \Gamma_{de}^{a}h_{\gamma\beta}{}^{e} - Q_{d\gamma}{}^{\varepsilon}h_{\varepsilon\beta}{}^{a} - Q_{d\beta}{}^{\varepsilon}h_{\gamma\varepsilon}{}^{a} , \tag{1.21}$$

$${}^{*}\nabla_{d}h_{\beta}{}^{\alpha}{}_{b} = \partial_{d}h_{\beta}{}^{\alpha}{}_{b} - \Gamma_{db}^{e}h_{\beta}{}^{\alpha}{}_{e} + Q_{d\varepsilon}{}^{\alpha}h_{\beta}{}^{\varepsilon}{}_{b} - Q_{d\beta}{}^{\varepsilon}h_{\varepsilon}{}^{\alpha}{}_{b} , \tag{1.22}$$

$Q_{c\beta}{}^{\alpha}$ being defined by

$$Q_{c\beta}{}^{\alpha} = P_{c\beta}{}^{\alpha} - h_{\beta}{}^{\alpha}{}_{c} ,$$

and

$${}^{**}\nabla_{\delta}L_{cb}{}^{\alpha} = \partial_{\delta}L_{cb}{}^{\alpha} + \bar{\Gamma}_{\delta\varepsilon}^{\alpha}L_{cb}{}^{\varepsilon} - L_{c}{}^{e}{}_{\delta}L_{eb}{}^{\alpha} - L_{b}{}^{e}{}_{\delta}L_{ce}{}^{\alpha} , \tag{1.23}$$

$${}^{**}\nabla_{\delta}L_{b}{}^{a}{}_{\beta} = \partial_{\delta}L_{b}{}^{a}{}_{\beta} - \bar{\Gamma}_{\delta\beta}^{\varepsilon}L_{b}{}^{a}{}_{\varepsilon} + L_{e}{}^{a}{}_{\delta}L_{b}{}^{e}{}_{\beta} - L_{b}{}^{e}{}_{\delta}L_{e}{}^{a\beta} , \tag{1.24}$$

$${}^{**}\nabla_{\delta}h_{\gamma\beta}{}^{a} = \partial_{\delta}h_{\gamma\beta}{}^{a} - \bar{\Gamma}_{\delta\gamma}^{\varepsilon}h_{\varepsilon\beta}{}^{a} - \bar{\Gamma}_{\delta\beta}^{\varepsilon}h_{\gamma\varepsilon}{}^{a} + L_{e}{}^{a}{}_{\delta}h_{\gamma\beta}{}^{e} , \tag{1.25}$$

$${}^{**}\nabla_{\delta}h_{\beta}{}^{\alpha}{}_{b} = \partial_{\delta}h_{\beta}{}^{\alpha}{}_{b} + \bar{\Gamma}_{\delta\varepsilon}^{\alpha}h_{\beta}{}^{\varepsilon}{}_{b} - \bar{\Gamma}_{\delta\beta}^{\varepsilon}h_{\varepsilon}{}^{\alpha}{}_{b} - L_{b}{}^{e}{}_{\delta}h_{\beta}{}^{\alpha}{}_{e} , \tag{1.26}$$

$$(1.27)\qquad L_{\delta\gamma b}{}^{a} = \partial_\delta L_b{}^a{}_\gamma - \partial_\gamma L_b{}^a{}_\delta + L_e{}^a{}_\delta L_b{}^e{}_\gamma - L_e{}^a{}_\gamma L_b{}^e{}_\delta\,,$$

$$(1.28)\qquad \bar{K}_{\delta\gamma\beta}{}^{\alpha} = \partial_\delta \bar{\Gamma}^\alpha_{\gamma\beta} - \partial_\gamma \bar{\Gamma}^\alpha_{\delta\beta} + \bar{\Gamma}^\alpha_{\delta\varepsilon}\bar{\Gamma}^\varepsilon_{\gamma\beta} - \bar{\Gamma}^\alpha_{\gamma\varepsilon}\bar{\Gamma}^\varepsilon_{\delta\beta}\,.$$

Among these, the functions $K_{dcb}{}^a$ are projectable in $\tilde{U}$ and its projections, denoted by $K_{dcb}{}^a$ too, are components of the curvature tensor of the base space $\{M, g\}$. On each fibre $\bar{M}$, the functions $\bar{K}_{\delta\gamma\beta}{}^\alpha$ are components of the curvature tensor of the induced Riemannian metric $\bar{g}$ and $L_{\delta\gamma b}{}^a$ those of the curvature tensor of the normal connection of $\bar{M}$ in $\tilde{M}$. The components $\tilde{K}_{DCB}{}^A$ satisfy the same algebraic equations as those $\tilde{K}_{kji}{}^h$ satisfy.

Denote by $\tilde{K}_{CB}$ components of the Ricci tensor of $\{\tilde{M}, \tilde{g}\}$ with respect to the basis (E_B) in $\tilde{U}$, and by K_{cb} and $\bar{K}_{\gamma\beta}$ components of the Ricci tensors of the base space $\{M, g\}$ in (U, x^a) and each fibre $\{\bar{M}, \bar{g}\}$ in $(\bar{U}, y^\alpha)$ respectively. Then we have

$$(1.29)\qquad \tilde{K}_{cb} = K_{cb} - 2L_{ce}{}^{\varepsilon}L_b{}^e{}_\varepsilon - h_\delta{}^\varepsilon{}_c h_\varepsilon{}^\delta{}_b + \tfrac{1}{2}({}^*\nabla_c h_\varepsilon{}^\varepsilon{}_b + {}^*\nabla_b h_\varepsilon{}^\varepsilon{}_c)\,,$$

$$(1.30)\qquad \tilde{K}_{\gamma b} = {}^{**}\nabla_\gamma h_\varepsilon{}^\varepsilon{}_b - {}^{**}\nabla_\varepsilon h_\gamma{}^\varepsilon{}_b + {}^*\nabla_e L_b{}^e{}_\gamma - 2h_\gamma{}^\varepsilon{}_e L_b{}^e{}_\varepsilon\,,$$

$$(1.31)\qquad \tilde{K}_{\gamma\beta} = \bar{K}_{\gamma\beta} - h_{\gamma\beta}{}^e h_\varepsilon{}^\varepsilon{}_e + {}^*\nabla_e h_{\gamma\beta}{}^e - L_a{}^e{}_\gamma L_e{}^a{}_\beta\,.$$

Denoting by $\tilde{K}$, K and $\bar{K}$ the scalar curvatures of $\tilde{M}$, M and each fibre $\bar{M}$ respectively, we have the relation

$$(1.32)\qquad \tilde{K} = K^L + \bar{K} - L_{cb\varepsilon}L^{cb\varepsilon} - h_{\gamma\beta e}h^{\gamma\beta e} - h_\gamma{}^\gamma{}_e h_\beta{}^{\beta e} + 2{}^*\nabla_e h_\varepsilon{}^{\varepsilon e}\,.$$

§2. Certain classes of almost contact structures

Let $\tilde{M}$ be an m-dimensional ($m =$ odd) real differentiable manifold and a tensor field $\tilde{\phi}$ of type (1, 1), $\tilde{\xi}$ a vector field and $\tilde{\eta}$ a 1-form on $\tilde{M}$ satisfying the equations

$$\tilde{\phi}^2 = -I + \tilde{\xi}\otimes\tilde{\eta}\,, \qquad \tilde{\phi}\tilde{\xi} = \tilde{\eta}\circ\tilde{\phi} = 0\,, \qquad \tilde{\eta}(\tilde{\xi}) = 1\,,$$

where I is the identity transformation. Then $\tilde{M}$ is said to have an almost contact structure. It is known that there is a positive definite Riemannian metric $\tilde{g}$ on $\tilde{M}$ such that

$$\tilde{g}(\tilde{\phi}\tilde{X}, \tilde{Y}) = -\tilde{g}(\tilde{X}, \tilde{\phi}\tilde{Y}) \qquad \text{and} \qquad \tilde{g}(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{\phi}\tilde{X}, \tilde{\phi}\tilde{Y}) + \tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y})\,,$$

where $\tilde{X}$ and $\tilde{Y}$ are arbitrary vectors on $\tilde{M}$. In this case, $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is called an almost contact metric structure. The fundamental 2-form $\tilde{\Theta}$ is defined by $\tilde{\Theta}(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{\phi}\tilde{X}, \tilde{Y})$. We denote by $\tilde{\nabla}$ the covariant differentiation with respect to the Riemannian connection of $\tilde{g}$ and put

$$A(\tilde{X}, \tilde{Y}) = (\tilde{\nabla}_{\tilde{X}}\tilde{\phi})\tilde{Y} + (\tilde{\nabla}_{\tilde{\phi}\tilde{X}}\tilde{\phi})\tilde{\phi}\tilde{Y} - \tilde{\eta}(\tilde{Y})\tilde{\nabla}_{\tilde{\phi}\tilde{X}}\tilde{\xi}\,. \tag{2.1}$$

An almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ on $\tilde{M}$ is said to be normal if the almost complex structure J on $\tilde{M} \times E^1$ given by

$$J\left(\tilde{X}, f\frac{d}{dt}\right) = \left(\tilde{\phi}\tilde{X} - f\tilde{\xi}, \tilde{\eta}(\tilde{X})\frac{d}{dt}\right),$$

f being a C^∞-function on $\tilde{M} \times E^1$, is integrable. The integrability is equivalent to the condition

$$N(\tilde{\phi}, \tilde{\phi}) + 2d\tilde{\eta} \otimes \tilde{\xi} = 0\,,$$

where $N(\tilde{\phi}, \tilde{\phi})$ denotes the Nijenhuis tensor

$$N(\tilde{\phi}, \tilde{\phi})(\tilde{X}, \tilde{Y}) = \tilde{\phi}^2[\tilde{X}, \tilde{Y}] + [\tilde{\phi}\tilde{X}, \tilde{\phi}\tilde{Y}] - \tilde{\phi}[\tilde{\phi}\tilde{X}, \tilde{Y}] - \tilde{\phi}[\tilde{X}, \tilde{\phi}\tilde{Y}]$$

of $\tilde{\phi}$. Define a Riemannian metric W on $\tilde{M} \times E^1$ by

$$W\left(\left(\tilde{X}, f\frac{d}{dt}\right), \left(\tilde{Y}, \bar{f}\frac{d}{dt}\right)\right) = \tilde{g}(\tilde{X}, \tilde{Y}) + f\bar{f},$$

where f and $\bar{f}$ are C^∞-functions on $\tilde{M} \times E^1$. A. Gray and L. M. Hervella [17] classified almost Hermitian manifolds and gave a class ω_4 of Hermitian manifolds containing locally conformal Kaehler manifolds.

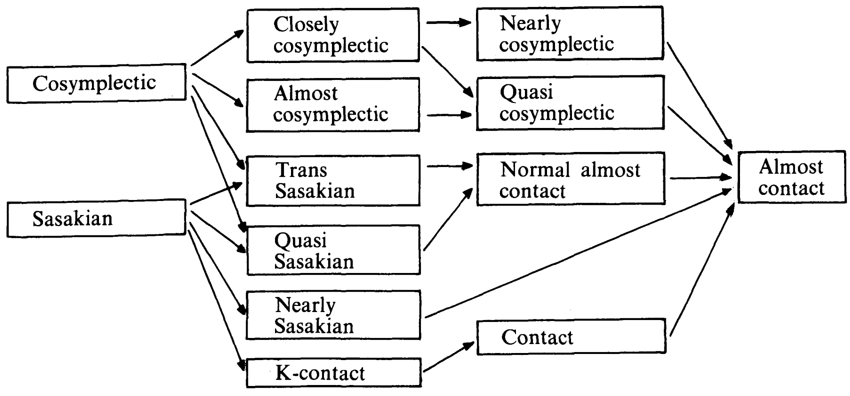
An almost contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on $\tilde{M}$ is said to be

(*a*) *quasi cosymplectic* [7] if $A(\tilde{X}, \tilde{Y}) = 0$,
(*b*) *closely cosymplectic* [5] if $\tilde{\phi}$ is Killing and $\tilde{\eta}$ is closed,
(*c*) *nearly cosymplectic* [3] if $\tilde{\phi}$ is Killing,
(*d*) *almost cosymplectic* [7, 9, 15, 26] if $\tilde{\Theta}$ and $\tilde{\eta}$ are closed,
(*e*) *nearly Sasakian* [6] if

$$(\tilde{\nabla}_{\tilde{X}}\tilde{\phi})\tilde{Y} + (\tilde{\nabla}_{\tilde{Y}}\tilde{\phi})\tilde{X} = -2\tilde{g}(\tilde{X}, \tilde{Y})\tilde{\xi} + \tilde{\eta}(\tilde{X})\tilde{Y} + \tilde{\eta}(\tilde{Y})\tilde{X}\,,$$

(*f*) *trans Sasakian* [28] if $(\tilde{M} \times E^1, J, W)$ belongs to the class ω_4 of Hermitian manifolds.
(*g*) *quasi Sasakian* [2, 16, 19, 20, 33] if $\tilde{\Theta}$ is closed and $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ is normal,
(*h*) *cosymplectic* if $\tilde{\Theta}$ and $\tilde{\eta}$ are closed and $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ is normal,
(*i*) *contact* if $d\tilde{\eta} = \tilde{\Theta}$,
(*j*) *K-contact* if $\tilde{\nabla}\tilde{\eta} = \tilde{\Theta}$,
(*k*) *Sasakian* if $\tilde{\Theta} = d\tilde{\eta}$ and $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ is normal.

Then a schematic arrangement of structures is illustrated by the following diagram:



where $A \to B$ means that the class A of structures is subordinate to the class B. Concerning properties and examples of the spaces pictured in this diagram, see [2, 3, 4, 5, 7, 9, 15, 16, 19, 20, 26, 28, 33 etc.].

Chapter II. Fibred Riemannian spaces with almost contact metric structure, the case where the structure vector is tangent to the fibre

§3. Almost contact structures in a fibred Riemannian space

In previous papers [22, 35], we have studied an almost contact metric structure on the total space of a fibred Riemannian space $\tilde{M}$ having an almost Hermitian space M as base space and almost contact spaces $\bar{M}$ as fibres.

Throughout this chapter we consider a fibred Riemannian space $\tilde{M}$ with such structure, and recall fundamental formulae and some results for the later use.

Denoting an almost Hermitian structure of M and its lift in the total space $\tilde{M}$ by (J, g) which is independent of the fibre and an almost contact metric structure of each fibre $\bar{M}$ by $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ which is in general dependent on points of the base space. If we put

$$\begin{aligned} \tilde{\phi} &= J_b{}^a E^b \otimes E_a + \bar{\phi}_\beta{}^\alpha C^\beta \otimes C_\alpha\,, \\ \tilde{\eta} &= \bar{\eta}_\alpha C^\alpha\,, \qquad \tilde{\xi} = \bar{\xi}^\alpha C_\alpha\,, \\ \tilde{g} &= g_{ba} E^b \otimes E^a + \bar{g}_{\beta\alpha} C^\beta \otimes C^\alpha\,, \end{aligned} \tag{3.1}$$

then $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ defines an almost contact metric structure on $\tilde{M}$. Conversely, if there is in $\tilde{M}$ an almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, $\tilde{\phi}$ is projectable and $\tilde{\xi}$ is always vertical, then the structure induces an almost Hermitian structure (J, g) in the base space M and an almost contact metric structure $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ in each fibre $\bar{M}$.

We shall denote the covariant differentiation with respect to the frame $(E_B) = (E_b, C_\beta)$ in the total space $\tilde{M}$ by $\tilde{\nabla}_B = E_B{}^i\tilde{\nabla}_i$. By means of (1.6) and (3.1), we get [22]

$$(\tilde{\nabla}_c\tilde{\phi})E_b = (\nabla_c J_b{}^a)E_a + (L_{cb}{}^\beta\bar{\phi}_\beta{}^\alpha - L_{ce}{}^\alpha J_b{}^e)C_\alpha\,, \tag{3.2}$$

$$(\tilde{\nabla}_c\tilde{\phi})C_\beta = (L_c{}^a{}_\gamma\bar{\phi}_\beta{}^\gamma - L_c{}^e{}_\beta J_e{}^a)E_a + ({}^*\nabla_c\bar{\phi}_\beta{}^\alpha)C_\alpha\,, \tag{3.3}$$

$$(\tilde{\nabla}_\gamma\tilde{\phi})E_b = ({}^{**}\nabla_\gamma J_b{}^a)E_a + (h_\gamma{}^\beta{}_b\phi_\beta{}^\alpha - h_\gamma{}^\alpha{}_a J_b{}^a)C_\alpha\,, \tag{3.4}$$

$$(\tilde{\nabla}_\gamma\tilde{\phi})C_\beta = (h_{\gamma\varepsilon}{}^a\bar{\phi}_\beta{}^\varepsilon - h_{\gamma\beta}{}^e J_e{}^a)E_a + (\bar{\nabla}_\gamma\bar{\phi}_\beta{}^\alpha)C_\alpha\,, \tag{3.5}$$

$$\tilde{\nabla}_c\tilde{\xi} = (L_c{}^a{}_\alpha\bar{\xi}^\alpha)E_a + ({}^*\nabla_c\bar{\xi}^\alpha)C_\alpha\,, \tag{3.6}$$

$$\tilde{\nabla}_\gamma\tilde{\xi} = (h_{\gamma\beta}{}^a\bar{\xi}^\beta)E_a + (\bar{\nabla}_\gamma\bar{\xi}^\alpha)C_\alpha\,, \tag{3.7}$$

$$\tilde{\nabla}_c\tilde{\eta} = (L_{cb}{}^\alpha\bar{\eta}_\alpha)E^b + ({}^*\nabla_c\bar{\eta}_\beta)C^\beta\,, \tag{3.8}$$

$$\tilde{\nabla}_\gamma\tilde{\eta} = (h_\gamma{}^\alpha{}_b\bar{\eta}_\alpha)E^b + (\bar{\nabla}_\gamma\bar{\eta}_\beta)C^\beta\,, \tag{3.9}$$

where we have put

$$\begin{aligned}
&\nabla_c J_b{}^a = \partial_c J_b{}^a + \Gamma^a_{cd}J_b{}^d - \Gamma^e_{cb}J_e{}^a\,,\\
&{}^*\nabla_c\bar{\phi}_\beta{}^\alpha = \partial_c\bar{\phi}_\beta{}^\alpha + Q_{c\varepsilon}{}^\alpha\bar{\phi}_\beta{}^\varepsilon - Q_{c\beta}{}^\varepsilon\bar{\phi}_\varepsilon{}^\alpha\,,\\
&{}^*\nabla_c\bar{\xi}^\alpha = \partial_c\bar{\xi}^\alpha + Q_{c\beta}{}^\alpha\bar{\xi}^\beta\,,\\
&{}^*\nabla_c\bar{\eta}_\beta = \partial_c\bar{\eta}_\beta - Q_{c\beta}{}^\alpha\bar{\eta}_\alpha\,,\\
&{}^{**}\nabla_\gamma J_b{}^a = \partial_\gamma J_b{}^a + L_d{}^a{}_\gamma J_b{}^d - L_b{}^d{}_\gamma J_d{}^a\,,\\
&\bar{\nabla}_\gamma\bar{\phi}_\beta{}^\alpha = \partial_\gamma\bar{\phi}_\beta{}^\alpha + \bar{\Gamma}^\alpha_{\gamma\varepsilon}\bar{\phi}_\beta{}^\varepsilon - \bar{\Gamma}^\varepsilon_{\gamma\beta}\bar{\phi}_\varepsilon{}^\alpha\,,\\
&\bar{\nabla}_\gamma\bar{\xi}^\alpha = \partial_\gamma\bar{\xi}^\alpha + \bar{\Gamma}^\alpha_{\gamma\beta}\bar{\xi}^\beta\,,\\
&\bar{\nabla}_\gamma\bar{\eta}_\beta = \partial_\gamma\bar{\eta}_\beta - \bar{\Gamma}^\alpha_{\gamma\beta}\bar{\eta}_\alpha\,,\\
&Q_{c\beta}{}^\alpha = P_{c\beta}{}^\alpha - h_\beta{}^\alpha{}_c\,.
\end{aligned} \tag{3.10}$$

By use of (3.1) ~ (3.9), we have

$$\begin{aligned}
&(\tilde{\nabla}_j\tilde{\phi}_i{}^h + \tilde{\nabla}_i\tilde{\phi}_j{}^h)E^j{}_c E^i{}_b = (\nabla_c J_b{}^e + \nabla_b J_c{}^e)E^h{}_e - (L_{ca}{}^\alpha J_b{}^a + L_{ba}{}^\alpha J_c{}^a)C^h{}_\alpha\,,\\
&(\tilde{\nabla}_j\tilde{\phi}_i{}^h + \tilde{\nabla}_i\tilde{\phi}_j{}^h)E^j{}_c C^i{}_\beta = (L_e{}^a{}_\beta J_c{}^e - 2L_c{}^e{}_\beta J_e{}^a + L_c{}^a{}_\alpha\bar{\phi}_\beta{}^\alpha)E^h{}_a\\
&\qquad + ({}^*\nabla_c\bar{\phi}_\beta{}^\alpha + h_\beta{}^\varepsilon{}_c\bar{\phi}_\varepsilon{}^\alpha - h_\beta{}^\alpha{}_e J_c{}^e)C^h{}_\alpha\,,\\
&(\tilde{\nabla}_j\tilde{\phi}_i{}^h + \tilde{\nabla}_i\tilde{\phi}_j{}^h)C^j{}_\gamma C^i{}_\beta = (h_{\gamma\varepsilon}{}^a\bar{\phi}_\beta{}^\varepsilon + h_{\beta\varepsilon}{}^a\bar{\phi}_\gamma{}^\varepsilon - 2h_{\gamma\beta}{}^e J_e{}^a)E^h{}_a\\
&\qquad + (\bar{\nabla}_\gamma\bar{\phi}_\beta{}^\alpha + \bar{\nabla}_\beta\bar{\phi}_\gamma{}^\alpha)C^h{}_\alpha\,.
\end{aligned} \tag{3.11}$$

If we put $\Theta(X, Y) = g(JX, Y)$ and $\bar{\Theta}(\bar{X}, \bar{Y}) = \bar{g}(\bar{\phi}\bar{X}, \bar{Y})$ for vector fields X, Y in M and $\bar{X}$, $\bar{Y}$ in $\bar{M}$, then we get

$$
(3.12)\qquad
\begin{aligned}
&(d\tilde{\Theta})^H = d\Theta\,,\\
&d\tilde{\Theta}(E_c, E_b, C_\alpha) = 2L_{cb}{}^{\beta}\bar{\phi}_{\beta\alpha}\,,\\
&d\tilde{\Theta}(E_c, C_\alpha, E_b) = 2(L_c{}^e{}_\alpha J_{be} + L_b{}^e{}_\alpha J_{ec})\,,\\
&d\tilde{\Theta}(E_c, C_\beta, C_\alpha) = {}^*\nabla_c\bar{\phi}_{\beta\alpha} + h_{\beta\gamma c}\bar{\phi}_\alpha{}^\gamma + h_\alpha{}^\gamma{}_c\bar{\phi}_{\gamma\beta}\,,\\
&(d\tilde{\Theta})^V = d\bar{\Theta}\,,\\
&d\tilde{\eta}(E_c, E_b) = 2L_{cb}{}^\alpha\bar{\eta}_\alpha\,,\\
&d\tilde{\eta}(E_c, C_\beta) = {}^*\nabla_c\bar{\eta}_\beta - h_\beta{}^\alpha{}_c\bar{\eta}_\alpha\,,\\
&(d\tilde{\eta})^V = d\bar{\eta}\,.
\end{aligned}
$$

§4. Nearly Sasakian structures

Y. Tashiro [34] showed that a hypersurface of an almost Hermitian manifold inherits an almost contact metric structure. D. E. Blair, D. K. Showers and K. Yano [6] proved that a quasi-umbilical hypersurface of a nearly Kaehler manifold admits a nearly Sasakian structure, that is,

$$
(4.1)\qquad (\tilde{\nabla}_{\tilde{X}}\tilde{\phi})\tilde{Y} + (\tilde{\nabla}_{\tilde{Y}}\tilde{\phi})\tilde{X} = -2\tilde{g}(\tilde{X}, \tilde{Y})\tilde{\xi} + \tilde{\eta}(\tilde{X})\tilde{Y} + \tilde{\eta}(\tilde{Y})\tilde{X}\,,
$$

and that a nearly Sasakian structure is Sasakian if it is normal, and the vector field $\tilde{\xi}$ is Killing. It is known that a five-sphere S^5 properly imbedded in S^6 admits a nearly Sasakian structure which is not Sasakian.

Now we suppose that a fibred space $\tilde{M}$ is nearly Sasakian. Then it follows from (3.11, 2) and (4.1) with $\tilde{X} = E_c$, $\tilde{Y} = C_\beta$ that

$$
(4.2)\qquad L_{eb\beta}J_c^{\ e} + 2L_{ce\beta}J_b^{\ e} + L_{cb\alpha}\bar{\phi}_\beta{}^\alpha = g_{cb}\bar{\eta}_\beta\,.
$$

By taking the symmetric part in b and c, we have

$$
L_{ce\beta}J_b^{\ e} + L_{be\beta}J_c^{\ e} = 2g_{cb}\bar{\eta}_\beta\,,
$$

and substituting this into (4.2),

$$
(4.3)\qquad 3L_{ce\beta}J_b^{\ e} + L_{cb\alpha}\bar{\phi}_\beta{}^\alpha = 3g_{cb}\bar{\eta}_\beta\,.
$$

Moreover, transvecting $J_a^{\ b}$ to (4.3), we obtain

$$
(4.4)\qquad 3L_{ac\beta} + L_{cb\alpha}J_a^{\ b}\bar{\phi}_\beta{}^\alpha = 3J_{ac}\bar{\eta}_\beta\,,
$$

and

$$L_{cb\alpha}\bar{\xi}^{\alpha} = J_{cb}\,. \tag{4.5}$$

On the other hand, transvecting $\bar{\phi}_{\alpha}{}^{\beta}$ to (4.3) and using (4.5), we get

$$L_{cb\alpha} - 3L_{ce\beta}J_b{}^e\bar{\phi}_{\alpha}{}^{\beta} = J_{cb}\bar{\eta}_{\alpha}\,. \tag{4.6}$$

By comparing of (4.4) with (4.6), it follows that

$$L_{ce\beta}J_b{}^e\bar{\phi}_{\alpha}{}^{\beta} = 0\,,$$

and, by virtue of (4.6), we have

$$L_{cb\alpha} = J_{cb}\bar{\eta}_{\alpha}\,.$$

Moreover we see from (3.11, 2) and (4.1) that

$$*\nabla_c\bar{\phi}_{\beta\alpha} + h_{\beta}{}^{\varepsilon}{}_c\bar{\phi}_{\varepsilon\alpha} - h_{\beta\alpha a}J_c{}^a = 0\,. \tag{4.7}$$

By use of (3.10, 2) and (3.10, 9), we see that the equation (4.7) is equivalent to

$$\partial_c\bar{\phi}_{\beta\alpha} - P_{c\beta}{}^{\varepsilon}\bar{\phi}_{\varepsilon\alpha} - P_{c\alpha}{}^{\varepsilon}\bar{\phi}_{\beta\varepsilon} + h_{\alpha\varepsilon c}\bar{\phi}_{\beta}{}^{\varepsilon} - 2h_{\beta\varepsilon c}\bar{\phi}_{\alpha}{}^{\varepsilon} - h_{\beta\alpha a}J_c{}^a = 0\,. \tag{4.8}$$

Taking the symmetric part in α and β, we get

$$h_{\beta\varepsilon}{}^c\bar{\phi}_{\alpha}{}^{\varepsilon} + h_{\alpha\varepsilon}{}^c\bar{\phi}_{\beta}{}^{\varepsilon} - 2h_{\beta\alpha}{}^eJ_e{}^c = 0\,, \tag{4.9}$$

the left hand side of which are identical with the coefficients of (3.11, 3). Substituting (4.9) into (4.8), we obtain the equation

$$\partial_c\bar{\phi}_{\beta\alpha} - P_{c\alpha}{}^{\varepsilon}\bar{\phi}_{\beta\varepsilon} - P_{c\beta}{}^{\varepsilon}\bar{\phi}_{\varepsilon\alpha} - \tfrac{3}{2}(h_{\beta\varepsilon c}\bar{\phi}_{\alpha}{}^{\varepsilon} - h_{\alpha\varepsilon}{}^c\bar{\phi}_{\beta}{}^{\varepsilon}) = 0\,. \tag{4.10}$$

Besides, we can see that the space M is nearly Kaehlerian from (3.11, 1) and each fibre $\bar{M}$ is nearly Sasakian from (3.11, 3). Contracting (4.7) with $\bar{g}^{\beta\alpha}$, we have $h_{\alpha\beta a}\bar{g}^{\alpha\beta} = 0$. Thus we have

PROPOSITION 4.1. *The almost contact metric structure* $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *on* $\tilde{M}$ *is nearly Sasakian if and only if*

(1) *M is nearly Kaehlerian,*
(2) *$\bar{M}$ is nearly Sasakian,*
(3) $L_{cb}{}^{\alpha} = J_{cb}\bar{\xi}^{\alpha}$,
(4) $\partial_c\bar{\phi}_{\beta\alpha} - P_{c\beta}{}^{\varepsilon}\bar{\phi}_{\varepsilon\alpha} - P_{c\alpha}{}^{\varepsilon}\bar{\phi}_{\beta\varepsilon} + h_{\alpha}{}^{\varepsilon}{}_c\bar{\phi}_{\beta\varepsilon} + 2h_{\beta}{}^{\varepsilon}{}_c\bar{\phi}_{\varepsilon\alpha} - h_{\beta\alpha a}J_c{}^a = 0.$

In this case, each fibre is minimal in $\tilde{M}$.

§5. Fibred Sasakian space

In this section, we consider the fibred Sasakian space with Einstein or η-Einstein metric. In [22], we proved

PROPOSITION 5.1. *The almost contact metric structure on $\tilde{M}$ is Sasakian if and only if*

(1) *M is Kaehlerian,*

(2) *each fibre is Sasakian,*

(3) $L_{cb}{}^{\alpha} = J_{cb}\bar{\xi}^{\alpha}$,

(4) $\partial_c\bar{\phi}_{\beta}{}^{\alpha} + P_{c\varepsilon}{}^{\alpha}\bar{\phi}_{\beta}{}^{\varepsilon} - P_{c\beta}{}^{\varepsilon}\bar{\phi}_{\varepsilon}{}^{\alpha} + 2h_{\beta}{}^{\varepsilon}{}_{c}\bar{\phi}_{\varepsilon}{}^{\alpha} = 0$,

(5) $h_{\gamma}{}^{\varepsilon}{}_{b}\bar{\phi}_{\varepsilon}{}^{\alpha} - h_{\gamma}{}^{\alpha}{}_{a}J_{b}{}^{a} = 0$.

In this case, each fibre is minimal in $\tilde{M}$ and $*\nabla_d L_{cb}{}^{\alpha} = 0$.

Next, we suppose that, in addition to the assumptions of Proposition 5.1, $\tilde{M}$ has conformal fibres and $\tilde{g}$ is an Einstein metric. Then we have $L_{cb}{}^{\alpha} = J_{cb}\bar{\xi}^{\alpha}$ and $h_{\beta\alpha}{}^{a} = 0$ because each fibre is minimal. By use of (1.29), (1.30) and (1.31), we get

$$K_{cb} = (\alpha + 2)g_{cb}\,, \qquad \bar{K}_{\gamma\beta} = \alpha\bar{g}_{\gamma\beta} - n\bar{\eta}_{\gamma}\bar{\eta}_{\beta}\,,$$

where $\alpha = \tilde{K}/m$. Hence, by use of (1.32), we get

$$K = n(\bar{K} + n + 2p)/p$$

and then

$$\tilde{K} = m(n + \bar{K})/p\,.$$

Conversely, if we substitute $h_{\beta\alpha}{}^{a} = 0$ and $L_{cb}{}^{\alpha} = J_{cb}\bar{\xi}^{\alpha}$ into (1.29) $\sim$ (1.31), we have

$$\tilde{K}_{cb} = K_{cb} - 2g_{cb}\,, \qquad \tilde{K}_{\gamma b} = 0\,, \qquad \tilde{K}_{\gamma\beta} = \bar{K}_{\gamma\beta} + n\bar{\eta}_{\gamma}\bar{\eta}_{\beta}\,.$$

Thus we have

PROPOSITION 5.2. *Let $\tilde{M}$ be a fibred Sasakian space with conformal fibres. Then $\tilde{M}$ is Einstein if and only if*

(1) *M is Einstein,*

(2) $\bar{K}_{\gamma\beta} = \alpha\bar{g}_{\gamma\beta} - n\bar{\eta}_{\gamma}\bar{\eta}_{\beta}$,

(3) $K = n(n + 2p + \bar{K})/p$,

where $\alpha = \tilde{K}/m$.

Now, we assume that a fibred Sasakian space with conformal fibres $\tilde{M}$ is an η-Einstein space, that is,

$$\tilde{K}_{ji} = a\tilde{g}_{ji} + b\tilde{\eta}_{j}\tilde{\eta}_{i}\,, \tag{5.1}$$

a and b being constant. Then we get

$$K_{cb} = (a + 2)g_{cb}\,, \qquad \bar{K}_{\gamma\beta} = a\bar{g}_{\gamma\beta} + (b - n)\bar{\eta}_{\gamma}\bar{\eta}_{\beta}\,. \tag{5.2}$$

and

(5.3) $$K = n(a+2)\,, \qquad \bar{K} = pa + b - n\,.$$

On the other hand, we have the relation

(5.4) $$a + b = m - 1$$

because $\tilde{\xi}$ satisfies $\tilde{K}_{ji}\tilde{\xi}^j\tilde{\xi}^i = m - 1$. By use of the relations (5.3) and (5.4), we obtain

(5.5) $$a = \frac{\bar{K}}{p-1} - 1\,, \qquad b = m - \frac{\bar{K}}{p-1}\,,$$

and the relation

(5.6) $$\frac{K}{n} = \frac{\bar{K}}{p-1} + 1\,.$$

Conversely, if the base space M is Einstein, each fibre is η-Einstein and the relation (5.6) is satisfied, then, by use of (1.29) $\sim$ (1.32) and $\tilde{K}_{\gamma b} = 0$, we see that $\tilde{M}$ is η-Einstein.

PROPOSITION 5.3. *Let $\tilde{M}$ be a fibred Sasakian space with conformal fibres and $p \neq 1$. Then $\tilde{M}$ is η-Einstein if and only if*

(1) *M is Einstein,*

(2) *each fibre is η-Einstein,*

(3) *$K = n\bar{K}/(p-1) + n$.*

In particular, if $p = 1$, then $\tilde{M}$ is η-Einstein if and only if M is Einstein.

Next, we assume that a fibred Sasakian space $\tilde{M}$ is conformally flat, then we get

(5.7)
$$\tilde{K}_{kji}{}^h = \frac{1}{m-2}(\delta_k^h\tilde{K}_{ji} - \delta_j^h\tilde{K}_{ki} + \tilde{K}_k{}^h\tilde{g}_{ji} - \tilde{K}_j{}^h\tilde{g}_{ki}) - \frac{\tilde{K}}{(m-1)(m-2)}(\delta_k^h\tilde{g}_{ji} - \delta_j^h\tilde{g}_{ki})\,.$$

Since the components $\tilde{K}_{dc\beta}{}^\alpha$ vanish and $L_{cb}{}^\alpha = J_{cb}\bar{\xi}^\alpha$ by Proposition 5.1, it follows from (1.12)

(5.8) $$*\nabla_c h_{\beta\alpha d} - *\nabla_d h_{\beta\alpha c} + 2(**\nabla_\beta J_{dc})\bar{\eta}_\alpha + 2J_{dc}\bar{\phi}_{\beta\alpha} - h_{\varepsilon\alpha d}h_{\beta}{}^{\varepsilon}{}_c + h_{\varepsilon\alpha c}h_{\beta}{}^{\varepsilon}{}_d = 0\,.$$

Transvecting this equation with $\bar{\phi}^{\beta\alpha}$ and making use of the equation (5) of Proposition 5.1, we have

(5.9) $$2(p-1)J_{dc} = h_{\beta\varepsilon a}J_c{}^a h^{\beta\varepsilon}{}_d - h_{\beta\varepsilon a}J_d{}^a h^{\beta\varepsilon}{}_c\,.$$

Moreover, transvecting this equation with J^{dc}, we obtain

$$n(p-1)+h_{\beta ab}h^{\beta ab}=0\,.$$

Hence we see that the dimension of fibre is equal to one and $h=(h_{\beta a}{}^{b})$ vanishes identically. From this fact and Proposition 5.1, we have $\tilde{K}_{cb}=K_{cb}-2g_{cb}$ by means of (1.29), and the equation (1.10) implies that

$$\begin{aligned}K_{dcba}=&\frac{1}{n-1}(g_{da}K_{cb}-g_{ca}K_{db}+g_{cb}K_{da}-g_{db}K_{ca}+4g_{ca}g_{db}-4g_{da}g_{cb})\\&-\frac{K-n}{n(n-1)}(g_{da}g_{cb}-g_{ca}g_{db})+J_{da}J_{cb}-J_{ca}J_{db}-2J_{dc}J_{ba}\,,\end{aligned}\tag{5.10}$$

by use of $m=n+p=n+1$ and (1.32). From (5.10), we see that M is Einstein, that is,

$$K_{cb}=\frac{K}{n}g_{cb}\,.\tag{5.11}$$

If we substitute (5.11) into (5.10), then the curvature tensor of M is reduced to

$$K_{dcba}=\frac{K-3n}{n(n-1)}(g_{da}g_{cb}-g_{ca}g_{db})+J_{da}J_{cb}-J_{ca}J_{db}-2J_{dc}J_{ba}\,.\tag{5.12}$$

Thus we have

THEOREM 5.4. *If the fibred Sasakian space $\tilde{M}$ is conformally flat, then we have*

(1) *$h=(h_{\beta a}{}^{b})$ vanishes identically,*
(2) dim $\bar{M}=1$,
(3) *the curvature tensor of the base space is of the form* (5.12).

In particular, the base space M is a complex space form if and only if $K=n(n+2)$.

Since the space of constant curvature is conformally flat, by use of Theorem 5.4, the fibred Sasakian space of constant curvature has 1-dimensional isometric fibres and $m=n+1$ and $\tilde{K}=K-n$. Therefore, if we compare the horizontal components of

$$\tilde{K}_{kji}{}^{h}=\frac{K-n}{n(n+1)}(\delta_k^h\tilde{g}_{ji}-\delta_j^h\tilde{g}_{ki})$$

with (1.10), then we get

$$K_{dcba}=\frac{K-n}{n(n+1)}(g_{da}g_{cb}-g_{ca}g_{db})+J_{da}J_{cb}-J_{ca}J_{db}-2J_{dc}J_{ba}\,.$$

From this equation, we see that $K = n(n+2)$ and then $(K-n)/n(n+1) = 1$. Hence the base space M is a complex space form with constant holomorphic sectional curvature 4. Thus we have

THEOREM 5.5. *If a fibred Sasakian space $\tilde{M}$ is of constant curvature, then the base space M is a complex space form with constant holomorphic sectional curvature* 4 *and $\tilde{M}$ has* 1-*dimensional isometric fibres.*

EXAMPLE 5.6. The Hopf fibering $\pi: S^{2n+1} \to CP(n)$ with totally geodesic fibre S^1 (cf. [10, 11]) is a typical example of fibred Sasakian space satisfying the conditions stated in Theorem 5.5, where $CP(n)$ is a complex projective space of complex dimension $n \geq 1$.

§6. Quasi Sasakian and cosymplectic structures

In [22], we proved

PROPOSITION 6.1. *If an almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on $\tilde{M}$ is normal, then the complex structure J on M is integrable and the almost contact structure $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ on $\bar{M}$ is normal.*

More precisely, putting

$$N_{c\beta}{}^{\alpha} = J_c{}^e(\partial_e \bar{\phi}_\beta{}^\alpha) + \bar{\phi}_\beta{}^\gamma(\partial_c \bar{\phi}_\gamma{}^\alpha),$$

we can see that

PROPOSITION 6.2. *An almost contact metric structure on $\tilde{M}$ is normal if and only if the almost complex structure on M is integrable, the almost contact metric structure on $\bar{M}$ is normal and $N_{c\beta}{}^{\alpha} = 0$.*

If the structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ in $\tilde{M}$ is quasi Sasakian, that is, $d\tilde{\Theta} = 0$ and the structure is normal, then, by means of (3.10, 5), (3.12) and Proposition 6.1, we can see that M is Kaehlerian, $\bar{M}$ is quasi Sasakian, and

$$L_{cb}{}^{\beta}\bar{\phi}_{\beta\alpha} = 0, \tag{6.1}$$

$$L_c{}^e{}_\alpha J_{be} + L_b{}^e{}_\alpha J_{ec} = 0, \tag{6.2}$$

$$*\nabla_c \bar{\phi}_{\beta\alpha} + h_{\beta\gamma c}\bar{\phi}_\alpha{}^\gamma + h_\alpha{}^\gamma{}_c \bar{\phi}_{\gamma\beta} = 0, \tag{6.3}$$

$$N_{c\beta}{}^{\alpha} = 0. \tag{6.4}$$

Hence, from (6.1) and (6.2), we see that the field $L_{cb}{}^{\alpha}$ is expressed as the form $L_{cb}{}^{\alpha} = \Lambda_{cb}\bar{\xi}^{\alpha}$, where Λ_{cb} is a skew symmetric (0, 2)-form and satisfy

$J_b{}^e \Lambda_e{}^a = \Lambda_b{}^e J_e{}^a$. Moreover, by means of (3.10, 2) and (3.10, 9), the relation (6.3) is equivalent to

$$\partial_c \overline{\phi}_{\beta\alpha} = P_{c\beta}{}^\gamma \overline{\phi}_{\gamma\alpha} + P_{c\alpha}{}^\gamma \overline{\phi}_{\beta\gamma} .$$

Thus, by means of (3.12) and Proposition 6.2, we can state

PROPOSITION 6.3. *The almost contact metric structure on $\tilde{M}$ is quasi Sasakian if and only if*

(1) *M is Kaehlerian, i.e. $d\Theta = 0$ and the structure J is integrable.*

(2) *$\overline{M}$ is quasi Sasakian, i.e. $d\overline{\Theta} = 0$ and the structure $(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is normal,*

(3) *the field $L_{cb}{}^\alpha$ is written in the form $L_{cb}{}^\alpha = \Lambda_{cb}\overline{\xi}^\alpha$ with Λ_{cb} satisfying $J_b{}^e \Lambda_e{}^a = \Lambda_b{}^e J_e{}^a$.*

(4) *$\partial_c \overline{\phi}_{\beta\alpha} = P_{c\beta}{}^\gamma \overline{\phi}_{\gamma\alpha} + P_{c\alpha}{}^\gamma \overline{\phi}_{\beta\gamma}$ and*

(5) *$N_{c\beta}{}^\alpha = 0$.*

Since the cosymplectic structure is characterized by $\tilde{\nabla}\tilde{\phi} = 0$ and $\tilde{\nabla}\tilde{\xi} = 0$, it follows from (3.6) and (3.12, 2) that

$$L_c{}^b{}_\alpha \overline{\xi}^\alpha = 0 \qquad \text{and} \qquad L_{cb}{}^\alpha \overline{\phi}_{\beta\alpha} = 0 .$$

Hence $L = (L_{cb}{}^\alpha)$ vanishes identically. Moreover, by means of the equations (3.2) $\sim$ (3.7), we see that M is Kaehlerian, $\overline{M}$ is cosymplectic and

$$h_\gamma{}^\beta{}_b \overline{\phi}_\beta{}^\alpha - h_\gamma{}^\alpha{}_a J_b{}^a = 0 , \tag{6.5}$$

$$*\nabla_c \overline{\phi}_\beta{}^\alpha = 0 , \tag{6.6}$$

$$*\nabla_c \overline{\xi}^\alpha = 0 , \tag{6.7}$$

$$h_{\gamma\beta}{}^a \overline{\xi}^\beta = 0 . \tag{6.8}$$

By use of (6.5), $h_\gamma{}^\beta{}_b \overline{\phi}_{\beta\alpha}$ is symmetric in α and γ and hence $*\nabla_c \overline{\phi}_\beta{}^\alpha = 0$ is equivalent to

$$\partial_c \overline{\phi}_\beta{}^\alpha = P_{c\beta}{}^\varepsilon \overline{\phi}_\varepsilon{}^\alpha - P_{c\varepsilon}{}^\alpha \overline{\phi}_\beta{}^\varepsilon - 2h_\beta{}^\varepsilon{}_c \overline{\phi}_\varepsilon{}^\alpha . \tag{6.9}$$

The relation $*\nabla_c \overline{\xi}^\alpha = 0$ with $h_{\gamma\beta}{}^a \overline{\xi}^\beta = 0$ is reduced to

$$\partial_c \overline{\xi}^\alpha + P_{c\gamma}{}^\alpha \overline{\xi}^\gamma = 0 . \tag{6.10}$$

Conversely, if the base space is Kaehlerian, each fibre is cosymplectic and the equations (6.5), (6.8) $\sim$ (6.10) and $L_{cb}{}^\alpha = 0$ are satisfied, then, by use of (3.2) $\sim$ (3.7), we can see that $\tilde{M}$ is cosymplectic. Thus we have

PROPOSITION 6.4. *The almost contact metric structure on M is cosymplectic if and only if*

(1) *M is Kaehlerian,*
(2) *$\overline{M}$ is cosymplectic,*
(3) $L_{cb}{}^{\alpha} = 0$,
(4) $\partial_c \overline{\phi}_\beta{}^\alpha = P_{c\beta}{}^\varepsilon \overline{\phi}_\varepsilon{}^\alpha - P_{c\varepsilon}{}^\alpha \overline{\phi}_\beta{}^\varepsilon - 2h_\beta{}^\varepsilon{}_c \overline{\phi}_\varepsilon{}^\alpha$,
(5) $h_\gamma{}^\beta{}_b \overline{\phi}_\beta{}^\alpha - h_\gamma{}^\alpha{}_a J_b{}^a = 0$,
(6) $\partial_c \overline{\xi}^\alpha + P_{c\gamma}{}^\alpha \overline{\xi}^\gamma = 0$, $h_{\gamma\beta c} \overline{\xi}^\beta = 0$.

In this case, each fibre is minimal in $\tilde{M}$.

§7. Examples

Now, we shall give new examples of cosymplectic, Sasakian and quasi Sasakian structures on a 5-dimensional Euclidean space E^5 as fibred Riemannian space. Of course, these examples can be extendable to E^{2n+1} by the same method of construction.

EXAMPLE 7.1. We denote Cartesian coordinates in E^5 by $(x_1, x_2, x_3, x_4, x_5)$ and define a symmetric tensor field $\tilde{g}$ by

$$\tilde{g} = \begin{pmatrix} 1+\tau^2 & 0 & \sigma\tau & 0 & -\tau \\ 0 & 1 & 0 & 0 & 0 \\ \sigma\tau & 0 & 1+\sigma^2 & 0 & -\sigma \\ 0 & 0 & 0 & 1 & 0 \\ -\tau & 0 & -\sigma & 0 & 1 \end{pmatrix}.$$

where σ and τ are functions on E^5. Then $\tilde{g}$ is a positive definite Riemannian metric. The inverse matrix of $\tilde{g}$ is given by

$$\tilde{g}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \tau \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \sigma \\ 0 & 0 & 0 & 1 & 0 \\ \tau & 0 & \sigma & 0 & 1+\sigma^2+\tau^2 \end{pmatrix}.$$

We define an almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ on E^5 by

$$\tilde{\phi} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\tau & 0 & -\sigma & 0 \end{pmatrix}.$$

$$\tilde{\eta} = (-\tau, 0, -\sigma, 0, 1),$$

$$\tilde{\xi} = {}^t(0, 0, 0, 0, 1),$$

where A^t is the transposed matrix of A. Then $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ constitutes an almost contact metric structure on E^5. The fundamental 1-form $\tilde{\eta}$ and 2-form $\tilde{\Theta}$ have the forms

$$\tilde{\eta} = -\tau dx^1 - \sigma dx^3 + dx^5 \qquad \text{and} \qquad \tilde{\Theta} = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

respectively, and hence

$$d\tilde{\eta} = dx^1 \wedge d\tau + dx_3 \wedge d\sigma \qquad \text{and} \qquad d\tilde{\Theta} = 0\,.$$

We seek for the conditions of normality in terms of the Nijenhuis tensor of $\tilde{\phi}$. The Nijenhuis tensor $N_{kj}{}^i$ can be written as ([31])

$$N_{kj}{}^i = \tilde{\phi}_k{}^h(\partial_h\tilde{\phi}_j{}^i - \partial_j\tilde{\phi}_h{}^i) - \tilde{\phi}_j{}^h(\partial_h\tilde{\phi}_k{}^i - \partial_k\tilde{\phi}_h{}^i) + \tilde{\eta}_k(\partial_j\tilde{\xi}^i) - \tilde{\eta}_j(\partial_k\tilde{\xi}^i)\,,$$

where the indices h, i, j, k run over the range $1, 2, \ldots, 5$ for this example. As components of $\tilde{\xi}$ are all constant, the third and fourth terms of the right hand side do not appear. Hence non-trivial components of $N_{kj}{}^i$ are given by

$$N_{13}{}^5 = \partial_3\tau - \partial_1\sigma\,,$$
$$N_{14}{}^5 = -\partial_2\sigma + \partial_4\tau = -N_{23}{}^5\,,$$
$$N_{15}{}^5 = \partial_5\tau\,,$$
$$N_{24}{}^5 = -\sigma\partial_5\tau + \tau\partial_5\sigma - \partial_3\tau + \partial_1\sigma\,,$$
$$N_{35}{}^5 = \partial_5\sigma\,.$$

Therefore, we can see that E^5 with $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is

(1) cosymplectic if $\tau = \tau(x_1, x_3)$, $\sigma = \sigma(x_1, x_3)$ and $\partial_3\tau = \partial_1\sigma$,

(2) Sasakian if $\tau = \tau(x_1, x_2, x_3)$, $\sigma = \sigma(x_1, x_3, x_4)$, $\partial_2\tau =$ constant $(=1) = \partial_4\sigma$, and $\partial_3\tau = \partial_1\sigma$,

(3) quasi Sasakian if $\tau = \tau(x_1, x_2, x_3, x_4)$, $\sigma = \sigma(x_1, x_2, x_3, x_4)$, $\partial_4\tau = \partial_2\sigma$ and $\partial_3\tau = \partial_1\sigma$,

where τ and σ are functions dependent only of the coordinates indicated in parentheses. For example, we can take $\tau = \sigma = \sin(x_1 + x_3)$ for a cosymplectic structure and $\tau = x_1 + x_2 + 2x_3$, $\sigma = 2x_1 + x_3 + x_4$ for a Sasakian one. By a choice of the functions τ and σ, we obtain pretty extensive examples of quasi Sasakian structure which are neither cosymplectic nor Sasakian, say $\tau = \sigma = \sin(x_1 + 2x_2 + x_3 + 2x_4)$.

If we take vector fields

$$E_1 = {}^t(1, 0, 0, 0, \tau)\,,$$
$$E_2 = {}^t(0, 1, 0, 0, 0)\,,$$
$$C_1 = (0, 0, 1, 0, 0)\,,$$

$$C_2 = (0, 0, 0, 1, 0)\,,$$

$$C_3 = (0, 0, 0, 0, 1)\,,$$

then these vector fields form a frame field in E^5 and we see that

$$J = \tilde{\phi}(E_b, E^a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$g = \tilde{g}(E_b, E_a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\bar{\phi} = \tilde{\phi}(C_\beta, C^\alpha) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\sigma & 0 \end{pmatrix},$$

$$\bar{g} = \tilde{g}(C_\beta, C_\alpha) = \begin{pmatrix} 1+\sigma^2 & 0 & -\sigma \\ 0 & 1 & 0 \\ -\sigma & 0 & 1 \end{pmatrix},$$

$$\bar{\xi} = \tilde{\xi}C^\alpha = {}^t(0, 0, 1)\,,$$

$$\bar{\eta} = \tilde{\eta}(C_\beta) = (-\sigma, 0, 1)\,,$$

where the indices a, b and α, β run over the ranges 1, 2 and 1, 2, 3 respectively. Hence the space E^5 with $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ becomes a fibred Riemannian space having $(E^2; J, g)$ as base space and $(E^3; \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ as fibre.

On the other hand, by the analogous way, $(E^3; \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ with local coordinates (x_3, x_4, x_5) becomes an almost contact metric space and we can see that $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ is normal if and only if $\partial_5\sigma = 0$. Since the fundamental 1-form $\bar{\eta}$ and 2-form $\bar{\Theta}$ of the fibre E^3 are given by

$$\bar{\eta} = -\sigma dx_3 + dx_5 \qquad \text{and} \qquad \bar{\Theta} = dx_3 \wedge dx_4$$

and

$$d\bar{\eta} = (\partial_4\sigma)dx_3 \wedge dx_4 + (\partial_5\sigma)dx_3 \wedge dx_5\,,$$

we can see that $(E^3; \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ becomes

(1) cosymplectic if $\sigma = \sigma(x_3)$, i.e. $\partial_4\sigma = 0 = \partial_5\sigma$,

(2) Sasakian if $\sigma = \sigma(x_3, x_4)$ and $\partial_4\sigma = \text{constant}\ (=1)$,

(3) quasi Sasakian if $\sigma = \sigma(x_3, x_4)$.

Henceforth, we have constructed a fibred Riemannian space with cosymplectic (Sasakian, quasi Sasakian) structure having a Kaehlerian space as base space and a cosymplectic (Sasakian, quasi Sasakian, respectively) space as fibre.

§8. Cosymplectic space form

We assume that $\tilde{M}$ is a cosymplectic space with constant $\tilde{\phi}$-holomorphic sectional curvature k. Then the curvature tensor has the form [12, 20, 24]

$$(8.1)\quad \begin{aligned}\tilde{K}_{kji}{}^{h} = \frac{k}{4}(&\tilde{g}_{ji}\delta_k^h - \tilde{g}_{ki}\delta_j^h + \tilde{\phi}_{ji}\tilde{\phi}_k{}^h - \tilde{\phi}_{ki}\tilde{\phi}_j{}^h - 2\tilde{\phi}_{kj}\tilde{\phi}_i{}^h \\ &- \tilde{\eta}_j\tilde{\eta}_i\delta_k^h + \tilde{\eta}_j\tilde{\xi}^h\tilde{g}_{ki} - \tilde{\eta}_k\tilde{\xi}^h\tilde{g}_{ji} + \tilde{\eta}_k\tilde{\eta}_i\delta_j^h)\,.\end{aligned}$$

Then, since $L_{cb}{}^{a} = 0$ in the cosymplectic case, the equations (1.10) and (1.14) $\sim$ (1.17) give rise to

$$(8.2)\quad K_{dcb}{}^{a} = \frac{k}{4}(\delta_d^a g_{cb} - g_{db}\delta_c^a + J_d{}^a J_{cb} - J_{db}J_c^a - 2J_{dc}J_b{}^a)\,,$$

$$(8.3)\quad h_{\gamma}{}^{\varepsilon}{}_{d}h_{\varepsilon}{}^{\alpha}{}_{b} - {}^{*}\nabla_d h_{\gamma}{}^{\alpha}{}_{b} = -\frac{k}{4}(g_{db}\delta_\gamma^\alpha + J_{db}\bar{\phi}_\gamma{}^\alpha - \bar{\eta}_\gamma\bar{\xi}^\alpha g_{db})\,,$$

$$(8.4)\quad h_{\delta}{}^{\varepsilon}{}_{b}h_{\gamma\varepsilon}{}^{a} - h_{\gamma}{}^{\varepsilon}{}_{b}h_{\delta\varepsilon}{}^{a} = -\frac{k}{2}\bar{\phi}_{\gamma\delta}J_b{}^a\,,$$

$$(8.5)\quad {}^{**}\nabla_\delta h_{\gamma\beta}{}^{a} - {}^{**}\nabla_\gamma h_{\delta\beta}{}^{a} = 0\,,$$

$$(8.5)\quad \begin{aligned}\bar{K}_{\delta\gamma\beta}{}^{\alpha} = \frac{k}{4}(&\bar{g}_{\gamma\beta}\delta_\delta^\alpha - \bar{g}_{\delta\beta}\delta_\gamma^\alpha + \bar{\phi}_{\gamma\beta}\bar{\phi}_\delta{}^\alpha - \bar{\phi}_{\delta\beta}\bar{\phi}_\gamma{}^\alpha - 2\bar{\phi}_{\delta\gamma}\bar{\phi}_\beta{}^\alpha \\ &- \bar{\eta}_\gamma\bar{\eta}_\beta\delta_\delta^\alpha + \bar{\eta}_\delta\bar{\eta}_\beta\delta_\gamma^\alpha + \bar{\eta}_\gamma\bar{\xi}^\alpha\bar{g}_{\delta\beta} - \bar{\eta}_\delta\bar{\xi}^\alpha\bar{g}_{\gamma\beta}) + h_{\gamma\beta}{}^{e}h_{\delta}{}^{\alpha}{}_{e} - h_{\delta\beta}{}^{e}h_{\gamma}{}^{\alpha}{}_{e}\,,\end{aligned}$$

(1.11) and (1.13) are trivial, and (1.12) is equivalent to (8.3). As seen from (8.2), the base space M is a complex space form and we have

$$K_{cb} = \tfrac{1}{4}(n+2)g_{cb}k\,,$$

and

$$(8.7)\quad \frac{k}{4} = K/n(n+2)\,.$$

Contracting α and γ in (8.3), using (8.7) and the minimality of $\overline{M}$, we get

$$(8.8)\quad h_{\alpha\beta b}h^{\alpha\beta}{}_{d} = -K(p-1)g_{db}/n(n+2)\,,$$

and so,

$$(8.9)\quad \|h_{\alpha\beta b}\|^2 = -(p-1)K/(n+2)\,.$$

Since $\tilde{K}$ **and** k is in the relation

$$\frac{k}{4} = \frac{\tilde{K}}{(m+1)(m-1)}, \tag{8.10}$$

we obtain, from (1.32) and Proposition 6.4,

$$\frac{k}{4} = \frac{K}{(m-1)(n+2)} + \frac{\bar{K}}{(m+1)(m-1)}.$$

By use of (8.7) and (8.10), we get

$$n(n+2)\bar{K} - (p-1)(m+1)K = 0, \tag{8.11}$$

from which and (8.6),

$$\bar{K}_{\gamma\beta} = \frac{(p+1)}{(p-1)(m+1)}(\bar{g}_{\gamma\beta} - \bar{\eta}_\gamma \bar{\eta}_\beta)\bar{K} - h_{\gamma\varepsilon e} h_\beta{}^{\varepsilon e}$$

and

$$\bar{K} = \frac{p^2-1}{4}k - \|h_{\alpha\beta b}\|^2.$$

Hence we see that

PROPOSITION 8.1. *If the fibred space $\tilde{M}$ is a cosymplectic space form, then the base space M is a complex space form.*

In the special case of the codimension $p = 1$, we have $\bar{K} = 0$ and $h_{\alpha\beta b} = 0$ from (8.9). Thus we have

COROLLARY 8.2. *If $\tilde{M}$ is a fibred Riemannian space with cosymplectic space form and one dimensional fibres, then $\tilde{M}$ is locally the product manifold of a complex space form and E^1.*

If $p \neq 1$ and $\tilde{M}$ has conformal fibres, then the second fundamental $h_{\gamma\beta}{}^a = 0$, because each fibre is minimal in $\tilde{M}$ by means of Proposition 6.4. Moreover we get $K = 0$ by use of (8.9), $k = 0$, $\bar{K} = 0$ and $\tilde{K} = 0$ by the equations (8.7), (8.10) and (8.11). Therefore we can state that

COROLLARY 8.3. *If a fibred Riemannian space $\tilde{M}$ is a cosymplectic space form and has conformal fibres of dimension $p \neq 1$, then $\tilde{M}$ itself, the base space and the fibres are locally Euclidean.*

Chapter III. Fibred Riemannian spaces with almost contact metric structure, the case where the structure vector is normal to the fibre

§9. Structures in the total, base and fibre spaces

In this chapter, we consider the case where each fibre is $\tilde{\phi}$-invariant and normal to $\tilde{\xi}$, and deal with quasi Sasakian or various cosymplectic structures in the total space.

We suppose that a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ has a projectable almost contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, each fibre $\overline{M}$ is $\tilde{\phi}$-invariant and normal to the structure vector $\tilde{\xi}$. Such a space is called a *fibred almost contact metric space with $\tilde{\phi}$-invariant fibres normal to $\tilde{\xi}$*. Then we can easily see that $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ has the form

$$(9.1)\qquad \begin{aligned} \tilde{\phi} &= \phi_b{}^a E^b \otimes E_a + \bar{J}_\beta{}^\alpha C^\beta \otimes C_\alpha\,, \\ \tilde{\xi} &= \xi^a E_a\,, \qquad \tilde{\eta} = \eta_a E^a\,, \\ \tilde{g} &= g_{ba} E^b \otimes E^a + \bar{g}_{\beta\alpha} C^\beta \otimes C^\alpha\,, \end{aligned}$$

(ϕ, ξ, η, g) defines an almost contact metric structure in the base space M and $(\bar{J}, \bar{g})$ an almost Hermitian structure in each fibre $\overline{M}$.

Conversely, if, in a fibred Riemannian space $\tilde{M}$, M has an almost contact metric structure (ϕ, ξ, η, g) the lift of which to the total space $\tilde{M}$ is denoted by the same letters (ϕ, ξ, η, g), and each fibre has an almost Hermitian structure $(\bar{J}, \bar{g})$, then we can construct an almost contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on the total space $\tilde{M}$ by putting (9.1).

If we apply the differential operator $\tilde{\nabla}_c = E^j{}_c \tilde{\nabla}_j$ and $\tilde{\nabla}_\gamma = C^j{}_\gamma \tilde{\nabla}_j$ to the equations (9.1) and take account of (1.6), then we obtain

$$(9.2)\qquad (\tilde{\nabla}_c \tilde{\phi}) E_b = (\nabla_c \phi_b{}^a) E_a + (L_{cb}{}^\beta \bar{J}_\beta{}^\alpha - L_{ce}{}^\alpha \phi_b{}^e) C_\alpha\,,$$

$$(9.3)\qquad (\tilde{\nabla}_c \tilde{\phi}) C_\beta = (L_c{}^a{}_\alpha \bar{J}_\beta{}^\alpha - L_c{}^b{}_\beta \phi_b{}^a) E_a + ({}^*\nabla_c \bar{J}_\beta{}^\alpha) C_\alpha\,,$$

$$(9.4)\qquad (\tilde{\nabla}_\gamma \tilde{\phi}) E_b = ({}^{**}\nabla_\gamma \phi_b{}^a) E_a + (h_\gamma{}^\beta{}_b \bar{J}_\beta{}^\alpha - h_\gamma{}^\alpha{}_e \phi_b{}^e) C_\alpha\,,$$

$$(9.5)\qquad (\tilde{\nabla}_\gamma \tilde{\phi}) C_\beta = (h_{\gamma\varepsilon}{}^a \bar{J}_\beta{}^\varepsilon - h_{\gamma\beta}{}^e \phi_e{}^a) E_a + (\bar{\nabla}_\gamma \bar{J}_\beta{}^\alpha) C_\alpha\,,$$

$$(9.6)\qquad \tilde{\nabla}_c \tilde{\xi} = (\nabla_c \xi^a) E_a - (L_{ce}{}^\alpha \xi^e) C_\alpha\,,$$

$$(9.7)\qquad \tilde{\nabla}_\gamma \tilde{\xi} = ({}^{**}\nabla_\gamma \xi^a) E_a - (h_\gamma{}^\alpha{}_e \xi^e) C_\alpha\,,$$

$$(9.8)\qquad {}^*\nabla_c \bar{g} = 0\,,$$

$$(9.9)\qquad {}^{**}\nabla_\gamma g = 0\,,$$

where we have put

$$
(9.10)\qquad
\begin{aligned}
&\nabla_c\phi_b{}^a = \partial_c\phi_b{}^a + \Gamma^a_{ce}\phi_b{}^e - \Gamma^e_{cb}\phi_e{}^a\,,\\
&\nabla_c\xi^a = \partial_c\xi^a + \Gamma^a_{cb}\xi^b\,,\\
&{}^*\nabla_c\bar{J}_\beta{}^\alpha = \partial_c\bar{J}_\beta{}^\alpha + Q_{c\varepsilon}{}^\alpha\bar{J}_\beta{}^\varepsilon - Q_{c\beta}{}^\varepsilon\bar{J}_\varepsilon{}^\alpha\,,\\
&{}^*\nabla_c\bar{g}_{\beta\alpha} = \partial_c\bar{g}_{\beta\alpha} - Q_{c\alpha}{}^\varepsilon\bar{g}_{\beta\varepsilon} - Q_{c\beta}{}^\varepsilon\bar{g}_{\varepsilon\alpha}\,,\\
&{}^{**}\nabla_\gamma\phi_b{}^a = \partial_\gamma\phi_b{}^a + L_e{}^a{}_\gamma\phi_b{}^e - L_b{}^e{}_\gamma\phi_e{}^a\,,\\
&{}^{**}\nabla_\gamma\xi^a = \partial_\gamma\xi^a + L_e{}^a{}_\gamma\xi^e\,,\\
&{}^{**}\nabla_\gamma g_{ba} = \partial_\gamma g_{ba} - L_b{}^e{}_\gamma g_{ea} - L_a{}^e{}_\gamma g_{be}\,,\\
&\bar{\nabla}_\gamma\bar{J}_\beta{}^\alpha = \partial_\gamma\bar{J}_\beta{}^\alpha - \bar{\Gamma}^\varepsilon_{\gamma\beta}\bar{J}_\varepsilon{}^\alpha + \bar{\Gamma}^\alpha_{\gamma\varepsilon}\bar{J}_\beta{}^\varepsilon\,,\\
&Q_{c\beta}{}^\alpha = P_{c\beta}{}^\alpha - h_\beta{}^\alpha{}_c\,.
\end{aligned}
$$

By use of (9.1), (9.6) and (9.7), we get

$$
(9.11)\qquad (d\tilde{\eta})^H = d\eta\,,\qquad d\tilde{\eta}(C_\beta, E_c) = 0\,,\qquad (d\tilde{\eta})^V = 0\,,
$$

because $\tilde{\eta}$ is projectable. By virtue of (9.1), we have $\tilde{\phi}(C_\beta, C^\alpha) = \bar{J}_\beta{}^\alpha$. Hence, we have

Proposition 9.1. *The almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ in a fibred almost contact metric space with $\tilde{\phi}$-invariant fibres normal to $\tilde{\xi}$ cannot be contact.*

§10. Almost cosymplectic structures

An almost contact structure with closed forms $\tilde{\Theta}$ and $\tilde{\eta}$ is said to be almost cosymplectic. S. I. Goldberg [15] have examined the integrability of almost cosymplectic structures and showed that a normal almost cosymplectic manifold is cosymplectic. Z. Olszak [26] gave certain sufficient conditions for an almost contact structure to be almost cosymplectic. In this section, we consider almost cosymplectic structures in fibred Riemannian spaces.

Taking account of (9.2) $\sim$ (9.5) and (9.10), we obtain the following relations among the fundamental 2-forms $\tilde{\Theta}$ in the total space, Θ in the base space M and Ω in the fibre spaces $\bar{M}$:

$$
(10.1)\qquad
\begin{aligned}
&(d\tilde{\Theta})^H = d\Theta\,,\\
&d\tilde{\Theta}(E_c, E_b, C_\alpha) = 2L_{cb}{}^\varepsilon\bar{J}_{\varepsilon\alpha}\,,\\
&d\tilde{\Theta}(E_c, C_\beta, C_\alpha) = {}^*\nabla_c\bar{J}_{\beta\alpha} - h_\beta{}^\varepsilon{}_c\bar{J}_{\varepsilon\alpha} - h_\alpha{}^\varepsilon{}_c\bar{J}_{\beta\varepsilon}\,,\\
&(d\tilde{\Theta})^V = d\Omega\,.
\end{aligned}
$$

Since $d\tilde{\eta} = 0$ and $d\tilde{\Theta} = 0$ for an almost cosymplectic structure, by means of (10.1, 2), we get $L_{cb}{}^{\alpha} = 0$. By use of (9.10, 3) and (9.10, 9), we see that $d\tilde{\Theta}(E_c, C_\beta, C_\alpha) = 0$ is equivalent to

$$\partial_c \bar{J}_{\beta\alpha} - P_{c\beta}{}^{\varepsilon} \bar{J}_{\varepsilon\alpha} - P_{c\alpha}{}^{\varepsilon} \bar{J}_{\beta\varepsilon} = 0 .$$

Thus, referring to (9.11) and (10.1), we have

PROPOSITION 10.1. *The almost contact metric structure* $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *on the fibred space* $\tilde{M}$ *is almost cosymplectic if and only if the following conditions are satisfied:*

(1) *the base space is almost cosymplectic,*
(2) *each fibre is almost Kaehlerian,*
(3) $L_{cb}{}^{\alpha} = 0$, *and*
(4) $\partial_c \bar{J}_{\beta\alpha} - P_{c\beta}{}^{\varepsilon} \bar{J}_{\varepsilon\alpha} - P_{c\alpha}{}^{\varepsilon} \bar{J}_{\beta\varepsilon} = 0.$

Moreover, if a fibred almost cosymplectic space $\tilde{M}$ has isometric fibres, then the second fundamental form $h = (h_{\alpha\beta}{}^{a})$ with respect to E_a vanishes identically. Therefore, by means of Lemma 1.1, the fibred space $\tilde{M}$ is locally a Riemannian product space of the base space and a fibre.

Conversely, if a space M is almost cosymplectic and $\bar{M}$ is almost Kaehlerian, then the product space $\tilde{M} = M \times \bar{M}$ becomes an almost cosymplectic space by means of Proposition 10.1 and the equations (9.11) and (10.1). Thus we have

COROLLARY 10.2. *If a fibred space* $\tilde{M}$ *with almost cosymplectic structure has isometric fibres, then* $\tilde{M}$ *is locally the product of an almost cosymplectic space and an almost Kaehlerian space, and vice versa.*

§11. Normality of the induced almost contact metric structure on $\tilde{M}$ and quasi Sasakian structures

The Nijenhuis tensor of an almost contact structure (ϕ, ξ, η) in the base space M is given by

$$N_{cb}{}^{a} = \phi_c{}^{e}(\partial_e \phi_b{}^{a} - \partial_b \phi_e{}^{a}) - \phi_b{}^{e}(\partial_e \phi_c{}^{a} - \partial_c \phi_e{}^{a}) + \eta_c(\partial_b \xi^{a}) - \eta_b(\partial_c \xi^{a}) . \tag{11.1}$$

On the other hand, the Nijenhuis tensor of an almost complex structure $\bar{J} = (\bar{J}_\beta{}^{\alpha})$ in the fibre is given by

$$\bar{N}_{\gamma\beta}{}^{\alpha} = \bar{J}_\gamma{}^{\varepsilon}(\partial_\varepsilon \bar{J}_\beta{}^{\alpha} - \partial_\beta \bar{J}_\varepsilon{}^{\alpha}) - \bar{J}_\beta{}^{\varepsilon}(\partial_\varepsilon \bar{J}_\gamma{}^{\alpha} - \partial_\gamma \bar{J}_\varepsilon{}^{\alpha}) .$$

In our envisaged case, the Nijenhuis tensor $\tilde{N}$ of the almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ of the total space $\tilde{M}$ is given by the expression (11.1) with

$(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ in place of (ϕ, ξ, η), and it splits into the following components:

$$(11.2)\qquad \begin{aligned} \tilde{N}_{cb}{}^{a} &= N_{cb}{}^{a}\,, \\ \tilde{N}_{c\beta}{}^{a} &= 0\,, \\ \tilde{N}_{c\beta}{}^{\gamma} &= \phi_c{}^e(\partial_e \bar{J}_\beta{}^\gamma) - \bar{J}_\beta{}^\alpha(\partial_c \bar{J}_\alpha{}^\gamma)\,, \\ \tilde{N}_{\gamma\beta}{}^{\alpha} &= \bar{N}_{\gamma\beta}{}^{\alpha}\,. \end{aligned}$$

Thus we have

PROPOSITION 11.1. *In order that the almost contact metric structure on $\tilde{M}$ is normal if and only if the almost contact structure on M is normal, the almost complex structure on $\overline{M}$ is integrable and $\tilde{N}_{c\beta}{}^{\gamma}$ vanishes identically.*

If the structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is quasi Sasakian, that is, the structure is normal and $d\tilde{\Theta} = 0$, then the structure vector $\tilde{\xi}$ is Killing [2, 33], i.e.,

$$\mathscr{L}_{\tilde{\xi}}\tilde{g}_{ji} = \tilde{\nabla}_j\tilde{\eta}_i + \tilde{\nabla}_i\tilde{\eta}_j = 0$$

where $\mathscr{L}_{\tilde{\xi}}$ indicates the Lie derivation with respect to $\tilde{\xi}$, and it is known that

LEMMA 11.2 [2]. *A quasi Sasakian manifold $\tilde{M}$ is cosymplectic if and only if $\tilde{\nabla}_j\tilde{\phi}_i{}^h = 0$.*

LEMMA 11.3 [33]. *In a quasi Sasakian manifold $\tilde{M}$, we have*

$$(\tilde{\nabla}_{\tilde{X}}\tilde{\Theta})(\tilde{Y}, \tilde{Z}) = \tilde{\eta}(\tilde{Y})(\tilde{\nabla}_{\tilde{X}}\tilde{\eta})\tilde{\phi}\tilde{Z} - \tilde{\eta}(\tilde{Z})(\tilde{\nabla}_{\tilde{X}}\tilde{\eta})\tilde{\phi}\tilde{Y}\,.$$

Assume that the almost contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ in $\tilde{M}$ is quasi Sasakian and projectable. In this case we call $\tilde{M}$ a *fibred quasi Sasakian space*. Then, by means of (9.1, 2), (9.3) and Lemma 11.3, we obtain

$$(11.3)\qquad {}^*\nabla_c \bar{J}_{\beta\gamma} = 0\,.$$

By means of (10.1, 3), it follows

$$(11.4)\qquad h_\beta{}^\varepsilon{}_c \bar{J}_{\varepsilon\alpha} + h_\alpha{}^\varepsilon{}_c \bar{J}_{\beta\varepsilon} = 0\,,$$

Hence, transvecting $\bar{J}^{\beta\alpha}$ with (11.4), we get

$$(11.5)\qquad h_{\alpha\beta b}\bar{g}^{\alpha\beta} = 0\,,$$

that is, each fibre is minimal in $\tilde{M}$. By means of (10.1), Proposition 11.1 and (11.5), we have

PROPOSITION 11.4. *The almost contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on $\tilde{M}$ is quasi Sasakian if and only if the following conditions are satisfied:*

(1) *the base space is quasi Sasakian,*
(2) *each fibre is Kaehlerian,*
(3) $L_{cb}{}^{\alpha} = 0$,
(4) $\partial_c \bar{J}_{\beta\alpha} - P_{c\beta}{}^{\varepsilon}\bar{J}_{\varepsilon\alpha} - P_{c\alpha}{}^{\varepsilon}\bar{J}_{\beta\varepsilon} = 0$, *and*
(5) $\tilde{N}_{c\beta}{}^{\gamma} = 0$.
In this case, each fibre is minimal in $\tilde{M}$.

In addition, if $\tilde{M}$ has conformal fibres, then the second fundamental form $h = (h_{\gamma\beta}{}^{\alpha})$ vanishes identically. We have seen the normal connection $L = (L_{cb}{}^{\alpha})$ of $\bar{M}$ in $\tilde{M}$ is zero. Consequently the fibred quasi Sasakian space $\tilde{M}$ is locally the Riemannian product space of the base space and the fibre by means Lemma 1.1.

Conversely, if a space M is quasi Sasakian and $\bar{M}$ is Kaehlerian, then the product space $\tilde{M} = M \times \bar{M}$ becomes a quasi Sasakian space by use of Proposition 11.1 and the equation (10.1). Thus we have

COROLLARY 11.5. *If a fibred quasi Sasakian space* $\tilde{M}$ *has conformal fibres, then* $\tilde{M}$ *is locally the product of a quasi Sasakian space and a Kaehlerian space, and vice versa.*

§12. Nearly cosymplectic and closely cosymplectic structures

An almost contact manifolds, whose almost contact structure tensor is Killing, is called a nearly cosymplectic manifold, which was introduced by D. E. Blair [3]. It is known that a five-sphere S^5 as a totally geodesic hypersurface of S^6 carries a non-normal nearly cosymplectic structure [4]. In particular, if a nearly cosymplectic structure is normal, then it is cosymplectic.

D. E. Blair and D. K. Showers [5] introduced the notion of closely cosymplectic structure by the condition that $\tilde{\phi}$ is Killing and $\tilde{\eta}$ is closed, and showed $S^6 \times S^1$ admits this structure. Clearly, a closely cosymplectic structure is nearly cosymplectic but the converse is not true, in general. For example, it is known that S^5 is a nearly cosymplectic manifold but not a closely cosymplectic manifold in view of the following

LEMMA 12.1 [5]. *Every 5-dimensional closely cosymplectic manifold is cosymplectic.*

By virtue of (9.2) $\sim$ (9.7), components of $\tilde{\nabla}_j\tilde{\phi}_{ih} + \tilde{\nabla}_i\tilde{\phi}_{jh}$ with respect to a frame $E_A = (E_a, C_\alpha)$ are given by

$$(\tilde{\nabla}_j\tilde{\phi}_{ih} + \tilde{\nabla}_i\tilde{\phi}_{jh})E^j{}_cE^i{}_bE^h{}_a = \nabla_c\phi_{ba} + \nabla_b\phi_{ca}\,,$$

$$(\tilde{\nabla}_j\tilde{\phi}_{ih} + \tilde{\nabla}_i\tilde{\phi}_{jh})E^j{}_cE^i{}_bC^h{}_\alpha = L_c{}^e{}_\alpha\phi_{eb} + L_b{}^e{}_\alpha\phi_{ec}\,,$$

$$
\begin{aligned}
&(\tilde{\nabla}_j\tilde{\phi}_{ih} + \tilde{\nabla}_i\tilde{\phi}_{jh})E^j{}_cC^i{}_\beta E^h{}_a = L_{ca}{}^\alpha\bar{J}_{\beta\alpha} - 2L_c{}^e{}_\beta\phi_{ea} - L_a{}^e{}_\beta\phi_{ce}\,,\\
&(\tilde{\nabla}_j\tilde{\phi}_{ih} + \tilde{\nabla}_i\tilde{\phi}_{jh})E^j{}_cC_\beta{}^iC_\alpha{}^h = {}^*\nabla_c\bar{J}_{\beta\alpha} + h_\beta{}^\varepsilon{}_c\bar{J}_{\varepsilon\alpha} - h_{\beta\alpha}{}^e\phi_{ce}\,,\\
&(\tilde{\nabla}_j\tilde{\phi}_{ih} + \tilde{\nabla}_i\tilde{\phi}_{jh})C^j{}_\gamma C^i{}_\beta E^h{}_a = h_\gamma{}^\varepsilon{}_a\bar{J}_{\beta\varepsilon} + h_\beta{}^\varepsilon{}_a\bar{J}_{\gamma\varepsilon} - 2h_{\gamma\beta}{}^e\phi_{ea}\,,\\
&(\tilde{\nabla}_j\tilde{\phi}_{ih} + \tilde{\nabla}_i\tilde{\phi}_{jh})C^j{}_\gamma C^i{}_\beta C^h{}_\alpha = \bar{\nabla}_\gamma\bar{J}_{\beta\alpha} + \bar{\nabla}_\beta\bar{J}_{\gamma\alpha}\,,
\end{aligned}
\tag{12.1}
$$

and components of $\mathscr{L}_{\tilde{\xi}}\tilde{g}$ by

$$
\begin{aligned}
&(\tilde{\nabla}_j\tilde{\eta}_i + \tilde{\nabla}_i\tilde{\eta}_j)E^j{}_cE^i{}_b = \nabla_c\eta_b + \nabla_b\eta_c\,,\\
&(\tilde{\nabla}_j\tilde{\eta}_i + \tilde{\nabla}_i\tilde{\eta}_j)E^j{}_cC^i{}_\beta = -2L_{ce\beta}\xi^e\,,\\
&(\tilde{\nabla}_j\tilde{\eta}_i + \tilde{\nabla}_i\tilde{\eta}_j)C^j{}_\gamma C^i{}_\beta = -2h_{\gamma\beta}{}^e\eta_e\,.
\end{aligned}
\tag{12.2}
$$

Assume that the total space $\tilde{M}$ admits a nearly Sasakian structure, which is characterized by the equation

$$
\tilde{\nabla}_j\tilde{\phi}_{ih} + \tilde{\nabla}_i\tilde{\phi}_{jh} = -2\tilde{g}_{ji}\tilde{\eta}_h + \tilde{\eta}_j\tilde{g}_{ih} + \tilde{\eta}_i\tilde{g}_{jh}\,.
\tag{12.3}
$$

The component of the right hand side of (12.3) with respect to $C_\gamma \otimes C_\beta \otimes E_a$ is reduced to $-2\bar{g}_{\gamma\beta}\eta_a$. Comparing this component with (12.1, 5), we get

$$
h_\gamma{}^\varepsilon{}_a\bar{J}_{\beta\varepsilon} + h_\beta{}^\varepsilon{}_a\bar{J}_{\gamma\varepsilon} - 2h_{\gamma\beta}{}^e\phi_{ea} = -2\bar{g}_{\gamma\beta}\eta_a\,.
\tag{12.4}
$$

Since the vector field $\tilde{\xi}$ is Killing in a nearly Sasakian manifold [6], by means of (12.2, 3), $h_{\gamma\beta a}\xi^a = 0$. Transvecting ξ^a with (12.4), we have $\bar{g}_{\gamma\beta} = 0$. This is a contradiction. Thus we can state

Proposition 12.2. *A fibred almost contact metric space with $\tilde{\phi}$-invariant fibres normal to $\tilde{\xi}$ does not admit a nearly Sasakian structure.*

If the structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is nearly cosymplectic, then $\tilde{\phi}$ is Killing by the definition and it is known [3] that the vector field $\tilde{\xi}$ is Killing. Hence we see that M is nearly cosymplectic and $\bar{M}$ is nearly Kaehlerian by means of (12.1, 1) and (12.1, 6) respectively. Moreover, by means of (12.2, 2) and (12.1, 2), we get

$$
L_{cb\beta}\xi^b = 0\,, \qquad L_c{}^e{}_\alpha\phi_{eb} = -L_b{}^e{}_\alpha\phi_{ec}
$$

and (12.1, 3) implies

$$
L_{cb\beta}\bar{J}_\gamma{}^\beta = 3L_c{}^e{}_\gamma\phi_{eb}\,,
$$

or equivalently,

$$
L_{cb\beta} = 3L_c{}^e{}_\gamma\bar{J}_\beta{}^\gamma\phi_{be}\,.
$$

By these relations, we obtain

$$L_{cb\beta}\bar{J}_\alpha^{\ \beta} = -3L_{ce\alpha}\phi_b^{\ e} = -9L_c^{\ d}{}_\gamma \bar{J}_\alpha^{\ \gamma}\phi_{ed}\phi_b^{\ e} = 9L_{cb\beta}\bar{J}_\alpha^{\ \beta}\,,$$

that is, $L_{cb\beta} = 0$.

From (12.2, 3), we have $h_{\beta\alpha}{}^e\eta_e = 0$ and hence

$$h_{\beta\alpha}{}^e\bar{g}^{\beta\alpha} = 0$$

by the contraction of (12.1, 5) in β and γ. By use of (9.10), we see that

$$*\nabla_c\bar{J}_{\beta\alpha} + h_\beta{}^\varepsilon{}_c\bar{J}_{\varepsilon\alpha} - h_{\beta\alpha}{}^e\phi_{ce} = 0 \tag{12.5}$$

is equivalent to

$$\partial_d\bar{J}_{\beta\alpha} - P_{d\beta}{}^\varepsilon\bar{J}_{\varepsilon\alpha} - P_{d\alpha}{}^\varepsilon\bar{J}_{\beta\varepsilon} + 2h_\beta{}^\varepsilon{}_d\bar{J}_{\varepsilon\alpha} + h_\alpha{}^\varepsilon{}_d\bar{J}_{\beta\varepsilon} - h_{\beta\alpha}{}^e\phi_{de} = 0\,.$$

Thus we have

PROPOSITION 12.3. *The almost contact metric structure on a fibred space $\tilde{M}$ is nearly cosymplectic if and only if the following conditions are satisfied*:

(1) *the base space is nearly cosymplectic,*
(2) *each fibre is nearly Kaehlerian,*
(3) $L_{cb}{}^\alpha = 0$ *and*
(4) $\partial_d\bar{J}_{\beta\alpha} - P_{d\beta}{}^\varepsilon\bar{J}_{\varepsilon\alpha} - P_{d\alpha}{}^\varepsilon\bar{J}_{\beta\varepsilon} + 2h_\beta{}^\varepsilon{}_d\bar{J}_{\varepsilon\alpha} + h_\alpha{}^\varepsilon{}_d\bar{J}_{\beta\varepsilon} - h_{\beta\alpha}{}^e\phi_{de} = 0.$

In this case, each fibre is minimal in $\tilde{M}$.

If $\tilde{M}$ has conformal fibres, then the second fundamental form $h = (h_{\gamma\beta}{}^a)$ vanishes identically. Consequently the nearly cosymplectic space $\tilde{M}$ is locally the Riemannian product of a nearly cosymplectic and a nearly Kaehlerin space by means of Lemma 1.1.

Conversely, if a space M is nearly cosymplectic and $\bar{M}$ is nearly Kaehlerian, then the product space $\tilde{M} = M \times \bar{M}$ becomes a nearly cosymplectic space by use of (12.1) and Proposition 12.3. Thus we have

COROLLARY 12.4. *Let $\tilde{M}$ be a fibred space with a nearly cosymplectic structure. If $\tilde{M}$ has conformal fibres, then $\tilde{M}$ is locally the product of a nearly cosymplectic space and a nearly Kaehlerian space, and vice versa.*

REMARK. To prove Corollary 12.4, we have introduced the fibred almost contact metric space with $\tilde{\phi}$-invariant fibres normal to $\tilde{\xi}$. But, even if we consider the case that each fibre is $\tilde{\phi}$-invariant and tangent to $\tilde{\xi}$, it is possible to prove Corollary 12.4 by exchanging the base space and the fibre.

If a structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is closely cosymplectic, then $\tilde{\phi}$ is Killing and $\tilde{\eta}$ is closed by definition. A closely cosymplectic structure is nearly cosymplectic, and $\tilde{\nabla}\tilde{\eta} = 0$ because $\tilde{\xi}$ is Killing in a nearly cosymplectic manifold [3]. Hence, by a similar method to that of the case of nearly cosymplectic structure, we have the following

PROPOSITION 12.5. *The almost contact metric structure on the fibred space $\tilde{M}$ is closely cosymplectic if and only if the following conditions are satisfied:*

(1) *the base space is closely cosymplectic,*

(2) *each fibre is nearly Kaehlerian,*

(3) $L_{cb}{}^{\alpha} = 0$ *and*

(4) $\partial_d \bar{J}_{\beta\alpha} - P_{d\beta}{}^{\varepsilon}\bar{J}_{\varepsilon\alpha} - P_{d\alpha}{}^{\varepsilon}\bar{J}_{\beta\varepsilon} + 2h_{\beta}{}^{\varepsilon}{}_{d}\bar{J}_{\varepsilon\alpha} + h_{\alpha}{}^{\varepsilon}{}_{d}\bar{J}_{\beta\varepsilon} - h_{\beta\alpha}{}^{e}\phi_{de} = 0.$

In this case, each fibre is minimal in $\tilde{M}$.

COROLLARY 12.6. *Let $\tilde{M}$ be a fibred space with a closely cosymplectic structure. If $\tilde{M}$ has conformal fibres, then $\tilde{M}$ is locally the product of a closely cosymplectic space and a nearly Kaehlerian space, and vice versa.*

§13. Quasi cosymplectic and cosymplectic structures

M. Capursi [7] has introduced the notion of quasi cosymplectic structure by the property $A(\tilde{X}, \tilde{Y}) = 0$, the tensor A defined in (2.1). The notion includes closely cosymplectic and almost cosymplectic ones, and the space $(S^2 \times E^4) \times E^1$ is an example of quasi cosymplectic manifold which is neither closely cosymplectic nor almost cosymplectic nor cosymplectic.

In this section, we state characterizations of fibred spaces with quasi cosymplectic structure. By use of (9.2) $\sim$ (9.7), components of the tensor A with respect to the frame (E_a, C_α) are given by

$$\begin{aligned}
&A_{ji}{}^{h}E^{j}{}_{c}E^{i}{}_{b}E_{h}{}^{a} = \nabla_c \phi_b{}^a + \phi_c{}^e \phi_b{}^d \nabla_e \phi_d{}^a - \eta_b \phi_c{}^e \nabla_e \xi^a ,\\
&A_{ji}{}^{h}E^{j}{}_{c}E^{i}{}_{b}C_{h}{}^{\alpha} = L_{cb}{}^{\beta}\bar{J}_{\beta}{}^{\alpha} - L_{ce}{}^{\alpha}\phi_b{}^e + L_{ed}{}^{\beta}\bar{J}_{\beta}{}^{\alpha}\phi_c{}^e\phi_b{}^d + L_{eb}{}^{\alpha}\phi_c{}^e - L_{ed}{}^{\alpha}\xi^d\phi_c{}^e\eta_b ,\\
&A_{ji}{}^{h}E^{j}{}_{c}C^{i}{}_{\beta}E_{h}{}^{a} = L_{c}{}^{a}{}_{\alpha}\bar{J}_{\beta}{}^{\alpha} - L_{e}{}^{a}{}_{\beta}\phi_c{}^e - L_{b}{}^{e}{}_{\alpha}\bar{J}_{\beta}{}^{\alpha}\phi_c{}^b\phi_e{}^a ,\\
(13.1)\quad &A_{ji}{}^{h}E^{j}{}_{c}C^{i}{}_{\beta}C_{h}{}^{\alpha} = {}^{*}\nabla_c \bar{J}_{\beta}{}^{\alpha} + \phi_c{}^e \bar{J}_{\beta}{}^{\gamma}({}^{*}\nabla_e \bar{J}_{\gamma}{}^{\alpha}) ,\\
&A_{ji}{}^{h}C^{j}{}_{\gamma}E^{i}{}_{b}E_{h}{}^{a} = L_{e}{}^{a}{}_{\gamma}\phi_b{}^e - L_{b}{}^{e}{}_{\gamma}\phi_e{}^a + (L_{d}{}^{a}{}_{\beta}\phi_e{}^d - L_{e}{}^{d}{}_{\beta}\phi_d{}^a)\bar{J}_{\gamma}{}^{\beta}\phi_b{}^e\\
&\qquad\qquad\qquad\qquad + \eta_b \bar{J}_{\gamma}{}^{\beta}L^{a}{}_{e\beta}\xi^e ,\\
&A_{ji}{}^{h}C^{j}{}_{\gamma}E^{i}{}_{b}C_{h}{}^{\alpha} = h_{\gamma}{}^{\varepsilon}{}_{b}\bar{J}_{\varepsilon}{}^{\alpha} - h_{\gamma}{}^{\alpha}{}_{d}\phi_b{}^d + h_{\delta}{}^{\varepsilon}{}_{d}\bar{J}_{\gamma}{}^{\delta}\bar{J}_{\varepsilon}{}^{\alpha}\phi_b{}^d + h_{\varepsilon}{}^{\alpha}{}_{b}\bar{J}_{\gamma}{}^{\varepsilon} ,\\
&A_{ji}{}^{h}C^{j}{}_{\gamma}C^{i}{}_{\beta}C_{h}{}^{\alpha} = \bar{\nabla}_{\gamma}\bar{J}_{\beta}{}^{\alpha} + \bar{J}_{\gamma}{}^{\varepsilon}\bar{J}_{\beta}{}^{\delta}\bar{\nabla}_{\varepsilon}\bar{J}_{\delta}{}^{\alpha} .
\end{aligned}$$

If $\tilde{M}$ is a quasi cosymplectic space, then (13.1, 1) and (13.1, 7) imply that the base space M is quasi cosymplectic and each fibre $\bar{M}$ is quasi Kaehlerian, that is, $\bar{\nabla}_{\gamma}\bar{J}_{\beta}{}^{\alpha} + \bar{J}_{\gamma}{}^{\varepsilon}\bar{J}_{\beta}{}^{\delta}\bar{\nabla}_{\varepsilon}\bar{J}_{\delta}{}^{\alpha} = 0$. Taking the symmetric part of the covariant component of (13.1, 3) with respect to the indices a and c, we have

$$L_{ae\beta}\phi_c{}^e + L_{ce\beta}\phi_a{}^e = 0\,. \tag{13.2}$$

Transvecting ξ^c and $\phi_b{}^a$ successively to this equation, we obtain

$$L_{ce\beta}\xi^e = 0\,. \tag{13.3}$$

Then it follows from the equation (13.1, 5) that

$$L_{ea\gamma}\phi_c{}^e = L_{ca\beta}\bar{J}_\gamma{}^\beta\,. \tag{13.4}$$

Substituting this into (13.1, 3), we obtain

$$L_b{}^e{}_\gamma \bar{J}_\beta{}^\gamma \phi_d{}^b \phi_e{}^a = 0\,,$$

and this together with (13.3) gives $L_{ce}{}^\alpha = 0$.

On the other hand, if we take the skew symmetric part of the covariant component of (13.1, 6) with respect to the indices γ and α, we get

$$h_\gamma{}^\varepsilon{}_e \bar{J}_{\beta\varepsilon} = h_\beta{}^\varepsilon{}_e \bar{J}_\gamma{}^\varepsilon\,, \tag{13.5}$$

or equivalently

$$h_{\gamma\beta}{}^b = h_\lambda{}^\alpha{}_b \bar{J}_\gamma{}^\lambda \bar{J}_{\alpha\beta}\,.$$

If we transvect $\bar{J}_\lambda{}^\beta$ to (13.5) and contract in the indices γ and λ, then $h_{\alpha\beta}{}^e \bar{g}^{\alpha\beta} = 0$. Thus we have

PROPOSITION 13.1. *The almost contact metric structure in a fibred space $\tilde{M}$ is quasi cosymplectic if and only if the following conditions are satisfied:*

(1) *the base space is quasi cosymplectic,*
(2) *each fibre is quasi Kaehlerian,*
(3) $L_{cb}{}^\alpha = 0$,
(4) $h_\gamma{}^\alpha{}_e \bar{J}_{\beta\alpha} = h_\beta{}^\alpha{}_e \bar{J}_{\gamma\alpha}$ *and*
(5) $({}^*\nabla_c \bar{J}_\beta{}^\varepsilon)\bar{J}_\varepsilon{}^\alpha + ({}^*\nabla_e \bar{J}_\beta{}^\alpha)\phi_c{}^e = 0$

In this case, each fibre is minimal in $\tilde{M}$.

If, in addition, $\tilde{M}$ has conformal fibres, then the second fundamental form $h = (h_{\gamma\beta}{}^a)$ vanishes identically. Hence, by the same argument to that of Corollary 11.5, we have the

COROLLARY 13.2. *If a fibred space $\tilde{M}$ with quasi cosymplectic structure has conformal fibres, then $\tilde{M}$ is locally the product of a quasi cosymplectic space and a quasi Kaehlerian space, and vice versa.*

To the contrary, if a fibred almost contact metric space with $\tilde{\phi}$-invariant fibres normal to $\tilde{\xi}$ is cosymplectic, that is, $\tilde{\nabla}_j\tilde{\phi}_i^h = 0$ and $\tilde{\nabla}_j\tilde{\xi}^h = 0$, then by a similar method to that of the case of nearly cosymplectic structure, we have the following

PROPOSITION 13.3. *The almost contact metric structure on a fibred space $\tilde{M}$ is cosymplectic if and only if the following conditions are satisfied:*
(1) *the base space is cosymplectic,*
(2) *each fibre is Kaehlerian,*
(3) $L_{cb}{}^{\alpha} = 0$,
(4) $h_{\gamma}{}^{\beta}{}_b\bar{J}_{\beta\alpha} - h_{\gamma\alpha e}\phi_b{}^e = 0$,
(5) $\partial_c\bar{J}_{\beta}{}^{\alpha} - P_{c\beta}{}^{\lambda}\bar{J}_{\lambda}{}^{\alpha} + P_{c\lambda}{}^{\alpha}\bar{J}_{\beta}{}^{\lambda} + 2h_{\beta}{}^{\lambda}{}_c\bar{J}_{\lambda}{}^{\alpha} = 0$,
(6) $h_{\gamma}{}^{\alpha}{}_e\xi^e = 0$.
In this case, each fibre is minimal in $\tilde{M}$.

COROLLARY 13.4. *If a fibred space $\tilde{M}$ with a cosymplectic structure has conformal fibres, then $\tilde{M}$ is locally the product of a cosymplectic space and a Kaehlerian space, and vice versa.*

Chapter IV. Cosymplectic structure and cosymplectic Bochner curvature tensor

§14. Closely cosymplectic and cosymplectic structures

Up to the present, we have considered various almost contact metric structures from the view point of fibred Riemannian space. However, if the structure vector field $\tilde{\xi}$ is parallel, $\tilde{\nabla}\tilde{\xi} = 0$, like a closely cosymplectic or cosymplectic structure, then the manifold is locally the product of a Riemannian manifold with a 1-dimensional Euclidean space. We have scarcely seen discussions on closely cosymplectic or cosymplectic manifolds from this view point in the literature. For this reason we give here a treatment of such manifolds as product Riemannian manifolds.

We assume that the structure vector field $\tilde{\xi}$ of an almost contact metric manifold $\tilde{M}$ is parallel. Then $\tilde{M}$ is locally the Riemannian product $\tilde{M} = M \times E^1$ of a Riemannian manifold M with a 1-dimensional Euclidean space E^1. The vector field $\tilde{\xi}$ is unit and tangent to E^1. Therefore, in the trivial sense, the manifold $\tilde{M}$ has a structure with 1-dimensional fibre $\bar{M} = E^1$ discussed in Chapter II. The almost contact metric structure is expressed in the form

$$\tilde{\phi} = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{\xi} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \tilde{\eta} = (0, 1), \qquad \tilde{g} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix},$$

and the horizontal part $J = \tilde{\phi}^H$ satisfies the equation $J^2 = -I$. It is noted that h and L vanish identically.

If the structure is closely cosymplectic, $\tilde{\nabla}_j\tilde{\phi}_i^h + \tilde{\nabla}_i\tilde{\phi}_j^h = 0$, then it follows from (3.3) and (3.4) that

$$\tilde{\nabla}_\gamma\tilde{\phi}^H = \partial_\gamma J = (\tilde{\nabla}_k\tilde{\phi}_{ji} + \tilde{\nabla}_j\tilde{\phi}_{ki})C^k{}_\gamma E^j{}_b E_i{}^a = 0\,,$$

hence $\tilde{\phi}$ is projectable and the projection J gives an almost complex structure on M. By means of (3.2) in the cosymplectic case or (3.11, 1) in the closely cosymplectic case, we can state the following

PROPOSITION 14.1. *A cosymplectic (resp. closely cosymplectic) manifold $\tilde{M}$ is the Riemannian product $M \times E^1$ of a Kaehlerian (resp. nearly Kaehlerian) manifold M with a 1-dimensional Euclidean space E^1.*

Therefore the nature of the manifold $\tilde{M}$ is characterized by that of the manifold M. If we take a local coordinate system (x^a, y) in $\tilde{M} = M \times E^1$, then non-trivial components of the curvature tensor of $\tilde{M}$ are only $\tilde{K}_{dcb}{}^a = K_{dcb}{}^a$.

The cosymplectic Bochner curvature $\tilde{B} = (\tilde{B}_{kjih})$ is defined by

$$\begin{aligned}\tilde{B}_{kjih} = \tilde{K}_{kjih} - \{&\tilde{K}_{ji}(\tilde{g}_{kh} - \tilde{\eta}_k\tilde{\eta}_h) - \tilde{K}_{ki}(\tilde{g}_{jh} - \tilde{\eta}_j\tilde{\eta}_h) + \tilde{K}_{kh}(\tilde{g}_{ji} - \tilde{\eta}_j\tilde{\eta}_i)\\ &- \tilde{K}_{jh}(\tilde{g}_{ki} - \tilde{\eta}_k\tilde{\eta}_i) - \tilde{\phi}_{kh}\tilde{H}_{ij} + \tilde{\phi}_{jh}\tilde{H}_{ik} - \tilde{\phi}_{ji}\tilde{H}_{hk} + \tilde{\phi}_{ki}\tilde{H}_{hj} + 2\tilde{\phi}_{ih}\tilde{H}_{jk}\\ &+ 2\tilde{\phi}_{kj}\tilde{H}_{hi}\}/(m+3) + \tilde{K}\{(\tilde{g}_{kh} - \tilde{\eta}_k\tilde{\eta}_h)(\tilde{g}_{ji} - \tilde{\eta}_j\tilde{\eta}_i) - (\tilde{g}_{jh} - \tilde{\eta}_j\tilde{\eta}_h)(\tilde{g}_{ki} - \tilde{\eta}_k\tilde{\eta}_i)\\ &- \tilde{\phi}_{kh}\tilde{\phi}_{ij} + \tilde{\phi}_{jh}\tilde{\phi}_{ik} + 2\tilde{\phi}_{kj}\tilde{\phi}_{hi}\}/(m+1)(m+3)\,,\end{aligned}$$

where $\tilde{H}_{ij} = \tilde{\phi}_i^k\tilde{K}_{kj} = -\tilde{H}_{ji}$. It is easily seen that $\tilde{B}$ has non-trivial components

$$\begin{aligned}\tilde{B}_{dcba} = K_{dcba} - \frac{1}{m+3}\{&K_{cb}g_{da} - K_{db}g_{ca} + K_{da}g_{cb} - K_{ca}g_{db} - J_{da}\tilde{H}_{bc}\\ &+ J_{ca}\tilde{H}_{bd} - J_{cb}\tilde{H}_{ad} + J_{db}\tilde{H}_{ac} + 2J_{ba}\tilde{H}_{cd} + 2J_{dc}\tilde{H}_{ab}\}\\ &+ \frac{\tilde{K}}{(m+1)(m+3)}\{g_{da}g_{cb} - g_{ca}g_{db} - J_{da}J_{bc} + J_{ca}J_{bd} + 2J_{dc}J_{ab}\}\,.\end{aligned}$$

Since $\tilde{H}_{cb} = J_c^e K_{eb}$, $\tilde{K} = K$ and $m = n + 1$, the components $\tilde{B}_{dcba}$ are the same as Bochner curvature tensor B defined on the Kaehler space M (for the detail of Bochner curvature, see [21, 23, 36] etc.). Hence we can state that

PROPOSITION 14.2. *A cosymplectic manifold $\tilde{M}$ with vanishing cosymplectic Bochner curvature tensor is the product of a Kaehlerian manifold M with vanishing Bochner curvature tensor and a 1-dimensional Euclidean space E^1.*

Let $\{e_1, \ldots, e_q, Je_1 = e_1*, \ldots, Je_q = e_q*\}$ be a J-basis in M with $2q = n$, and the indices λ, μ and $\lambda*$, $\mu*$ run over the ranges 1, ..., q and $1*$, ..., $q*$ respectively. The symbol $|a|$ means that

$$|a| = \lambda \qquad \text{for} \qquad a = \lambda \text{ or } \lambda* ,$$

and $|a|, |b|, |c|, |d| \neq$ means that $|a|, |b|, |c|, |d|$ differ from one another.

For the Bochner curvature tensor, T. Kashiwada [21] proved

PROPOSITION 14.3. *Let M be a Kaehler manifold with $n = 2q \geqq 8$. If the relation*

$$K_{dcba} = 0 \qquad (|a|, |b|, |c|, |d| \neq) \tag{14.1}$$

holds for every J-basis $\{e_\lambda, e_\lambda*\}$, then the Bochner curvature vanishes. The converse is true.

PROPOSITION 14.4. *Let M be a Kaehler manifold with $n = 2q \geqq 8$. Then the necessary and sufficient condition in order that the Bochner curvature tensor vanishes identically is*

$$H(e_\lambda, e_\lambda*) + H(e_\mu, e_\mu*) = 8H(e_\lambda, e_\mu) \qquad (\lambda \neq \mu) \tag{14.2}$$

for every J-basis $(e_\lambda, e_\lambda*)$, where $H(X, Y)$ means that the sectional curvature with respect to the plane spanned by X and Y.

Combining Propositions 14.2, 14.3 and 14.4, we have

COROLLARY 14.5. *Let $\tilde{M}$ be a cosymplectic manifold with dimension $m \geqq 9$. If the relation* (14.1) *is satisfied for every $\tilde{\phi}$-basis $\{e_\lambda, e_\lambda*, \tilde{\xi}\}$, then the cosymplectic Bochner curvature tensor of $\tilde{M}$ vanishes and vice versa.*

COROLLARY 14.6. *In a cosymplectic manifold $\tilde{M}$ with dimension $m \geqq 9$, in order that the cosymplectic Bochner curvature tensor vanishes identically, it is necessary and sufficient that the relation* (14.2) *holds for every $\tilde{\phi}$-basis $\{e_\lambda, e_\lambda*, \tilde{\xi}\}$.*

EXAMPLE 14.7. The locally product manifold $M_1(c) \times M_2(-c)$ of constant holomorphic sectional curvature $c(\geqq 0)$ and $-c$ with $\dim M_1 + \dim M_2 = 2q \geqq 4$ and $\min\{\dim M_1, \dim M_2\} \geqq 2$ is a Kaehler manifold with vanishing Bochner curvature [25]. Hence the space $\tilde{M} = M_1(c) \times M_2(-c) \times E^1$ is a cosymplectic space with vanishing cosymplectic Bochner curvature tensor by means of Proposition 14.2. But, by means of Proposition 8.2 or Corollary 8.3, such a space $\tilde{M}$ does not become a cosymplectic space form because $M_1(c) \times M_2(-c)$ is not a complex space form.

§15. Cosymplectic conformal connection

In this section, we consider a cosymplectic conformal connection and give a sufficient condition in order that a cosymplectic manifold has the vanishing cosymplectic Bochner curvature tensor.

Let $\tilde{M}$ be a cosymplectic manifold with structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, and D an affine connection with coefficients $\Lambda_{ji}{}^h$ which satisfies

$$D_k(e^{2\alpha}\tilde{g}_{ji}) = 2e^{2\alpha}p_k\tilde{\eta}_j\tilde{\eta}_i\,,$$
$$D_k\tilde{\phi} = 0\,, \qquad D_k\tilde{\xi} = 0$$

and

$$\Lambda_{ji}{}^h - \Lambda_{ij}{}^h = -2\tilde{\phi}_{ji}u^h\,,$$

where α is a scalar field, p_k the gradient of a scalar field p and u^h a vector field. Denoting by Γ_{ji}^h the Riemannian connection of $\tilde{M}$, $\Lambda_{ji}{}^h$ are given by

$$\Lambda_{ji}{}^h = \Gamma_{ji}^h + (\delta_j^h - \tilde{\eta}_j\tilde{\xi}^h)p_i + (\delta_i^h - \tilde{\eta}_i\tilde{\xi}^h)p_j - p^h(\tilde{g}_{ji} - \tilde{\eta}_j\tilde{\eta}_i) + \tilde{\phi}_j{}^h q_i + \tilde{\phi}_i{}^h q_j - \tilde{\phi}_{ji}q^h\,,$$

where we have put

$$p^h = p_k\tilde{g}^{kh}\,, \qquad q^h = p^k\tilde{\phi}_k{}^h\,, \qquad q_i = q^h\tilde{g}_{hi} \tag{15.1}$$

and p satisfies

$$\mathscr{L}_{\tilde{\xi}}p = p_k\tilde{\xi}^k = 0\,. \tag{15.2}$$

In this case, an affine connection D is said to be *cosymplectic conformal connection* [14]. From (15.1) and (15.2), we have

$$\begin{gathered} p_k\tilde{\phi}_i^k = -q_i\,, \qquad q_k\tilde{\phi}_i^k = p_i\,, \qquad \tilde{\phi}_k{}^h p^k = q^h\,, \qquad \tilde{\phi}_k{}^h q^k = -p^h \\ p_k q^k = 0\,, \qquad p_k p^k = q_k q^k\,. \end{gathered} \tag{15.3}$$

Now, the curvature tensor of $\Lambda_{ji}{}^h$ is defined by

$$R_{kji}{}^h = \partial_k\Lambda_{ji}{}^h - \partial_j\Lambda_{ki}{}^h + \Lambda_{kl}{}^h\Lambda_{ji}{}^l - \Lambda_{jl}{}^h\Lambda_{ki}{}^l\,.$$

By a straightforward computation, we have the expression

$$\begin{aligned} R_{kji}{}^h = \tilde{K}_{kji}{}^h &- (\delta_k^h - \tilde{\eta}_k\tilde{\xi}^h)p_{ji} + (\delta_j^h - \tilde{\eta}_j\tilde{\xi}^h)p_{ki} - p_k{}^h(\tilde{g}_{ji} - \tilde{\eta}_j\tilde{\eta}_i) \\ &+ p_j{}^h(\tilde{g}_{ki} - \tilde{\eta}_k\tilde{\eta}_i) - \tilde{\phi}_k{}^h q_{ji} + \tilde{\phi}_j{}^h q_{ki} - q_k{}^h\tilde{\phi}_{ji} + q_j{}^h\tilde{\phi}_{ki} + (\tilde{\nabla}_k q_j - \tilde{\nabla}_j q_k)\tilde{\phi}_i{}^h \\ &+ 2\tilde{\phi}_{kj}(q_i p^h - p_i q^h)\,, \end{aligned}$$

where we have put

$$p_{ji} = \tilde{\nabla}_j p_i - p_j p_i + q_j q_i + \lambda(\tilde{g}_{ji} - \tilde{\eta}_j \tilde{\eta}_i)/2\,,$$
$$q_{ji} = \tilde{\nabla}_j q_i - p_j q_i - p_i q_j + \lambda \tilde{\phi}_{ji}/2\,,$$

and $\lambda = p_k p^k$.

Then we can see that the curvature tensor $R_{kji}{}^h$ satisfies the relations

$$R_{kjih} = -R_{jkih} = -R_{kjhi}$$

and

$$R_{kji}{}^h + R_{jik}{}^h + R_{ikj}{}^h = 2\tilde{\phi}_{kj}(2q_i p^h - \tilde{\nabla}_i q^h - \lambda \tilde{\phi}_i{}^h) + 2\tilde{\phi}_{ji}(2q_k p^h - \tilde{\nabla}_k q^h - \lambda \tilde{\phi}_k{}^h)$$
$$+ 2\tilde{\phi}_{ik}(2q_j p^h - \tilde{\nabla}_j q^h - \lambda \tilde{\phi}_j{}^h)\,.$$

Hence if we assume that

$$\tilde{\nabla}_j q^h - 2q_j p^h + \lambda \tilde{\phi}_j{}^h = 0\,,$$

then the curvature tensor R satisfies the first Bianchi identity and we get $R_{kjih} = R_{ihkj}$.

For any $\tilde{\phi}$-holomorphic section $\sigma = (\tilde{X}, \tilde{\phi}\tilde{X})$, the $\tilde{\phi}$-holomorphic sectional curvature with respect to the cosymplectic conformal connection is defined by

$$H(\sigma) = H(\tilde{X}) = -\frac{\tilde{g}(R(\tilde{\phi}\tilde{X}, \tilde{X})\tilde{\phi}\tilde{X}, \tilde{X})}{\tilde{g}(\tilde{X}, \tilde{X})\tilde{g}(\tilde{\phi}\tilde{X}, \tilde{\phi}\tilde{X})}\,.$$

Then $H(\sigma)$ is uniquely determined by the holomorphic section σ and is independent of the choice of $\tilde{X}$ on σ. If this holomorphic sectional curvature is independent of the holomorphic section at each point of $\tilde{M}$, then a cosymplectic conformal connection D is said to be of constant $\tilde{\phi}$-holomorphic sectional curvature.

If a cosymplectic conformal connection D is of constant holomorphic sectional curvature, then we have

$$\begin{aligned} R_{kjih} = c\{&(\tilde{g}_{kh} - \tilde{\eta}_k \tilde{\eta}_h)(\tilde{g}_{ji} - \tilde{\eta}_j \tilde{\eta}_i) - (\tilde{g}_{jh} - \tilde{\eta}_j \tilde{\eta}_h)(\tilde{g}_{ki} - \tilde{\eta}_k \tilde{\eta}_i) \\ &+ \tilde{\phi}_{kh}\tilde{\phi}_{ji} - \tilde{\phi}_{jh}\tilde{\phi}_{ki} - 2\tilde{\phi}_{kj}\tilde{\phi}_{ih}\}\,, \end{aligned} \tag{15.5}$$

c being a scalar. By use of (15.4) and (15.5), we obtain

$$\tilde{K}_{dcba} = 0 \qquad (|a|, |b|, |c|, |d| \neq)\,,$$

for every $\tilde{\phi}$-basis $\{e_\lambda, e_{\lambda^*}, \tilde{\xi}\}$. Thus, by virtue of Corollary 14.5, we have

THEOREM 15.1. *If an $m(\geqq 9)$-dimensional cosymplectic manifold $\tilde{M}$ admits a cosymplectic conformal connection D which satisfies*

$$\tilde{\nabla}_i q^h - 2q_i p^h + \lambda \tilde{\phi}_i{}^h = 0$$

and is of constant $\tilde{\phi}$-holomorphic sectional curvature, then the cosymplectic Bochner curvature tensor of $\tilde{M}$ vanishes.

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