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Large deviations in the Langevin dynamics of a short-range spin glass

GÉRARD BEN AROUS[1] and MICHEL SORTAIS[2]

[1]*Courant Institute of Mathematical Sciences, 251 Mercer Street, New York NY 10012, USA. E-mail: benarous@cims.nyu.edu*
[2]*T.U. Berlin, Institut für Mathematik, MA 7-4, Strasse des 17. Juni 136, D-10623 Berlin, Germany. E-mail: sortais@math.tu-berlin.de*

We consider a Langevin dynamics associated with a d-dimensional Edwards–Anderson model having Gaussian coupling variables, and show that the averaged law of the empirical process satisfies a large-deviation principle according to a good rate functional $\mathcal{I}^a$ having a unique minimizer Q_∞. The asymptotic dynamics Q_∞ may be characterized as the unique weak solution corresponding to a non-Markovian system of interacting diffusions having an infinite range of interaction. We then establish that the quenched law of the empirical process also obeys a large-deviation process, according to a (deterministic) good rate functional $\mathcal{I}^q$ satisfying $\mathcal{I}^q \geqslant \mathcal{I}^a$, so that, for a typical realization of the disorder variables, the quenched law of the empirical process also converges exponentially fast to a Dirac mass concentrated at Q_∞.

Keywords: disordered systems; interacting diffusion processes; large deviations; statistical mechanics

1. Introduction and statement of main results

The Edwards–Anderson model is a disordered spin system which was proposed in the 1970s as a mathematical model describing the magnetic behaviour of certain metallic alloys known as 'spin glasses' (see Edwards and Anderson 1975). For a fixed cubic volume $\Lambda \subset\subset \mathbb{Z}^d$, it has a *random* energy Hamiltonian defined on the space $\{\pm 1\}^\Lambda$ by the expression

$$H_\Lambda^{\boldsymbol{J}}(\boldsymbol{\sigma}) = -\sum_{\{i,j\}\in\Lambda^*} J_{\{i,j\}}\sigma_i\sigma_j.$$

Here Λ^* denotes the set consisting of all bonds $\{i, j\}$ in Λ (equipped, for example, with periodic boundary conditions), and $\boldsymbol{J} = (J_{\{i,j\}})_{\{i,j\}\in(\mathbb{Z}^d)^*}$ is a fixed realization corresponding to an independent and identically distributed (i.i.d.) family of standard Gaussian random variables indexed by all bonds in $\mathbb{Z}^d$.

Hitherto, studying the equilibrium properties of this short-range spin glass model has led to conflicting predictions in the theoretical physics literature, and rigorous mathematical results are quite rare. Indeed, there are several opinions as regards the number of ground states that should emerge in such a disordered spin system considered in dimension $d \geqslant 3$ and in the low-temperature regime; during the last decade, C.M. Newman and D.L. Stein

1350–7265

have given a precise mathematical formulation for some of these predictions, which we shall now briefly present; see Newman and Stein (1998) for a clear survey containing precise definitions and mathematical statements, as well as appropriate references to the physics literature.

(i) According to the 'droplet/scaling' heuristic arguments developed mostly by Fisher and Huse, there should be a convergence of the finite-volume Gibbs measure (with periodic boundary conditions) corresponding to such spin system towards a convex combination

$$\frac{1}{2}\rho_{\boldsymbol{J}}^{*} + \frac{1}{2}\rho_{\boldsymbol{J}}^{*\prime},$$

$\rho_{\boldsymbol{J}}^{*}$ and $\rho_{\boldsymbol{J}}^{*\prime}$ being the only two pure Gibbs states in infinite volume corresponding to the realization $\boldsymbol{J}$ of all couplings, and these two pure states being related by the global spin flip symmetry changing $\boldsymbol{\sigma}$ into $-\boldsymbol{\sigma}$.

(ii) On the other hand, the mean-field prediction of G. Parisi and his co-workers asserts that the structure of all such pure states should be roughly the same as the structure appearing when considering the statics of the Sherrington–Kirkpatrick (mean-field) model, so that the Gibbs measure in a large finite volume should asymptotically resemble a countable sum

$$\sum_{\alpha} W_{\boldsymbol{J}}^{\alpha}\rho_{\boldsymbol{J}}^{\alpha},$$

$(\rho_{\boldsymbol{J}}^{\alpha})$ being a countable family of infinite-volume pure Gibbs states associated with the couplings $\boldsymbol{J}$, and $(W_{\boldsymbol{J}}^{\alpha})$ being a (random) sequence of weights.

(iii) Finally, in the 'chaotic pairs of pure states' prediction proposed by Newman and Stein, there are also infinitely many pure states $\rho_{\boldsymbol{J}}^{\alpha}$ corresponding to a given typical realization $\boldsymbol{J}$ of the couplings, but these pure states do not appear in the same way when one considers a large, finite-volume Gibbs measure $p_{\Lambda}^{\boldsymbol{J}}$; in this prediction, for a given increasing sequence of volumes $\Lambda_L \nearrow \mathbb{Z}^d$ one instead has an approximate decomposition,

$$p_{\Lambda_L}^{\boldsymbol{J}} \approx \frac{1}{2}(\rho_{\boldsymbol{J}}^{\alpha_L}) + \frac{1}{2}(\rho_{\boldsymbol{J}}^{\alpha_L})',$$

for two pure states $(\rho_{\boldsymbol{J}}^{\alpha_L})$ and $(\rho_{\boldsymbol{J}}^{\alpha_L})'$ related by spin flip symmetry and *depending upon* L.

Moreover, it is now widely believed that such disordered spin systems display a complex dynamical behaviour in the low-temperature regime, with relaxation times that may become astronomically large. In this respect, it has been argued that studying the *equilibrium* properties of a spin glass might not be so relevant and that one should instead study their *dynamical* behaviour, since the equilibrium properties might well never be observed in the laboratory. At a theoretical level, such studies began essentially with the seminal paper by Sompolinsky and Zippelius (1982), in which the authors considered a relaxational Langevin dynamics scheme for the Sherrington–Kirkpatrick model and derived the limiting evolution of a single spin in the large-volume limit. Ben Arous and Guionnet (1997) and Guionnet (1997) were then able to confirm some of the main predictions contained in Sompolinsky and Zippelius (1982) by considering a system of interacting diffusions $\boldsymbol{x} = \{(x_t^i)_{0\leq t\leq T};\ i = 1, \ldots, N\}$ evolving according to such a Langevin dynamics scheme and establishing the convergence of the law of the empirical measure

$$\hat{\mu}_{\boldsymbol{x}}^{(N)} = \frac{1}{N}\sum_{i=1}^{N}\delta_{x^i}$$

towards a Dirac mass δ_{μ_∞}, with μ_∞ a probability measure on path space $\mathcal{C}([0, T]; \mathbb{R})$. To be more precise, Ben Arous and Guionnet (1997) derive a large-deviation upper bound for the law of the empirical measure $\hat{\mu}_{\boldsymbol{x}}^{(N)}$ considered in the *averaged regime*, that is, when expected values are taken not only with respect to the thermal noise driving the diffusions but also with respect to the possible realizations of the disorder variables. This large-deviation upper bound was established via a suitable adaptation of the Laplace–Varadhan method, and a careful study of the corresponding variational principle then revealed that there can only be one minimizing measure μ_∞, which may be characterized as the unique weak solution of a highly nonlinear stochastic differential equation in dimension 1. Unfortunately, such a large-deviation bound could only be established under an assumption of 'high temperature or short terminal time'; nevertheless, Guionnet (1997) removed this technical obstacle by establishing the validity of a law of large numbers for $\hat{\mu}_{\boldsymbol{x}}^{(N)}$ at arbitrarily low temperatures and with arbitrarily large terminal times.

At the same time, large-deviation techniques were also used by Grunwald (1996; 1998) in his mathematical study of the asymptotic behaviour of a Glauber dynamics scheme associated with the Sherrington–Kirkpatrick model. Instead of a high-dimensional diffusion, Grunwald considered a continuous-time Markov chain $\boldsymbol{\sigma} = \{(\sigma_t^i)_{0\leq t\leq T};\ i = 1, \ldots, N\}$ with values in $\{\pm 1\}^N$, and, using an alternative approach – mixtures of large-deviation systems, as introduced by Dinwoodie and Zabell (1992) – he was able to come to an analogous conclusion, namely the validity of a large-deviation upper bound for the empirical measure $\hat{\mu}_{\boldsymbol{\sigma}}^{(N)}$ considered in the averaged regime. There again, it was proved that the corresponding rate functional has a unique minimizer μ_∞ that may be viewed as the law of the asymptotic single-spin dynamics for the Sherrington–Kirkpatrick model; moreover, choosing such bounded discrete spin variables enabled Grunwald to formulate a technically simpler proof, and he was able, in particular, to establish the validity of such a large-deviation upper bound for arbitrary temperatures and terminal times.

Our aim in the present paper is to return to the original short-ranged spin glass, namely the Edwards–Anderson model introduced earlier, and to carry out again a large-deviations analysis for the joint behaviour of such an assembly of spins considered in a large volume and in a fixed, finite time horizon. For this purpose, we consider a Langevin dynamics scheme where each spin variable is a diffusion process on the unit circle S^1; this choice may seem artificial at first, but it enables us to deal with compact spin variables instead of unbounded ones, and at the same time one ends up with an asymptotic dynamics that may be viewed as a non-Markovian diffusion in infinite dimensions, whose drift coefficient may be computed explicitly. Our main result asserts that the empirical process associated with such interacting diffusions satisfies a full large-deviation principle (LDP), both in the *quenched* regime (i.e., for a fixed, typical realization of the coupling variables $\boldsymbol{J}$) and in the *averaged* regime, for arbitrary low temperatures and arbitrarily large values of the fixed terminal time T. We come to this conclusion by considering these interacting diffusions first in the averaged regime, where a Picard iteration method enables us to prove the validity of a full LDP via an appropriate *contraction principle* (cf. Dembo and Zeitouni, 1998, Section 4.2). Such a contraction principle has the advantage of providing us immediately with a characterization of the unique minimizer Q_∞ associated with the corresponding rate functional $\mathcal{I}^a$: Q_∞ may be explicitly described as the unique weak solution corresponding to an infinite-dimensional system of interacting diffusions. The validity of such an LDP in the quenched regime then follows from earlier work by (Comets 1989), and we also know that the corresponding quenched rate functional $\mathcal{I}^q$ is bounded from below by $\mathcal{I}^a$, which enables us to assert that Q_∞ is also the unique minimizer associated with $\mathcal{I}^q$.

We now come to a precise statement of our main results. Replacing the original discrete spin variables $\sigma^i \in \{\pm 1\}$ by circular spin variables $x^i = (\cos\theta^i, \sin\theta^i)$, we shall be considering a disordered energy landscape given by the Hamiltonian[1]

$$H_\Lambda^{\boldsymbol{J}}(\boldsymbol{x}) = H_\Lambda^{\boldsymbol{J}}(\boldsymbol{\theta}) = -\sum_{\{i,j\}\in\Lambda^*} J_{\{i,j\}} \cos(\theta^i - \theta^j).$$

Differentiating with respect to θ^i then gives

$$\partial_{\theta_i} H_\Lambda^{\boldsymbol{J}}(\boldsymbol{\theta}) = \sum_{j\sim i} J_{\{i,j\}} \sin(\theta^i - \theta^j),$$

the sum $\sum_{j\sim i}$ running over all nearest neighbours of i in Λ, so that the system of short-range interacting diffusions $(\mathcal{S}_\Lambda^{\boldsymbol{J}})$ given by

$$\mathrm{d}\theta_t^i = \mathrm{d}w_t^i + \beta \sum_{j\sim i} J_{\{i,j\}} \sin(\theta_t^j - \theta_t^i)\mathrm{d}t, \qquad i \in \Lambda,\ 0 \leqslant t \leqslant T,$$

has the Λ-dimensional Gibbs measure (proportional to $\exp(-\beta H_\Lambda^{\boldsymbol{J}}(\boldsymbol{\theta}))\prod_{i\in\Lambda} d\theta^i$) as unique invariant reversible measure. For the sake of simplicity, assume further that the vector of diffusions $\{(\theta_t^i)_{0\leqslant t\leqslant T};\ i\in\Lambda\}$ solving $(\mathcal{S}_\Lambda^{\boldsymbol{J}})$ has a 'deep quench' initial condition:

$$\mathrm{Law}(\boldsymbol{\theta}|_{t=0}) = u_0^{\otimes\Lambda},$$

where u_0 is, for example, the uniform probability distribution on the interval $]-\pi;\pi[$, and let $P_\Lambda^{\boldsymbol{J}}$ denote the law of $\{(\theta_t^i)_{0\leqslant t\leqslant T};\ i\in\Lambda\}$ ($P_\Lambda^{\boldsymbol{J}}$ is a probability measure on W_T^Λ, W_T being the Wiener space of all continuous functions $\omega : [0; T] \to \mathbb{R}$).

The empirical process $\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}$ associated with a Λ-dimensional configuration of diffusions $\boldsymbol{\theta} \in W_T^\Lambda$ is then defined as the following spatial average of Dirac masses:

[1] Adding an appropriate single site potential, one could also consider circular spin variables having a much better resemblance with the original ones (see Section 4).

$$\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)} = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \delta_{\boldsymbol{\theta}^{(\Lambda),(i)}},$$

$\boldsymbol{\tau} = \boldsymbol{\theta}^{(\Lambda)} \in W_T^{\mathbb{Z}^d}$ being the infinite-dimensional vector of diffusions obtained from $\boldsymbol{\theta}$ by periodically reproducing on the lattice the information contained in the box Λ, and $\boldsymbol{\tau}^{(i)} \in \Omega$ being the new configuration obtained from $\boldsymbol{\tau}$ by 'shifting the origin of the lattice at site i':

$$(\boldsymbol{\tau}^{(i)})_j = \tau_{j+i}, \qquad \forall j \in \mathbb{Z}^d$$

($\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}$ is a shift-invariant probability measure on $\Omega = W_T^{\mathbb{Z}^d}$ whose one site marginal is the empirical measure associated with $\boldsymbol{\theta}$).

Consider then the average of all probability measures $P_\Lambda^{\boldsymbol{J}}$ when $\boldsymbol{J}$ varies at random:

$$P_\Lambda(\cdot) = \int \mathrm{d}\gamma(\boldsymbol{J}) P_\Lambda^{\boldsymbol{J}}(\cdot),$$

and let $\mathcal{M}_s(\Omega)$ denote the set consisting of all shift-invariant probability measures on $\Omega = W_T^{\mathbb{Z}^d}$. Our main results may be summarized as follows:

Theorem 1.1. *(i) The law of the empirical process $\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}$ considered in the averaged regime (i.e., under $\mathrm{d}P_\Lambda(\boldsymbol{\theta})$) obeys a large-deviation principle on $\mathcal{M}_s(\Omega)$, on the scale $|\Lambda|$ and according to a good rate function $\mathcal{I}^a : \mathcal{M}_s(\Omega) \to [0; +\infty]$ having a unique minimizer Q_∞.*

(ii) Furthermore, almost surely in the realizations of the disorder variables $\boldsymbol{J}$, the law of the empirical process $\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}$ considered in the quenched regime (i.e., under $\mathrm{d}P_\Lambda^{\boldsymbol{J}}(\boldsymbol{\theta})$) also obeys a large-deviation principle on $\mathcal{M}_s(\Omega)$, on the scale $|\Lambda|$ and according to a good rate function $\mathcal{I}^q$ satisfying $\mathcal{I}^q \geqslant \mathcal{I}^a$.

As a simple consequence of the preceding large-deviations results, one may, for example, fix some bounded continuous functionals $\varphi_i : W_T \to \mathbb{R}$ $(1 \leqslant i \leqslant n)$ and some bounded continuous $F : \mathbb{R}^n \to \mathbb{R}$ to state that, for a typical realization of the disordered couplings $\boldsymbol{J}$, the distribution of

$$F\left(\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\varphi_1(\theta^i), \ldots, \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\varphi_n(\theta^i)\right)$$

under $\mathrm{d}P_\Lambda^{\boldsymbol{J}}(\boldsymbol{\theta})$ converges exponentially fast to a Dirac mass concentrated at

$$F\left(\int\varphi_1 \,\mathrm{d}Q_\infty, \ldots, \int\varphi_n \,\mathrm{d}Q_\infty\right)$$

when $\Lambda \nearrow \mathbb{Z}^d$, with an exponential speed of convergence that may be bounded from below uniformly in $\boldsymbol{J}$ by using the averaged large-deviations rate functional $\mathcal{I}^a$.

The asymptotic dynamics Q_∞ appearing above may be characterized as the unique weak solution corresponding to an infinite-dimensional system of interacting diffusions that is non-Markovian and has infinite range. This infinite-dimensional system of interacting diffusions has a stochastic differential at site $i \in \mathbb{Z}^d$ which may be expressed as

$$\mathrm{d}\theta_t^i = \mathrm{d}w_t^i + \beta^2 \langle F_t(\boldsymbol{\theta}_{[0;t]}^{(i)});\ G_t(\boldsymbol{\theta}_{[0;t]}^{(i)})\rangle_{l^\infty, l^1}\,\mathrm{d}t,$$

$F_t : \Omega \to l^\infty((\mathbb{Z}^d)^*)$, $0 \leqslant t \leqslant T$, being a family of $\mathbb{R}^{(\mathbb{Z}^d)^*}$-valued functionals such that

$$\forall t \in [0;\, T],\ \forall \boldsymbol{\omega} \in \Omega, \qquad \sup_{\{i,j\} \in (\mathbb{Z}^d)^*} |(F_t(\boldsymbol{\omega}))^{\{i,j\}}| < +\infty,$$

and $G_t : \Omega \to l^1((\mathbb{Z}^d)^*)$, $0 \leqslant t \leqslant T$, being a family of $\mathbb{R}^{(\mathbb{Z}^d)^*}$-valued functionals for which

$$\forall t \in [0;\, T],\ \forall \boldsymbol{\omega} \in \Omega, \qquad \sum_{\{i,j\} \in (\mathbb{Z}^d)^*} |(G_t(\boldsymbol{\omega}))^{\{i,j\}}| < +\infty.$$

Precise expressions for the functionals F_t and G_t may be found at the end of Section 2.1; both functionals actually depend on the shifted configuration of diffusions $\boldsymbol{\theta}^{(i)}$ considered during the whole time interval $[0;\, t]$, which we emphasize here by writing $F_t(\boldsymbol{\theta}^{(i)}_{[0;t]})$ and $G_t(\boldsymbol{\theta}^{(i)}_{[0;t]})$.

In order to establish these large-deviations results, we proceed as follows. In Section 2 we show that the finite-volume, averaged probability P_Λ may be viewed as the law of a system of long-range interacting diffusions that is spatially homogeneous. The next step consists in proposing an appropriate extension of such a system of interacting diffusions to infinite dimensions, the corresponding infinite-dimensional diffusion then being a natural candidate for the asymptotic dynamics associated with our disordered spin system. Surprisingly enough, there are actually several reasonable ways of extending the finite-dimensional averaged dynamics to infinite dimensions; we opt for the most robust extension, which may be considered without any restriction on the values of β and T, and develop a Picard iteration method for the corresponding infinite-dimensional diffusion. Such a method gives rise to some rather lengthy computations, but in the end it enables us to establish the validity of a contraction principle for the law of the empirical process $\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta}}$ considered in the averaged regime, and the corresponding good rate functional $\mathcal{I}^a : \mathcal{M}_s(\Omega) \to [0;\, +\infty]$ may then easily be seen to have a unique minimizer Q_∞, which we characterize as the unique weak solution associated with some infinite-dimensional system of long-range interacting diffusions.

In Section 3 we establish a quenched LDP for the empirical process $\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta}}$; our method of proof follows Comets (1989). We show that, for a given typical $\boldsymbol{J}$, the law of $\boldsymbol{\theta}$ under $\mathrm{d}P^{\boldsymbol{J}}_\Lambda(\boldsymbol{\theta})$ may be described by making use of a Gibbsian interaction $\Psi = (\psi_A)_{A \subset\subset \mathbb{Z}^d}$ defined on $(W_T \times \mathbb{R}^d)^{\mathbb{Z}^d}$. The Laplace–Varadhan method may then be applied quite straightforwardly in order to prove a quenched LDP for the empirical process; the drawback here is that the resulting (deterministic) rate functional $\mathcal{I}^q : \mathcal{M}_s(\Omega) \to [0;\, +\infty]$ has a rather complex expression, so that one is not in a position to state immediately that the set of all minimizers associated with $\mathcal{I}^q$ is again reduced to $\{Q_\infty\}$. This last statement follows, however, from the contraction principle developed in Section 2, combined with the uniform domination $\mathcal{I}^q \geqslant \mathcal{I}^a$, which is a general fact for large-deviations asymptotics in random media or disordered systems; see, for example, the introduction to Zeitouni (2001) for a short proof.

Finally, in Section 4, we briefly show that such LDPs may also be proved in a more general context: one may change to a certain extent the initial and boundary conditions entering in the definition of $P^{\boldsymbol{J}}_\Lambda$; a self potential term such as

$$-K\sum_{i\in\Lambda}\sin(\theta^i)^{2n},\qquad K\geqslant 0,\ n\in\mathbb{N},$$

may also be added to the original energy Hamiltonian $H^{\boldsymbol{J}}_\Lambda(\boldsymbol{\theta})$ in order to gain a better resemblance with the original 'hard spin' situation, as well as a magnetic field term

$$-\kappa\sum_{i\in\Lambda}\cos(\theta^i).$$

2. Averaged large-deviations estimates

2.1. A convenient expression for the dynamics in the averaged regime

Recall that W_T denotes the Wiener space consisting of all continuous functions $\omega:[0;T]\to\mathbb{R}$ (equipped with the topology of uniform convergence), and denote by R_T the Wiener measure on W_T having initial condition u_0 such that

$$R_T\{a\leqslant\omega(0)\leqslant b\}=u_0(]a;b[)=\frac{b-a}{2\pi},\qquad \forall-\pi<a\leqslant b<\pi.$$

Recall also that $P^{\boldsymbol{J}}_\Lambda$ is a (spatially inhomogeneous) probability measure on W_T^Λ defined as the weak solution corresponding to the short-range interacting diffusions system $(\mathcal{S}^{\boldsymbol{J}}_\Lambda)$ given by

$$\mathrm{d}\theta^i_t=\mathrm{d}w^i_t+\beta\sum_{j\sim i}J_{\{i,j\}}\sin(\theta^j_t-\theta^i_t)\mathrm{d}t,$$

$$\mathrm{Law}(\boldsymbol{\theta}|_{t=0})=u_0^{\otimes\Lambda},\qquad i\in\Lambda,\ 0\leqslant t\leqslant T.$$

Now the first step in the identification of the asymptotic dynamics Q_∞ will consist in viewing the finite-volume, averaged probability measure

$$P_\Lambda(.)=\mathbb{E}_{\boldsymbol{J}}[P^{\boldsymbol{J}}_\Lambda(.)]$$

as the law of a new stochastic differential system $(\mathcal{S}_\Lambda)$ that is non-Markovian and has a long-range interaction.

Proposition 2.1. *The probability measure P_Λ may be viewed as the law of the following system $(\mathcal{S}_\Lambda)$ of long-range interacting diffusions:*

$$\mathrm{d}\theta^i_t=\mathrm{d}w^i_t+\beta^2\sum_{j\sim i}F_t^{\{i,j\}}(\boldsymbol{\theta})\sin(\theta^j_t-\theta^i_t)\mathrm{d}t$$

$$\mathrm{Law}(\boldsymbol{\theta}|_{t=0})=u_0^{\otimes\Lambda},\qquad i\in\Lambda,\ 0\leqslant t\leqslant T.$$

where $F_t^{\{i,j\}}(\boldsymbol{\theta})$ is the $\{i,j\}$ coordinate of the Λ^*-dimensional real vector

$$F_t(\boldsymbol{\theta})=C_{\beta,t}(\boldsymbol{\theta})^{-1}A_t(\boldsymbol{\theta}),$$

$A_t(\boldsymbol{\theta})$ is the Λ^*-dimensional real vector with $\{i,j\}$ coordinate

$$A_t^{\{i,j\}}(\boldsymbol{\theta}) = [\cos(\theta_t^i - \theta_t^j) - \cos(\theta_0^i - \theta_0^j)] + \int_0^t \cos(\theta_s^i - \theta_s^j)\mathrm{d}s,$$

and $C_{\beta,t}(\boldsymbol{\theta}) = (Id + \beta^2 B_t(\boldsymbol{\theta}))$ *is the* $\Lambda^* \times \Lambda^*$ *real matrix such that*

$$B_t^{\{i,j\},\{k,l\}}(\boldsymbol{\theta}) = 0, \qquad \textit{if } \{i, j\} \textit{ and } \{k, l\} \textit{ are not adjacent},$$

$$B_t^{\{i,j\},\{i,j'\}}(\boldsymbol{\theta}) = \int_0^t \sin(\theta_s^j - \theta_s^i)\sin(\theta_s^{j'} - \theta_s^i)\mathrm{d}s, \qquad \textit{for } j' \neq j,$$

$$B_t^{\{i,j\},\{i,j\}}(\boldsymbol{\theta}) = 2\int_0^t \sin^2(\theta_s^j - \theta_s^i)\mathrm{d}s.$$

Proof. Let $P_\Lambda^{\boldsymbol{J}}$ and $R_t^{\otimes\Lambda}$ be the law of the system considered during time $[0, t]$, for $\beta > 0$ fixed and $\beta = 0$, respectively. According to Girsanov's theorem:

$$\begin{aligned}\frac{\mathrm{d}P_\Lambda^J}{\mathrm{d}R_t^{\otimes\Lambda}}(\boldsymbol{\theta}) &= \exp\left(\beta\sum_{i\in\Lambda}\int_0^t\left\{\sum_{j\sim i} J_{\{i,j\}}\sin(\theta_s^j - \theta_s^i)\right\}\mathrm{d}w_s^i\right.\\ &\qquad \left. -\frac{\beta^2}{2}\sum_{i\in\Lambda}\int_0^t\left\{\sum_{j\sim i} J_{\{i,j\}}\sin(\theta_s^j - \theta_s^i)\right\}^2\mathrm{d}s\right)\\ &= \exp\left(\beta\sum_{\{i,j\}} J_{\{i,j\}}\int_0^t \sin(\theta_s^j - \theta_s^i)(\mathrm{d}w_s^i - \mathrm{d}w_s^j)\right.\\ &\qquad \left. -\frac{\beta^2}{2}\sum_{\{i,j\}} J_{\{i,j\}}\sum_{\{k,l\}\sim\{i,j\}} J_{\{k,l\}}\int_0^t \sin(\theta_s^j - \theta_s^i)\sin(\theta_s^l - \theta_s^k)\mathrm{d}s\right),\end{aligned}$$

the sum $\sum_{\{k,l\}\sim\{i,j\}}$ running over all bonds $\{k, l\}$ in Λ that are adjacent to $\{i, j\}$ (so that $k = i$ or $j = l$). By Fubini's theorem,

$$\frac{\mathrm{d}P_\Lambda}{\mathrm{d}R_t^{\otimes\Lambda}}(\boldsymbol{\theta}) = M_t^\Lambda(\boldsymbol{\theta}) = \mathbb{E}\left[\frac{\mathrm{d}P_\Lambda^J}{dR_t^{\otimes\Lambda}}(\boldsymbol{\theta})\right].$$

$(M_t^\Lambda)_{0\leqslant t\leqslant T}$ is a non-negative martingale under $R_T^{\otimes\Lambda}$, having mean 1, and since the $J_{\{i,j\}}$s have a Gaussian distribution,

$$M_t^\Lambda(\boldsymbol{\theta}) = \mathbb{E}\left[\exp\left\{-\frac{\beta^2}{2}(\boldsymbol{J};\ B_t(\boldsymbol{\theta})\boldsymbol{J})_{\mathbb{R}^{\Lambda^*}}\right\}\right]$$

$$\cdot\ \mathbb{E}\left[\exp\{\beta(\boldsymbol{J};\ A_t(\boldsymbol{\theta}))_{\mathbb{R}^{\Lambda^*}}\}\frac{\exp\left\{-\dfrac{\beta^2}{2}(\boldsymbol{J};\ B_t(\boldsymbol{\theta})\boldsymbol{J})_{\mathbb{R}^{\Lambda^*}}\right\}}{\mathcal{Z}_t^\Lambda(\boldsymbol{\theta})}\right],$$

$$\log M_t^\Lambda(\boldsymbol{\theta}) =_{\mathrm{mart}} \frac{\beta^2}{2}\mathbb{E}\left[((\boldsymbol{J};\ A_t(\boldsymbol{\theta}))_{\mathbb{R}^{\Lambda^*}})^2 \times \frac{\exp\left\{-\dfrac{\beta^2}{2}(\boldsymbol{J};\ B_t(\boldsymbol{\theta})\boldsymbol{J})_{\mathbb{R}^{\Lambda^*}}\right\}}{\mathcal{Z}_t^\Lambda(\boldsymbol{\theta})}\right],$$

where:

- here and in the sequel, the sign $=_{\mathrm{mart}}$ means that the two semimartingales under consideration (on the left-hand side and on the right-hand side of the equality) have the same martingale part;
- $A_t(\boldsymbol{\theta}) \in \mathbb{R}^{\Lambda^*},\ B_t(\boldsymbol{\theta}) \in \mathbb{R}^{(\Lambda^*\otimes\Lambda^*)}$,

$$A_t^{\{i,j\}}(\boldsymbol{\theta}) = \int_0^t \sin(\theta_s^j - \theta_s^i)(\mathrm{d}w_s^i - \mathrm{d}w_s^j),$$

$$B_t^{\{i,j\},\{k,l\}}(\boldsymbol{\theta}) = 0, \qquad \text{if } \{i,\ j\} \text{ and } \{k,\ l\} \text{ are not adjacent},$$

$$B_t^{\{i,j\},\{i,j'\}}(\boldsymbol{\theta}) = \int_0^t \sin(\theta_s^j - \theta_s^i)\sin(\theta_s^{j'} - \theta_s^i)\mathrm{d}s, \qquad \text{for } j' \neq j,$$

$$B_t^{\{i,j\},\{i,j\}}(\boldsymbol{\theta}) = 2\int_0^t \sin^2(\theta_s^j - \theta_s^i)\mathrm{d}s;$$

- $\mathcal{Z}_t^\Lambda(\boldsymbol{\theta}) = \mathbb{E}[\exp\{-(\beta^2/2)(\boldsymbol{J};\ B_t(\boldsymbol{\theta})\boldsymbol{J})_{\mathbb{R}^{\Lambda^*}}\}]$.

According to Itô's formula, we have, under $R_t^{\otimes\Lambda}$,

$$\begin{aligned}
\sin(\theta_s^j - \theta_s^i)(\mathrm{d}w_s^i - \mathrm{d}w_s^j) &= \sin\theta_s^j(\cos\theta_s^i\,\mathrm{d}w_s^i) + \cos\theta_s^i(-\sin\theta_s^j\,\mathrm{d}w_s^j) \\
&\quad + \sin\theta_s^i(\cos\theta_s^j\,\mathrm{d}w_s^j) + \cos\theta_s^j(-\sin\theta_s^i\,\mathrm{d}w_s^i) \\
&= \sin\theta_s^j(\mathrm{d}(\sin\theta^i)_s + \tfrac{1}{2}\sin\theta_s^i\,\mathrm{d}s) + \cos\theta_s^i(\mathrm{d}(\cos\theta^j)_s + \tfrac{1}{2}\cos\theta_s^j\,\mathrm{d}s) \\
&\quad + \sin\theta_s^i(\mathrm{d}(\sin\theta^j)_s + \tfrac{1}{2}\sin\theta_s^j\,\mathrm{d}s) + \cos\theta_s^j(\mathrm{d}(\cos\theta^i)_s + \tfrac{1}{2}\cos\theta_s^i\,\mathrm{d}s),
\end{aligned}$$

so that

$$A_t^{\{i,j\}}(\boldsymbol{\theta}) = [\cos(\theta_t^i - \theta_t^j) - \cos(\theta_0^i - \theta_0^j)] + \int_0^t \cos(\theta_s^i - \theta_s^j)\mathrm{d}s.$$

Carrying on our computation of the martingale part of $\log M_t^\Lambda(\boldsymbol{\theta})$, we then have

$$\log M_t^\Lambda(\boldsymbol{\theta}) =_{\mathrm{mart}} \frac{\beta^2}{2}\sum_{\{i,j\}} A_t^{\{i,j\}}(\boldsymbol{\theta})\left\{\sum_{\{k,l\}} \mathbb{E}\left[J_{\{i,j\}}J_{\{k,l\}}\frac{\exp\{-\frac{\beta^2}{2}(\boldsymbol{J};\, B_t(\boldsymbol{\theta})\boldsymbol{J})_{\mathbb{R}^{\Lambda^*}}\}}{\mathcal{Z}_t^\Lambda(\boldsymbol{\theta})}\right] A_t^{\{k,l\}}(\boldsymbol{\theta})\right\}.$$

But, for each $t \in \mathbb{R}_+$, for each $\boldsymbol{\theta} \in \mathbb{R}^\Lambda$, the matrix $B_t(\boldsymbol{\theta})$ is in fact such that

$$\forall \boldsymbol{\lambda} \in \mathbb{R}^{\Lambda^*}, \qquad (\boldsymbol{\lambda};\, B_t(\boldsymbol{\theta})\boldsymbol{\lambda})_{\mathbb{R}^{\Lambda^*}} = \sum_{i\in\Lambda}\int_0^t \left\{\sum_{j\sim i}\lambda_{\{i,j\}}\sin(\theta_s^j - \theta_s^i)\right\}^2 \mathrm{d}s,$$

so that $B_t(\boldsymbol{\theta})$ is (symmetric) non-negative definite (in $\mathbb{R}^{(\Lambda^*\otimes\Lambda^*)}$), and

$$\mathbb{E}\left[J_{\{i,j\}}J_{\{k,l\}}\frac{\exp\left\{-\frac{\beta^2}{2}(\boldsymbol{J};\, B_t(\boldsymbol{\theta})\boldsymbol{J})_{\mathbb{R}^{\Lambda^*}}\right\}}{\mathcal{Z}_t^\Lambda(\boldsymbol{\theta})}\right]$$

is just the $(\{i, j\}, \{k, l\})$th coefficient of the inverse matrix of

$$C_{\beta,t}(\boldsymbol{\theta}) = (I + \beta^2 B_t(\boldsymbol{\theta})).$$

Hence,

$$\log M_t^\Lambda(\boldsymbol{\theta}) =_{\mathrm{mart}} \frac{\beta^2}{2}\sum_{\{i,j\}} A_t^{\{i,j\}}(\boldsymbol{\theta})((C_{\beta,t}(\boldsymbol{\theta}))^{-1}A_t(\boldsymbol{\theta}))^{\{i,j\}}$$

$$= \frac{\beta^2}{2}(A_t(\boldsymbol{\theta});\, C_{\beta,t}(\boldsymbol{\theta})^{-1}A_t(\boldsymbol{\theta}))_{\mathbb{R}^{\Lambda^*}}.$$

Since the family of matrices $\{(C_{\beta,t}(\boldsymbol{\theta}))^{-1}\}_{0\leqslant t\leqslant T}$ is differentiable in t,

$$\log M_t^\Lambda(\boldsymbol{\theta}) =_{\mathrm{mart}} \beta^2\int_0^t (C_{\beta,s}(\boldsymbol{\theta})^{-1}A_s(\boldsymbol{\theta});\, \mathrm{d}A_s(\boldsymbol{\theta}))_{\mathbb{R}^{\Lambda^*}}$$

$$=_{\mathrm{mart}} \beta^2\sum_{i\in\Lambda}\int_0^t\left\{\sum_{j\sim i}[C_{\beta,s}(\boldsymbol{\theta})^{-1}A_s(\boldsymbol{\theta})]^{\{i,j\}}\sin(\theta_s^j - \theta_s^i)\right\}\mathrm{d}w_s^i,$$

and the proposition is proved. □

Interestingly, there are several ways of extending the system $(\mathcal{S}_\Lambda)$ to infinite dimensions. Bearing in mind the methods devised by Ben Arous and Guionnet (1997; 1998) in the mean-field context, one may first note that, for general configurations $\boldsymbol{\theta} \in W_T^\Lambda$ (showing no periodicity), the empirical process

$$\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)} = \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\delta_{\boldsymbol{\theta}^{(\Lambda),(i)}}$$

is a purely atomic measure with $|\Lambda|$ distinct masses, so that the Hilbert spaces $\mathbb{R}^{\Lambda^*}$ and $L^2_{\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}}(\Omega;\, \mathbb{R}^d)$ are canonically isomorphic; the canonical isomorphism ϕ transforms a Λ-dimensional real vector A into an $L^2_{\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}}$ functional $a : \Omega \to \mathbb{R}^d$ through the identities

$$a_k(\boldsymbol{\theta}^{(\Lambda),(i)}) = A^{\{i,i+e_k\}}$$

holding for each $i \in \Lambda$ and each $1 \leqslant k \leqslant d$, $\{e_1, \ldots, e_k, \ldots, e_d\}$ being the canonical basis in $\mathbb{Z}^d$.

So $\phi : \mathbb{R}^{\Lambda^*} \to L^2_{\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta}}}(\Omega; \mathbb{R}^d)$ provides us with a way of viewing the inverse matrix $C_{\beta,t}(\boldsymbol{\theta})^{-1}$ as the inverse of a symmetric, positive definite operator

$$\mathcal{C}_{\beta,t} : L^2_{\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta}}}(\Omega; \mathbb{R}^d) \to L^2_{\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta}}}(\Omega; \mathbb{R}^d),$$

and the real vector $A_t(\boldsymbol{\theta})$ may also be viewed as an $L^2_{\hat{\pi}_{\boldsymbol{\theta}}}$ functional $a_t : \Omega \to \mathbb{R}^d$, a_t being given by

$$a_t(\boldsymbol{\omega}) = \begin{pmatrix} [\cos(\omega_u^{e_1} - \omega_u^O)]_0^t + \int_0^t \cos(\omega_u^{e_1} - \omega_u^O)\mathrm{d}u \\ \vdots \\ [\cos(\omega_u^{e_d} - \omega_u^O)]_0^t + \int_0^t \cos(\omega_u^{e_d} - \omega_u^O)\mathrm{d}u \end{pmatrix}.$$

One is then in a position to give a new expression for the stochastic differential at site i in the system $(\mathcal{S}_\Lambda)$, of the type

$$\mathrm{d}\theta_t^i = \mathrm{d}w_t^i + \beta^2 \left\{ (s_t(\boldsymbol{\theta}^{(i)}); [\mathcal{C}_{\beta,t}^{-1} a_t]_{\mathcal{L}_b}(\boldsymbol{\theta}^{(i)}))_{\mathbb{R}^d} + \sum_{k=1}^d (s_t^{(k)}(\boldsymbol{\theta}^{(i)}); [\mathcal{C}_{\beta,t}^{-1} a_t]_{\mathcal{L}_b}(\boldsymbol{\theta}^{(i-e_k)}))_{\mathbb{R}^d} \right\} \mathrm{d}t,$$

the functionals s_t, $s_t^{(k)} \in L^2_{\hat{\pi}_{\boldsymbol{\theta}}}$ being simply given by

$$s_t(\boldsymbol{\omega}) = \begin{pmatrix} \sin(\omega_t^{e_1} - \omega_t^O) \\ \vdots \\ \sin(\omega_t^{e_d} - \omega_t^O) \end{pmatrix} \quad \text{and} \quad s_t^{(k)}(\boldsymbol{\omega}) = \begin{pmatrix} \vdots \\ \sin(\omega_t^{-e_k} - \omega_t^O) \\ \vdots \end{pmatrix}, \qquad 1 \leqslant k \leqslant d.$$

At this stage one should also note that both a_t and $s_t : \Omega \to \mathbb{R}^d$ define bounded Lipschitz functionals on Ω, when one is considering a reasonable distance on Ω (metrizing the product topology), for example $d_{t=T} : \Omega \to \mathbb{R}_+$, the family of semidistances $(d_t)_{0 \leqslant t \leqslant T}$ being defined by

$$\forall \boldsymbol{\omega}, \boldsymbol{\tau} \in \Omega, \qquad d_t(\boldsymbol{\omega}, \boldsymbol{\tau}) = \sum_{i \in \mathbb{Z}^d} \rho^{|i|}(\|\omega^i - \tau^i\|_{\infty,t} \wedge 2),$$

for some fixed $\rho \in]0; 1[$ ($\|.\|_{\infty,t}$ denotes the seminorm of uniform convergence in $[0; t]$, and $|i| = \max(|i_1|, \ldots, |i_k|, \ldots, |i_d|)$).

On the other hand, the covariance operator

$$\mathcal{C}_{\beta,t} : L^2_{\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta}}}(\Omega; \mathbb{R}^d) \to L^2_{\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta}}}(\Omega; \mathbb{R}^d)$$

is given as a sum

$$\mathcal{C}_{\beta,t} = (\mathcal{I}d + \beta^2 \mathcal{B}_t),$$

and the symmetric, non-negative definite operator $\mathcal{B}_t : L^2_{\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}} \to L^2_{\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}}$ may also be defined on the Banach space $\mathcal{L}_b$ consisting of all bounded Lipschitz observables $a : \Omega \to \mathbb{R}^d$. So for small β and for t not too large, the covariance operators $\mathcal{C}_{\beta,t}$ may also be viewed as perturbations of the identity on $\mathcal{L}_b$, and the drift term at site i may be given an expression of the type

$$\mathrm{d}\theta^i_t = \mathrm{d}w^i_t + \beta^2 \left\{ \langle [\mathcal{C}^{-1}_{\beta,t} a_t]_{\mathcal{L}_b}(\boldsymbol{\theta}^{(\Lambda),(i)});\ s_t(\boldsymbol{\theta}^{(\Lambda),(i)})\rangle_{\mathbb{R}^d}\,\mathrm{d}t + \sum_{k=1}^{d} (s^{(k)}_t(\boldsymbol{\theta}^{(i)});\ [\mathcal{C}^{-1}_{\beta,t} a_t]_{\mathcal{L}_b}(\boldsymbol{\theta}^{(i-e_k)}))_{\mathbb{R}^d} \right\} \mathrm{d}t.$$

Such a finite-dimensional system of interacting diffusions has an obvious extension to infinite dimensions: one should simply let i vary in $\mathbb{Z}^d$ and consider the system $(\mathcal{S}_\infty)$ given by

$$\begin{cases} \mathrm{d}\theta^i_t = \mathrm{d}w^i_t + \beta^2 \left\{ (s_t(\boldsymbol{\theta}^{(i)});\ [\mathcal{C}^{-1}_{\beta,t} a_t]_{\mathcal{L}_b}(\boldsymbol{\theta}^{(i)}))_{\mathbb{R}^d} + \displaystyle\sum_{k=1}^{d} (s^{(k)}_t(\boldsymbol{\theta}^{(i)});\ [\mathcal{C}^{-1}_{\beta,t} a_t]_{\mathcal{L}_b}(\boldsymbol{\theta}^{(i-e_k)}))_{\mathbb{R}^d} \right\} \mathrm{d}t, \\ \mathrm{Law}(\boldsymbol{\theta}|_{t=0}) = u_0^{\otimes \mathbb{Z}^d}. \end{cases}$$

This extension of $(\mathcal{S}_\Lambda)$ to infinite dimensions has the advantage of providing us with a reasonable estimation of the regularity of the drift term associated with the stochastic differential $\mathrm{d}\theta^i_t$; but for $\beta^2 \cdot t \gg 1$ the covariance operators $\mathcal{C}_{\beta,t}$ might well become singular when viewed as bounded linear operators on the Banach space $\mathcal{L}_b$, so that one has to look for another extension to infinite dimensions in the general situation where β and T take arbitrarily large values.

The following elementary remarks are of great help in finding an alternative expression for the stochastic differential corresponding to an infinite-dimensional extension of $\mathcal{S}_\Lambda$. First of all, in the original expression for the stochastic differential corresponding to a finite volume, averaged dynamics,

$$\mathrm{d}\theta^i_t = \mathrm{d}w^i_t + \beta^2 \sum_{j\sim i} (C_{\beta,t}(\boldsymbol{\theta})^{-1} \cdot A_t(\boldsymbol{\theta}))^{\{i,j\}} \sin(\theta^j_t - \theta^i_t)\mathrm{d}t,$$

the covariance matrix $C_{\beta,t}(\boldsymbol{\theta})^{-1}$ is symmetric, so that one might just as well write

$$\mathrm{d}\theta^i_t = \mathrm{d}w^i_t + \beta^2 \langle C_{\beta,t}(\boldsymbol{\theta})^{-1} \cdot S^{[i]}_t(\boldsymbol{\theta});\ A_t(\boldsymbol{\theta})\rangle_{\mathbb{R}^{\Lambda^*}}\,\mathrm{d}t,$$

$S^{[i]}_t(\boldsymbol{\theta})$ being the Λ^*-dimensional real vector such that

$$(S^{[i]}_t(\boldsymbol{\theta}))^{\{k,l\}} = 0 \qquad \text{whenever } k \neq i,\ l \neq i,$$

$$(S^{[i]}_t(\boldsymbol{\theta}))^{\{i,j\}} = \sin(\theta^j_t - \theta^i_t).$$

Secondly, for any $\boldsymbol{\omega} \in \Omega$ and for $i \in \mathbb{Z}^d$, $0 \leqslant t \leqslant T$, one may define infinite-dimensional vectors $A_t(\boldsymbol{\omega})$, $S^{[i]}_t(\boldsymbol{\omega}) \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ just as in the finite-dimensional setting:

$$(A_t(\boldsymbol{\omega}))^{\{i,j\}} = [\cos(\omega_u^j - \omega_u^i)]_0^t + \int_0^t \cos(\omega_u^j - \omega_u^i)\mathrm{d}u,$$

$$(S_t^{[i]}(\boldsymbol{\omega}))^{\{k,l\}} = 0 \qquad \text{whenever } k \neq i,\ l \neq i,$$

$$(S_t^{[i]}(\boldsymbol{\omega}))^{\{i,j\}} = \sin(\omega_t^j - \omega_t^i),$$

and observe that

$$\forall i \in \mathbb{Z}^d,\ \forall t \in [0;\ T], \qquad A_t(\boldsymbol{\omega}) \in l^\infty((\mathbb{Z}^d)^*), \quad S_t^{[i]}(\boldsymbol{\omega}) \in l^1((\mathbb{Z}^d)^*).$$

Similarly, the symmetric matrices B_t may be extended to infinite dimensions by letting $B_t(\boldsymbol{\omega}) : l^2((\mathbb{Z}^d)^*) \to l^2((\mathbb{Z}^d)^*)$ denote the symmetric, non-negative definite operator such that

$$\langle B_t(\boldsymbol{\omega}) \cdot f_{\{i,j\}};\ f_{\{k,l\}} \rangle_{l^2((\mathbb{Z}^d)^*)} = \begin{cases} 0 & \text{if } \{i,\ j\} \text{ and } \{k,\ l\} \\ & \text{are not adjacent,} \\ \displaystyle\int_0^t \sin(\omega_s^j - \omega_s^i)\sin(\omega_s^l - \omega_s^i)\mathrm{d}s & \text{for } k = i,\ l \neq j, \\ \displaystyle 2\int_0^t \sin^2(\omega_s^j - \omega_s^i)\mathrm{d}s & \text{for } k = i,\ l = j. \end{cases}$$

$((f_{\{i,j\}})_{\{i,j\} \in (\mathbb{Z}^d)^*}$ is the canonical basis in $l^2((\mathbb{Z}^d)^*))$.

Thirdly, taking into account the fact that the vectors $S_t^{[i]}(\boldsymbol{\omega}) \in l^1((\mathbb{Z}^d)^*)$ have finite support and that the infinite-dimensional matrices

$$C_{\beta,t}(\boldsymbol{\omega}) = (Id + \beta^2 B_t(\boldsymbol{\omega})) : l^2((\mathbb{Z}^d)^*) \to l^2((\mathbb{Z}^d)^*)$$

are sparse, one may then prove that, for any $\boldsymbol{\omega}, \boldsymbol{\tau} \in \Omega$ and for any $i \in \mathbb{Z}^d$, $0 \leqslant t \leqslant T$, the $l^2((\mathbb{Z}^d)^*)$ vector

$$C_{\beta,t}(\boldsymbol{\omega})^{-1} \cdot S_t^{[i]}(\boldsymbol{\tau})$$

actually lies in $l^1((\mathbb{Z}^d)^*)$, with an $l^1((\mathbb{Z}^d)^*)$ norm that may be conveniently controlled. To be more precise:

Lemma 2.1. *For any $\boldsymbol{\omega}, \boldsymbol{\tau} \in \Omega$, the $l^2((\mathbb{Z}^d)^*)$ vector*

$$C_{\beta,t}(\boldsymbol{\omega})^{-1} \times S_t^{[i]}(\boldsymbol{\tau})$$

lies in $l^1((\mathbb{Z}^d)^*)$ and has an $l^1((\mathbb{Z}^d)^*)$ norm that is bounded from above by

$$K_{\beta,t}^d = \sqrt{2d}(2(1+\lambda))^d$$

for $\lambda = 2 + 4\beta^2 t$.

Proof. For the sake of simplicity, we assume that $i = O$. We also let $|V|^{\{k,l\}}$ denote the absolute value of the $\{k,\ l\}$ coordinate of a vector $V \in l^2((\mathbb{Z}^d)^*)$ and $P_{\{k,l\}}$ denote the projection of $l^2((\mathbb{Z}^d)^*)$ corresponding to the $\{k,\ l\}$ coordinate:

$$P_{\{k,l\}}(V) = \langle V;\ f_{\{k,l\}} \rangle_{l^2((\mathbb{Z}^d)^*)} \cdot f_{\{k,l\}}.$$

We then have

$$
\begin{aligned}
|C_{\beta,t}(\boldsymbol{\omega})^{-1} \times S_t^{[O]}(\boldsymbol{\tau})|^{\{k,l\}} &= \|P_{\{k,l\}}[C_{\beta,t}(\boldsymbol{\omega})^{-1} \times S_t^{[O]}(\boldsymbol{\tau})]\|_{l^2((\mathbb{Z}^d)^*)} \\
&= \left\| P_{\{k,l\}} \left[\sum_{n=0}^{\infty} \frac{(\lambda - \beta^2 B_t(\boldsymbol{\omega}))^n}{(1+\lambda)^{n+1}} S_t^{[O]}(\boldsymbol{\tau}) \right] \right\|_{l^2((\mathbb{Z}^d)^*)} \\
&= \left\| \sum_{n=0}^{\infty} P_{\{k,l\}} \left[\frac{(\lambda - \beta^2 B_t(\boldsymbol{\omega}))^n}{(1+\lambda)^{n+1}} S_t^{[O]}(\boldsymbol{\tau}) \right] \right\|_{l^2((\mathbb{Z}^d)^*)},
\end{aligned}
$$

and at this stage one should note that

$$
P_{\{k,l\}} \left[\frac{(\lambda - \beta^2 B_t(\boldsymbol{\omega}))^n}{(1+\lambda)^{n+1}} S_t^{[O]}(\boldsymbol{\tau}) \right] = 0 \qquad \text{whenever } n < |\{k,\, l\}| = (|k| \wedge |l|),
$$

so that

$$
\begin{aligned}
|C_{\beta,t}(\boldsymbol{\omega})^{-1} \times S_t^{[O]}(\boldsymbol{\tau})|^{\{k,l\}} &= \left\| P_{\{k,l\}} \left[\sum_{n \geqslant |\{k,l\}|} \frac{(\lambda - \beta^2 B_t(\boldsymbol{\omega}))^n}{(1+\lambda)^{n+1}} S_t^{[O]}(\boldsymbol{\tau}) \right] \right\|_{l^2((\mathbb{Z}^d)^*)} \\
&\leqslant \sum_{n \geqslant |\{k,l\}|} \left\| \frac{(\lambda - \beta^2 B_t(\boldsymbol{\omega}))^n}{(1+\lambda)^{n+1}} S_t^{[O]}(\boldsymbol{\tau}) \right\|_{l^2((\mathbb{Z}^d)^*)} \\
&\leqslant \frac{\sqrt{2d}}{(1+\lambda)} \sum_{n \geqslant |\{k,l\}|} \left(\frac{\lambda}{1+\lambda} \right)^n \\
&= \sqrt{2d} \left(\frac{\lambda}{1+\lambda} \right)^{|\{k,l\}|}.
\end{aligned}
$$

We then have

$$
\begin{aligned}
\sum_{\{k,l\}} |C_{\beta,t}(\boldsymbol{\omega})^{-1} \times S_t^{[O]}(\boldsymbol{\tau})|^{\{k,l\}} &= \sum_{n=0}^{\infty} \sum_{|\{k,l\}|=n} |C_{\beta,t}(\boldsymbol{\omega})^{-1} \times S_t^{[O]}(\boldsymbol{\tau})|^{\{k,l\}} \\
&\leqslant \sqrt{2d} \sum_{n=0}^{\infty} \sharp(\{k,\, l\} : |\{k,\, l\}| = n) \cdot \left(\frac{\lambda}{1+\lambda} \right)^n
\end{aligned}
$$

and

$$
\sharp(\{k,\, l\} : |\{k,\, l\}| = n) = 2^d \binom{n+d-1}{n},
$$

so that, finally,

$$\sum_{\{k,l\}} |C_{\beta,t}(\boldsymbol{\omega})^{-1} \times S_t^{[O]}(\boldsymbol{\tau})|^{\{k,l\}} \leqslant \frac{2^d\sqrt{2d}}{(d-1)!}\sum_{n=0}^{\infty}(n+d-1)\dots(n+1)\left(\frac{\lambda}{1+\lambda}\right)^n$$

$$= 2^d\sqrt{2d}(1+\lambda)^d.$$

□

Taking into account the preceding observations, one is then in a position to state that, for arbitrarily large values of the inverse temperature parameter β and of the terminal time T, the infinite-dimensional system of long-range interacting diffusions $(\mathcal{S}_\infty)$ given by

$$\mathrm{d}\theta_\mathrm{t}^\mathrm{i} = \mathrm{d}w_t^i + \beta^2\langle C_{\beta,t}(\boldsymbol{\theta})^{-1}\cdot S_t^{[i]}(\boldsymbol{\theta});\ A_t(\boldsymbol{\theta})\rangle_{l^1,l^\infty}\,\mathrm{d}t,$$

$$\mathrm{Law}(\boldsymbol{\theta}|_{t=0}) = u_0^{\otimes\mathbb{Z}^d}, \qquad i\in\mathbb{Z}^d,\ 0\leqslant t\leqslant T,$$

should have a unique weak solution $Q_\infty \in \mathcal{M}_s(\Omega)$. The next subsection is devoted to the proof of a much stronger statement concerning the existence and uniqueness of a solution for $(\mathcal{S}_\infty)$.

2.2. Construction of an Itô map for $(\mathcal{S}_\infty)$

Our aim here is to establish that, for arbitrary values of $\beta \geqslant 0$ and $T > 0$, the infinite-dimensional system of long-range interacting diffusions $(\mathcal{S}_\infty)$ has a unique strong solution $\boldsymbol{\theta} = \{(\theta_t^i)_{0\leqslant t\leqslant T};\ i\in\mathbb{Z}^d\}$, and that this solution may be constructed as the image

$$\boldsymbol{\theta} = \Phi(\boldsymbol{w})$$

of the original infinite-dimensional Brownian motion $\boldsymbol{w}$ through a measurable transformation Φ on $\Omega = W_T^{\mathbb{Z}^d}$ that is *Lipschitz continuous* with respect to the metric d_T introduced earlier.

We use a natural fixed-point method to construct the Itô map Φ: it consists in introducing a measurable transformation $\phi : \Omega\times\Omega \to \Omega\times\Omega$ defined through the identities

$$\{\phi(\boldsymbol{w};\,\boldsymbol{\theta})\}_t^i = \left(w_t^i;\ w_t^i + \beta^2\int_0^t \langle A_u(\boldsymbol{\theta});\ C_{\beta,u}(\boldsymbol{\theta})^{-1}S_u^{[i]}(\boldsymbol{\theta})\rangle_{l^\infty,l^1}\,\mathrm{d}u\right)$$

and showing that the successive iterates $\phi^{(n)}(\boldsymbol{w};\,\boldsymbol{\alpha})$ of ϕ on some initial configuration $(\boldsymbol{w};\,\boldsymbol{\alpha})$ converge to some $(\boldsymbol{w};\,\boldsymbol{\theta})\in\Omega^2$, the limiting configuration $(\boldsymbol{w};\,\boldsymbol{\theta})\in\Omega^2$ depending on $(\boldsymbol{w};\,\boldsymbol{\alpha})$ only through $\boldsymbol{w}$. One may then define $\Phi : \Omega\to\Omega$ simply through the equality

$$\Phi(\boldsymbol{w}) = \boldsymbol{\theta}$$

and establish the Lipschitz continuity of Φ on $(\Omega;\ d_T)$.

As a first step towards this goal, we introduce the 'essential part' of ϕ, namely $\varphi : \Omega\to\Omega$ defined by

$$\{\varphi(\boldsymbol{\theta})\}_t^i = \beta^2\int_0^t \langle A_u(\boldsymbol{\theta});\ C_{\beta,u}(\boldsymbol{\theta})^{-1}S_u^{[i]}(\boldsymbol{\theta})\rangle_{l^\infty,l^1}\,\mathrm{d}u,$$

and show that φ satisfies a kind of Gronwall inequality; we shall make use of the semidistances

$$d_t(\boldsymbol{\omega}, \boldsymbol{\tau}) = \sum_{i \in \mathbb{Z}^d} \rho^{|i|}(\|\omega^i - \tau^i\|_{\infty,t} \wedge 2)$$

introduced after Proposition 2.1, the value ρ being carefully chosen so that

$$\rho_0 < \rho < 1, \qquad \text{for } \rho_0 = \left(\frac{\lambda}{1+\lambda}\right) \quad \text{and} \quad \lambda = \lambda(\beta;\ T) = 2 + 4\beta^2 T.$$

Proposition 2.2. *There exists a positive constant $K = K(d, \beta, T)$ for which*

$$d_t(\varphi(\boldsymbol{\theta});\ \varphi(\boldsymbol{\tau})) \leqslant K \cdot \int_0^t d_u(\boldsymbol{\theta};\ \boldsymbol{\tau})\mathrm{d}u, \qquad \forall t \in [0;\ T], \forall \boldsymbol{\theta}, \boldsymbol{\tau} \in \Omega.$$

Obviously,

$$|\{\varphi(\boldsymbol{\theta})\}_t^i - \{\varphi(\boldsymbol{\tau})\}_t^i| \leqslant (a)_t^i + (b)_t^i + (c)_t^i$$

for

$$(a)_t^i = \int_0^t |\langle A_u(\boldsymbol{\theta}) - A_u(\boldsymbol{\tau});\ C_{\beta,u}(\boldsymbol{\theta})^{-1} S_u^{[i]}(\boldsymbol{\theta})\rangle_{l^\infty, l^1}|\mathrm{d}u = \int_0^t \eta_u^i(\boldsymbol{\theta};\ \boldsymbol{\tau})\mathrm{d}u,$$

$$(b)_t^i = \int_0^t |\langle A_u(\boldsymbol{\tau});\ (C_{\beta,u}(\boldsymbol{\theta})^{-1} - C_{\beta,u}(\boldsymbol{\tau})^{-1}) S_u^{[i]}(\boldsymbol{\theta})\rangle_{l^\infty, l^1}|\mathrm{d}u = \int_0^t \xi_u^i(\boldsymbol{\theta};\ \boldsymbol{\tau})\mathrm{d}u,$$

$$(c)_t^i = \int_0^t |\langle A_u(\boldsymbol{\tau});\ C_{\beta,u}(\boldsymbol{\tau})^{-1} (S_u^{[i]}(\boldsymbol{\theta}) - S_u^{[i]}(\boldsymbol{\tau}))\rangle_{l^\infty, l^1}|\mathrm{d}u = \int_0^t \zeta_u^i(\boldsymbol{\theta};\ \boldsymbol{\tau})\mathrm{d}u,$$

and we shall now give appropriate upper bounds enabling us to control η_u^i, ξ_u^i and ζ_u^i suitably.

Lemma 2.2. *For any choices of $i \in \mathbb{Z}^d$, $\boldsymbol{\theta}, \boldsymbol{\tau} \in \Omega$,*

$$|\langle A_u(\boldsymbol{\theta}) - A_u(\boldsymbol{\tau});\ C_{\beta,u}(\boldsymbol{\theta})^{-1} S_u^{[i]}(\boldsymbol{\theta})\rangle_{l^\infty, l^1}|$$
$$\leqslant \sqrt{2d}(2+T) \sum_{\{k,l\} \in (\mathbb{Z}^d)^*} \rho_0^{|k-i| \wedge |l-i|} \cdot ((\|\theta^k - \tau^k\|_{\infty,t} \wedge 2) + (\|\theta^l - \tau^l\|_{\infty,t} \wedge 2)).$$

Proof. This first inequality may be established by using the regularity of each functional $A_u^{\{k,l\}} : \Omega \to \mathbb{R}$, as well as the fact that $B_u(\boldsymbol{\theta})$ is an ‘infinite band matrix’, whereas $S_u^{[i]}(\boldsymbol{\theta})$ has finite support, so that

$$
\begin{aligned}
|C_{\beta,u}(\boldsymbol{\theta})^{-1}S_u^{[i]}(\boldsymbol{\theta})|^{\{k,l\}} &= \left\| P_{\{k,l\}}\left[\sum_{n\geqslant|k-i|\wedge|l-i|}\frac{(\lambda-\beta^2 B_u(\boldsymbol{\omega}))^n}{(1+\lambda)^{n+1}}S_u^{[i]}(\boldsymbol{\tau})\right]\right\|_{l^2((\mathbb{Z}^d)^*)} \\
&\leqslant \sum_{n\geqslant|k-i|\wedge|l-i|}\left\|\frac{(\lambda-\beta^2 B_u(\boldsymbol{\omega}))^n}{(1+\lambda)^{n+1}}S_u^{[i]}(\boldsymbol{\tau})\right\|_{l^2((\mathbb{Z}^d)^*)} \\
&\leqslant \frac{\sqrt{2d}}{(1+\lambda)}\sum_{n\geqslant|k-i|\wedge|l-i|}\left(\frac{\lambda}{1+\lambda}\right)^n \\
&= \sqrt{2d}\rho_0^{|k-i|\wedge|l-i|} \qquad \square
\end{aligned}
$$

The following inequality may be proved along the same lines:

Lemma 2.3. *For all $i \in \mathbb{Z}^d$, for all $\{k, l\} \in (\mathbb{Z}^d)^*$, for all $\boldsymbol{\theta}, \boldsymbol{\tau} \in \Omega$,*

$$
\begin{aligned}
|C_{\beta,u}(\boldsymbol{\tau})^{-1}\times(S_u^{[i]}(\boldsymbol{\theta})-S_u^{[i]}(\boldsymbol{\tau}))|^{\{k,l\}} &\leqslant \rho_0^{|k-i|\wedge|l-i|} \\
&\quad\cdot\left(2d(\|\theta^i-\tau^i\|_{\infty,u}\wedge 2)+\sum_{j\sim i}(\|\theta^j-\tau^j\|_{\infty,u}\wedge 2)\right).
\end{aligned}
$$

Proof. Here again, one may observe that

$$
|C_{\beta,u}(\boldsymbol{\tau})^{-1}\times(S_u^{[i]}(\boldsymbol{\theta})-S_u^{[i]}(\boldsymbol{\tau}))|^{\{k,l\}} \leqslant \|S_u^{[i]}(\boldsymbol{\theta})-S_u^{[i]}(\boldsymbol{\tau})\|_{l^2((\mathbb{Z}^d)^*)}\cdot\sum_{n\geqslant|k-i|\wedge|l-i|}\frac{\lambda^n}{(1+\lambda)^{n+1}}.
$$

Using Minkowski's inequality in $\mathbb{R}^{2d+1}$, we also have that

$$
\|S_u^{[i]}(\boldsymbol{\theta})-S_u^{[i]}(\boldsymbol{\tau})\|_{l^2((\mathbb{Z}^d)^*)} \leqslant \left(2d(\|\theta^i-\tau^i\|_{\infty,u}\wedge 2)+\sum_{j\sim i}(\|\theta^j-\tau^j\|_{\infty,u}\wedge 2)\right),
$$

which finishes the proof of the lemma. $\square$

For fixed $i \in \mathbb{Z}^d$ and $N \geqslant 1$, we next define the linear operator $P_N^{[i]}$: $l^2((\mathbb{Z}^d)^*) \to l^2((\mathbb{Z}^d)^*)$ as the sum of projectors

$$
P_N^{[i]} = \sum_{|\{k,l\}|_i\leqslant N} P_{\{k,l\}},
$$

where

$$
|\{k, l\}|_i = |k-i| \wedge |l-i|.
$$

We now come to the control of $\xi_u^i(\boldsymbol{\theta}; \boldsymbol{\tau})$.

Lemma 2.4. *There exists a positive constant C_1 (depending only on β, d and T) for which*

$$|(C_{\beta,u}(\boldsymbol{\theta})^{-1} - C_{\beta,u}(\boldsymbol{\tau})^{-1})S_u^{[i]}(\boldsymbol{\theta})|^{\{k,l\}}$$
$$\leqslant C_1 \sum_{n\geqslant|\{k,l\}|_i} n\rho_0^n \sum_{|\{k',l'\}|_i\leqslant n+1} ((\|\theta^{k'} - \tau^{k'}\|_{\infty,u} \wedge 2) + (\|\theta^{l'} - \tau^{l'}\|_{\infty,u} \wedge 2)).$$

Proof. Considering Taylor expansions around $\lambda \cdot Id$ and using a chaining argument enables one to express the difference

$$C_{\beta,u}(\boldsymbol{\theta})^{-1} - C_{\beta,u}(\boldsymbol{\tau})^{-1}$$

as a series

$$\sum_{n\geqslant 1} \frac{1}{(1+\lambda)^{n+1}} \sum_{m=0}^{n-1} (\lambda - \beta^2 B_t(\boldsymbol{\theta}))^{n-1-m} \cdot \beta^2 (B_t(\boldsymbol{\tau}) - B_t(\boldsymbol{\theta})) \cdot (\lambda - \beta^2 B_t(\boldsymbol{\tau}))^m$$

converging absolutely with respect to the operator norm $|||\cdot|||_{l^2((\mathbb{Z}^d)^*)}$. For each $n \geqslant 1$ and each $0 \leqslant m \leqslant n-1$, we also know that

$$P_{\{k,l\}}[((\lambda - \beta^2 B_t(\boldsymbol{\theta}))^{n-1-m} \cdot \beta^2 (B_t(\boldsymbol{\tau}) - B_t(\boldsymbol{\theta})) \cdot (\lambda - \beta^2 B_t(\boldsymbol{\tau}))^m) \cdot S_u^{[i]}(\boldsymbol{\theta})] = 0$$

whenever $n < |\{k, l\}|_i$ (since $(\lambda - \beta^2 B_t(\boldsymbol{\theta}))$ and $(\lambda - \beta^2 B_t(\boldsymbol{\tau}))$ both have a 'short-range property'). Hence:

$$|(C_{\beta,u}(\boldsymbol{\theta})^{-1} - C_{\beta,u}(\boldsymbol{\tau})^{-1})S_u^{[i]}(\boldsymbol{\theta})|^{\{k,l\}}$$
$$\leqslant \sum_{n\geqslant|\{k,l\}|_i} \frac{1}{(1+\lambda)^{n+1}} \sum_{m=0}^{n-1} \|(\lambda - \beta^2 B_u(\boldsymbol{\theta}))^{n-1-m} \cdot \beta^2 (B_u(\boldsymbol{\tau}) - B_u(\boldsymbol{\theta})) \cdot (\lambda - \beta^2 B_u(\boldsymbol{\tau}))^m \cdot S_u^{[i]}(\boldsymbol{\theta})\|_{l^2}$$
$$\leqslant \sum_{n\geqslant|\{k,l\}|_i} \frac{\beta^2}{(1+\lambda)^{n+1}} \sum_{m=0}^{n-1} \lambda^{n-1-m} \|(B_u(\boldsymbol{\tau}) - B_u(\boldsymbol{\theta})) \cdot (\lambda - \beta^2 B_u(\boldsymbol{\tau}))^m \cdot S_u^{[i]}(\boldsymbol{\theta})\|_{l^2}.$$

At this stage one should also note that the infinite band matrix

$$(B_u(\boldsymbol{\tau}) - B_u(\boldsymbol{\theta})) : l^2((\mathbb{Z}^d)^*) \to l^2((\mathbb{Z}^d)^*)$$

satisfies

$$\|(B_u(\boldsymbol{\tau}) - B_u(\boldsymbol{\theta})) \cdot V\|_{l^2}$$
$$\leqslant \sqrt{4d-1} \sqrt{\sum_{|\{k',l'\}|_i\leqslant N+1} \left(\int_0^u \sin(\theta_v^{k'} - \theta_v^{l'}) \mathrm{d}v - \int_0^u \sin(\tau_v^{k'} - \tau_v^{l'}) \mathrm{d}v \right)^2} \times \|V\|_{l^2}$$

for any finitely supported vector V lying in $P_N^{[i]}[l^2((\mathbb{Z}^d)^*)]$.

On the other hand, we certainly have

$$\|(\lambda - \beta^2 B_u(\boldsymbol{\tau}))^m \cdot S_u^{[i]}(\boldsymbol{\theta})\|_{l^2} \leqslant \lambda^m \sqrt{d}$$

as well as

$$(\lambda - \beta^2 B_u(\boldsymbol{\tau}))^m \cdot S_u^{[i]}(\boldsymbol{\theta}) = P_{m+1}^{[i]}[(\lambda - \beta^2 B_u(\boldsymbol{\tau}))^m \cdot S_u^{[i]}(\boldsymbol{\theta})].$$

Using the elementary estimate

$$\left(\int_0^t \sin(\theta_u^{k'} - \theta_u^{l'})\mathrm{d}u - \int_0^t \sin(\tau_u^{k'} - \tau_u^{l'})\mathrm{d}u\right)^2 \leqslant t^2((\|\theta^{k'} - \tau^{k'}\|_{\infty,t} \wedge 2) + (\|\theta^{l'} - \tau^{l'}\|_{\infty,t} \wedge 2))^2$$

and then Minkowski's inequality, we obtain

$$\begin{aligned}&\|(B_u(\boldsymbol{\tau}) - B_u(\boldsymbol{\theta}))\cdot(\lambda - \beta^2 B_u(\boldsymbol{\tau}))^m \cdot S_u^{[i]}(\boldsymbol{\theta})\|_{l^2}\\ &\qquad\leqslant 2\lambda^m d\cdot u\cdot \sum_{|\{k',l'\}|_i \leqslant m+2}((\|\theta^{k'} - \tau^{k'}\|_{\infty,u} \wedge 2) + (\|\theta^{l'} - \tau^{l'}\|_{\infty,u} \wedge 2)).\end{aligned}$$

Altogether, we now have:

$$\begin{aligned}&|(C_{\beta,u}(\boldsymbol{\theta})^{-1} - C_{\beta,u}(\boldsymbol{\tau})^{-1})S_u^{[i]}(\boldsymbol{\theta})|^{\{k,l\}}\\ &\quad\leqslant \frac{2\beta^2 ud}{\lambda(1+\lambda)}\sum_{n\geqslant|\{k,l\}|_i}\rho_0^n\sum_{m=0}^{n-1}\sum_{|\{k',l'\}|_i\leqslant m+2}((\|\theta^{k'} - \tau^{k'}\|_{\infty,u} \wedge 2) + (\|\theta^{l'} - \tau^{l'}\|_{\infty,u} \wedge 2))\\ &\quad\leqslant \frac{2\beta^2 ud}{\lambda(1+\lambda)}\sum_{n\geqslant|\{k,l\}|_i} n\rho_0^n \sum_{|\{k',l'\}|_i\leqslant n+1}((\|\theta^{k'} - \tau^{k'}\|_{\infty,u} \wedge 2) + (\|\theta^{l'} - \tau^{l'}\|_{\infty,u} \wedge 2))\end{aligned}$$

so that the inequality stated in the theorem holds true for $C_1 = 2\beta^2 ud/(\lambda(1+\lambda))$. □

Corollary 2.1. *There exists a positive $C_2 = C_2(\beta, d, T)$ for which*

$$|\langle A_u(\boldsymbol{\tau}); (C_{\beta,u}(\boldsymbol{\theta})^{-1} - C_{\beta,u}(\boldsymbol{\tau})^{-1})S_u^{[i]}(\boldsymbol{\theta})\rangle_{l^\infty,l^1}| \leqslant C_2 \sum_{k'\in\mathbb{Z}^d}(\|\theta^{k'} - \tau^{k'}\|_{\infty,u} \wedge 2)\sum_{n\geqslant|k'-i|-2} n^{d+1}\rho_0^n.$$

Proof. From Lemma 2.4 we obtain

$$\begin{aligned}&|\langle A_u(\boldsymbol{\tau}); (C_{\beta,u}(\boldsymbol{\theta})^{-1} - C_{\beta,u}(\boldsymbol{\tau})^{-1})S_u^{[i]}(\boldsymbol{\theta})\rangle_{l^\infty,l^1}|\\ &\quad\leqslant C_1\cdot(2+T)\sum_{\{k,l\}}\sum_{n\geqslant|\{k,l\}|_i} n\rho_0^n\sum_{|\{k',l'\}|_i\leqslant n+1}((\|\theta^{k'} - \tau^{k'}\|_{\infty,u} \wedge 2) + (\|\theta^{l'} - \tau^{l'}\|_{\infty,u} \wedge 2))\\ &\quad\leqslant C_1\cdot(2+T)\cdot(2d)\sum_{k'\in\mathbb{Z}^d}(\|\theta^{k'} - \tau^{k'}\|_{\infty,u} \wedge 2)\cdot\sum_{\{k,l\}\in(\mathbb{Z}^d)^*}\left(\sum_{(n\geqslant|\{k,l\}|_i)\&(n\geqslant|k'-i|-2)} n\rho_0^n\right),\end{aligned}$$

and the inequality stated in the theorem follows from the elementary bound

$$\sharp(\{k, l\}: \quad |\{k, l\}|_i \leqslant n) \leqslant 2^d\cdot n^d.$$ □

Proof of Proposition 2.2. Let us first give an appropriate estimate for

$$\sum_{i\in\mathbb{Z}^d}\rho^{|i|}\xi_u^i(\boldsymbol{\theta}; \boldsymbol{\tau}).$$

Remembering that the parameter ρ (from the definition of d_t) has been chosen in the interval $]\rho_0; 1[$, we may fix an auxiliary parameter $\rho_1 \in]\rho_0; \rho[$ and state that

$$\sum_{n\geqslant N} n^{d+1}\rho_0^n \leqslant C_3\rho_1^N$$

for some $C_3 < +\infty$ independent of N. Corollary 2.1 may thus be modified into

$$|\langle A_u(\boldsymbol{\tau});\,(C_{\beta,u}(\boldsymbol{\theta})^{-1} - C_{\beta,u}(\boldsymbol{\tau})^{-1})S_u^{[i]}(\boldsymbol{\theta})\rangle_{l^\infty,l^1}| \leqslant C_4 \sum_{k'\in\mathbb{Z}^d} \rho_1^{|k'-i|}(\|\theta^{k'} - \tau^{k'}\|_{\infty,u} \wedge 2),$$

and we then have

$$\begin{aligned}
\sum_{i\in\mathbb{Z}^d} \rho^{|i|}\xi_u^i(\boldsymbol{\theta};\boldsymbol{\tau}) &\leqslant C_4 \sum_{i\in\mathbb{Z}^d}\rho^{|i|} \sum_{k'\in\mathbb{Z}^d} \rho_1^{|k'-i|}(\|\theta^{k'} - \tau^{k'}\|_{\infty,u}\wedge 2)\\
&= C_4 \sum_{k'\in\mathbb{Z}^d} (\|\theta^{k'} - \tau^{k'}\|_{\infty,u}\wedge 2)\left(\sum_{i\in\mathbb{Z}^d}\rho_1^{|k'-i|}\rho^{|i|}\right)\\
&\leqslant C_5 \sum_{k'\in\mathbb{Z}^d} \rho^{|k'|}\cdot(\|\theta^{k'} - \tau^{k'}\|_{\infty,u}\wedge 2)\\
&= C_5 d_u(\boldsymbol{\theta};\boldsymbol{\tau}),
\end{aligned}$$

the last inequality following from the existence of a $C_6 < +\infty$ such that

$$\forall k' \in \mathbb{Z}^d, \qquad \sum_{i\in\mathbb{Z}^d} \rho^{|i|}\rho_1^{|k'-i|} \leqslant C_6\cdot\rho^{|k'|}$$

($(\rho^{|k|})_{k\in\mathbb{Z}^d}$ is subharmonic with respect to the kernel $(\rho_1^{|l|})_{l\in\mathbb{Z}^d}$).

According to Lemma 2.2,

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}^d} \rho^{|i|}\xi_u^i(\boldsymbol{\theta};\boldsymbol{\tau})\\
&\leqslant \sqrt{2d}(2+T)\sum_{i\in\mathbb{Z}^d}\rho^{|i|} \sum_{\{k,l\}\in(\mathbb{Z}^d)^*} \rho_0^{|k-i|\wedge|l-i|}\cdot((\|\theta^k - \tau^k\|_{\infty,u}\wedge 2) + (\|\theta^l - \tau^l\|_{\infty,u}\wedge 2))\\
&\leqslant (2d)^{3/2}\sum_{k\in\mathbb{Z}^d}(\|\theta^k - \tau^k\|_{\infty,u}\wedge 2)\left(\sum_{i\in\mathbb{Z}^d}\rho_0^{|k-i|\wedge|l-i|}\rho^{|i|}\right).
\end{aligned}$$

Using a subharmonic inequality for the sequence $(\rho^{|i|})_{i\in\mathbb{Z}^d}$ and the kernel $(\rho_1^{|j|})_{j\in\mathbb{Z}^d}$, we then obtain

$$\sum_{i\in\mathbb{Z}^d} \rho^{|i|}\xi_u^i(\boldsymbol{\theta};\boldsymbol{\tau}) \leqslant D\cdot\sum_{k\in\mathbb{Z}^d}\rho^{|k|}\cdot(\|\theta^k - \tau^k\|_{\infty,u}\wedge 2) = D\cdot d_u(\boldsymbol{\theta};\boldsymbol{\tau}).$$

Finally, according to Lemma 2.3,

$$\sum_{i\in\mathbb{Z}^d} \rho^{|i|}\zeta_u^i(\boldsymbol{\theta};\boldsymbol{\tau}) \leqslant E_1\sum_{i\in\mathbb{Z}^d}\rho^{|i|}(\|\theta^i - \tau^i\|_{\infty,u}\wedge 2)\left(\sum_{\{k,l\}\in(\mathbb{Z}^d)^*}\rho_0^{|\{k,l\}|_i}\right)$$

for some $E_1 > 0$, so that

$$\sum_{i \in \mathbb{Z}^d} \rho^{|i|} \zeta_u^i(\boldsymbol{\theta}; \boldsymbol{\tau}) \leqslant E_2 \cdot d_u(\boldsymbol{\theta}; \boldsymbol{\tau})$$

for $E_2 = E_1 \cdot (\sum_{\{k,l\} \in (\mathbb{Z}^d)^*} \rho_0^{|\{k,l\}|_i})$. Remembering that

$$d_t(\varphi(\boldsymbol{\theta}); \varphi(\boldsymbol{\tau})) \leqslant \int_0^t \left[\sum_{i \in \mathbb{Z}^d} \rho^{|i|} (\eta_u^i(\boldsymbol{\theta}; \boldsymbol{\tau}) + \xi_u^i(\boldsymbol{\theta}; \boldsymbol{\tau}) + \zeta_u^i(\boldsymbol{\theta}; \boldsymbol{\tau})) \right] \mathrm{d}u$$

finishes the proof. $\square$

The existence and Lipschitz continuity of the Itô map $\Phi : \Omega \to \Omega$ are now a straightforward consequence of Proposition 2.2.

Introducing the semidistances $d_t^{(2)} : \Omega^2 \to \mathbb{R}_+$ defined by

$$d_t^{(2)}((\boldsymbol{w}; \boldsymbol{\alpha}), (\boldsymbol{w}'; \boldsymbol{\alpha}')) = d_t(\boldsymbol{w}, \boldsymbol{w}') + d_t(\boldsymbol{\alpha}, \boldsymbol{\alpha}'),$$

we have

$$d_t^{(2)}(\phi(\boldsymbol{w}; \boldsymbol{\alpha}), \phi(\boldsymbol{w}'; \boldsymbol{\alpha}')) \leqslant 2d_t(\boldsymbol{w}, \boldsymbol{w}') + K \int_0^t d_u(\boldsymbol{\alpha}, \boldsymbol{\alpha}')\mathrm{d}u,$$

and successive iterations of the preceding inequality yield

$$\begin{aligned}
&d_t^{(2)}(\phi^{(n)}(\boldsymbol{w}; \boldsymbol{\alpha}), \phi^{(n)}(\boldsymbol{w}'; \boldsymbol{\alpha}')) \\
&\leqslant 2d_t(\boldsymbol{w}, \boldsymbol{w}') + K \int_0^t \left[2d_{t_1}(\boldsymbol{w}, \boldsymbol{w}') + K \int_0^{t_1} \left[2d_{t_2}(\boldsymbol{w}, \boldsymbol{w}') + \ldots + K \int_0^{t_{n-1}} d_{t_n}(\boldsymbol{\alpha}, \boldsymbol{\alpha}')\mathrm{d}t_n \right] \ldots \mathrm{d}t_2 \right] \mathrm{d}t_1 \\
&\leqslant 2d_t(\boldsymbol{w}, \boldsymbol{w}') \left(1 + Kt + \frac{K^2 t^2}{2} + \ldots + \frac{(Kt)^{n-1}}{(n-1)!} \right) + \frac{(Kt)^n}{n!} d_t(\boldsymbol{\alpha}, \boldsymbol{\alpha}').
\end{aligned}$$

We may thus state the following:

Corollary 2.2. *For any values of $\beta \geqslant 0$ and $T > 0$, the Itô map $\Phi : \Omega \to \Omega$ is well defined and satisfies*

$$d_T(\Phi(\boldsymbol{w}), \Phi(\boldsymbol{w}')) \leqslant (2\,\mathrm{e}^{KT} - 1) d_T(\boldsymbol{w}, \boldsymbol{w}').$$

So for arbitrarily large values of the inverse temperature parameter β and of the terminal time T, the infinite-dimensional system of long-range interacting diffusions $(\mathcal{S}_\infty)$ does indeed have a unique strong solution $\boldsymbol{\theta} = \{(\theta_t^i)_{0 \leqslant t \leqslant T};\ i \in \mathbb{Z}^d\}$.

2.3. Averaged large deviations of the empirical process

The first part of Theorem 1.1 is now a straightforward consequence of the existence and continuity of Φ. Indeed, in the i.i.d. case (where $\beta = 0$), we know that the law of the empirical process $\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}$ considered under $\mathrm{d}R_T^{\otimes\Lambda}(\boldsymbol{\theta})$ satisfies an LDP on $\mathcal{M}_s(\Omega)$, on the scale $|\Lambda|$ and according to a good rate functional

$$\mathcal{H}_{R_T^{\otimes\mathbb{Z}^d}} : \mathcal{M}_s(\Omega) \to [0; +\infty]$$

known as the specific entropy functional (relative to $R_T^{\otimes\mathbb{Z}^d}$) and defined by

$$\mathcal{H}_{R_T^{\otimes\mathbb{Z}^d}}(Q) = \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{|\Lambda|} H(Q_\Lambda | R_T^{\otimes\Lambda}) = \sup_{\Lambda\subset\subset\mathbb{Z}^d} \frac{1}{|\Lambda|} H(Q_\Lambda | R_T^{\otimes\Lambda}),$$

Q_Λ denoting a projection of $Q \in \mathcal{M}_s(\Omega)$ with respect to the 'W_T^Λ-measurable' events, and $H(Q_\Lambda | R_T^{\otimes\Lambda})$ denoting relative entropy with respect to $R_T^{\otimes\Lambda}$. A short proof for this LDP may be found in Comets (1986), and Chapter 15 in Georgii (1988) establishes some of the most important properties of the specific entropy functional $\mathcal{H}$ in a general setting.

Now for $\beta > 0$, we also know that the law of $\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}$ considered in the averaged regime (i.e., under $\mathrm{d}P_\Lambda(\boldsymbol{\theta})$) coincides with the law of $\hat{\pi}_{\Phi(\mathbf{w}^{(\Lambda)})}^{(\Lambda)}$ when $\mathbf{w}$ is distributed under $\mathrm{d}R_T^{\otimes\Lambda}(\mathbf{w})$. We thus have

$$\mathrm{Law}_{\mathrm{d}P_\Lambda(\boldsymbol{\theta})}(\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}) = \mathrm{Law}_{\mathrm{d}R_T^{\otimes\Lambda}(\mathbf{w})}(\tilde{\Phi}(\hat{\pi}_{\mathbf{w}}^{(\Lambda)})),$$

$\tilde{\Phi} : \mathcal{M}_s(\Omega) \to \mathcal{M}_s(\Omega)$ being simply defined by

$$\tilde{\Phi}(\mu) = \mu \circ \Phi^{-1}.$$

$\tilde{\Phi}$ certainly defines a continuous transformation on $\mathcal{M}_s(\Omega)$ (equipped with its topology of weak convergence) and we may now apply the contraction principle (a proof of which may be found in Dembo and Zeitouni 1998, Section 4.2) in order to state that the law of the empirical process $\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}$ under $\mathrm{d}P_\Lambda(\boldsymbol{\theta})$ obeys an LDP on $\mathcal{M}_s(\Omega)$, on the scale $|\Lambda|$ and according to the good rate functional $\mathcal{I}^a : \Omega \to [0; +\infty]$ defined by

$$\mathcal{I}^a(\mu) = \inf_{\tilde{\Phi}(\nu)=\mu} \mathcal{H}_{R_T^{\otimes\mathbb{Z}^d}}(\nu).$$

The good rate functional $\mathcal{I}^a$ certainly has $Q_\infty = \tilde{\Phi}(R_T^{\otimes\mathbb{Z}^d})$ as its unique minimizer (since $R_T^{\otimes\mathbb{Z}^d}$ is the unique minimizer corresponding to $\mathcal{H}_{R_T^{\otimes\mathbb{Z}^d}}$), and, as a straightforward application of the Borel–Cantelli lemma (see Ben Arous and Guionnet 1995), one may then state that, almost surely in the realizations of the disordered couplings $\boldsymbol{J} \in \mathbb{R}^{(\mathbb{Z}^d)^*}$, the law of the empirical process $\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}$ considered in the quenched regime (i.e., under $\mathrm{d}P_\Lambda^{\boldsymbol{J}}(\boldsymbol{\theta})$) converges weakly to the Dirac mass δ_{Q_∞}. As we shall see in the next section, such a strong law of large numbers may actually be reinforced into a quenched LDP for the empirical process.

3. Quenched large deviations estimates

Let us now consider a fixed (typical) realization $\boldsymbol{J}_0$ of the coupling variables; for convenience' sake we may also view $\boldsymbol{J}_0$ as an infinite configuration of d-dimensional vectors indexed by the vertices of the lattice, so that

$$\boldsymbol{J}_0 = (J_0(i))_{i\in\mathbb{Z}^d} \in (\mathbb{R}^d)^{(\mathbb{Z}^d)}$$

for $J_0(i) = (J_0(\{i, i+e_1\}), \ldots, J_0(\{i, i+e_d\}))$; so we may now consider any couple $(\boldsymbol{\theta}; \boldsymbol{J})$ as an infinite-dimensional spin configuration where each spin variable $(\theta(i); J(i))$ lies in $W_T \times \mathbb{R}^d$.

Let $\mathcal{R}_\Lambda^{\boldsymbol{J}_0}$ denote the joint law of $\boldsymbol{\theta}$ and $\boldsymbol{J}$ when $\boldsymbol{\theta}$ is distributed according to $R_T^{\otimes\Lambda}$ and $\boldsymbol{J}$ is fixed at the value $\boldsymbol{J}_0$ (or rather, at the projection of $\boldsymbol{J}_0$ onto $\mathbb{R}^{\Lambda^*}$, Λ still being equipped with its periodic boundary conditions). According to Theorem III.1 in Comets (1989), we know that, almost surely in the realization $\boldsymbol{J}_0$, the law of the 'joint empirical process' $\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta},\boldsymbol{J}}$ considered under $\mathrm{d}\mathcal{R}_\Lambda^{\boldsymbol{J}_0}(\boldsymbol{\theta}; \boldsymbol{J})$ obeys an LDP on $\mathcal{M}_s((W_T \times \mathbb{R}^d)^{(\mathbb{Z}^d)})$, on the scale $|\Lambda|$ and according to the deterministic rate functional $\mathcal{H}^q$ given by

$$\mathcal{H}^q(\mu) = \begin{cases} \mathcal{H}_{R_T^{\otimes\mathbb{Z}^d}}(\mu_{\boldsymbol{\theta}}) & \text{if } \mu \text{ has a second marginal } \mu_J \text{ coinciding with } (\mathcal{N}(0;1)^{\otimes d})^{\otimes\mathbb{Z}^d}, \\ +\infty & \text{otherwise.} \end{cases}$$

Let us now denote by $\mathcal{P}_\Lambda^{\boldsymbol{J}_0}$ the joint law of $\boldsymbol{\theta}$ and $\boldsymbol{J}$ when $\boldsymbol{\theta}$ is distributed according to $\mathrm{d}P_\Lambda^{\boldsymbol{J}_0}$ and $\boldsymbol{J}$ is fixed at the value $\boldsymbol{J}_0$. We also know that

$$\begin{aligned} \frac{\mathrm{d}\mathcal{P}_\Lambda^{\boldsymbol{J}_0}}{\mathrm{d}\mathcal{R}_\Lambda^{\boldsymbol{J}_0}} &= \exp\left\{\frac{\beta}{2}\sum_{i\in\Lambda}\left(\sum_{j\sim i} J_{\{i,j\}}\left([\cos(\theta_t^j - \theta_t^i)]_0^T + \int_0^T \cos(\theta_t^j - \theta_t^i)\mathrm{d}t\right)\right)\right. \\ &\qquad \left. -\frac{\beta^2}{4}\sum_{i\in\Lambda}\left(\sum_{j\sim i}\sum_{k\sim i} J_{\{i,j\}}J_{\{i,k\}}\int_0^T \sin(\theta_t^j - \theta_t^i)\sin(\theta_t^k - \theta_t^i)\mathrm{d}t\right)\right\} \\ &= \exp\left\{\sum_{i\in\Lambda}\mathcal{F}^i(\boldsymbol{\theta}; \boldsymbol{J})\right\}, \end{aligned}$$

the functional $\mathcal{F}^i : (W_T \times \mathbb{R}^d)^{(\mathbb{Z}^d)} \to \mathbb{R}$ being of the form

$$\mathcal{F}^i = -\sum_{A\ni i}\frac{\psi_A}{|A|}$$

for some translation-invariant Gibbsian interaction $\Psi = (\psi_A)_{A\subset\subset\mathbb{Z}^d}$ on $(W_T \times \mathbb{R}^d)^{(\mathbb{Z}^d)}$. A precise definition of translation-invariant Gibbsian interactions may be found in Comets (1989); in the present case, Ψ is such that

$$\psi_{\{i,j\}}(\boldsymbol{\theta}; \boldsymbol{J}) = -\beta J_{\{i,j\}}\left([\cos(\theta_t^j - \theta_t^i)]_0^T + \int_0^T \cos(\theta_t^j - \theta_t^i)\mathrm{d}t\right) + \beta^2 J_{\{i,j\}}^2\int_0^T \sin^2(\theta_t^j - \theta_t^i)\mathrm{d}t \qquad \text{if } j \sim i,$$

$$\psi_{\{i,j,k\}}(\boldsymbol{\theta}; \boldsymbol{J}) = \frac{\beta^2}{2}J_{\{i,j\}}J_{\{i,k\}}\int_0^T \sin(\theta_t^j - \theta_t^i)\sin(\theta_t^k - \theta_t^i)\mathrm{d}t, \qquad \text{if } j \sim i,\ k \sim i \text{ and } k \neq j,$$

$$\psi_A \equiv 0, \qquad \text{for any other } A.$$

Just as in Comets (1989), we may now use the fact that

$$\frac{\mathrm{d}\mathcal{P}_{\Lambda}^{\boldsymbol{J}_0}}{\mathrm{d}\mathcal{R}_{\Lambda}^{\boldsymbol{J}_0}}(\boldsymbol{\theta};\,\boldsymbol{J}) = \exp\left\{|\Lambda|\int \mathcal{F}^O \mathrm{d}\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta},\boldsymbol{J}}\right\},$$

and applying the Laplace–Varadhan method in this context leads us to the statement that, a.s. ($\boldsymbol{J}_0$), the law of the 'joint empirical process' $\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta},\boldsymbol{J}}$ under $\mathrm{d}\mathcal{P}_{\Lambda}^{\boldsymbol{J}_0}(\boldsymbol{\theta};\,\boldsymbol{J})$ should satisfy an LDP on the scale $|\Lambda|$ and according to the good rate function $\mathcal{J}^q : \mathcal{M}_s((W_T \times \mathbb{R}^d)^{(\mathbb{Z}^d)}) \longrightarrow [0;\,+\infty]$ given by

$$\mathcal{J}^q(\mu) = \begin{cases} \mathcal{H}^q(\mu) - \int \mathcal{F}^O \mathrm{d}\mu & \text{if } \mathcal{H}^q(\mu) < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

However, since $\mathcal{F}^O$ is not bounded, some verifications are needed in order to apply the Laplace–Varadhan method. The validity of the upper bound

$$(U): \qquad \limsup_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \mathcal{P}_{\Lambda}^{\boldsymbol{J}_0}\{\hat{\pi}^{(\Lambda)}_{\cdot} \in \mathcal{C}\} \leqslant -\inf_{\mu \in \mathcal{C}} \mathcal{J}^q(\mu),$$

for any closed set $\mathcal{C} \subset \mathcal{M}_s((W_T \times \mathbb{R}^d)^{(\mathbb{Z}^d)})$, follows from the existence of a $\delta > 1$ for which

$$A_\delta = \limsup_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \int \mathrm{d}\mathcal{P}_{\Lambda}^{\boldsymbol{J}_0} \exp\{\delta \cdot |\Lambda| \langle \mathcal{F}^O;\, \hat{\pi}^{(\Lambda)} \rangle\} < +\infty.$$

The finiteness of A_δ may actually be proved for any $\delta > 1$ after decomposing the exponent $\delta \cdot |\Lambda| \langle \mathcal{F}^O;\, \hat{\pi}^{(\Lambda)} \rangle$ into

$$\begin{aligned} &\left\{ \sum_{i \in \Lambda} \left(\frac{\delta\beta}{2} \sum_{j \sim i} J_{\{i,j\}} \int_0^T \sin(\theta_t^j - \theta_t^i)(\mathrm{d}w_t^j - \mathrm{d}w_t^i) \right.\right. \\ &\qquad \left.\left. - \frac{\delta^2\beta^2}{4} \sum_{j \sim i} \sum_{k \sim i} J_{\{i,j\}} J_{\{i,k\}} \int_0^T \sin(\theta_t^j - \theta_t^i)\sin(\theta_t^k - \theta_t^i)\mathrm{d}t \right) \right\} \\ &\qquad + \left\{ \frac{(\delta^2 - \delta)\beta^2}{4} \sum_{i \in \Lambda} \left(\sum_{j \sim i} \sum_{k \sim i} J_{\{i,j\}} J_{\{i,k\}} \int_0^T \sin(\theta_t^j - \theta_t^i)\sin(\theta_t^k - \theta_t^i)\mathrm{d}t \right) \right\} \end{aligned}$$

and observing that the first term in the preceding sum is the logarithm of a unit martingale while the second term is obviously bounded from above by

$$(\delta^2 - \delta)\beta^2 T \sum_{\{i,j\} \in \Lambda^*} J_{\{i,j\}}^2.$$

Integrating with respect to first $\boldsymbol{\theta}$ first and then $\boldsymbol{J}$, we then obtain

$$\int \mathrm{d}\mathcal{P}_{\Lambda}^{\boldsymbol{J}_0} \exp\{\delta \cdot |\Lambda| \langle \mathcal{F}^O;\, \hat{\pi}^{(\Lambda)} \rangle\} \leqslant \int \mathrm{d}\mathcal{P}_{\Lambda}^{\boldsymbol{J}_0} \exp\{(\delta^2 - \delta)\beta^2 T \sum_{\{i,j\} \in \Lambda^*} J_{\{i,j\}}^2\},$$

so that

$$A_\delta \leqslant (\delta^2 - \delta)\beta^2 T \cdot d, \qquad \text{a.s. } (\boldsymbol{J}_0).$$

A few remarks should also be made before establishing that the lower bound

$$(L): \qquad \liminf_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \mathcal{P}_\Lambda^{\boldsymbol{J}_0}\{\hat{\pi}^{(\Lambda)}_{\cdot} \in \mathcal{O}\} \geqslant -\inf_{\mu \in \mathcal{O}} \mathcal{J}^q(\mu)$$

holds for any open set $\mathcal{O} \subset \mathcal{M}_s((W_T \times \mathbb{R}^d)^{(\mathbb{Z}^d)})$.

First, it suffices to prove

$$(L'): \qquad \liminf_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \mathcal{P}_\Lambda^{\boldsymbol{J}_0}\{\hat{\pi}^{(\Lambda)}_{\cdot} \in \mathcal{N}\} \geqslant -\mathcal{J}^q(\mu)$$

whenever $\mathcal{N}$ is an open neighbourhood of μ, μ being such that $\mathcal{J}^q(\mu) < +\infty$.

Secondly, since each of the functionals

$$F^{O,j}(\boldsymbol{\theta}) = \left([\cos(\theta_t^j - \theta_t^O)]_0^T + \int_0^T \cos(\theta_t^j - \theta_t^O)\mathrm{d}t \right), \qquad \text{for } j \sim O,$$

and

$$F^{O,j,k}(\boldsymbol{\theta}) = \int_0^T \sin(\theta_t^j - \theta_t^O)\sin(\theta_t^k - \theta_t^O)\mathrm{d}t, \qquad \text{for } j \sim O,\ k \sim O,$$

is uniformly bounded and continuous on Ω, we may consider an open neighbourhood $\mathcal{U}_\epsilon$ of $\mu_{\boldsymbol{\theta}}$ (first marginal of μ) such that:

$$\left| \int F^{O,j}\mathrm{d}\nu - \int F^{O,j}\mathrm{d}\mu \right| < \epsilon \quad \text{and} \quad \left| \int F^{O,j,k}\mathrm{d}\nu - \int F^{O,j,k}\mathrm{d}\mu \right| < \epsilon, \qquad \forall j,\ k \sim O,$$

whenever ν has a $\boldsymbol{\theta}$-marginal lying in $\mathcal{U}_\epsilon$ ($\epsilon > 0$ is arbitrarily small and fixed).

Thirdly, since $\boldsymbol{J}_0$ is typical we may also choose $\Lambda_0 \subset\subset \mathbb{Z}^d$ such that

$$\left| \frac{1}{|\Lambda^*|} \sum_{\{i,j\} \in \Lambda^*} J_{\{i,j\}} \right| < \epsilon \quad \text{and} \quad \left| \frac{1}{|\Lambda|} \sum_{i \in \Lambda} J_{\{i,i+e_k\}} J_{\{i,i+e_l\}} - \delta_{k,l} \right| < \epsilon$$

whenever $\Lambda \supseteq \Lambda_0$, for all $1 \leqslant k,\ l \leqslant d$.

Setting

$$N_\epsilon = \{(\boldsymbol{\theta}, \boldsymbol{J}) | \hat{\pi}^{(\Lambda)}_{(\boldsymbol{\theta},\boldsymbol{J})} \in \mathcal{N} \text{ and } \hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta}} \in \mathcal{U}_\epsilon\}$$

and decomposing the difference $(\langle \mathcal{F}^O; \nu \rangle - \langle \mathcal{F}^O; \mu \rangle)$ into

$$\int \mathrm{d}(\nu \otimes \mu)(\boldsymbol{\theta}^1, \boldsymbol{J}^1; \boldsymbol{\theta}^2, \boldsymbol{J}^2)\{(A) + (A') + (B) + (B')\},$$

where

$$(A) = \frac{\beta}{2}\sum_{j\sim O}(J^1_{\{O,j\}} - J^2_{\{O,j\}}) \cdot F^{O,j}(\boldsymbol{\theta}^1),$$

$$(A') = \frac{\beta}{2}\sum_{j\sim O} J^2_{\{O,j\}} \cdot (F^{O,j}(\boldsymbol{\theta}^1) - F^{O,j}(\boldsymbol{\theta}^2)),$$

$$(B) = -\frac{\beta^2}{4}\sum_{j\sim O}\sum_{k\sim O}(J^1_{\{O,j\}}J^1_{\{O,k\}} - J^2_{\{O,j\}}J^2_{\{O,k\}}) \cdot F^{O,j,k}(\boldsymbol{\theta}^1),$$

$$(B') = -\frac{\beta^2}{4}\sum_{j\sim O}\sum_{k\sim O} J^2_{\{O,j\}}J^2_{\{O,k\}} \cdot (F^{O,j,k}(\boldsymbol{\theta}^1) - F^{O,j,k}(\boldsymbol{\theta}^2)),$$

one may then check that the inequality

$$|\langle \mathcal{F}^O; \hat{\pi}^{(\Lambda)}_{(\boldsymbol{\theta},\boldsymbol{J})}\rangle - \langle \mathcal{F}^O; \mu\rangle| \leqslant K\epsilon$$

holds for some constant K depending only on β, T and d, whenever $(\boldsymbol{\theta}, \boldsymbol{J})$ is chosen in N_ϵ and $\Lambda \supseteq \Lambda_0$.

Defining

$$\mathcal{N}_\epsilon = \mathcal{N} \bigcap \{\nu | \nu \text{ has a first marginal lying in } \mathcal{U}_\epsilon\},$$

we thus have

$$\begin{aligned}
\liminf_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\ln \mathcal{P}^{\boldsymbol{J}_0}_\Lambda\{\hat{\pi}^{(\Lambda)}_\cdot \in \mathcal{N}\} &\geqslant \liminf_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\ln \mathcal{P}^{\boldsymbol{J}_0}_\Lambda\{\hat{\pi}^{(\Lambda)}_\cdot \in \mathcal{N}_\epsilon\}\\
&= \liminf_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\ln \int_{N_\epsilon} \exp\{|\Lambda| \cdot \langle \mathcal{F}^O; \hat{\pi}^{(\Lambda)}_{(\boldsymbol{\theta},\boldsymbol{J})}\rangle\} \mathrm{d}\mathcal{R}^{\boldsymbol{J}_0}_\Lambda(\boldsymbol{\theta}, \boldsymbol{J})\\
&= \langle \mathcal{F}^O; \mu\rangle + \liminf_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\ln \int_{N_\epsilon}\\
&\qquad \exp\{|\Lambda| \cdot (\langle \mathcal{F}^O; \hat{\pi}^{(\Lambda)}_{(\boldsymbol{\theta},\boldsymbol{J})}\rangle - \langle \mathcal{F}^O; \mu\rangle)\} \mathrm{d}\mathcal{R}^{\boldsymbol{J}_0}_\Lambda(\boldsymbol{\theta}, \boldsymbol{J})\\
&\geqslant \langle \mathcal{F}^O; \mu\rangle - K\epsilon + \liminf_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\ln \mathcal{R}^{\boldsymbol{J}_0}_\Lambda\{\hat{\pi}^{(\Lambda)}_\cdot \in \mathcal{N}_\epsilon\}\\
&\geqslant \langle \mathcal{F}^O; \mu\rangle - K\epsilon - \left(\inf_{\nu\in\mathcal{N}_\epsilon} \mathcal{H}^q(\nu)\right)\\
&\geqslant -(\mathcal{H}^q(\mu) - \langle \mathcal{F}^O; \mu\rangle + K\epsilon).
\end{aligned}$$

Letting $\epsilon \searrow 0$ we obtain (L'), and the lower bound (L) is now also proved.

Such a quenched LDP for the joint empirical process $\hat{\pi}^{(\Lambda)}_{(\boldsymbol{\theta},\boldsymbol{J})}$ may naturally be contracted to an LDP for the empirical process $\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta}}$ still considered in the quenched regime, that is, under $\mathrm{d}P_{\Lambda}^{\boldsymbol{J}_0}$; one thus obtains part (ii) of Theorem 1.1: a.s. ($\boldsymbol{J}_0$), the law of the empirical process $\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)}$ under $\mathrm{d}P_{\Lambda}^{\boldsymbol{J}_0}$ obeys an LDP on $\mathcal{M}_s(\Omega)$, on the scale $|\Lambda|$ and according to the (deterministic) good rate functional $\mathcal{I}^q : \mathcal{M}_s(\Omega) \longrightarrow [0; +\infty]$ given by

$$\mathcal{I}^q(\eta) = \inf_{1\text{st marg}(\mu)=\eta} (\mathcal{J}^q(\mu)).$$

The preceding expression for $\mathcal{I}^q$ does not enable one to see immediately that Q_∞ is also the unique minimizer associated with $\mathcal{I}^q$; this fact follows, however, from the inequality

$$\mathcal{I}^q \geqslant \mathcal{I}^a \geqslant 0,$$

a proof of which may be found in Zeitouni (2001).

4. Miscellaneous generalizations

4.1. Addition of a self potential and of an external magnetic field

In this section we would first like to point out that one may choose a Langevin dynamics framework using soft spins that have a better resemblance to the hard spins $\sigma^i = \pm 1$, and still obtain the large deviation results described in Theorem 1.1. This may be done by a change of the reference measure on S^1 corresponding to our circular spins; for any positive integer n, we may, for example, replace the uniform measure on S^1 by

$$\mathrm{d}m(\theta^i) = \mathrm{e}^{-2V(\theta^i)}\mathrm{d}\theta^i,$$

with $V(\theta^i) = K(\sin\theta^i)^{2n}$, for some $K > 0$ and $n \in \mathbb{N}$. The reference measure R_T on path space W_T is then replaced by the probability R_T^V corresponding to the stochastic differential equation

$$\mathrm{d}\theta_t^i = \mathrm{d}w_t^i - 2Kn\sin(\theta_t^i)^{2n-1}\cos(\theta_t^i)\mathrm{d}t$$

and having the initial condition

$$\mathrm{d}m_0(\theta^i) = \frac{\mathbb{1}_{-\pi \leqslant \theta^i \leqslant \pi}}{Z_n}\mathrm{d}m(\theta^i), \qquad Z_n = \int_{-\pi}^{\pi} \mathrm{e}^{-2V(\theta)}\mathrm{d}\theta.$$

In this context, the Langevin dynamics corresponding to the random Hamiltonian H_Λ^J is given by the system of short-range interacting diffusions

$$\begin{cases} \mathrm{d}\theta_t^i = \mathrm{d}w_t^i - 2Kn\sin(\theta_t^i)^{2n-1}\cos(\theta_t^i)\mathrm{d}t + \beta\sum\limits_{j\sim i} J_{\{i,j\}}\sin(\theta_t^j - \theta_t^i)\mathrm{d}t, \\ \mathrm{Law}(\boldsymbol{\theta}|_{t=0}) = m_0^{\otimes\Lambda}, \qquad i \in \Lambda,\ 0 \leqslant t \leqslant T, \end{cases}$$

and one obtains the same expression when computing the Radon–Nikodym derivative corresponding to the averaged regime,

$$M_t^\Lambda(\boldsymbol{\theta}) = \mathbb{E}\left[\frac{\mathrm{d}P_\Lambda^J}{\mathrm{d}(R_t^V)^{\otimes\Lambda}}(\boldsymbol{\theta})\right]$$

$$= \mathbb{E}\left[\exp\left\{-\frac{\beta^2}{2}(\boldsymbol{J};\ B_t(\boldsymbol{\theta})\boldsymbol{J})_{\mathbb{R}^{\Lambda^*}}\right\}\right]$$

$$\times\, \mathbb{E}\left[\exp\{\beta(\boldsymbol{J};\ A_t(\boldsymbol{\theta}))_{\mathbb{R}^{\Lambda^*}}\}\frac{\exp\{-\frac{\beta^2}{2}(\boldsymbol{J};\ B_t(\boldsymbol{\theta})\boldsymbol{J})_{\mathbb{R}^{\Lambda^*}}\}}{\mathcal{Z}_t^\Lambda(\boldsymbol{\theta})}\right],$$

although $(M_t^\Lambda)_{0\leqslant t\leqslant T}$ has now to be viewed as an $(R_T^V)^{\otimes\Lambda}$-martingale instead of an $R_T^{\otimes\Lambda}$-martingale.

Using the very same methods as in the preceding section, we obtain an LDP for the empirical process considered in the averaged regime, with a good rate function having a unique minimizer Q_∞^V. This time Q_∞^V may be presented as a continuous transform of the new reference dynamics $(R_t^V)^{\otimes\mathbb{Z}^d}$; under the very same condition of high temperature or short terminal time as before, Q_∞^V may thus be viewed as the unique weak solution corresponding to the infinite-dimensional system

$$\begin{cases} \mathrm{d}\theta_t^i = \mathrm{d}w_t^i - 2Kn\sin(\theta_t^i)^{2n-1}\cos(\theta_t^i)\mathrm{d}t + \beta^2\{(s_t(\boldsymbol{\theta}^{(i)});\ [\mathcal{C}_{\beta,t}^{-1}a_t]_{\mathcal{L}_b}(\boldsymbol{\theta}^{(i)}))_{\mathbb{R}^d}. \\ \quad \left.+\displaystyle\sum_{k=1}^d (s_t^{(k)}(\boldsymbol{\theta}^{(i)});\ [\mathcal{C}_{\beta,t}^{-1}a_t]_{\mathcal{L}_b}(\boldsymbol{\theta}^{(i-e_k)}))_{\mathbb{R}^d}\right\}\mathrm{d}t, \qquad i\in\mathbb{Z}^d,\ 0\leqslant t\leqslant T, \end{cases}$$

whereas for arbitrarily large β and T, Q_∞^V may also be presented as the unique weak solution corresponding to

$$\mathrm{d}\theta_t^i = \mathrm{d}w_t^i - 2Kn\sin(\theta_t^i)^{2n-1}\cos(\theta_t^i)\mathrm{d}t + \beta^2\langle A_t(\boldsymbol{\theta});\ C_{\beta,t}(\boldsymbol{\theta})^{-1}S_t^{(i)}(\boldsymbol{\theta})\rangle_{l^\infty,l^1}\mathrm{d}t,$$
$$i\in\mathbb{Z}^d,\ 0\leqslant t\leqslant T.$$

Let us now consider systems submitted to an external magnetic field. The Hamiltonian H_Λ^J considered since the beginning of Section 1 is now replaced by

$$H_\Lambda^{J,\kappa}(\boldsymbol{\sigma}) = -\sum_{\{i,j\}\in\Lambda^*} J_{\{i,j\}}\sigma^i.\sigma^j - \kappa\sum_{i\in\Lambda}\sigma^i$$

or, in the case of circular spins,

$$H_\Lambda^{J,\kappa}(\boldsymbol{\theta}) = -\sum_{\{i,j\}\in\Lambda^*} J_{\{i,j\}}\cos(\theta^i-\theta^j) - \kappa\sum_{i\in\Lambda}\cos\theta^i,$$

κ being some fixed real number.

Differentiating $H_\Lambda^{J,\kappa}$ with respect to each of the θ^i, we obtain the following system of short-range interacting diffusions:

$$\mathrm{d}\theta_t^i = \mathrm{d}w_t^i + \beta\{-\kappa\sin\theta_t^i + \sum_{j\sim i} J_{\{i,j\}}\sin(\theta_t^j - \theta_t^i)\}\mathrm{d}t,$$
$$\mathrm{Law}(\boldsymbol{\theta}|_{t=0}) = u_0^{\otimes\Lambda}, \qquad i\in\Lambda,\ 0\leqslant t\leqslant T).$$

This time, the Radon-Nikodym derivative $M_t^{\Lambda,\kappa} = \mathrm{d}P_\Lambda^\kappa/\mathrm{d}R_t^{\otimes\Lambda}$ is such that

$$\begin{aligned}M_t^{\Lambda,\kappa}(\boldsymbol{\theta}) &= \mathbb{E}\Bigg[\exp\Bigg\{\beta\sum_{i\in\Lambda}\int_0^t\Bigg(-\kappa\sin\theta_u^i + \sum_{j\sim i}J_{\{i,j\}}\sin(\theta_u^j-\theta_u^i)\Bigg)\mathrm{d}w_u^i \\ &\qquad -\frac{\beta^2}{2}\sum_{i\in\Lambda}\int_0^t\Bigg(-\kappa\sin\theta_u^i + \sum_{j\sim i}J_{\{i,j\}}\sin(\theta_u^j-\theta_u^i)\Bigg)^2\mathrm{d}u\Bigg\}\Bigg] \\ &= \exp\Bigg(-\beta\kappa\sum_{i\in\Lambda}\int_0^t\sin\theta_u^i\,\mathrm{d}w_u^i - \frac{\beta^2\kappa^2}{2}\sum_{i\in\Lambda}\int_0^t\sin^2\theta_u^i\mathrm{d}u\Bigg) \\ &\qquad\times\mathbb{E}\Bigg[\exp\Bigg\{\beta(\boldsymbol{J};\,A_{\kappa,t}(\boldsymbol{\theta}))_{\mathbb{R}^{\Lambda^*}} - \frac{\beta^2}{2}(\boldsymbol{J};\,B_t(\boldsymbol{\theta})\boldsymbol{J})_{\mathbb{R}^{\Lambda^*}}\Bigg\}\Bigg]\end{aligned}$$

for

$$A_{\kappa,t}(\boldsymbol{\theta})^{\{i,j\}} = \beta\int_0^t\sin(\theta_u^j-\theta_u^i)(\mathrm{d}w_u^i-\mathrm{d}w_u^j) + \beta^2\kappa\int_0^t\sin(\theta_u^j-\theta_u^i)(\sin\theta_u^i - \sin\theta_u^j)\mathrm{d}u,$$

so that

$$\begin{aligned}\log M_t^{\Lambda,\kappa}(\boldsymbol{\theta}) &=_{\mathrm{mart}} -\beta\kappa\sum_{i\in\Lambda}\int_0^t\sin\theta_u^i\mathrm{d}w_u^i + \frac{1}{2}(A_{\kappa,t}(\boldsymbol{\theta});\,C_{\beta,t}^{-1}(\boldsymbol{\theta})A_{\kappa,t}(\boldsymbol{\theta}))_{\mathbb{R}^{\Lambda^*}} \\ &=_{\mathrm{mart}}\sum_{i\in\Lambda}\int_0^t\Bigg\{-\beta\kappa\sin\theta_u^i + \beta\sum_{j\sim i}(C_{\beta,u}^{-1}(\boldsymbol{\theta})A_{\kappa,u}(\boldsymbol{\theta}))^{\{i,j\}}\sin(\theta_u^j-\theta_u^i)\Bigg\}\mathrm{d}w_u^i.\end{aligned}$$

Hence, the covariance matrices $(C_{\beta,t}(\boldsymbol{\theta}))_{0\leqslant t\leqslant T}$ are just the same as in the case of a zero external magnetic field, and we simply need to define new $l^\infty((\mathbb{Z}^d)^*)$-valued functionals $A_t^\kappa(\boldsymbol{\theta})$, given by

$$A_t^\kappa(\boldsymbol{\theta})^{\{i,j\}} = \beta A_t(\boldsymbol{\theta})^{\{i,j\}} + \beta^2\kappa\int_0^t\sin(\omega_u^j-\omega_u^i)(\sin\omega_u^i - \sin\omega_u^j)\mathrm{d}u.$$

For arbitrarily large β and T one may then define an Itô map $\Phi^\kappa:\Omega\to\Omega$ similar to Φ, except for the functionals $A_t(\boldsymbol{\theta})$ which have to be replaced by $A_t^\kappa(\boldsymbol{\theta})$.

We may thus state the following proposition:

Proposition 4.1. *Fix $K>0$, $\kappa\in\mathbb{R}$, $n\in\mathbb{N}$, and consider the system of randomly interacting diffusions $\mathcal{S}_\Lambda^{\boldsymbol{J},\kappa}$ given by*

$$\mathrm{d}\theta_t^i = \mathrm{d}w_t^i - 2Kn\sin(\theta_t^i)^{2n-1}\cos(\theta_t^i)\mathrm{d}t - \beta\kappa\sin\theta_t^i\mathrm{d}t + \beta\sum_{j\sim i} J_{\{i,j\}}\sin(\theta_t^j - \theta_t^i)\mathrm{d}t$$

$$\mathrm{Law}(\boldsymbol{\theta}|_{t=0}) = m_0^{\otimes\Lambda}, \qquad i\in\Lambda,\ 0\leqslant t\leqslant T).$$

Let $P_\Lambda^{\boldsymbol{J},\kappa}$ be the law of $\mathcal{S}_\Lambda^{\boldsymbol{J},\kappa}$, define P_Λ^κ as the average of the $P_\Lambda^{\boldsymbol{J},\kappa}$ when $\boldsymbol{J}$ varies at random,

$$P_\Lambda^\kappa = \int \mathrm{d}\gamma(\boldsymbol{J})P_\Lambda^{\boldsymbol{J},\kappa}$$

and let Π_Λ^κ (or $\Pi_\Lambda^{\boldsymbol{J},\kappa}$) denote the law of the empirical process under P_Λ^κ (or $P_\Lambda^{\boldsymbol{J},\kappa}$).

$\{\Pi_\Lambda^\kappa\}_{\Lambda\subset\subset\mathbb{Z}^d}$ and $\Pi_\Lambda^{\boldsymbol{J},\kappa}$ both satisfy a large-deviation principle on $\mathcal{M}_s(\Omega)$, on the scale $|\Lambda|$ and according to some good rate functionals $\mathcal{I}_\kappa^a$ and $\mathcal{I}_\kappa^q$ such that

$$\mathcal{I}_\kappa^q \geqslant \mathcal{I}_\kappa^a \geqslant 0.$$

$\mathcal{I}_\kappa^a$ and $\mathcal{I}_\kappa^a$ have a (common) unique minimizer Q_∞^κ, which may be explicitly presented as the unique weak solution corresponding to the following infinite-dimensional system of interacting diffusions:

$$\begin{cases} \mathrm{d}\theta_t^i = \mathrm{d}w_t^i - 2Kn\sin(\theta_t^i)^{2n-1}\cos(\theta_t^i)\mathrm{d}t - \beta\kappa\sin\theta_t^i\,\mathrm{d}t + \beta^2\langle A_t^\kappa(\boldsymbol{\theta});\, C_{\beta,t}(\boldsymbol{\theta})^{-1}S_t^{(i)}(\boldsymbol{\theta})\rangle_{l^\infty,l^1}\,\mathrm{d}t \\ \mathrm{Law}(\boldsymbol{\theta}|_{t=0}) = m_0^{\otimes\mathbb{Z}^d}, \qquad i\in\mathbb{Z}^d,\ 0\leqslant t\leqslant T. \end{cases}$$

4.2. Change of the initial and boundary conditions

Let us briefly mention that the original quenched and averaged LDPs stated in Theorem 1.1 hold under much broader hypotheses on the initial and boundary conditions.

As regards the initial conditions, we may consider, for example, a summable, translation-invariant interaction $\mathcal{P}$ on $\mathbb{R}^{\mathbb{Z}^d}$, that is, a family $\mathcal{P} = (P_A)_{A\subset\subset\mathbb{Z}^d}$ of continuous, real-valued functionals defined on $\mathbb{R}^{\mathbb{Z}^d}$ and such that:

- $\forall A\subset\subset\mathbb{Z}^d,\ P_A:\mathbb{R}^{\mathbb{Z}^d}\to\mathbb{R}$ depends on $\boldsymbol{\omega}\in\mathbb{R}^{\mathbb{Z}^d}$ only through its projection on $\mathbb{R}^A$;
- $\forall A\subset\subset\mathbb{Z}^d,\ \forall i\in\mathbb{Z}^d,\ \forall\boldsymbol{\omega}\in\mathbb{R}^{\mathbb{Z}^d},\ P_{i+A}(\boldsymbol{\omega}) = P_A(\boldsymbol{\omega}^{(i)})$;
- $\sum_{A\ni O}(\sup_{\boldsymbol{\omega}\in\mathbb{R}^{\mathbb{Z}^d}}|P_A(\boldsymbol{\omega})|) < +\infty$.

Fixing a finite volume $\Lambda = [-N;\,N]^d\bigcap\mathbb{Z}^d$, we may then replace the original 'deep quench' initial condition $m_0^{\otimes\Lambda}$ by the finite-volume Gibbs measure μ_Λ corresponding to the interaction $\mathcal{P}$ and to the reference measure $m_0^{\otimes\mathbb{Z}^d}$, Λ still being equipped with its periodic boundary conditions. We thus obtain a new 'quenched' system of interacting diffusions $(\mathcal{S}_\Lambda^{\boldsymbol{J}})'$ having initial condition μ_Λ; we then let $(P_\Lambda^{\boldsymbol{J}})'$ denote the corresponding probability law on W_T^Λ and $P'_\Lambda = \mathbb{E}_{\boldsymbol{J}}[(P_\Lambda^{\boldsymbol{J}})']$.

Letting Π'_Λ denote the law of the empirical process considered under P'_Λ, we then have

$$\frac{\mathrm{d}\Pi'_\Lambda}{\mathrm{d}\Pi_\Lambda}(\pi) = \exp\left\{|\Lambda|\int_{\mathbb{R}^{\mathbb{Z}^d}} v(\boldsymbol{\omega})\mathrm{d}\pi^0(\boldsymbol{\omega})\right\}$$

whenever $\pi = \hat{\pi}_{\mathbf{x}}^{(\Lambda)}$ for some $\mathbf{x}\in\Omega$, π^0 denoting the projection of π at time $t=0$ and v being simply defined as

$$\upsilon(\boldsymbol{\omega}) = -\sum_{A \ni O} \frac{P_A(\boldsymbol{\omega})}{|A|}.$$

υ defines a bounded continuous functional on $\mathbb{R}^{\mathbb{Z}^d}$ (since $\mathcal{P}$ is summable) so that Varadhan's lemma may be applied directly in order to establish that (Π'_Λ) also obeys an LDP on $\mathcal{M}_s(\Omega)$, on the scale $|\Lambda|$ and according to the good rate function $\mathcal{I}' : \mathcal{M}_s(\Omega) \to [0 + \infty]$ given by

$$\mathcal{I}'(\pi) = \mathcal{I}(\pi) - \int_{\mathbb{R}^{\mathbb{Z}^d}} \upsilon \, \mathrm{d}\pi^0 - K,$$

where

$$K = \inf_{\eta \in \mathcal{M}_s(\Omega)} \left\{ \mathcal{I}(\eta) - \int_{\mathbb{R}^{\mathbb{Z}^d}} \upsilon \, \mathrm{d}\eta^0 \right\}.$$

Here again, the set of all minimizers corresponding to $\mathcal{I}'$ will be reduced to a single asymptotic dynamics $\{Q'_\infty\}$ whenever the set of all infinite-volume Gibbs measures corresponding to $\mathcal{P}$ and $m_0^{\mathbb{Z}^d}$ is reduced to a singleton $\{\mu_\infty\}$, and we may also establish a quenched LDP for the empirical process.

Let us finally show that these large-deviations results also hold when considering other boundary conditions. For the sake of simplicity, we shall first consider the case where the periodic boundary conditions on $\Lambda = [-N;\, N]^d \bigcap \mathbb{Z}^d$ are being replaced by zero boundary conditions outside Λ. We then have to deal with a new 'quenched' system of interacting diffusions $(\mathcal{S}_\Lambda^{\boldsymbol{J}})''$, whose probability law may be denoted by $(P_\Lambda^{\boldsymbol{J}})''$. We then let $P''_\Lambda = \mathbb{E}_{\boldsymbol{J}}[(P_\Lambda^{\boldsymbol{J}})'']$ and $\Pi''_\Lambda = \mathrm{Law}_{\mathrm{d}P''_\Lambda(\mathbf{x})}(\hat{\pi}_{\mathbf{x}}^{(\Lambda)})$, and we shall make an additional remark before computing the Radon-Nikodym derivative $\mathrm{d}\Pi''_\Lambda/\mathrm{d}\Pi_\Lambda(\pi)$ corresponding to some $\pi = \hat{\pi}_{\mathbf{x}}^{(\Lambda)}$: letting $\partial\Lambda$ denote the set consisting of all $i \in \Lambda$ such that $j \notin \Lambda$ for some $j \sim i$, we know that under the (usual) periodic boundary conditions, the contribution of the 'boundary terms' to the Girsanov exponent appearing in $\mathrm{d}\Pi_\Lambda/\mathrm{d}\Pi_\Lambda^{\beta=0}(\pi)$ has a martingale part summing to

$$\sum_{i \in \partial\Lambda} \int_0^T \left(\sum_{j \sim i, j \notin \Lambda} J_{\{i'', j''\}} \sin(\theta_t^{j'''} - \theta_t^i) \right) \mathrm{d}\theta_t^i,$$

i'', j'' and j''' being given respectively by $i'' = i$, $j'' = j$ and $j''' = j - 2N \cdot e_k$ in the case where $j = i + e_k$, whereas $i'' = j''' = j + 2N \cdot e_k$ and $j'' = j + (2N+1) \cdot e_k$ in the case where $j = i - e_k$, for some $1 \leqslant k \leqslant d$ (a convention of this kind is implicit in the whole paper, but the sites i'', j'' and j''' were not introduced earlier in order to make the notation lighter).

On the other hand, in the case of zero boundary conditions on Λ, the aforementioned martingale part is just

$$\sum_{i \in \partial\Lambda} \int_0^T \left(\sum_{j \sim i, j \notin \Lambda} J_{\{i,j\}} \sin(-\theta_t^i) \right) \mathrm{d}\theta_t^i,$$

so that

$$\frac{\mathrm{d}\Pi''_\Lambda}{\mathrm{d}\Pi_\Lambda}(\pi) = \mathbb{E}\left[\exp\left\{\beta \sum_{i\in\partial\Lambda}\int_0^T \sum_{j\sim i, j\notin\Lambda} (J_{\{i,j\}}\sin(-\theta^i_t) - J_{\{i,j''\}}\sin(\theta^{j'''}_t - \theta^i_t))\mathrm{d}\theta^i_t \right.\right.$$

$$\left.\left. -\frac{\beta^2}{2}\sum_{i\in\partial\Lambda}\int_0^T\left(\sum_{j\sim i, j\notin\Lambda}(J_{\{i,j\}}\sin(-\theta^i_t) - J_{\{i,j''\}}\sin(\theta^{j'''}_t - \theta^i_t))\right)^2 \mathrm{d}t\right\}\right]$$

Introducing, further, the set $\Lambda^{**} \subset (\mathbb{Z}^d)^*$ consisting of all bonds $\{i, j\}$ for which $i \in \partial\Lambda$ and $j \notin \Lambda$, we then have

$$\frac{\mathrm{d}\Pi''_\Lambda}{\mathrm{d}\Pi_\Lambda}(\pi) = \exp\{\langle A_T(\boldsymbol{\theta});\, C_{\beta,T}(\boldsymbol{\theta})^{-1}A_T(\boldsymbol{\theta})\rangle_{\mathbb{R}^{\Lambda^{**}}}\},$$

$A_T(\boldsymbol{\theta})$ being the Λ^{**}-dimensional vector such that

$$A_T^{\{i,j\}}(\boldsymbol{\theta}) = \beta\int_0^T - \sin(\theta^i_t)\mathrm{d}\theta^i_t - \beta\int_0^T \sin(\theta^{j'''}_t - \theta^i_t)(\mathrm{d}\theta^{j'''}_t - \mathrm{d}\theta^i_t)$$

$$= \beta[\cos\theta^i_t]_0^T + \frac{\beta}{2}\int_0^T \cos\theta^i_t \mathrm{d}t + \beta[\cos(\theta^{j'''}_t - \theta^i_t)]_0^T + \beta\int_0^T \cos(\theta^{j'''}_t - \theta^i_t)\mathrm{d}t$$

in the case where $j = i + e_k$ for some $1 \leqslant k \leqslant d$, whereas

$$A_T^{\{i,j\}}(\boldsymbol{\theta}) = \beta\int_0^T - \sin(\theta^i_t)\mathrm{d}\theta^i_t$$

$$= \beta[\cos\theta^i_t]_0^T + \frac{\beta}{2}\int_0^T \cos\theta^i_t \mathrm{d}t$$

in the case where $j = i - e_k$ for some $1 \leqslant k \leqslant d$, and $C_{\beta,T}(\boldsymbol{\theta})^{-1}$ being again a symmetric, non-negative definite matrix whose eigenvalues lie in [0; 1].

We may subsequently observe that

$$\forall\Lambda \subset\subset \mathbb{Z}^d, \forall\boldsymbol{\theta} \in W_T^\Lambda, \qquad \frac{\mathrm{d}\Pi''_\Lambda}{\mathrm{d}\Pi_\Lambda}(\hat{\pi}^{(\Lambda)}_{\boldsymbol{\theta}}) \leqslant \exp\{K\cdot|\Lambda^{**}|\} = \exp\{2Kd\cdot(2N+1)^{d-1}\}$$

for some constant K depending only on β and T. This last fact is certainly sufficient to guarantee that $(\Pi''_\Lambda)_{\Lambda\subset\subset\mathbb{Z}^d}$ satisfies the very same LDP as $(\Pi_\Lambda)_{\Lambda\subset\subset\mathbb{Z}^d}$; we may observe, for example, that $(\Pi''_\Lambda)_{\Lambda\subset\subset\mathbb{Z}^d}$ remains exponentially tight and that, for any bounded continuous functional $\mathcal{F}$ on $\mathcal{M}_s(\Omega)$,

$$\lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\log\int_{\mathcal{M}_s(\Omega)} \mathrm{e}^{|\Lambda|\cdot\mathcal{F}(\pi)}\mathrm{d}\Pi''_\Lambda(\pi)$$

exists and coincides with

$$\lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\log\int_{\mathcal{M}_s(\Omega)} \mathrm{e}^{|\Lambda|\cdot\mathcal{F}(\pi)}\mathrm{d}\Pi_\Lambda(\pi) = \sup_{\pi\in\mathcal{M}_s(\Omega)}\{\mathcal{F}(\pi) - \mathcal{I}(\pi)\},$$

so that Bryc's inverse Varadhan lemma (cf. Dembo and Zeitouni 1998, Section 4.4) applies to $(\Pi''_\Lambda)_{\Lambda\subset\subset\mathbb{Z}^d}$.

In the preceding arguments we have only been using the fact that each of the coordinates appearing in the Λ^{**}-dimensional vector $A_T(\boldsymbol{\theta})$ may be integrated by parts, so that $|A_T^{\{i,j\}}(\boldsymbol{\theta})|$ has a uniform upper bound. This fact still holds when considering other (non-zero) boundary conditions, for example, a boundary condition made of a fixed, typical realization $(\tau_t^j)_{0\leqslant t\leqslant T}^{j\in\Lambda^c}$ of i.i.d. Brownian motions.

Finally, we could also consider the situation where the 'quenched' system of interacting diffusions has a stochastic differential at site $i\in\partial\Lambda$ such that

$$\mathrm{d}\theta_t^i = \mathrm{d}w_t^i + \beta\sum_{j\sim i, j\in\Lambda} J_{\{i,j\}}\sin(\theta_t^j-\theta_t^i)\mathrm{d}t + \beta\sum_{j\sim i, j\notin\Lambda} J_{\{i,j\}}\sin(\zeta_t^j-\theta_t^i)\mathrm{d}t$$

for some auxiliary Brownian motions $(\zeta_t^j)_{0\leqslant t\leqslant T}^{j\in\Lambda^c}$. In this situation the Radon–Nikodym derivative

$$\frac{\mathrm{d}\Pi''_\Lambda}{\mathrm{d}\Pi_\Lambda}(\pi=\hat{\pi}_{\boldsymbol{\theta}}^{(\Lambda)})$$

may be expressed as

$$\begin{aligned}\frac{\mathrm{d}\Pi''_\Lambda}{\mathrm{d}\Pi_\Lambda}(\pi) &= \mathbb{E}_\zeta\mathbb{E}_{\boldsymbol{J}}\Bigg[\exp\Bigg\{\beta\sum_{i\in\partial\Lambda}\int_0^T\sum_{j\sim i, j\notin\Lambda}(J_{\{i,j\}}\sin(\zeta_t^j-\theta_t^i)-J_{\{i,j''\}}\sin(\theta_t^{j'''}-\theta_t^i))\mathrm{d}\theta_t^i\\ &\qquad -\frac{\beta^2}{2}\sum_{i\in\partial\Lambda}\int_0^T\Bigg(\sum_{j\sim i, j\notin\Lambda}(J_{\{i,j\}}\sin(\zeta_t^j-\theta_t^i)-J_{\{i,j''\}}\sin(\theta_t^{j'''}-\theta_t^i))\Bigg)^2\mathrm{d}t\Bigg\}\Bigg]\\ &= \mathbb{E}_\zeta[\exp\{\langle A_T(\boldsymbol{\theta});\, C_{\beta,T}(\boldsymbol{\theta})^{-1}A_T(\boldsymbol{\theta})\rangle_{\mathbb{R}^{\Lambda^{**}}}\}]\end{aligned}$$

for some Λ^{**}-dimensional vector $A_T(\boldsymbol{\theta})=A_T(\boldsymbol{\theta};\boldsymbol{\zeta})$ whose coordinates may be bounded uniformly in $\boldsymbol{\theta}$ and $\boldsymbol{\zeta}$ and some symmetric, non-negative definite matrix $C_{\beta,T}(\boldsymbol{\theta})^{-1}=C_{\beta,T}(\boldsymbol{\theta};\boldsymbol{\zeta})^{-1}$ whose eigenvalues lie in [0; 1], which again shows that $(\Pi''_\Lambda)_{\Lambda\subset\subset\mathbb{Z}^d}$ satisfies the very same LDP as $(\Pi_\Lambda)_{\Lambda\subset\subset\mathbb{Z}^d}$.

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