

Consistency of Bayes factor for nonnested model selection when the model dimension grows

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Zellner's g -prior is a popular prior choice for the model selection problems in the context of normal regression models. Wang and Sun [*J. Statist. Plann. Inference* **147** (2014) 95–105] recently adopt this prior and put a special hyper-prior for g , which results in a closed-form expression of Bayes factor for *nested* linear model comparisons. They have shown that under very general conditions, the Bayes factor is consistent when two competing models are of order $O(n^\tau)$ for $\tau < 1$ and for $\tau = 1$ is almost consistent except a small inconsistency region around the null hypothesis. In this paper, we study Bayes factor consistency for *nonnested* linear models with a growing number of parameters. Some of the proposed results generalize the ones of the Bayes factor for the case of nested linear models. Specifically, we compare the asymptotic behaviors between the proposed Bayes factor and the intrinsic Bayes factor in the literature.

Keywords: Bayes factor; growing number of parameters; model selection consistency; nonnested linear models; Zellner's g -prior

1. Introduction

We reconsider the classical linear regression model

$$\mathbf{Y} = \mathbf{1}_n \alpha + \mathbf{X}_p \boldsymbol{\beta}_p + \boldsymbol{\varepsilon}, \quad (1.1)$$

where $\mathbf{Y} = (y_1, \dots, y_n)'$ is an n -vector of responses, \mathbf{X}_p is an $n \times p$ design matrix of full column rank, containing all potential predictors, $\mathbf{1}_n$ is an $n \times 1$ vector of ones, α is an unknown intercept, and $\boldsymbol{\beta}_p$ is a p -vector of unknown regression coefficients. Throughout the paper, it is assumed that the random error for all models follows the multivariate normal distribution, denoted by $\boldsymbol{\varepsilon} \sim N(\mathbf{0}_n, \sigma^2 \mathbf{I}_n)$, where $\mathbf{0}_n$ is an $n \times 1$ vector of zeros, σ^2 is an unknown positive scalar, and \mathbf{I}_n is an n -dimensional identity matrix. Without loss of generality, we also assume that the columns of \mathbf{X}_p have been centered, so that each column has mean zero.

In the class of linear regression models, we often assume that there is an unknown subset of the important predictors which contributes to the prediction of \mathbf{Y} or has an impact on the response variable \mathbf{Y} . This is by natural a model selection problem where we would like to select a linear model by identifying the important predictors in this subset. Suppose that we have two such

linear regression models M_j and M_i , with dimensions j and i ,

$$M_j : \mathbf{Y} = \mathbf{1}_n \alpha + \mathbf{X}_j \boldsymbol{\beta}_j + \boldsymbol{\varepsilon}, \tag{1.2}$$

$$M_i : \mathbf{Y} = \mathbf{1}_n \alpha + \mathbf{X}_i \boldsymbol{\beta}_i + \boldsymbol{\varepsilon}, \tag{1.3}$$

where \mathbf{X}_i is an $n \times i$ submatrix of \mathbf{X}_p and $\boldsymbol{\beta}_i$ is an $i \times 1$ vector of unknown regression coefficients. As commented by Kass and Raftery [11], a natural way to compare the two competing models is the Bayes factor, which has nice model selection consistency properties. Here, consistency means that the true model will be eventually selected if enough data is provided, assuming that the true model exists. Our particular interest in this paper is to study the model selection consistency of Bayes factor when the model dimension grows with the sample size. To be more specific, we consider the following three asymptotic scenarios:

- Scenario 1. $i = O(n^{a_1})$ and $j = O(n^{a_2})$ with $0 \leq a_1 \leq a_2 < 1$.
- Scenario 2. $i = O(n^{a_1})$ and $j = O(n^{a_2})$ with $0 \leq a_1 < a_2 = 1$.
- Scenario 3. $i = O(n^{a_1})$ and $j = O(n^{a_2})$ with $a_1 = a_2 = 1$.

When the two models M_i and M_j are nested, Moreno, Girón and Casella [18] study the consistency of the intrinsic Bayes factor under the three asymptotic scenarios. Later on, Wang and Sun [22] derive an explicit closed-form Bayes factor associated with Zellner’s g -prior for comparing the two models. They show that under very general conditions, the Bayes factor is consistent when the two models are of order $O(n^\tau)$ for $\tau < 1$ and for $\tau = 1$ is almost consistent except a small inconsistency region around the null hypothesis. Such a small set of models around the null hypothesis can be characterized in terms of a pseudo-distance between models defined by Moreno and Girón [17]. Finally, Wang and Sun [22] compare the proposed results with the ones for the intrinsic Bayes factor due to [18].

It should be noted that M_i and M_j are not necessarily nested in many practical situations. As commented by Pesaran and Weeks [20], “*in econometric analysis, nonnested models arise naturally when rival economic theories are used to explain the same phenomenon, such as unemployment, inflation or output growth.*” In fact, the problem of comparing nonnested models has been studied in a fairly large body of econometric and statistical literature from both practical and theoretical viewpoints, dating back to [10]. For instance, Cox [4] develops a likelihood ratio testing procedure and shows that under appropriate conditions, the proposed approach and its variants have well-behaved asymptotic properties. Watnik and Johnson [25] consider the asymptotic behavior of three different testing procedures (the J-test, the JA-test, and the modified Cox test) for the analysis of nonnested linear models under the alternative hypothesis. The interested reader is referred to [24] and [25] for detailed descriptions of the three testing procedures.

Girón et al. [7] consider the intrinsic Bayes factor for comparing pairs of nonnested models based on the two different encompassing criteria: *encompassing from above* and *encompassing from below*. Later on, Moreno and Girón [17] present a comparative analysis of the intrinsic Bayes factor under the two criteria in linear regression models. Recently, Girón et al. [8] study the consistency of the intrinsic Bayes factor for the case of nonnested linear models under the first two asymptotic scenarios above. The latter two papers mainly focus on the consistency of the intrinsic Bayes factor when the model dimension grows with the sample size, whereas under the same asymptotic scenario, the researchers should also be interested in the consistency of Bayes

factor based on Zellner’s g -prior, which is a popular prior choice for the model selection problems in linear regression models. To the best of our knowledge, the latter has just received little attention over the years, even though it is of the utmost importance to address the consistency issue for nonnested models.

In this paper, we investigate Bayes factor consistency associated with Zellner’s g -prior for the problem of comparing nonnested models under the three asymptotic scenarios above. Specifically, we compare the asymptotic results between the proposed Bayes factor and the intrinsic Bayes factor due to [8]. The results show that the asymptotic behaviors of the two Bayes factors are quite comparable in the first two scenarios. It is remarkable that we also study the consistency of the proposed Bayes factor under Scenario 3, whereas such a scenario is still an open problem for the intrinsic Bayes factor highlighted by Girón et al. [8].

The remainder of this paper is organized as follows. In Section 2, we present an explicit closed-form expression of Bayes factor based on the null-based approach. In Section 3, we address the consistency of Bayes factor for nonnested models under the three asymptotic scenarios. Additionally, we compare the proposed results with the ones of the intrinsic Bayes factor. An application of the results in Section 3 to the ANOVA models is provided in Section 4. Some concluding remarks are presented in Section 5, with additional proofs given in the Appendix.

2. Bayes factor

Within a Bayesian framework, one of the common ways for the model selection problems is to compare models in terms of their posterior probabilities given by

$$P(M_j|\mathbf{Y}) = \frac{p(M_j)p(\mathbf{Y}|M_j)}{\sum_i p(M_i)p(\mathbf{Y}|M_i)} = \frac{p(M_j)\text{BF}[M_j : M_b]}{\sum_i p(M_i)\text{BF}[M_i : M_b]}, \tag{2.1}$$

where $p(M_j)$ is the prior probability for model M_j and $p(M_j|\mathbf{Y})$ is the marginal likelihood of \mathbf{Y} given M_j , and $\text{BF}[M_j : M_b]$ is the *Bayes factor*, which compares each model M_j to the base model M_b and is defined as

$$\text{BF}[M_j : M_b] = \frac{p(\mathbf{Y}|M_j)}{p(\mathbf{Y}|M_b)}. \tag{2.2}$$

The Bayes factor in (2.2) depends on the base model M_b , which is often chosen arbitrarily in practical situations. There are two common choices for M_b : one is *the null-based approach* by using the null model (M_0), the other is *the full-based approach* by choosing the full model (M_F). This paper focuses on the null-based approach because (i) the null model is commonly used as the base model when using Zellner’s g -priors in most of the literature [14] and (ii) unlike the full model, the dimension of the null model is independent of the sample size. This is crucial in addressing the consistency of Bayes factor with an increasing model dimension. Accordingly, we compare the reducing model M_j with M_0 :

$$M_j : \mathbf{Y} = \mathbf{1}_n\alpha + \mathbf{X}_j\boldsymbol{\beta}_j + \boldsymbol{\varepsilon}, \tag{2.3}$$

$$M_0 : \mathbf{Y} = \mathbf{1}_n\alpha + \boldsymbol{\varepsilon}. \tag{2.4}$$

Zellner’s g -prior [27] is often to choose the same noninformative priors for the common parameters that appear in both models and to assign Zellner’s g -prior for others that are only in the larger model. The reasonability of this choice is that if the common parameters are orthogonal (i.e., the expected Fisher information matrix is diagonal) to the new parameters in the larger model, the Bayes factor is quite robust to the choice of the same (even improper) priors for the common parameters; see [12]. Since α and σ^2 are the common orthogonal parameters in (2.3) and (2.4), we consider the following prior distributions for $(\alpha, \sigma^2, \beta_j)$

$$\begin{aligned}
 M_0 : p(\alpha, \sigma^2) &\propto \frac{1}{\sigma^2}, \\
 M_j : p(\alpha, \sigma^2, \beta_j) &\propto \frac{1}{\sigma^2} \quad \text{and} \quad \beta_j | \sigma^2 \sim N(\mathbf{0}, g\sigma^2(\mathbf{X}'_j \mathbf{X}_j)^{-1}).
 \end{aligned}
 \tag{2.5}$$

The amount of information in Zellner’s g -prior is controlled by a scaling factor g , and thus the choice of g is quite critical. A nice review of various choices of g -priors was provided by Liang et al. [14] and later discussed further by Ley and Steel [13]. In most of the developments of the g -priors, the expression of Bayes factor may not have an analytically tractable form, so numerical approximations will generally be employed, whereas it may not be an easy task for practitioners to choose an appropriate one. In particular, standard approximation, such as Laplace approximation, becomes quite challenging when the number of parameters grows with the sample size.

It is remarkable that Maruyama and George [16] propose an explicit closed-form expression of Bayes factor based on combined use of a generalization of Zellner’s g -prior and the beta-prime prior for g :

$$\pi(g) = \frac{g^b(1+g)^{-a-b-2}}{B(a+1, b+1)} I_{(0,\infty)}(g), \tag{2.6}$$

where $a > -1$, $b > -1$, and $B(\cdot, \cdot)$ is a beta function. Noting that Zellner’s g -prior is a special case of the generalization of Zellner’s g -prior in [16], we obtain the following result and the proof directly follows Theorem 3.1 of [16] and is thus omitted for simplicity.

Theorem 1. *Under the prior in (2.6) with $b = (n - j - 1)/2 - a - 2$, the Bayes factor for comparing M_j and M_0 can be simplified as*

$$\text{BF}[M_j : M_0] = \frac{\Gamma(j/2 + a + 1)\Gamma((n - j - 1)/2)}{\Gamma(a + 1)\Gamma((n - 1)/2)} (1 - R_j^2)^{-(n-j-1)/2+a+1}, \tag{2.7}$$

where R_j^2 is the usual coefficient of the determination of model M_j .

The Bayes factor in (2.7) is very attractive for practitioners because of its explicit expression without integral representation, which is not available for other choices of the hyperparameter b . One may argue that such an expression comes at a certain cost on interpreting the role of the prior for g , since this prior depends on both the sample size and the model size through the hyperparameter b . It is noteworthy that this type of the prior has been studied in the literature. For example, Bayarri et al. [1] propose a truncated version of the beta-prime prior for g , such

that $g > (n + 1)/(j + 3) - 1$. A similar type of the prior has also been considered by Ley and Steel [13].

At this point, we provide several arguments justifying the specification of the hyperparameters as follows. (i) The choice of $b = (n - j - 1)/2 - a - 2$ yields an implicit $O(n)$ choice of g [16], that is, $g = O(n)$, which will prevent the hyper- g prior from asymptotically dominating the likelihood function; (ii) as the sample size grows, the right tail of the beta-prime prior behaves like $g^{-(a+2)}$, leading to a very fat tail for small values of a , an attractive property suggested by Gustafson, Hossain and MacNab [9]; (iii) with a choice of $a = -1/2$ and some transformation $\theta = (\mathbf{X}'\mathbf{X})^{1/2}\beta$, the prior makes the asymptotic tail behavior of

$$p(\theta|\sigma^2) = \int_0^\infty p(\theta|\sigma^2, g)\pi(g) dg \tag{2.8}$$

become the multivariate Cauchy for sufficient large $\theta \in R^p$, recommended by Zellner [27]; (iv) the resulting Bayes factor in (2.7) enjoys nice theoretical properties and good performances in practical applications; see, for example, [16,22,23], among others, and (v) when the model dimension j is bounded, the Bayes factor in (2.7) is asymptotically equivalent to the Schwarz approximation.

Theorem 2. *When the model dimension j is fixed, for large sample sizes n , the Bayes factor in (2.7) is equivalent to the Schwarz approximation given by*

$$\text{BF}[M_j : M_0] \approx \exp\left[-\frac{j}{2} \log n - \frac{n}{2} \log(1 - R_j^2)\right]. \tag{2.9}$$

Proof. See the [Appendix](#). □

One of the most attractive properties in the Bayesian approaches is the model selection consistency, which means the true model (assuming it exists) will be selected if enough data is provided. This property has been intensively studied under different asymptotic scenarios as the sample size approaches infinity. For example, when the model dimension is fixed, see [3,13,14,16], to name just a few. Of particular note is that the consistency of various Bayes factors in the listed references behaves very similarly, because for sufficiently large values of n , the intrinsic Bayes factor and Bayes factors associated with mixtures of g -priors (e.g., $g = n$ and Zellner–Siow prior) can all be approximated by the Schwarz approximation in (2.9); see Theorem 2 of [19]. Also, we can show that this approximation is valid for the Bayes factor with the hyper- g prior in [14].

When the model dimension grows with the sample size, Moreno, Girón and Casella [18] study the consistency of the intrinsic Bayes factors for comparing nested models, and a generalization of the consistency to *nonnested* models has been addressed by Girón et al. [8]. More recently, Wang and Sun [22] address the consistency of Bayes factor associated with Zellner’s g -prior for nested models, whereas its consistency for the case of nonnested models is also of the utmost importance. We shall particularly be interested in comparing the asymptotic behaviors between the proposed Bayes factor and the intrinsic Bayes factor under the same asymptotic scenario. The presented results provide researchers a valuable theoretical base for the comparison among nested and nonnested models, which naturally appears in practical situations.

3. Bayes factor consistency for nonnested linear models

In this section, we consider the model selection consistency of Bayes factor for comparing nonnested models under the three asymptotic scenarios. The Bayes factor in (2.7) may not be directly applied to the problem of comparing nonnested models, whereas we can calculate the Bayes factor between M_j and M_0 , $\text{BF}[M_j : M_0]$, and the Bayes factor between M_i and M_0 , $\text{BF}[M_i : M_0]$. Thereafter, the Bayes factor for comparing M_j and M_i can be formulated as

$$\text{BF}[M_j : M_i] = \frac{\text{BF}[M_j : M_0]}{\text{BF}[M_i : M_0]}. \tag{3.1}$$

The Bayes factor for comparing M_j and M_i in (1.2) and (1.3) is thus given by

$$\text{BF}[M_j : M_i] = \frac{\Gamma(j/2 + a + 1)\Gamma((n - j - 1)/2) (1 - R_j^2)^{-(n-j-1)/2+a+1}}{\Gamma(i/2 + a + 1)\Gamma((n - i - 1)/2) (1 - R_i^2)^{-(n-i-1)/2+a+1}}. \tag{3.2}$$

Let M_T stand for the true model

$$M_T : \mathbf{Y} = \mathbf{1}_n\alpha + \mathbf{X}_T\boldsymbol{\beta}_T + \boldsymbol{\varepsilon}.$$

According to [5], the Bayes factor is said to be consistent when

$$\text{plim}_{n \rightarrow \infty} \text{BF}[M_j : M_i] = \infty,$$

if M_j is the true model M_T , whereas

$$\text{plim}_{n \rightarrow \infty} \text{BF}[M_j : M_i] = 0,$$

if M_i is the true model M_T , where ‘plim’ stands for convergence in probability and the probability distribution is the sampling distribution under M_T . For notational simplicity, let

$$\delta_{ji} = \frac{1}{\sigma^2} \boldsymbol{\beta}'_j \frac{\mathbf{X}'_j(\mathbf{I}_n - \mathbf{H}_i)\mathbf{X}_p}{n} \boldsymbol{\beta}_j,$$

where $\mathbf{H}_i = \mathbf{X}_i(\mathbf{X}'_i\mathbf{X}_i)^{-1}\mathbf{X}_i$ with \mathbf{X}_i being an $n \times i$ submatrix of \mathbf{X}_p . According to [8], the value of δ_{ji} can be viewed as a pseudo-distance between M_j and M_i , in which the two models are not necessarily nested. Such a pseudo-distance has the following properties: (i) it is always equal to 0 from any model M_j to itself, that is, $\delta_{jj} = 0$; (ii) if M_i is nested in M_j , it is also equal to 0, that is, $\delta_{ij} = 0$, and (iii) for any model M_k , we have $\delta_{ki} \geq \delta_{kj}$ if M_i is nested in M_j . To study the model selection consistency, it is usually assumed that when the sample size approaches infinity, the limiting value of δ_{ji} , denoted by δ_{ji}^* , always exists, where

$$\delta_{ji}^* = \lim_{n \rightarrow \infty} \frac{1}{\sigma^2} \boldsymbol{\beta}'_j \frac{\mathbf{X}'_j(\mathbf{I}_n - \mathbf{H}_i)\mathbf{X}_j}{n} \boldsymbol{\beta}_j. \tag{3.3}$$

In what follows, let $\lim_{n \rightarrow \infty} [M]Z_n$ represent the limit in probability of the random sequence $\{Z_n : n \geq 1\}$ under the assumption that we are sampling from model M . We present one useful lemma which is critical for deriving the main theorems in this paper, and the proof of the lemma is directly from Lemma 1 of [8] and is not shown here for simplicity.

Lemma 1. *Suppose that we are interested in comparing two models M_i and M_p with dimensions i and p , respectively, where M_i is nested in M_p . As n approaches infinity, both i and p grow with n as $i = O(n^{a_1})$ and $p = O(n^{a_2})$ for $0 \leq a_1 \leq a_2 \leq 1$. When sampling from the true model M_T ,*

(i) *if $0 \leq a_1 \leq a_2 < 1$, it follows that*

$$\lim_{n \rightarrow \infty} [M_T] \left\{ \frac{1 - R_p^2}{1 - R_i^2} \right\} = \frac{1 + \delta_{ip}^*}{1 + \delta_{ii}^*}.$$

(ii) *If $0 \leq a_1 < a_2 = 1$, it follows that*

$$\lim_{n \rightarrow \infty} [M_T] \left\{ \frac{1 - R_p^2}{1 - R_i^2} \right\} = \frac{1 + \delta_{ip}^* - 1/r}{1 + \delta_{ii}^*},$$

where $r = \lim_{n \rightarrow \infty} n/p > 1$.

(iii) *If $a_1 = a_2 = 1$, it follows that*

$$\lim_{n \rightarrow \infty} [M_T] \left\{ \frac{1 - R_p^2}{1 - R_i^2} \right\} = \frac{1 + \delta_{ip}^* - 1/r}{1 + \delta_{ii}^* - 1/s},$$

where $r = \lim_{n \rightarrow \infty} n/p > 1$ and $s = \lim_{n \rightarrow \infty} n/i > 1$.

We are now in a position to characterize the consistency of Bayes factor in (3.2) for comparing nonnested linear models. We begin with Scenario 1, that is, the dimensions of models M_i and M_j are $i = O(n^{a_1})$ and $j = O(n^{a_2})$ with $0 \leq a_1 \leq a_2 < 1$, respectively. The following theorem summarizes Bayes factor consistency when either of the two models is the true model.

Theorem 3. *Let M_0 be the null model nested in both nonnested models M_i and M_j , whose dimensions are i and j , respectively. Suppose that $i = O(n^{a_1})$ and $j = O(n^{a_2})$ with $0 \leq a_1 \leq a_2 < 1$ and that $\delta_{ij}^* > 0$ and $\delta_{ji}^* > 0$. The Bayes factor in (3.2) is consistent whichever the true model is.*

Proof. See the [Appendix](#). □

Under the same asymptotic scenario, Girón et al. [8] also conclude that the intrinsic Bayes factor is consistent whichever the true model is when $\delta_{ij}^* > 0$ and $\delta_{ji}^* > 0$. Such an agreement of the consistency between the two Bayes factors is due to the fact that the dominated term is exactly the same on their asymptotic approximations under Scenario 1. It is noteworthy that Theorem 3 is also valid for other chosen base model nested in both models M_i and M_j , even though the main result of the theorem is derived based on the null-based approach. Moreover, Theorem 3 can be directly applied to the case in which the dimensions of the two competing models are fixed,

because it can be viewed as a limiting case with both $\lim_{n \rightarrow \infty} n/j$ and $\lim_{n \rightarrow \infty} n/i$ approaching infinity.

Corollary 1. *Suppose we are interested in comparing two models M_i and M_j with dimensions i and j , respectively, and that both dimensions are fixed. The Bayes factor in (3.2) is consistent under both models provided that $\delta_{ij}^* > 0$ and $\delta_{ji}^* > 0$.*

We now investigate Bayes factor consistency when the dimension of one of the nonnested models is of order $O(n)$. The main results are provided in the following theorem.

Theorem 4. *Let M_0 be the null model nested in both nonnested models M_i and M_j whose dimensions are i and j , respectively. Suppose that $i = O(n^{a_1})$ and $j = O(n^{a_2})$ with $0 \leq a_1 < a_2 = 1$ and that there exists a positive constant r such that $r = \lim_{n \rightarrow \infty} n/j > 1$.*

- (a) *The Bayes factor in (3.2) is consistent under M_i , provided that $\delta_{ij}^* > 0$.*
- (b) *The Bayes factor in (3.2) is consistent under M_j provided that*

$$\delta_{ji}^* \in (\kappa(r, \delta_{j0}^*), \delta_{j0}^*], \tag{3.4}$$

and $\delta_{j0}^* > \delta(r)$, where $\kappa(r, s) = [r(1 + s)]^{1/r} - 1$ and

$$\delta(r) = r^{1/(r-1)} - 1. \tag{3.5}$$

Proof. See the [Appendix](#). □

Some of the interesting findings can be drawn from the theorem as follows. First, the lower bound of δ_{j0}^* , denoted by $\delta(r)$, is exactly the same as the one in Theorem 2 of [22] for comparing nested linear models. Second, Theorem 4 can be extended to the case of nested model comparisons (i.e., M_i is nested in M_j) by assuming that $M_0 = M_i$. Third, the Bayes factor depends on the choice of the base model through the value of δ_{j0}^* , and therefore, to enlarge the consistency region in (3.4), we need to make δ_{j0}^* be as large as possible. This justifies that the null model M_0 would be the best choice as the base model. Fourth, the lower bound of δ_{ji}^* , denoted by $\kappa(r, \delta_{j0}^*)$, is a bounded decreasing function in r and satisfies that for any $\delta_{j0}^* > 0$,

$$\lim_{r \rightarrow \infty} \kappa(r, \delta_{j0}^*) = 0.$$

Finally, under the same scenario, Girón et al. [8] consider the consistency of the intrinsic Bayes factor and conclude that the intrinsic Bayes factor is consistent under M_i if $\delta_{ij}^* > 0$ and is consistent under M_j , provided that $\delta_{j0}^* > \xi(r)$ with

$$\xi(r) = \frac{r - 1}{(r + 1)^{(r-1)/r} - 1} - 1, \tag{3.6}$$

and

$$\delta_{ji}^* \in (\eta(r, \delta_{j0}^*), \delta_{j0}^*], \tag{3.7}$$

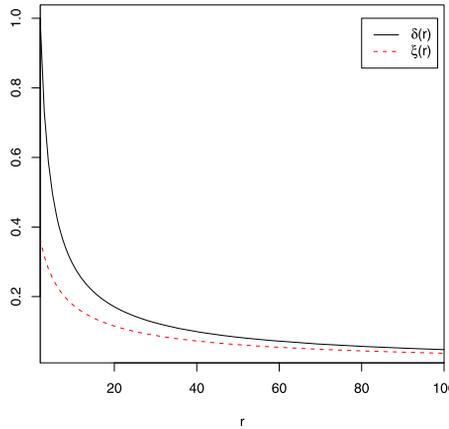


Figure 1. The inconsistency region comparisons (below the curves) for the proposed Bayes factor and the intrinsic Bayes factor under Scenario 2.

where $\eta(r, s) = \frac{r+s}{(1+r)^{r-1}/r} - 1$.

It is interesting to observe that the asymptotic behaviors of the two Bayes factors depend on the pseudo-distance between models δ_{jj}^* bounded by δ_{j0}^* . Figure 1 shows that the upper bounds of their inconsistency regions tend to each other as r increases. Moreover, Figure 2 provides their lower bounds with different values of δ_{j0}^* . When δ_{j0}^* is small, the consistency region of the proposed Bayes factor is included by the one of the intrinsic Bayes factor, whereas the difference between the two regions is small; see Figure 2(a). However, when δ_{j0}^* gets larger, the consistency

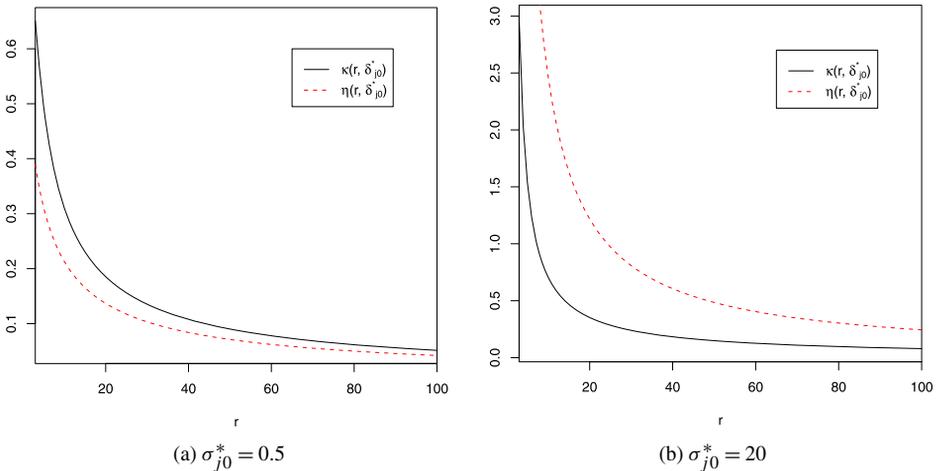


Figure 2. The lower bounds of the consistency regions in (3.4) and (3.7) with different limiting values of δ_{j0}^* under Scenario 2.

region of the proposed Bayes factor will contain the one of the intrinsic Bayes factor, whereas the difference between the two regions becomes significantly as δ_{j0}^* increases; see Figure 2(b). Thus, we may conclude that as δ_{j0}^* increases, the proposed Bayes factor outperforms the intrinsic Bayes factor from a theoretical viewpoint.

It deserves mentioning that the existence of an inconsistency region around the null hypothesis is quite reasonable from a practical point of view, because the nontrue smaller model M_i is parsimonious under large- p situation and is generally selected when conducting model selection, if the true larger model M_j is not so distinguishable from M_i . From the prediction view of point, Maruyama [15] has demonstrated the reasonability of the inconsistency region for the one-way fixed-effect ANOVA model, which could be viewed as a special case of the classical linear models in (1.1) after some reparameterization. A theoretical justification of this line of thought for a more general model is still under investigation and will be reported elsewhere.

The first two theorems mainly focus on the consistency of Bayes factor for the case in which at least one model is of order $O(n^\alpha)$ for $\alpha < 1$. It is worthy of investigating the consistency issue for the case where both models are of order $O(n)$: the growth rates of the two model dimensions are as fast as n . Such a scenario remains an open problem for the intrinsic Bayes factor commented by Girón et al. [8]. We summarize the consistency of the proposed Bayes factor under this scenario in the following theorem.

Theorem 5. *Let M_0 be the null model nested in both nonnested models M_i and M_j with dimensions $i = O(n)$ and $j = O(n)$, respectively. Suppose that there exist positive constants r and s such that $r = \lim_{n \rightarrow \infty} n/j > 1$ and $s = \lim_{n \rightarrow \infty} n/i > 1$. Without loss of generality, we assume that $r \leq s$.*

(a) *The Bayes factor in (3.2) is consistent under M_i provided that*

$$\delta_{ij}^* \in \left(\frac{r-1}{r} \left\{ \left[\frac{s^{1/s}}{r^{1/r}} (1 + \delta_{i0}^*)^{1/s-1/r} \right]^{r/(r-1)} - 1 \right\}, \delta_{i0}^* \right], \tag{3.8}$$

and that $\delta_{i0}^* > 0$ satisfying

$$\left(1 + \frac{\delta_{i0}^*}{1 - 1/r} \right)^{1-1/r} > \frac{(1/r)^{1/r}}{(1/s)^{1/s}} (1 + \delta_{i0}^*)^{1/s-1/r}. \tag{3.9}$$

(b) *The Bayes factor in (3.2) is consistent under M_j provided that*

$$\delta_{ji}^* \in (\phi(r, s, \delta_{j0}^*), \delta_{j0}^*], \tag{3.10}$$

where

$$\phi(a, b, c) = \frac{b-1}{b} \left[\frac{a^{1/a}}{b^{1/b}} (1+c)^{1/a-1/b} - 1 \right]^{b/(b-1)},$$

and that $\delta_{j0}^* > 0$ satisfying

$$\left(1 + \frac{\delta_{j0}^*}{1 - 1/s} \right)^{1-1/s} > \frac{r^{1/r}}{s^{1/s}} (1 + \delta_{j0}^*)^{1/r-1/s}. \tag{3.11}$$

Proof. See the Appendix. □

Unlike the first two asymptotic scenarios, Theorem 5(a) shows that under Scenario 3, there exists an inconsistency region around the alternative hypothesis when M_i is true and that the consistency under M_i depends on the chosen base model M_0 through the distance δ_{i0}^* only. The existence of the inconsistency region is quite reasonable because there are many candidates to be the base model, which could have a dimension of order $O(n^{a_1})$ with $a_1 \leq 1$. In particular, we observe that the inconsistency region disappears for the case in which $r = s$. This is also very understandable, because with the same growth rates, the parsimonious model is typically preferred in terms of model selection. Furthermore, it can be easily shown that the inequality in (3.9) and the lower bound of the consistency region in (3.8) are both valid for any $\delta_{i0}^* > 0$ if $s^{1/s} \leq r^{1/r}$, indicating that for any $\delta_{i0}^* > 0$, the inconsistency region disappears whenever $s \geq r \geq e \approx 2.718$. In order to enlarge the consistency region in (3.8), we need to choose a base model to maximize the distance δ_{i0}^* . Finally, when s tends to infinity, the inconsistency region disappears for any $\delta_{i0}^* > 0$ and $r > 1$, which shows that Theorem 5(a) just reduces to Theorem 4(a).

Theorem 5(b) shows that the consistency region under M_j depends on the chosen base model through δ_{j0}^* only. Thus, the base model should be chosen as small as possible to maximize the value of δ_{j0}^* . Note that when $r = s$, the inconsistency region disappears under M_j . Also, if the rate of growth of M_i is smaller than that of M_j (i.e., s tends to infinity), then with $\lim_{s \rightarrow \infty} s^{1/s} = 1$, the inequality in (3.11) turns to be

$$\delta_{j0}^* > r^{1/(r-1)} - 1 = \delta(r), \tag{3.12}$$

which becomes inequality in (3.5) in Theorem 4, and the lower bound in (3.10) is

$$\begin{aligned} \lim_{s \rightarrow \infty} \phi(r, s, \delta_{j0}^*) &= \lim_{s \rightarrow \infty} \frac{s-1}{s} \left[\frac{r^{1/r}}{s^{1/s}} (1 + \delta_{j0}^*)^{1/r-1/s} - 1 \right]^{s/(s-1)} \\ &= [r(1 + \delta_{j0}^*)]^{1/r} - 1 = \kappa(r, \delta_{j0}^*). \end{aligned}$$

This illustrates that Theorem 4(b) is just a special of Theorem 5(b) when s approaches infinity. We may thus conclude that when s tends to infinity, Theorem 5 reduces to Theorem 4.

We have compared the consistency of the proposed Bayes factor with the one of the intrinsic Bayes factor due to [8] under the first two asymptotic scenarios above. A brief summary of comparisons between the two Bayes factors is presented in Table 1. We observe that the consistency

Table 1. The consistency regions of the Bayes factor in (3.2) and the intrinsic Bayes factor due to [8] for different choices of a_1 and a_2

Rate of divergence	The proposed Bayes factor	The intrinsic Bayes factor
$0 < a_1 = a_2 = 1$	$M_j: \delta_{j0}^* > \psi(r)$ and $\delta_{ji}^* \in (\phi(r, s, \delta_{j0}^*), \delta_{j0}^*]$	$M_j: \text{unknown}$
$0 \leq a_1 < a_2 = 1$	$M_j: \delta_{j0}^* > \delta(r)$ and $\delta_{ji}^* \in (\kappa(r, \delta_{j0}^*), \delta_{j0}^*]$	$M_j: \delta_{j0}^* > \xi(r)$ and $\delta_{ji}^* \in (\eta(r, \delta_{j0}^*), \delta_{j0}^*]$
$0 \leq a_1 \leq a_2 < 1$	$M_j: \delta_{ij}^* > 0$ and $\delta_{ji}^* > 0$	$M_j: \delta_{ij}^* > 0$ and $\delta_{ji}^* > 0$

results presented here are similar to the ones for the intrinsic Bayes factor studied by Girón et al. [8]. The similarity occurs, mainly because the asymptotic behaviors of the two Bayes factors depend on a limiting value of $(1 - R_j^2)/(1 - R_i^2)$ summarized in Lemma 1. The consistency of the intrinsic Bayes factor is still an open problem under Scenario 3. We presume that under Scenario 3, the consistency of the intrinsic Bayes factor also behaves similarly with the one of the proposed Bayes factor, but some further investigation about this presumption is required.

4. Application

It is well known that the ANalysis Of VAriance (ANOVA) models are extremely important in exploratory and confirmatory data analysis in various fields, including agriculture, biology, ecology, and psychology studies. One major difference between the ANOVA models and the classical linear model is that the matrix $[1_n, X_p]$ does not necessarily have full column rank in ANOVA setting. Some constraints are thus required for making the model be identifiable. Here, under the sum-to-zero constraint [6], the ANOVA model with constraints for uniqueness can be reparameterized into the classical linear model without constraints; see [26].

As an illustration, Maruyama [15] and Wang and Sun [21] reparameterize the ANOVA models with the sum-to-zero constraint into the classical linear model in (1.1). Thereafter, based on Zellner’s g -prior with the beta-prime prior for g , they obtain an explicit closed-form Bayes factor, which can be treated as a special case of the Bayes factor in (2.7). Consequently, the asymptotic results of the proposed Bayes factor can be easily applied to various ANOVA models. The application to the one-way ANOVA model is straightforward and is thus omitted here for simplicity. In this section, we mainly consider the results for the two-way balanced ANOVA model with the same number of observations per cell. It deserves mentioning that the results can also be generalized to cover the unbalanced case.

Consider a factorial design with two treatment factors A and B having p and q levels, respectively, with a total of pq factorial cells. Suppose y_{ijl} is the l th observation in the (i, j) th cell defined by the i th level of A and the j th level of B , satisfying the following model

$$y_{ijl} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijl}, \quad \varepsilon_{ijl} \sim N(0, \sigma^2), \tag{4.1}$$

for $i = 1, \dots, p, j = 1, \dots, q$, and $l = 1, \dots, r$. The number of parameters is pqr . We shall be interested in the following five submodels:

- M_0 : No effect of A and no effect of B , that is, $\alpha_i = 0, \beta_j = 0$, and $\gamma_{ij} = 0$ for all i and j .
- M_1 : Only effect of A , that is, $\beta_j = 0$ and $\gamma_{ij} = 0$ for all i and j .
- M_2 : Only effect of B , that is, $\alpha_i = 0$ and $\gamma_{ij} = 0$ for all i and j .
- M_3 : The additive model (without interaction), that is, $\gamma_{ij} = 0$ for all i and j .
- M_4 : The full model (with interaction).

By using the sum-to-zero constraint, Maruyama derives an explicit closed-form Bayes factor associated with Zellner’s g -prior for the regression coefficients of the reparameterized model (i.e., equation (4.7) of [15]) and the beta-prime distribution for the scaling factor g . Moreover, Maruyama studies the consistency of Bayes factor under different asymptotic scenarios. When both p and q approach infinity and r is fixed, Maruyama concludes that the Bayes factor is

consistent except under the full model M_4 , and that when sampling from M_4 , the Bayes factor is consistent only if

$$\delta_{43}^* > H(r, \delta_{10}^* + \delta_{20}^*), \tag{4.2}$$

where δ_{ji}^* is equal to the limit of the sum of squares of the differences between the coefficients of model M_i and the coefficients of model M_j as n tends to infinity, and $H(r, c)$ with positive c is the (unique) positive solution of

$$\frac{(x + 1)^r}{r} - (x + 1) - c = 0. \tag{4.3}$$

Such an inconsistency region occurs due to the model comparison between M_4 and M_3 . Of particular note is that when comparing M_4 and M_3 , we are in the case of Theorem 4 with $a_2 = 1$ and that any null hypothesis will result in a model M_i with a reduced set of parameters that will satisfy $a_1 < a_2$ of Theorem 4. Consequently, when sampling from the full model M_4 , the Bayes factor in (3.2) is consistent only if $\delta_{4i}^* \leq \delta_{40}^*$ and

$$\delta_{4i}^* > [r(1 + \delta_{40}^*)]^{1/r} - 1. \tag{4.4}$$

When comparing models M_4 and M_3 , the consistency region in (4.4) becomes

$$\delta_{43}^* > [r(1 + \delta_{10}^* + \delta_{20}^* + \delta_{43}^*)]^{1/r} - 1,$$

which is equivalent to

$$\frac{(\delta_{43}^* + 1)^r}{r} - (\delta_{43}^* + 1) - (\delta_{10}^* + \delta_{20}^*) = 0. \tag{4.5}$$

This is exactly coincident with equation (4.3) provided by Maruyama [15]. It deserves mentioning that an extension of the results of the preceding section to higher-order designs is straightforward.

5. Concluding remarks

In this paper, we have investigated the consistency of Bayes factor for nonnested linear models for the case in which the model dimension grows with the sample size. It has been shown that in some cases, the proposed Bayes factor is consistent whichever the true model is, and that in others, the consistency depends on the pseudo-distance between the larger model and the base model. Specifically, the pseudo-distance can be used to characterize the inconsistency region of Bayes factor. By comparing the consistency issues between the proposed Bayes factor and the intrinsic Bayes factor, we observe that the asymptotic results presented here are similar to the ones for the intrinsic Bayes factor. It would be interesting to see the finite sample performance of the two Bayes factors, which is currently under investigation and will be reported elsewhere.

The consistency of Bayes factor further indicates that besides the three commonly used families of hyper- g priors in [14], the beta-prime prior is also a good candidate for the scaling factor g in Zellner’s g -prior. Such a comment has also been claimed by Wang and Sun [22] when studying Bayes factor consistency for nested linear models with a growing number of parameters. From a theoretical point of view, we may conclude that like the intrinsic Bayes factor, the proposed Bayes factor should also serve as a powerful tool for model selection in the class of normal regression models due to its comparable asymptotic performance.

It is worth investigating the issues of consistency of Bayes factor based on mixtures of g -priors due to [14] under the three asymptotic scenarios. However, in most of the developments of the g -priors, the expression of Bayes factor may not have an analytically tractable form, and some efficient approximations are required. Standard approximation technique, such as Laplace approximation, becomes quite challenging when the number of parameters grows with the sample size, because the error in approximations needs to be uniformly small over the class of all possible models. Such a situation has also been encountered by Berger, Ghosh and Mukhopadhyay [2] when studying the ANOVA models. We plan to address these issues in our future work.

Finally, it deserves mentioning that we mainly address Bayes factor consistency based on a special choice of the hyperparameter b in the beta-prime prior, which results in an explicit closed-form expression of Bayes factor. In an ongoing project, we investigate the effects of b on the consistency of Bayes factor, especially for the case when b does not actually depend on n .

Appendix

It is well known that the asymptotic approximation of the gamma function, given by Stirling’s formula, can be approximated by

$$\Gamma(\gamma_1 x + \gamma_2) \approx \sqrt{2\pi} e^{-\gamma_1 x} (\gamma_1 x)^{\gamma_1 x + \gamma_2 - 1/2}, \tag{A.1}$$

when x is sufficiently large. Here, “ $f \approx g$ ” is used to indicate that the ratio of the two sides approaches one as x tends to infinity, that is,

$$\lim_{x \rightarrow \infty} \frac{\Gamma(\gamma_1 x + \gamma_2)}{\sqrt{2\pi} e^{-\gamma_1 x} (\gamma_1 x)^{\gamma_1 x + \gamma_2 - 1/2}} = 1.$$

Proof of Theorem 2. When the model dimension is j is bounded and the sample size n is large, it follows directly from Stirling’s formula that

$$\Gamma\left(\frac{n-j-1}{2}\right) \approx \sqrt{2\pi} e^{-n/2} \left(\frac{n}{2}\right)^{(n-j)/2-1} \quad \text{and} \quad \Gamma\left(\frac{n-1}{2}\right) \approx \sqrt{2\pi} e^{-n/2} \left(\frac{n}{2}\right)^{n/2-1}.$$

The Bayes factor in (2.7) is asymptotically equivalent

$$\begin{aligned} \text{BF}[M_j : M_i] &\approx \frac{\sqrt{2\pi} e^{-n/2} (n/2)^{(n-j)/2-1}}{\sqrt{2\pi} e^{-n/2} (n/2)^{n/2-1}} (1 - R_j^2)^{-(n-j-1)/2+a+1} \\ &\approx \left(\frac{n}{2}\right)^{-j/2} (1 - R_j^2)^{-n/2} \approx \exp\left[-\frac{j}{2} \log n - \frac{n}{2} \log(1 - R_j^2)\right]. \end{aligned}$$

This completed the proof. □

We now investigate the model selection consistency of Bayes factor in (3.2) under the three different asymptotic scenarios mentioned above. For simplicity of notation, let c_i represent a finite constant for $i = 1, 2, \dots, 5$ throughout the following proofs. When $(j/2 + a + 1)$ and $(n - j - 1)/2$ are sufficiently large, it follows directly from Stirling's formula that

$$\Gamma\left(\frac{j}{2} + a + 1\right) \approx \sqrt{2\pi} e^{-j/2} \left(\frac{j}{2}\right)^{j/2+a+1/2}$$

and

$$\Gamma\left(\frac{n - j - 1}{2}\right) \approx \sqrt{2\pi} e^{-(n-j)/2} \left(\frac{n - j}{2}\right)^{(n-j)/2-1}.$$

Proof of Theorem 3. Under Scenario 1, $i = O(n^{a_1})$ and $j = O(n^{a_2})$ with $0 \leq a_1 \leq a_2 < 1$, by using the two approximation equations above, it follows that

$$\begin{aligned} \text{BF}[M_j : M_i] &= \frac{\Gamma(j/2 + a + 1)\Gamma((n - j - 1)/2)}{\Gamma(i/2 + a + 1)\Gamma((n - i - 1)/2)} \frac{(1 - R_j^2)^{-(n-j-1)/2+a+1}}{(1 - R_i^2)^{-(n-i-1)/2+a+1}} \\ &= c_1 \frac{j^{j/2+a+1}(n - j)^{(n-j)/2+1}}{i^{i/2+a+1}(n - i)^{(n-i)/2+1}} \frac{(1 - R_j^2)^{-(n-j)/2}}{(1 - R_i^2)^{-(n-i)/2}} \tag{A.2} \\ &= c_1 \frac{(j/n)^{j/2}}{(i/n)^{i/2}} \left(\frac{j}{i}\right)^{a+1} \left(\frac{1 - j/n}{1 - i/n}\right) \left[\frac{(1 - j/n)^{1-j/n}}{(1 - i/n)^{1-i/n}} \frac{(1 - R_j^2)^{-(1-j/n)}}{(1 - R_i^2)^{-(1-i/n)}}\right]^{n/2}. \end{aligned}$$

(a) We first show the Bayes factor consistency when the true model is M_i . As n tends to infinity, we observe that the dominated term in brackets of equation (A.2) can be approximated by

$$\frac{(1 - j/n)^{1-j/n}}{(1 - i/n)^{1-i/n}} \frac{(1 - R_j^2)^{-(1-j/n)}}{(1 - R_i^2)^{-(1-i/n)}} \approx \left(\frac{1 - R_j^2}{1 - R_i^2}\right)^{-1},$$

because of j/n and i/n approaching to zero as n approaches infinity. From Lemma 1(a) and the fact that $\delta_{ii} = 0$, we observe that under M_i , it follows

$$\begin{aligned} \text{BF}[M_j : M_i] &= c_2 \frac{(j/n)^{j/2}}{(i/n)^{i/2}} \left(\frac{j}{i}\right)^{a+1} \left(\frac{1 - j/n}{1 - i/n}\right) \left(\frac{1 + \delta_{ij}}{1 + \delta_{ii}}\right)^{-n/2} \\ &= c_2 \frac{(j/n)^{j/2}}{(i/n)^{i/2}} \left(\frac{j}{i}\right)^{a+1} \left(\frac{1 - j/n}{1 - i/n}\right) (1 + \delta_{ij})^{-n/2}, \end{aligned}$$

which approaches zero as $\delta_{ij} > 0$, indicating that the Bayes factor in (3.2) is consistent when M_i is true.

(b) Consistency under M_j is provided as follows. By using Lemma 1(a), it follows that under model M_j , the Bayes factor in (3.2) can be further approximated by

$$\begin{aligned} \text{BF}[M_j : M_i] &= c_3 \frac{(j/n)^{j/2}}{(i/n)^{i/2}} \left(\frac{j}{i}\right)^{a+1} \left(\frac{1-j/n}{1-i/n}\right) \left(\frac{1+\delta_{jj}}{1+\delta_{ji}}\right)^{-n/2} \\ &= c_3 \frac{(j/n)^{j/2}}{(i/n)^{i/2}} \left(\frac{j}{i}\right)^{a+1} \left(\frac{1-j/n}{1-i/n}\right) (1+\delta_{ji})^{n/2}, \end{aligned}$$

because $\delta_{jj} = 0$. It should be noted that as n tends to infinity, the fifth dominated term approaches infinity if $\delta_{ji} > 0$. Therefore, the Bayes factor also approaches infinity as $\delta_{ji} > 0$, proving the consistency under M_j . This completes the proof the theorem. \square

Proof of Theorem 4. Under Scenario 2, $i = O(n^{a_1})$ and $j = O(n^{a_2})$ with $0 \leq a_1 < a_2 = 1$, by using the two approximation equations above, it follows that

$$\begin{aligned} \text{BF}[M_j : M_i] &= \frac{\Gamma(j/2 + a + 1)\Gamma((n - j - 1)/2)}{\Gamma(i/2 + a + 1)\Gamma((n - i - 1)/2)} \frac{(1 - R_j^2)^{-(n-j-1)/2+a+1}}{(1 - R_i^2)^{-(n-i-1)/2+a+1}} \\ &= c_4 \frac{(j/i)^{a+1}}{(i/n)^{i/2}} \left(\frac{1-j/n}{1-i/n}\right) \\ &\quad \times \left[\left(\frac{j}{n}\right)^{j/n} \frac{(1-j/n)^{1-j/n}}{(1-i/n)^{1-i/n}} \frac{(1 - R_j^2)^{-(1-j/n)}}{(1 - R_i^2)^{-(1-i/n)}} \right]^{n/2}. \end{aligned} \tag{A.3}$$

(a) If the true model is M_i , from Lemma 1(b) and the fact that $\delta_{ii} = 0$, we observe that the dominated term in brackets of (A.3) can be approximated by

$$\begin{aligned} &\left(\frac{j}{n}\right)^{j/n} \frac{(1-j/n)^{1-j/n}}{(1-i/n)^{1-i/n}} \frac{(1 - R_j^2)^{-(1-j/n)}}{(1 - R_i^2)^{-(1-i/n)}} \\ &\approx \left(\frac{1}{r}\right)^{1/r} \left(1 - \frac{1}{r}\right)^{1-1/r} \left(\frac{1 - R_j^2}{1 - R_i^2}\right)^{-(1-1/r)} (1 - R_i^2)^{1/r} \\ &\approx \left(\frac{1}{r}\right)^{1/r} \left(\frac{1 - 1/r}{1 - 1/r + \delta_{ij}}\right)^{1-1/r} \left(\frac{1}{1 + \delta_{i0}}\right)^{1/r}. \end{aligned}$$

Accordingly, the approximation of Bayes factor in (3.2) is given by

$$\text{BF}[M_j : M_i] \approx c_4 \frac{(j/i)^{a+1}}{(i/n)^{i/2}} \left[\left(\frac{1}{r}\right)^{1/r} \left(\frac{1 - 1/r}{1 - 1/r + \delta_{ij}}\right)^{1-1/r} \left(\frac{1}{1 + \delta_{i0}}\right)^{1/r} \right]^{n/2},$$

which approaches zero as n tends to infinity, and therefore, the consistency under M_i is proved.

(b) If the true model is M_j , from Lemma 1(b) and the fact that $\delta_{jj} = 0$, we observe that the dominated term in brackets of (A.3) can be approximated by

$$\begin{aligned} & \left(\frac{j}{n}\right)^{j/n} \frac{(1-j/n)^{1-j/n} (1-R_j^2)^{-(1-j/n)}}{(1-i/n)^{1-i/n} (1-R_i^2)^{-(1-i/n)}} \\ & \approx \left(\frac{1}{r}\right)^{1/r} \left(1-\frac{1}{r}\right)^{1-1/r} \left(\frac{1-R_j^2}{1-R_i^2}\right)^{-1} (1-R_j^2)^{1/r} \\ & \approx \left(\frac{1}{r}\right)^{1/r} \left(1-\frac{1}{r}\right)^{1-1/r} \left(\frac{1-1/r}{1+\delta_{ji}}\right)^{-1} \left(\frac{1-1/r}{1+\delta_{j0}}\right)^{1/r} \\ & \approx \left(\frac{1}{r}\right)^{1/r} (1+\delta_{ji}) \left(\frac{1}{1+\delta_{j0}}\right)^{1/r}. \end{aligned}$$

Therefore, the Bayes factor in (3.2) under M_j turns out to be

$$\text{BF}[M_j : M_i] = c_5 \frac{(j/i)^{a+1}}{(i/n)^{i/2}} \left[\left(\frac{1}{r}\right)^{1/r} (1+\delta_{ji}) \left(\frac{1}{1+\delta_{j0}}\right)^{1/r} \right]^{n/2}. \tag{A.4}$$

To show the consistency under M_j , it is sufficient to show that the dominated term in brackets of (A.4) is strictly larger than one when n tends to infinity. This is equivalent to

$$\left(\frac{1}{r}\right)^{1/r} (1+\delta_{ji}) \left(\frac{1}{1+\delta_{j0}}\right)^{1/r} > 1,$$

which gives that

$$\delta_{ji} > [r(1+\delta_{j0})]^{1/r} - 1.$$

On the other hand, we have $\delta_{ji} \leq \delta_{j0}$, which provides that

$$\delta_{j0} \geq \delta_{ji} > [r(1+\delta_{j0})]^{1/r} - 1,$$

indicating that

$$\delta_{j0} > r^{1/(r-1)} - 1 = \delta(r).$$

In order for the interval where the distance δ_{ji} should lie

$$\delta_{ji} \in ([r(1+\delta_{j0})]^{1/r} - 1, \delta_{j0}]$$

to be nonempty, a necessary and sufficient condition is that $\delta_{j0} > \delta(r)$. This completes the proof. □

Proof of Theorem 5. Under Scenario 3, $i = O(n^{a_1})$ and $j = O(n^{a_2})$ with $a_1 = a_2 = 1$, by using the two approximations equations, it follows that

$$\begin{aligned} \text{BF}[M_j : M_i] &= \frac{\Gamma(j/2 + a + 1)\Gamma((n - j - 1)/2) (1 - R_j^2)^{-(n-j-1)/2+a+1}}{\Gamma(i/2 + a + 1)\Gamma((n - i - 1)/2) (1 - R_i^2)^{-(n-i-1)/2+a+1}} \\ &= c_5 \left(\frac{j}{i}\right)^{a+1} \left(\frac{1 - j/n}{1 - i/n}\right) \\ &\quad \times \left[\frac{(j/n)^{j/n} (1 - j/n)^{1-j/n} (1 - R_j^2)^{-(1-j/n)}}{(i/n)^{i/n} (1 - i/n)^{1-i/n} (1 - R_i^2)^{-(1-i/n)}} \right]^{n/2}. \end{aligned} \tag{A.5}$$

(a) If the true model is M_i , from Lemma 1(c) and the fact that $\delta_{ii} = 0$, we observe that the dominated term in brackets of (A.5) can be approximated by

$$\begin{aligned} &\frac{(j/n)^{j/n} (1 - j/n)^{1-j/n} (1 - R_j^2)^{-(1-j/n)}}{(i/n)^{i/n} (1 - i/n)^{1-i/n} (1 - R_i^2)^{-(1-i/n)}} \\ &\approx \frac{(1/r)^{1/r} (1 - 1/r)^{1-1/r} \left(\frac{1 - R_j^2}{1 - R_i^2}\right)^{-(1-1/r)}}{(1/s)^{1/s} (1 - 1/s)^{1-1/s}} (1 - R_i^2)^{1/r-1/s} \\ &\approx \frac{(1/r)^{1/r} (1 - 1/r)^{1-1/r} \left(\frac{1 + \delta_{ij} - 1/r}{1 - 1/s}\right)^{-(1-1/r)} \left(\frac{1 - 1/s}{1 + \delta_{i0}}\right)^{1/r-1/s}}{(1/s)^{1/s} (1 - 1/s)^{1-1/s}} \\ &\approx \frac{(1/r)^{1/r} [1 + \delta_{ij}/(1 - 1/r)]^{-(1-1/r)}}{(1/s)^{1/s} (1 + \delta_{i0})^{1/r-1/s}}. \end{aligned} \tag{A.6}$$

For the Bayes factor to be consistent, it is sufficient to show that the dominated term in (A.6) is strictly less than 1 as n approaches infinity. This is equivalent to

$$\left(1 + \frac{\delta_{ij}}{1 - 1/r}\right)^{1-1/r} > \frac{(1/r)^{1/r}}{(1/s)^{1/s}} (1 + \delta_{i0})^{1/s-1/r},$$

which implies that

$$\delta_{ij} > \frac{r - 1}{r} \left\{ \left[\frac{s^{1/s}}{r^{1/r}} (1 + \delta_{i0})^{1/s-1/r} \right]^{r/(r-1)} - 1 \right\}.$$

In addition, from the property of the pseudo-distance, we have $\delta_{i0} \geq \delta_{ij}$. Therefore, it follows that

$$\delta_{i0} \geq \delta_{ij} > \frac{r - 1}{r} \left\{ \left[\frac{s^{1/s}}{r^{1/r}} (1 + \delta_{i0})^{1/s-1/r} \right]^{r/(r-1)} - 1 \right\},$$

indicating that the value of δ_{ij} must satisfy

$$\left(1 + \frac{\delta_{i0}}{1 - 1/r}\right)^{1-1/r} > \frac{(1/r)^{1/r}}{(1/s)^{1/s}}(1 + \delta_{i0})^{1/s-1/r}.$$

Under the conditions stated in the theorem, we take limits and obtain that the Bayes factor tends to zero, and thus, the Bayes factor is consistent under M_i .

(b) If the true model is M_j , from Lemma 1(c) and the fact that $\delta_{jj} = 0$, we observe that the dominated term in brackets of (A.5) can be approximated by

$$\begin{aligned} & \frac{(j/n)^{j/n} (1 - j/n)^{1-j/n} (1 - R_j^2)^{-(1-j/n)}}{(i/n)^{i/n} (1 - i/n)^{1-i/n} (1 - R_i^2)^{-(1-i/n)}} \\ & \approx \frac{(1/r)^{1/r} (1 - 1/r)^{1-1/r}}{(1/s)^{1/s} (1 - 1/s)^{1-1/s}} \left(\frac{1 - R_j^2}{1 - R_i^2}\right)^{-(1-1/s)} (1 - R_j^2)^{1/r-1/s} \\ & \approx \frac{(1/r)^{1/r} (1 - 1/r)^{1-1/r}}{(1/s)^{1/s} (1 - 1/s)^{1-1/s}} \left(\frac{1 - 1/r}{1 + \delta_{ji} - 1/s}\right)^{-(1-1/s)} \left(\frac{1 - 1/r}{1 + \delta_{j0}}\right)^{1/r-1/s} \\ & \approx \frac{(1/r)^{1/r} [1 + \delta_{ji}/(1 - 1/s)]^{1-1/s}}{(1/s)^{1/s} (1 + \delta_{j0})^{1/r-1/s}}. \end{aligned} \tag{A.7}$$

For the Bayes factor to be consistent, it is sufficient to show that the dominated term in (A.7) is strictly larger than one as n approaches infinity. This is equivalent to

$$\frac{(1/r)^{1/r} [1 + \delta_{ji}/(1 - 1/s)]^{1-1/s}}{(1/s)^{1/s} (1 + \delta_{j0})^{1/r-1/s}} > 1.$$

Simple algebra shows that

$$\delta_{ji} > \frac{s - 1}{s} \left\{ \left[\frac{r^{1/r}}{s^{1/s}} (1 + \delta_{j0})^{1/r-1/s} \right]^{s/(s-1)} - 1 \right\}.$$

On the other hand, we also have $\delta_{j0} \geq \delta_{ji}$, which provides that

$$\delta_{j0} \geq \delta_{ji} > \frac{s - 1}{s} \left\{ \left[\frac{r^{1/r}}{s^{1/s}} (1 + \delta_{j0})^{1/r-1/s} \right]^{s/(s-1)} - 1 \right\}, \tag{A.8}$$

indicating that

$$\left(1 + \frac{\delta_{j0}}{1 - 1/s}\right)^{1-1/s} > \frac{r^{1/r}}{s^{1/s}} (1 + \delta_{j0})^{1/r-1/s}.$$

In order for the interval where the distance δ_{ji} should lie

$$\delta_{ji} \in \left(\frac{s - 1}{s} \left[\frac{r^{1/r}}{s^{1/s}} (1 + \delta_{j0})^{1/r-1/s} - 1 \right]^{s/(s-1)}, \delta_{j0} \right]$$

to be nonempty, a necessary and sufficient condition is that δ_{j_0} satisfies inequality in (A.8). This completes the proof. \square

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References

- [1] Bayarri, M.J., Berger, J.O., Forte, A. and García-Donato, G. (2012). Criteria for Bayesian model choice with application to variable selection. *Ann. Statist.* **40** 1550–1577. [MR3015035](#)
- [2] Berger, J.O., Ghosh, J.K. and Mukhopadhyay, N. (2003). Approximations and consistency of Bayes factors as model dimension grows. *J. Statist. Plann. Inference* **112** 241–258. [MR1961733](#)
- [3] Casella, G., Girón, F.J., Martínez, M.L. and Moreno, E. (2009). Consistency of Bayesian procedures for variable selection. *Ann. Statist.* **37** 1207–1228. [MR2509072](#)
- [4] Cox, D.R. (1962). Further results on tests of separate families of hypotheses. *J. Roy. Statist. Soc. Ser. B* **24** 406–424. [MR0156409](#)
- [5] Fernández, C., Ley, E. and Steel, M.F.J. (2001). Benchmark priors for Bayesian model averaging. *J. Econometrics* **100** 381–427. [MR1820410](#)
- [6] Fujikoshi, Y. (1993). Two-way ANOVA models with unbalanced data. *Discrete Math.* **116** 315–334. [MR1222130](#)
- [7] Girón, F.J., Martínez, M.L., Moreno, E. and Torres, F. (2006). Objective testing procedures in linear models: Calibration of the p -values. *Scand. J. Stat.* **33** 765–784. [MR2300915](#)
- [8] Girón, F.J., Moreno, E., Casella, G. and Martínez, M.L. (2010). Consistency of objective Bayes factors for nonnested linear models and increasing model dimension. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM* **104** 57–67. [MR2666441](#)
- [9] Gustafson, P., Hossain, S. and MacNab, Y.C. (2006). Conservative prior distributions for variance parameters in hierarchical models. *Canad. J. Statist.* **34** 377–390. [MR2328550](#)
- [10] Hoel, P.G. (1947). On the choice of forecasting formulas. *J. Amer. Statist. Assoc.* **42** 605–611.
- [11] Kass, R.E. and Raftery, A.E. (1995). Bayes factors. *J. Amer. Statist. Assoc.* **90** 773–795.
- [12] Kass, R.E. and Vaidyanathan, S.K. (1992). Approximate Bayes factors and orthogonal parameters, with application to testing equality of two binomial proportions. *J. Roy. Statist. Soc. Ser. B* **54** 129–144. [MR1157716](#)
- [13] Ley, E. and Steel, M.F.J. (2012). Mixtures of g -priors for Bayesian model averaging with economic applications. *J. Econometrics* **171** 251–266. [MR2991863](#)
- [14] Liang, F., Paulo, R., Molina, G., Clyde, M.A. and Berger, J.O. (2008). Mixtures of g priors for Bayesian variable selection. *J. Amer. Statist. Assoc.* **103** 410–423. [MR2420243](#)
- [15] Maruyama, Y. (2013). A Bayes factor with reasonable model selection consistency for ANOVA model. Available at [arXiv:0906.4329v2](#) [stat.ME].
- [16] Maruyama, Y. and George, E.I. (2011). Fully Bayes factors with a generalized g -prior. *Ann. Statist.* **39** 2740–2765. [MR2906885](#)
- [17] Moreno, E. and Girón, F.J. (2008). Comparison of Bayesian objective procedures for variable selection in linear regression. *TEST* **17** 472–490. [MR2470092](#)
- [18] Moreno, E., Girón, F.J. and Casella, G. (2010). Consistency of objective Bayes factors as the model dimension grows. *Ann. Statist.* **38** 1937–1952. [MR2676879](#)

- [19] Moreno, E., Girón, F.J. and Casella, G. (2014). Posterior model consistency in variable selection as the model dimension grows. Preprint.
- [20] Pesaran, M.H. and Weeks, M. (1999). Non-nested hypothesis testing: An overview. Cambridge Working Papers in Economics 9918, Faculty of Economics, Univ. of Cambridge.
- [21] Wang, M. and Sun, X. (2013). Bayes factor consistency for unbalanced ANOVA models. *Statistics* **47** 1104–1115. [MR3175737](#)
- [22] Wang, M. and Sun, X. (2014). Bayes factor consistency for nested linear models with a growing number of parameters. *J. Statist. Plann. Inference* **147** 95–105. [MR3151848](#)
- [23] Wang, M., Sun, X. and Lu, T. (2015). Bayesian structured variable selection in linear regression models. *Comput. Statist.* **30** 205–229. [MR3334718](#)
- [24] Watnik, M., Johnson, W. and Bedrick, E.J. (2001). Nonnested linear model selection revisited. *Comm. Statist. Theory Methods* **30** 1–20. [MR1862585](#)
- [25] Watnik, M.R. and Johnson, W.O. (2002). The behaviour of linear model selection tests under globally non-nested hypotheses. *Sankhyā Ser. A* **64** 109–138. [MR1968378](#)
- [26] Wetzels, R., Grasman, R.P.P.P. and Wagenmakers, E.-J. (2012). A default Bayesian hypothesis test for ANOVA designs. *Amer. Statist.* **66** 104–111. [MR2968006](#)
- [27] Zellner, A. (1986). On assessing prior distributions and Bayesian regression analysis with g -prior distributions. In *Bayesian Inference and Decision Techniques. Stud. Bayesian Econometrics Statist.* **6** 233–243. Amsterdam: North-Holland. [MR0881437](#)

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