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BIHARMONIC ORBITS OF ISOTROPY REPRESENTATIONS OF SYMMETRIC SPACES

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Abstract

In this paper, we give a necessarly and sufficient condition for orbits of linear isotropy representations of Riemannian symmetric spaces are biharmonic submanifolds in hyperspheres in Euclidean spaces. In particular, we obtain examples of biharmonic submanifolds in hyperspheres whose co-dimension is greater than one.

1. Introduction

J. Eells and L. Lemaire ([6]) introduced the notion of biharmonic map as a generalization of the notion of harmonic map. For a smooth map φ from a Riemannian manifold (M, g) into another Riemannian manifold (N, h), φ is said to be harmonic if it is a critical point of the energy functional defined by

$$E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 \, d\mu_g.$$

The Euler-Lagrange equation is given by the vanishing of the tension field τ_φ. Harmonic maps are studied by many mathematicians (cf. [5], [7], [9]).

The biharmonic maps, which is a generalization of the harmonic map, is defined as a critical point of bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M \|\tau_\varphi\|^2 \, d\mu_g.$$

Similar to harmonic maps, biharmonic maps are characterized by the Euler-Lagrange equation $\tau_{2,\varphi} = 0$ where $\tau_{2,\varphi}$ is the bitension field of φ. It is known that the equation $\tau_{2,\varphi} = 0$ is a fourth order partial differential equation. By definition, harmonic maps are biharmonic maps.

On the other hand, a biharmonic map is not necessary harmonic. The B. Y. Chen's conjecture is to ask whether every biharmonic submanifold of the Euclidean space $\mathbf{R}^n$ must be harmonic, i.e., minimal ([4]). It was partially solved positively. For example, K. Akutagawa and Sh. Maeta showed ([1]) that every

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complete properly immersed biharmonic submanifold in the Euclidean space $\mathbf{R}^n$ must be minimal. Furthermore, it is known (cf. [14], [15], [16]) that every biharmonic map of a complete Riemannian manifold into another Riemannian manifold of non-positive sectional curvature with finite energy and finite bienergy must be harmonic.

On the contrary, for the target Riemannian manifold (N, h) of non-negative sectional curvature, there exist examples of biharmonic submanifolds which are not harmonic. A biharmonic submanifold is called proper if it is not harmonic. T. Ichiyama, J. Inoguchi and H. Urakawa ([10]) classified homogeneous hypersurfaces which are proper biharmonic in the hypersphere in Euclidean spaces. More generally, biharmonic homogeneous hypersurfaces in compact symmetric spaces are studied in [17] and [11]. Furthermore, S. Ohno, T. Sakai and H. Urakawa construct higher co-dimensional biharmonic submanifolds in compact symmetric spaces as orbits of Hermann actions which are generalizations of isotropy actions of compact symmetric spaces ([18]). However, since the rank of hyperspheres are one, the cohomogeneity of Hermann actions on hyperspheres are one. Therefore, in orbits of Hermann actions on hyperspheres, there is no proper biharmonic submanifolds whose co-dimension is greater than one.

A. Blamuş, S. Montaldo and C. Oniciuc give new examples of proper biharmonic submanifolds in spheres and classification of biharmonic submanifolds which are the direct products of some spheres in the unit sphere in [2] and [3].

In this paper, using root systems, we describe a necessary and sufficient condition for an orbit of the linear isotropy representation of a Riemannian symmetric space to be biharmonic in the hypersphere, and give examples of proper biharmonic submanifolds in the hypersphere whose co-dimension is greater than one.

The organization of this paper is as follows. In Section 2, we recall the foundation for the following sections. In 2.1, we describe biharmonic isometric immersions. In particular, we explain that for an isometric immersion whose tension field is parallel, the biharmonic property is characterized by a condition of the second fundamental form of the isometric immersion (Theorem 2.4). In 2.2, we examine the linear isotropy representations of Riemannian symmetric spaces. We state that the second fundamental form of an orbit of the linear isotropy representation of a Riemanniam symmetric space is described by the root system of the Riemannian symmetric space. Moreover, we show the tension field of an orbit of the linear isotropy representation of a Riemanniam symmetric space is parallel with respect to the normal connection. In Section 3, we state and prove our main theorem (Theorem 3.1) and give new examples of proper biharmonic submanifolds of hyperspheres.

2. Preliminaries

2.1. Biharmonic isometric immersions. In this section, we describe biharmonic isometric immersions. Let (M, g) and (N, h) be Riemannian manifolds, and φ be a smooth map from M into N. We denote by ∇, ∇^h the Levi-Civita connections on TM, TN of (M,g), (N,h), and by $\overline{\nabla}$ the induced connection on $\varphi^{-1}TN$ respectively. Let B_φ denotes the second fundamental form of φ, i.e.

$$B_\varphi(X,Y) = \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y)$$

for $X, Y \in \mathfrak{X}(M)$. For $p \in M$, $B_\varphi(X,Y)_p$ depends only on the vectors X_p, $Y_p \in T_pM$. Then we define the tension field τ_φ of φ by

$$(\tau_\varphi)_p = \sum_{i=1}^{m} B_\varphi(e_i, e_i)_p \quad (p \in M)$$

where $\{e_i\}_{i=1}^m$ is an orthonormal basis of T_pM. Then the tension field τ_φ is a smooth section on $\varphi^{-1}TN$.

Definition 2.1. A smooth map φ is called harmonic if $\tau_\varphi = 0$. If a harmonic map φ is an isometric embedding, then the image $\varphi(M) \subset N$ is called a harmonic submanifold.

Remark 2.2. When a smooth map φ is an isometric immersion, the definition of τ_φ coincides with the definition of mean curvature vector field of φ. Then, a harmonic map is a minimal immersion, and a harmonic submanifold is a minimal submanifold.

There are articles whose mean curvature vector field is defined by dividing the trace of the second fundamental form by the dimension of the submanifold. The reference [12] is one of them. Even if either definition is adopted, since the mean curvature vector field coincides with the exception of the difference in the scalar multiplication, the definition of the minimality does not change.

To define the notion of biharmonic maps, we define the Jacobi operator J. For $V \in \Gamma(\varphi^{-1}TN)$

$$J(V) := \overline{\Delta}V - \mathscr{R}(V),$$

where $\overline{\Delta}V = \overline{\nabla}^*\overline{\nabla}V = \sum_{i=1}^m \{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}V - \overline{\nabla}_{\nabla_{e_i}e_i}V\}$, $\mathscr{R}(V) = \sum_{i=1}^m R^h(V, d\varphi(e_i)) \cdot d\varphi(e_i)$. Here R^h is the curvature tensor field of N. Then we set

$$\tau_{2,\varphi} = J(\tau_\varphi).$$

The vector field $\tau_{2,\varphi}$ is called a bitention field of φ.

Definition 2.3. A smooth map φ is called biharmonic if $\tau_{2,\varphi} = 0$. If a biharmonic map φ is an isometric embedding, then the image $\varphi(M) \subset N$ is called a biharmonic submanifold.

Then we have the following theorem.

Theorem 2.4 ([17]). *Let $\varphi : M \to N$ be a isometric immersion which satisfies that $\nabla_X^\perp \tau_\varphi = 0$ for all $X \in \mathfrak{X}(M)$. Here $\nabla^\perp$ is the normal connection of φ. Then*

φ is biharmonic if and only if for any $p \in M$,

$$\sum_{i=1}^{m} R^h((\tau_\varphi)_p, d\varphi(e_i))\, d\varphi(e_i) = \sum_{i,j=1}^{m} h((\tau_\varphi)_p, B_\varphi(e_i, e_j)_p) B_\varphi(e_i, e_j)_p \tag{2.1}$$

holds, where $\{e_i\}_{i=1}^m$ is an orthonormal basis of T_pM.

Remark 2.5. The condition (2.1) is equivalent to the following equation,

$$\sum_{i=1}^{m} R^h(\tau_\varphi, d\varphi(e_i))\, d\varphi(e_i) = \sum_{i=1}^{m} B_\varphi(A_{\tau_\varphi} e_i, e_i)). \tag{2.2}$$

Here A_{τ_φ} is the shape operator of φ with respect to τ_φ. It holds $g(A_{\tau_\varphi}X, Y) = h(B_\varphi(X, Y), \tau_\varphi)$.

2.2. Compact symmetric pair and the second fundamental form of R-spaces in spheres. In this section, we express the second fundamental form of orbits of the linear isotropy representations of Riemannian symmetric spaces in hyperspheres in terms of root systems.

Let G be a compact connected semisimple Lie group and σ an involutive automorphism of G. We take a subgroup K of G which satisfies $\mathrm{Fix}(\sigma, G)_0 \subset K \subset \mathrm{Fix}(\sigma, G)$, where $\mathrm{Fix}(\sigma, G)$ is the subgroup of the fixed point set of σ and $\mathrm{Fix}(\sigma, G)_0$ is the identity component of $\mathrm{Fix}(\sigma, G)$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. The involutive automorphism of $\mathfrak{g}$ induced from σ will be also denoted by σ. Then, by definition of K, we have $\mathfrak{k} = \{X \in \mathfrak{g} \mid \sigma(X) = X\}$. Take an $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

is an orthogonal direct sum decomposition of $\mathfrak{g}$ where $\mathfrak{m} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$.

Let π denotes the natural projection from G onto the coset manifold G/K. The tangent space $T_{\pi(e)}G/K$ of G/K at the origin $\pi(e)$ is identified with $\mathfrak{m}$ in a natural way, where e is the identity element of G. Then the inner product $\langle \cdot, \cdot \rangle$ induces a G-invariant Riemannian metric on G/K. We denote the Riemannian metric on G/K by the same symbol $\langle \cdot, \cdot \rangle$. Then G/K is a compact Riemannian symmetric space with respect to $\langle \cdot, \cdot \rangle$.

The group G acts on G/K isometrically by $L_y(xK) := yxK$ $(x, y \in G)$. Thus the subgroup K acts on G/K isometrically, and the action is called the isotropy action of G/K. Since for any $k \in K$, the isometry L_k fixes $o := eK \in G/K$, the differential dL_k of L_k at o gives a linear transformation on T_oG/K. For each $k, k' \in K$, $L_k \circ L_{k'} = L_{kk'}$ holds. Thus, K has a representation on T_oG/K, and this representation on T_oG/K is called the linear isotropy representation of G/K.

On the other hand, the differential $\mathrm{Ad}(x)$ of an inner automorphism I_x at e is an automorphism on $\mathfrak{g}$ for $x \in G$, where $\mathrm{I}_x(y) = xyx^{-1}$ $(y \in G)$. Then we have

$$\mathrm{Ad}(k)\mathfrak{k} = \mathfrak{k}, \quad \mathrm{Ad}(k)\mathfrak{m} = \mathfrak{m} \tag{2.3}$$

for any $k \in K$. Therefore, K has a representation on $\mathfrak{m}$. It is well known that

$$(d\pi)_e(\mathrm{Ad}(k)X) = (dL_k)_o((d\pi)_e(X)) \quad (k \in K, X \in \mathfrak{m}).$$

Thus, the linear isotropy representation and the adjoint representation on $\mathfrak{m}$ are equivalent as an orthogonal representation. Hence we identify the linear isotropy representation and the adjoint representation on $\mathfrak{m}$. Hereafter we consider the representation K on $\mathfrak{m}$.

Take and fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$. Then it is known

$$\mathrm{Ad}(K)\mathfrak{a} = \mathfrak{m}.$$

Since

$$\langle \mathrm{Ad}(k)X, \mathrm{Ad}(k)Y \rangle = \langle X, Y \rangle \quad (X, Y \in \mathfrak{m})$$

holds for all $k \in K$, $\mathrm{Ad}(k)$ preserving the unit sphere S in $\mathfrak{m}$.

For each $H \in S$, the orbit $\mathrm{Ad}(K)H$ in S is a submanifold of S, and $\mathrm{Ad}(K)H$ is called an R-space. In particular, if $\mathrm{Ad}(K)H$ in S is a minimal submanifold, then it is called a minimal R-space.

We would like to examine a necessary and sufficient condition for an R-space $\mathrm{Ad}(K)H$ in S is a biharmonic submanifold.

In order to apply Theorem 2.4 to $\mathrm{Ad}(K)H$ in S, we calculate the second fundamental form of $\mathrm{Ad}(K)H$ in S, using the root system of G/K. We define subspaces of $\mathfrak{g}$ as follows:

$$\mathfrak{k}_0 = \{X \in \mathfrak{k} \,|\, [H', X] = 0 \ \ (H' \in \mathfrak{a})\},$$

for each $\lambda \in \mathfrak{a} \backslash \{0\}$,

$$\mathfrak{k}_\lambda = \{X \in \mathfrak{k} \,|\, [H', [H', X]] = -\langle \lambda, H' \rangle^2 X \ \ (H' \in \mathfrak{a})\},$$

$$\mathfrak{m}_\lambda = \{X \in \mathfrak{m} \,|\, [H', [H', X]] = -\langle \lambda, H' \rangle^2 X \ \ (H' \in \mathfrak{a})\}.$$

We set $\Sigma = \{\lambda \in \mathfrak{a} \backslash \{0\} \,|\, \mathfrak{k}_\lambda \neq \{0\}\}$ and $m(\lambda) = \dim \mathfrak{k}_\lambda$. The subset Σ in $\mathfrak{a}$ is called the root system of G/K (cf. [8]). Since $\mathfrak{k}_\lambda = \mathfrak{k}_{-\lambda}$, if $\lambda \in \Sigma$, then $-\lambda \in \Sigma$. Fix a basis of $\mathfrak{a}$ and define a lexicographic ordering $>$ on $\mathfrak{a}$ with respect to the basis of $\mathfrak{a}$, and set $\Sigma^+ = \{\lambda \in \Sigma \,|\, \lambda > 0\}$.

Hereafter, we assume $H \in \mathfrak{a} \cap S$. In order to compute the second fundamental form of $\mathrm{Ad}(K)H$ in S, we use the following lemma.

Lemma 2.6 ([8]). *For each $\lambda \in \Sigma^+$, there exist orthonormal bases $\{S_{\lambda,i}\}_{i=1}^{m(\lambda)}$ and $\{T_{\lambda,i}\}_{i=1}^{m(\lambda)}$ of $\mathfrak{k}_\lambda$ and $\mathfrak{m}_\lambda$ respectively such that for any $H' \in \mathfrak{a}$,*

$$[H', S_{\lambda,i}] = \langle \lambda, H' \rangle T_{\lambda,i}, \quad [H', T_{\lambda,i}] = -\langle \lambda, H' \rangle S_{\lambda,i}, \quad [S_{\lambda,i}, T_{\lambda,i}] = \lambda,$$

$$\mathrm{Ad}(\exp(H'))S_{\lambda,i} = \cos\langle \lambda, H' \rangle S_{\lambda,i} + \sin\langle \lambda, H' \rangle T_{\lambda,i}$$

$$\mathrm{Ad}(\exp(H'))T_{\lambda,i} = -\sin\langle \lambda, H' \rangle S_{\lambda,i} + \cos\langle \lambda, H' \rangle T_{\lambda,i}$$

holds.

By Lemma 2.6, we have the following direct sum decompositions:

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda = \mathfrak{k}_0 \oplus \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} \mathbf{R} \cdot S_{\lambda, i},$$

$$\mathfrak{m} = \mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{m}_\lambda = \mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} \mathbf{R} \cdot T_{\lambda, i}.$$

The tangent space $T_H(\mathrm{Ad}(K)H)$ and the normal space $T_H^\perp(\mathrm{Ad}(K)H)$ in S of $\mathrm{Ad}(K)H$ at the point $H \in \mathfrak{a} \cap S$ is given as

$$\begin{aligned} T_H(\mathrm{Ad}(K)H) &= \left\{ \frac{d}{dt} \mathrm{Ad}(\exp(tX))H \Big|_{t=0} \,\Big|\, X \in \mathfrak{k} \right\} = \{[X, H] \mid X \in \mathfrak{k}\} = [\mathfrak{k}, H] \\ &= \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} \mathbf{R} \cdot (\langle \lambda, H \rangle T_{\lambda, i}) = \sum_{\lambda \in \Sigma^+, \langle \lambda, H \rangle \neq 0} \sum_{i=1}^{m(\lambda)} \mathbf{R} \cdot T_{\lambda, i} \\ &= \sum_{\lambda \in \Sigma^+, \langle \lambda, H \rangle \neq 0} \mathfrak{m}_\lambda, \end{aligned}$$

$$T_H^\perp(\mathrm{Ad}(K)H) = \left(\mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+, \langle \lambda, H \rangle = 0} \mathfrak{m}_\lambda \right) \cap T_H S.$$

For $H \in \mathfrak{a} \cap S$, we set $\Sigma_H = \{\lambda \in \Sigma \mid \langle \lambda, H \rangle = 0\}$. Let X^T denotes the tangent vector in $T_H S = \{Y \in \mathfrak{m} \mid \langle Y, H \rangle = 0\}$ which is defined as

$$X^T = X - \langle X, H \rangle H$$

for $X \in \mathfrak{m}$. The vector X^T depends on $H \in \mathfrak{a} \cap S$.

Then we compute the covariant derivative of the orbit $\mathrm{Ad}(K)H$ in S. Let $\nabla^{\mathfrak{m}}$, ∇^S and ∇ denote the Levi-Civita connections of $\mathfrak{m}$, S and $\mathrm{Ad}(K)H$, respectively. For each $\lambda \in \Sigma^+ \backslash \Sigma_H$, $1 \leq i \leq m(\lambda)$, we define a vector field $(T_{\lambda,i})^*$ on $\mathfrak{m}$ by

$$\begin{aligned} (T_{\lambda,i})^*_X &= \frac{d}{dt} \mathrm{Ad}\left(\exp\left(-\frac{tS_{\lambda,i}}{\langle \lambda, H \rangle} \right) \right) X \Big|_{t=0} \\ &= -\frac{[S_{\lambda,i}, X]}{\langle \lambda, H \rangle} \quad (X \in \mathfrak{m}) \end{aligned}$$

Then $(T_{\lambda,i})^*_H = T_{\lambda,i}$ holds. Moreover, for each $X \in \mathrm{Ad}(K)H$ and $Y \in S$, $(T_{\lambda,i})^*_X \in T_X \mathrm{Ad}(K)H$ and $(T_{\lambda,i})^*_Y \in T_Y S$ holds. Hence $(T_{\lambda,i})^*$ gives a tangent vector field on $\mathrm{Ad}(K)H$ and a tangent vector field on S.

Using $(T_{\lambda,i})^*$, we compute the covariant derivative of $\mathrm{Ad}(K)H$. By the following lemma, it is sufficient to compute the covariant derivative on $\mathfrak{m}$.

Lemma 2.7 ([13]). *Let* $(N, \langle\,,\rangle)$ *be a Riemannian manifold and* M *be a submanifold of* N. *Let* ∇^N *and* ∇^M *denote the Levi-Civita connection of* N *and* M, *respectively. Then, we have*:

(1) $\nabla^N_X Y = \nabla^M_X Y + B(X, Y)$ $(X, Y \in \mathfrak{X}(M))$,

(2) $\nabla^N_X \xi = -A_\xi X + \nabla^\perp_X \xi$ $(X \in \mathfrak{X}(M), \xi \in \Gamma(T^\perp M))$.

Here B *and* A *denote second fundamental form on* $M \subset N$ *and shape operator of* $M \subset N$, *respectively.*

Moreover, we can compute the covariant derivative of $\mathfrak{m}$.

Proposition 2.8. *For each* $\lambda, \mu \in \Sigma^+ \backslash \Sigma_H$, $1 \leq i \leq m(\lambda)$, $1 \leq j \leq m(\mu)$, *we have*

$$(\nabla^{\mathfrak{m}}_{(T_{\lambda,i})^*}(T_{\mu,j})^*)_H = -\frac{[S_{\mu,j}, T_{\lambda,i}]}{\langle \mu, H\rangle}.$$

Proof. For $\lambda, \mu \in \Sigma^+ \backslash \Sigma_H$, $1 \leq i \leq m(\lambda)$, $1 \leq j \leq m(\mu)$, we set a smooth curve

$$c(t) = \mathrm{Ad}\left(\exp\left(-\frac{tS_{\lambda,i}}{\langle \lambda, H\rangle}\right)\right)H \tag{2.4}$$

in $\mathrm{Ad}(K)H$. Since $dc/dt(0) = (T_{\lambda,i})^*_H = T_{\lambda,i}$, we have

$$\begin{aligned}\frac{d}{dt}(T_{\mu,j})^*_{c(t)}\Big|_{t=0} &= \frac{d}{dt}\frac{-1}{\langle \mu, H\rangle}[S_{\mu,j}, c(t)]\Big|_{t=0} \\ &= -\frac{1}{\langle \mu, H\rangle}\left[S_{\mu,j}, -\frac{1}{\langle \lambda, H\rangle}[S_{\lambda,i}, H]\right] \\ &= -\frac{1}{\langle \mu, H\rangle}[S_{\mu,j}, T_{\lambda,i}]. \end{aligned}$$

□

By using Proposition 2.8 and Lemma 2.7, we can express the tension field τ_H of $\mathrm{Ad}(K)H$ in S. In [12], the mean curvature vector field calculated by using the lemma corresponding to Proposition 2.8. The result of [12] using the symbol in this paper is as follows.

Corollary 2.9 ([12]). *Let* $\widetilde{\tau_H}$ *be the tension field of* $\mathrm{Ad}(K)H$ *in* $\mathfrak{m}$. *Then,*

$$(\widetilde{\tau_H})_H = -\sum_{\lambda \in \Sigma^+ \backslash \Sigma_H} \frac{m(\lambda)}{\langle \lambda, H\rangle}\lambda \tag{2.5}$$

holds. In particular, $(\widetilde{\tau_H})_H \in \mathfrak{a}$ *holds.*

By the above corollary, we have the following.

Corollary 2.10. *Let* τ_H *be the tension field of* $\mathrm{Ad}(K)H$ *in* S. *Then,*

$$(\tau_H)_H = -\left(\sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \frac{m(\lambda)}{\langle \lambda, H\rangle}\lambda\right)^T \tag{2.6}$$

holds. In particular, $(\tau_H)_H \in \mathfrak{a}$ *holds.*

Proof. We can see that

$$\mathfrak{m} = T_H \,\mathrm{Ad}(K)H \oplus (T_H^\perp \,\mathrm{Ad}(K)H \cap T_H S) \oplus (T_H^\perp \,\mathrm{Ad}(K)H \cap T_H^\perp S).$$

We set $V = T_H^\perp \,\mathrm{Ad}(K)H \cap T_H S$. Since S is the unit sphere in $\mathfrak{m}$, we can apply Lemma 2.7. By applying Lemma 2.7 to $\mathrm{Ad}(K)H \subset S$, $\mathrm{Ad}(K)H \subset \mathfrak{m}$ and $S \subset \mathfrak{m}$, we can compute $(\tau_H)_H$ as follows:

$$\begin{aligned}
(\tau_H)_H &= \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \sum_{i=1}^{m(\lambda)} (\nabla^S_{T_{\lambda,i}} (T_{\lambda,i})^* - \nabla_{T_{\lambda,i}} (T_{\lambda,i})^*) \\
&= \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \sum_{i=1}^{m(\lambda)} (\nabla^S_{T_{\lambda,i}} (T_{\lambda,i})^*)_{V\text{-part}} \\
&= \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \sum_{i=1}^{m(\lambda)} \{(\nabla^{\mathfrak{m}}_{T_{\lambda,i}} (T_{\lambda,i})^*)_{T_H S\text{-part}}\}_{V\text{-part}} \\
&= \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \sum_{i=1}^{m(\lambda)} (\nabla^{\mathfrak{m}}_{T_{\lambda,i}} (T_{\lambda,i})^*)_{V\text{-part}} \\
&= ((\widetilde{\tau_H})_H)_{V\text{-part}} = (\widetilde{\tau_H})_H^T.
\end{aligned}$$

Therefore, by Corollary 2.9, we have the consequence. □

Since $(\widetilde{\tau_H})_H^T = (\tau_H)_H$ and

$$\langle (\widetilde{\tau_H})_H, H\rangle = -\sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \frac{m(\lambda)}{\langle \lambda, H\rangle}\langle \lambda, H\rangle = -\sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} m(\lambda) = -\dim(\mathrm{Ad}(K)H)$$

holds. Therefore, we have

$$(\widetilde{\tau_H})_H = (\tau_H)_H - \dim(\mathrm{Ad}(K)H)H. \tag{2.7}$$

Moreover, since $\mathrm{Ad}(K)H$ is homogeneous, we have

$$(\widetilde{\tau_H})_X = (\tau_H)_X - \dim(\mathrm{Ad}(K)H)X \quad (X \in \mathrm{Ad}(K)H). \tag{2.8}$$

Thus, $(\tau_H)_H = 0$ if and only if $(\widetilde{\tau_H})_H = -\dim(\mathrm{Ad}(K)H)H$. Y. Kitagawa and Y. Ohnita prove that $\widetilde{\tau_H}$ is parallel with respect to the normal connection of

$\mathrm{Ad}(K)H$ in $\mathfrak{m}$. Using this fact, we can prove the following lemma by a simple calculation.

LEMMA 2.11. *For any* $X \in T_H \mathrm{Ad}(K)H$,

$$\nabla^{\perp}_X \tau_H = 0 \tag{2.9}$$

holds. Here $\nabla^{\perp}$ *is the normal connection of* $\mathrm{Ad}(K)H$ *in* S.

The above lemma shows that the orbit $\mathrm{Ad}(K)H$ in S holds the assumption of Theorem 2.3.

3. Main theorem and examples

In this section, under the same condition in Section 2.2, we prove our main theorems (Theorems 3.1 and 3.5).

According to Corollary 2.10, the tension field τ_H of $\mathrm{Ad}(K)H$ in S can be calculated using the root system. Hence, Theorem 3.1 gives a necessary and sufficient condition for orbits of linear isotropy representations of Riemannian symmetric spaces are biharmonic submanifolds in unit spheres in terms of root systems.

THEOREM 3.1. *Let* $H \in \mathfrak{a} \cap S$. *Then*, $\mathrm{Ad}(K)H$ *is biharmonic in* S *if and only if*

$$\dim(\mathrm{Ad}(K)H)(\tau_H)_H = \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} m(\lambda) \frac{\langle \lambda, (\tau_H)_H \rangle}{\langle \lambda, H \rangle^2} (\lambda)^T. \tag{3.1}$$

Here, for $X \in \mathfrak{m}$, X^T *denotes the tangent vector in* $T_H S$ *defined as* $X^T = X - \langle X, H \rangle H$.

Proof. By Lemma 2.11, we can apply Theorem 2.4 to the orbit $\mathrm{Ad}(K)H$ in S. We compute both sides of the equation (2.2). If $(\tau_H)_H = 0$, then the equation (2.2) holds. Thus we suppose $(\tau_H)_H \neq 0$.

Let R denotes the curvature tensor of S. Since S is the unit sphere, we can easily calculate R (see [8]). In particular, for each orthonormal frame $\{X, Y\}$ in $T_H S$, $R(X, Y)Y = X$ holds. Thus, for each $\lambda \in \Sigma^+ \setminus \Sigma_H$, $1 \leq i \leq m(\lambda)$, we have

$$R((\tau_H)_H, T_{\lambda,i})T_{\lambda,i} = (\tau_H)_H.$$

Then we have

$$\begin{aligned} \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \sum_{i=1}^{m(\lambda)} R((\tau_H)_H, T_{\lambda,i})T_{\lambda,i} &= \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} m(\lambda)(\tau_H)_H \\ &= \dim(\mathrm{Ad}(K)H)(\tau_H)_H \end{aligned}$$

holds.

Let $B(\cdot,\cdot)$ denotes the second fundamental form of $\mathrm{Ad}(K)H$ in S. By Lemmas 2.7 and 2.8, for $\lambda,\mu\in\Sigma^+\backslash\Sigma_H$, $1\le i\le m(\lambda)$, $1\le j\le m(\mu)$,

$$\begin{aligned}\langle A_{(\tau_H)_H}T_{\lambda,i},T_{\mu,j}\rangle &= \langle(\tau_H)_H,B(T_{\lambda,i},T_{\mu,j})\rangle = \langle(\tau_H)_H,(\nabla^{\mathfrak{m}}_{(T_{\lambda,i})^*}(T_{\mu,j})^*)^T_H\rangle\\ &= \left\langle(\tau_H)_H,-\frac{1}{\langle\mu,H\rangle}[S_{\mu,j},T_{\lambda,i}]\right\rangle = -\frac{1}{\langle\mu,H\rangle}\langle(\tau_H)_H,[S_{\mu,j},T_{\lambda,i}]\rangle\\ &= -\frac{1}{\langle\mu,H\rangle}\langle-[S_{\mu,j},(\tau_H)_H],T_{\lambda,i}\rangle = \frac{1}{\langle\mu,H\rangle}\langle\langle\mu,(\tau_H)_H\rangle T_{\mu,j},T_{\lambda,i}\rangle\\ &= \frac{\langle\mu,(\tau_H)_H\rangle}{\langle\mu,H\rangle}\delta_{\mu,\lambda}\delta_{j,i}.\end{aligned}$$

Hence we obtain

$$A_{(\tau_H)_H}T_{\lambda,i} = \frac{\langle\lambda,(\tau_H)_H\rangle}{\langle\lambda,H\rangle}T_{\lambda,i}\quad(\lambda\in\Sigma^+\backslash\Sigma_H,\ 1\le i\le m(\lambda)).$$

Thus,

$$\begin{aligned}B(A_{(\tau_H)_H}T_{\lambda,i},T_{\lambda,i}) &= \frac{\langle\lambda,(\tau_H)_H\rangle}{\langle\lambda,H\rangle}B(T_{\lambda,i},T_{\lambda,i})\\ &= \frac{\langle\lambda,(\tau_H)_H\rangle}{\langle\lambda,H\rangle}\frac{[S_{\lambda,i},T_{\lambda,i}]^T}{\langle\lambda,H\rangle} = \frac{\langle\lambda,(\tau_H)_H\rangle}{\langle\lambda,H\rangle^2}\lambda^T.\end{aligned}$$

Therefore, we have the consequence. □

COROLLARY 3.2. *We set*

$$(T_{2,H})_H = 2\dim(\mathrm{Ad}(K)H)(\tau_H)_H - \sum_{\lambda\in\Sigma^+\backslash\Sigma_H}\left(m(\lambda)\frac{\langle\lambda,(\widetilde{\tau_H})_H\rangle}{\langle\lambda,H\rangle^2}(\lambda)^T\right),$$

$$(\widetilde{T_{2,H}})_H = 2\dim(\mathrm{Ad}(K)H)(\widetilde{\tau_H})_H - \sum_{\lambda\in\Sigma^+\backslash\Sigma_H}\left(m(\lambda)\frac{\langle\lambda,(\widetilde{\tau_H})_H\rangle}{\langle\lambda,H\rangle^2}\lambda\right)$$

Then, we have the following;

(1) *the orbit* $\mathrm{Ad}(K)H$ *in* S *is biharmonic if and only if* $(T_{2,H})_H = 0$.

(2) *the orbit* $\mathrm{Ad}(K)H$ *in* S *is biharmonic if and only if there exists some constant* $c\in\mathbf{R}$, $(\widetilde{T_{2,H}})_H = cH$ *holds.*

Proof. The equation (3.1) is equivalent to,

$$\begin{aligned}0 &= \dim(\mathrm{Ad}(K)H)(\tau_H)_H - \sum_{\lambda\in\Sigma^+\backslash\Sigma_H}m(\lambda)\frac{\langle\lambda,(\tau_H)_H\rangle}{\langle\lambda,H\rangle^2}(\lambda)^T\\ &= \dim(\mathrm{Ad}(K)H)(\tau_H)_H - \sum_{\lambda\in\Sigma^+\backslash\Sigma_H}m(\lambda)\left(\frac{\langle\lambda,(\widetilde{\tau_H})_H\rangle}{\langle\lambda,H\rangle^2} - \frac{\langle H,(\widetilde{\tau_H})_H\rangle}{\langle\lambda,H\rangle}\right)(\lambda)^T\end{aligned}$$

$$= \dim(\mathrm{Ad}(K)H)(\tau_H)_H - \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \left(m(\lambda) \frac{\langle \lambda, (\widetilde{\tau_H})_H \rangle}{\langle \lambda, H \rangle^2} (\lambda)^T \right) + \dim(\mathrm{Ad}(K)H)(\tau_H)_H$$

$$= 2 \dim(\mathrm{Ad}(K)H)(\tau_H)_H - \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \left(m(\lambda) \frac{\langle \lambda, (\widetilde{\tau_H})_H \rangle}{\langle \lambda, H \rangle^2} (\lambda)^T \right). \qquad \square$$

Remark 3.3. The vector $(T_{2,H})_H$ is not necessarily the bitension field of $\mathrm{Ad}(K)H$ in S, but the condition $(T_{2,H})_H = 0$ is a necessary and sufficient condition for $\mathrm{Ad}(K)H$ to be biharmonic in S.

Remark 3.4. Let $\Pi = \{\alpha_1, \dots \alpha_r\}$ be a set of simple roots of Σ where $r = \dim \mathfrak{a}$. For $1 \le i \le r$, we define $H_{\alpha_i} \in \mathfrak{a}$ by

$$\langle H_{\alpha_i}, \alpha_j \rangle = \delta_{i,j} \quad (1 \le j \le r).$$

Here, $\delta_{i,j}$ is the Kronecker delta. Since Π is a basis of $\mathfrak{a}$, $\{H_{\alpha_1}, \dots, H_r\}$ is also a basis of $\mathfrak{a}$. Using Π and $\{H_{\alpha_1}, \dots, H_{\alpha_r}\}$, we set an open subset $\mathscr{C}$ of $\mathfrak{a}$ as follows:

$$\mathscr{C} = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle > 0 \ \ (\alpha \in \Pi)\} = \left\{ \sum_{i=1}^r x_i H_{\alpha_i} \,\middle|\, x_i > 0 \right\}.$$

The closure $\overline{\mathscr{C}}$ of $\mathscr{C}$ is given as

$$\overline{\mathscr{C}} = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle \ge 0 \ \ (\alpha \in \Pi)\} = \left\{ \sum_{i=1}^r x_i H_{\alpha_i} \,\middle|\, x_i \ge 0 \right\}.$$

Then,

$$\mathrm{Ad}(K)\overline{\mathscr{C}} = \mathfrak{m} \tag{3.2}$$

holds.

For each subset $\Delta \subset \Pi$, we set

$$\mathscr{C}^\Delta = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle > 0, \ \langle \beta, H \rangle = 0 \ \ (\alpha \in \Delta, \ \beta \in \Pi \setminus \Delta)\}.$$

Then we have the cell decomposition of $\overline{\mathscr{C}}$

$$\overline{\mathscr{C}} = \bigcup_{\Delta \subset \Pi} \mathscr{C}^\Delta \quad \text{(disjoint union)}. \tag{3.3}$$

The set $\overline{\mathscr{C}}$ is the orbit space of the representation of $\mathrm{Ad}(K)$ on $\mathfrak{m}$. Moreover the cell decomposition (3.3) is a decomposition of orbits type of R-spaces.

Biharmonic orbits can be given by solving Equation (3.1) for H. However, it is difficult to solve this equation in general. In [9], by using a convex function on $\mathscr{C}^\Delta \cap S$ which satisfy $(\mathrm{grad}\, F)_H = (\tau_H)_H$, they show that there exists a unique $H \in \mathscr{C}^\Delta \cap S$ such that $(\tau_H)_H = 0$ as a critical point of the function. Even if such a function f on $\mathscr{C}^\Delta \cap S$ exists for $(T_{2,H})_H$, it is difficult to decide whether a critical point of f gives a proper biharmonic submanifold or a harmonic submanifold. Therefore we add some assumptions for Σ and H and discuss the equation $(T_{2,H})_H = 0$.

Hereafter, we consider the case where the root system Σ is reducible. This assumption means that the representation of K on $\mathfrak{m}$ is reducible. Thus, the orbit $\mathrm{Ad}(K)H$ is a direct product of some R-spaces.

Let us take (G_1, K_1) and (G_2, K_2) as symmetric pairs with connected semi-simple Lie groups G_1 and G_2. We consider the case of $(G, K) = (G_1 \times G_2, K_1 \times K_2)$ We write $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{m}_i$ for the decomposition of Lie algebra $\mathfrak{g}_i$ of G_i with respect to the symmetric pair (G_i, K_i) and fix a $\mathrm{Ad}(G_i)$-invariant inner product on $\mathfrak{m}_i$ for each $i = 1, 2$.

The unit spheres in $\mathfrak{m}_1$, that in $\mathfrak{m}_2$ that in $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ are denoted by S_1, S_2 and S, respectively. Note that the unit sphere S in $\mathfrak{m}$ is stable by the adjoint representation of $K_1 \times K_2$.

For $i = 1, 2$, fix a maximal abelian subspace $\mathfrak{m}_i$. The root system of (G_i, K_i) denoted by Σ_i. Then the root system Σ of (G, K) is decomposed as $\Sigma = \Sigma_1 \sqcup \Sigma_2$, For $\lambda \in \Sigma_1$, $\mu \in \Sigma_2$, $\langle \lambda, \mu \rangle = 0$ and $\Sigma = \Sigma_1 \cup \Sigma_2$ hold. We take $H_i \in S_i$ for $i = 1, 2$. Then we have

$$\dim \mathrm{Ad}(K)H_i = \sum_{\lambda \in \Sigma_i^+ \setminus \Sigma_H} m(\lambda).$$

Since $\mathrm{Ad}(K_1 \times \{e\})H_2 = \{H_2\}$ and $\mathrm{Ad}(\{e\} \times K_2)H_1 = \{H_1\}$, $\mathrm{Ad}(K)H_i$ is isometric to $\mathrm{Ad}(K_i)H_i \subset S_i$ for $i = 1, 2$.

For $\theta \in (0, \pi/2)$, we set $H = \cos\theta H_1 + \sin\theta H_2$. Then the tension field of the orbit $\mathrm{Ad}(K)H$ in S is given as

$$\begin{aligned}
(\widetilde{\tau_H})_H &= -\sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} m(\lambda) \frac{\lambda}{\langle \lambda, H \rangle} \\
&= -\left(\frac{1}{\cos\theta} \sum_{\lambda \in \Sigma_1^+ \setminus \Sigma_H} m(\lambda) \frac{\lambda}{\langle \lambda, H_1 \rangle} + \frac{1}{\sin\theta} \sum_{\mu \in \Sigma_2^+ \setminus \Sigma_H} m(\mu) \frac{\mu}{\langle \mu, H_2 \rangle} \right) \\
&= \frac{1}{\cos\theta} (\widetilde{\tau_{H_1}})_{H_1} + \frac{1}{\sin\theta} (\widetilde{\tau_{H_2}})_{H_2}.
\end{aligned}$$

The following theorem gives new examples of proper biharmonic submanifolds of the unit sphere which are direct products of two R-spaces.

THEOREM 3.5. *Let we take $H_1 \in S_1$ and $H_2 \in S_2$ satisfying that the R-spaces $\mathrm{Ad}(K)H_1$ and $\mathrm{Ad}(K)H_2$ are harmonic (or equivalently, minimal) in S_1 and S_2, respectively. The dimension of $\mathrm{Ad}(K)H_i$ denoted by n_i for $i = 1, 2$. For each $\theta \in (0, \pi/2)$, we set $H_\theta = \cos\theta H_1 + \sin\theta H_2$.*

(1) *The following two conditions on θ are equivalent:*

(a) *The R-space $\mathrm{Ad}(K)H_\theta$ is harmonic in S.*

(b) $\cos\theta = n_1/(n_1 + n_2)$.

(2) *The following two conditions on* θ *are equivalent*:

(a) *The* R-*space* $\mathrm{Ad}(K)H_\theta$ *is biharmonic in* S.

(b) $\cos\theta = n_1/(n_1+n_2)$ *or* $1/2$.

In particular, if $n_1 \neq n_2$, *then the* R-*space* $\mathrm{Ad}(K)H_{\pi/4}$ *is proper biharmonic in* S.

Proof. Since

$$(\widetilde{\tau_H})_H = -\left(\frac{n_1}{\cos\theta}H_1 + \frac{n_2}{\sin\theta}H_2\right),$$

$(\tau_H)_H = 0$ if and only if

$$-\left(\frac{n_1}{\cos\theta}H_1 + \frac{n_2}{\sin\theta}H_2\right) = -(n_1+n_2)(\cos\theta H_1 + \sin\theta H_2).$$

Thus we have

$$\begin{aligned}
0 &= \left\{\frac{n_1}{\cos\theta} - (n_1+n_2)\cos\theta\right\}H_1 + \left\{\frac{n_2}{\sin\theta} - (n_1+n_2)\sin\theta\right\}H_2 \\
&= \frac{1}{\cos\theta}\{n_1(\sin\theta)^2 - n_2(\cos\theta)^2\}H_1 + \frac{1}{\sin\theta}\{n_2(\cos\theta)^2 - n_1(\sin\theta)^2\}H_2.
\end{aligned}$$

The solution of the above equation is

$$(\cos\theta)^2 = \frac{n_1}{n_1+n_2}.$$

Then $(\sin\theta)^2 = n_2/(n_1+n_2)$ holds.

A necessary and sufficient condition for an orbit $\mathrm{Ad}(K)H \subset S$ to be biharmonic is there exists $c \in \mathbf{R}$, such that $(\widetilde{T_{2,H}})_H = cH$. To examine the condition $(\widetilde{T_{2,H}})_H = cH$, we compute $(\widetilde{T_{2,H}})_H$. Then we have

$$\begin{aligned}
(\widetilde{T_{2,H}})_H &= 2\dim(\mathrm{Ad}(K)H)(\widetilde{\tau_H})_H - \sum_{\lambda\in\Sigma^+\setminus\Sigma_H}\left(m(\lambda)\frac{\langle\lambda,(\widetilde{\tau_H})_H\rangle}{\langle\lambda,H\rangle^2}\lambda\right) \\
&= 2(n_1+n_2)(\widetilde{\tau_H})_H - \sum_{\lambda\in\Sigma_1^+\setminus\Sigma_H}\left(m(\lambda)\frac{-n_1}{(\cos\theta)^3}\frac{\langle\lambda,H_1\rangle}{\langle\lambda,H_1\rangle^2}\lambda\right) \\
&\quad - \sum_{\mu\in\Sigma_2^+\setminus\Sigma_H}\left(m(\mu)\frac{-n_2}{(\sin\theta)^3}\frac{\langle\mu,H_2\rangle}{\langle\mu,H_2\rangle^2}\mu\right) \\
&= 2(n_1+n_2)(\widetilde{\tau_H})_H - \frac{n_1}{(\cos\theta)^3}(\widetilde{\tau_{H_1}})_{H_1} - \frac{n_2}{(\sin\theta)^3}(\widetilde{\tau_{H_2}})_{H_2} \\
&= -2(n_1+n_2)\left(\frac{n_1}{\cos\theta}H_1 + \frac{n_2}{\sin\theta}H_2\right) + \frac{n_1^2}{(\cos\theta)^3}H_1 + \frac{n_2^2}{(\sin\theta)^3}H_2
\end{aligned}$$

$$= \frac{1}{\cos\theta}\left\{-2(n_1+n_2)n_1 + \frac{n_1^2}{(\cos\theta)^2}\right\}H_1 + \frac{1}{\sin\theta}\left\{-2(n_1+n_2)n_2 + \frac{n_2^2}{(\sin\theta)^2}\right\}H_2.$$

Since $H = \cos\theta H_1 + \sin\theta H_2$, a necessary and sufficient condition for an orbit $\mathrm{Ad}(K)H \subset S$ to be biharmonic is there exists $c \in \mathbf{R}$, such that

$$\left\{\begin{array}{l} \dfrac{1}{\cos\theta}\left\{-2(n_1+n_2)n_1 + \dfrac{n_1^2}{(\cos\theta)^2}\right\} = c\cos\theta \\ \dfrac{1}{\sin\theta}\left\{-2(n_1+n_2)n_2 + \dfrac{n_2^2}{(\sin\theta)^2}\right\} = c\sin\theta. \end{array}\right. \tag{3.4}$$

The above equation holds if and only if

$$\frac{1}{(\cos\theta)^2}\left\{-2(n_1+n_2)n_1 + \frac{n_1^2}{(\cos\theta)^2}\right\} - \frac{1}{(\sin\theta)^2}\left\{-2(n_1+n_2)n_2 + \frac{n_2^2}{(\sin\theta)^2}\right\} = 0 \tag{3.5}$$

holds. Then, we can calculate the left side of Equation (3.5).

$$\frac{1}{(\cos\theta)^2}\left\{-2(n_1+n_2)n_1 + \frac{n_1^2}{(\cos\theta)^2}\right\} - \frac{1}{(\sin\theta)^2}\left\{-2(n_1+n_2)n_2 + \frac{n_2^2}{(\sin\theta)^2}\right\}$$
$$= \frac{(n_1(\sin\theta)^2 - n_2(\cos\theta)^2)^2((\sin\theta)^2 - (\cos\theta)^2)}{(\cos\theta)^4(\sin\theta)^4}.$$

Hence the solutions of Equation (3.5) are

$$(\cos\theta)^2 = \frac{n_1}{n_1+n_2},\ \frac{1}{2}. \tag{3.6}$$ □

Finally, we introduce concrete examples of biharmonic submanifolds in the unit sphere which given by Theorem 3.5. We consider the case of $(G_1, K_1) = (G_2, K_2) = (\mathrm{SU}(n), \mathrm{SO}(n))$ $(n > 3)$. In this case, we can see that

$$\mathrm{Lie}(G_1) = \mathfrak{g}_1 = \{X \in \mathrm{M}(n, \mathbf{C}) \mid {}^t\overline{X} + X = 0\}.$$

Here, $\overline{X}$ and ${}^tX$ denote the complex conjugation and the transpose of $X \in \mathrm{M}(n, \mathbf{C})$, respectively. For $X \in \mathfrak{g}$, we set $\sigma(X) = \overline{X}$. Then

$$\mathfrak{k}_1 = \{X \in \mathfrak{g} \mid X = \overline{X}\} = \{X \in \mathrm{M}(n, \mathbf{R}) \mid X = -{}^tX\},$$
$$\mathfrak{m}_1 = \{X \in \mathfrak{g} \mid -X = \overline{X}\} = \sqrt{-1}\{X \in \mathrm{M}(n, \mathbf{R}) \mid X = {}^tX,\ \mathrm{trace}(X) = 0\}.$$

It is known that $\mathrm{Ad}(k)X = kXk^{-1}$ for $k \in K_1$, $X \in \mathfrak{m}_1$. We set $\langle X, Y \rangle = -\mathrm{trace}({}^t\overline{X}Y)$ for $X, Y \in \mathfrak{g}_1$. Then $\langle X, Y \rangle$ is a $\mathrm{Ad}(G_1)$-invariant inner product of $\mathfrak{g}_1$. We define a subspace $\mathfrak{a}_1$ of $\mathfrak{m}_1$ by

$$\mathfrak{a}_1 = \{H = \sqrt{-1}\,\mathrm{diag}(h_1, \dots, h_n) \mid h_1, \dots, h_n \in \mathbf{R},\ \mathrm{trace}(H) = 0\}.$$

Then $\mathfrak{a}_1$ is a maximal abelian subspace of $\mathfrak{m}_1$. A simple calculation shows that the root system Σ_1 of (G_1, K_1) with respect to $\mathfrak{a}_1$ is given as

$$\Sigma_1 = \{\pm(E_i^i - E_j^j) \mid 1 \le i < j \le n\},$$

where E_i^j denotes the $n \times n$-matrix whose (i, j)-entry is one and all the other entries are zero. The set $\Pi_1 = \{\alpha_i = E_i^i - E_{i+1}^{i+1} \mid 1 \le i \le n-1\}$ is a set of simple roots in Σ_1.

For $1 \le i \le n-1$, we set

$$H_{\alpha_i} = \frac{1}{n}\left((n-i)\sum_{j=1}^{i} E_j^j - i\sum_{j=i+1}^{n} E_j^j\right).$$

Then, $\langle \alpha_i, H_{\alpha_j} \rangle = \delta_{i,j}$ $(i, j \in \{1, \dots, n-1\})$ holds. We set $H_i = H_{\alpha_i}/\|H_{\alpha_i}\|$ for $1 \le i \le n-1$. Then, by Corollary 2.10, the orbit $\mathrm{Ad}(K_1)H_i$ is a minimal submanifold of S for $1 \le i \le n-1$.

We can see that the isotropy subgroup of $\mathrm{Ad}(K_1)H_i$ at H_i is isomorphic to $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$. Therefore, $\mathrm{Ad}(K_1)H_i$ is diffeomorphic to the Grassmannian manifold $G_i(\mathbf{R}^n)$. In particular, $\dim \mathrm{Ad}(K_1)H_i = i(n-i)$. Hence for $1 \le i, j \le n-1$, if $\dim \mathrm{Ad}(K_1)H_i = \dim \mathrm{Ad}(K_1)H_j$, then $i = j, n-j$.

By the above argument, we can apply Theorem 3.5 for $(G, K) = (G_1 \times G_2, K_1 \times K_2)$. By Theorem 3.5, for $1 \le i, j \le n-1$ if $i \ne j, (n-j)$, then $(\mathrm{Ad}(K_1)H_i/\sqrt{2}) \times (\mathrm{Ad}(K_2)H_j/\sqrt{2}) \subset S^{n(n+1)-3}(1)$ is a proper biharmonic submanifold.

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