T. Y. CAO, Z. K. LIU AND X. Y. YANG
KODAI MATH. J.
41 (2018), 579–590

DERIVED CATEGORY WITH RESPECT TO GORENSTEIN AC-PROJECTIVE MODULES

TIANYA CAO, ZHONGKUI LIU AND XIAOYAN YANG

Abstract

The aim of this paper is to study the derived category with respect to Gorenstein AC-projective modules. We characterize the bounded Gorenstein AC derived category and obtain some triangle equivalences. We also establish a right recollement related with Gorenstein AC derived category.

1. Introduction

Throughout this paper, R denotes a ring with unity, all modules are left R-modules. Recall that an R-module M is Gorenstein projective if there exists an exact sequence $P^{\bullet} = \cdots \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$ of projective modules, which remains exact after applying $\mathrm{Hom}_R(-, P)$ for every projective module P, such that $M = \mathrm{Ker}(P^0 \to P^1)$. Gorenstein projective modules receive a lot of attention: for example, they form the basis of Gorenstein algebra (see e.g. [1, 8, 19, 20]), they play an important role in the Tate cohomology (see e.g. [2, 4]), they are widely used in the theory of stable and singularity categories (see e.g. [4, 11]).

Gorenstein rings were introduced by Iwanage and subsequently studied by many authors. Over such rings there is a complete and hereditary cotorsion pair $(\mathscr{GP}, \mathscr{GP}^{\perp})$, where $\mathscr{GP}$ denotes the class of Gorenstein projective modules. Hovey [13] established a Quillen model structure in view of the cotorsion pair $(\mathscr{GP}, \mathscr{GP}^{\perp})$. The homotopy category of this model category is a generalization of the stable module category over a quasi-Frobenius ring. To generalize Gorenstein homological algebra to more general rings, Bravo et al. [3] introduced the notion of Gorenstein AC-projective modules. They call a module N to be of type FP_{∞} if N has a projective resolution by finitely generated projective modules; a module L is level if $\mathrm{Tor}_R^1(N, L) = 0$ for all right R-modules N of type FP_{∞}. If the above complex $P^{\bullet}$ stays exact after applying $\mathrm{Hom}_R(-, L)$ for every level module L, then the module $M = \mathrm{Ker}(P^0 \to P^1)$ is called Gorenstein

2010 *Mathematics Subject Classification*. 18G35; 18E30.
Key words and phrases. Gorenstein AC-projective module, homotopy category, right recollement.
Received June 12, 2017; revised January 16, 2018.

AC-projective. They proved that $(\mathscr{GAP}, \mathscr{GAP}^{\perp})$ is a complete and hereditary cotorsion pair, and then established the Gorenstein AC-projective model structure over arbitrary rings, where $\mathscr{GAP}$ denotes the class of Gorenstein AC-projective modules. When R is (left) coherent for which all flat modules have finite projective dimension, Gorenstein AC-projective modules coincide with Gorenstein projective modules. Thus the cotorsion pair and the model structure involving modules built from the Gorenstein projective modules. Further, the class of Gorenstein AC-projective modules is closed under extensions, direct summands and kernels of epimorphisms by [3, Lemma 8.6].

As a Gorenstein version of the derived category $\mathbf{D}^*(R)$ with $* \in \{blank, -, +, b\}$, Gao and Zhang [9] introduced and studied Gorenstein derived category $\mathbf{D}^*_{\mathscr{GP}}(R)$, which is defined as the Verdier quotient of the homotopy category $\mathbf{K}^*(R)$ with respect to the thick subcategory $\mathbf{K}^*_{\mathscr{GP}\text{-}ac}(R)$ of Gorenstein projective acyclic complexes. In this paper, we are inspired to investigate Gorenstein AC-projective derived category. This has some advantages in studying Gorenstein AC-homological algebra of [3]. For example, the relative derived functors with respect to Gorenstein AC-projective modules can be interpreted as the Hom functors of the Gorenstein AC derived category.

The paper is organized as follows: In section 2 we introduce Gorenstein AC derived category and show intimate connections with derived category and Gorenstein derived category. Meanwhile, we give a new characterization of relative derived functor of Hom with respect to Gorenstein AC-projective modules as morphisms in the corresponding Gorenstein AC derived category. In section 3, the bounded Gorenstein AC derived category $\mathbf{D}^b_{\mathscr{GAP}}(R)$ is studied. Moreover, we give a sufficient condition of triangle equivalence $\mathbf{D}^b_{\mathscr{GAP}}(R) \cong \mathbf{D}^b_{\mathscr{GAP}}(S)$ for rings R and S. Finally, we obtain a right recollement related with Gorenstein AC derived category.

2. Derived category with respect to Gorenstein AC-projective modules

In this section, we introduce derived category with respect to Gorenstein AC-projective modules.

Definition 2.1. Let $\mathscr{X}$ be a class of R-modules. An R-complex X is called $\mathscr{X}$-acyclic, if $\mathrm{Hom}_R(G, X)$ is acyclic for every $G \in \mathscr{X}$. A morphism $f : X \to Y$ of R-complexes is called a $\mathscr{X}$-quasi-isomorphism, if $\mathrm{Hom}_R(G, f)$ is a quasi-isomorphism for every $G \in \mathscr{X}$. Throughout, $\mathbf{K}^*(\mathscr{X})$ denote the homotopy category of complexes constructed by modules in $\mathscr{X}$, and $\mathbf{K}^*_{\mathscr{X}\text{-}ac}(R)$ is the subcategory of $\mathbf{K}^*(R)$ consisting of $\mathscr{X}$-acyclic complexes, in particular $\mathbf{K}^*_{ac}(R)$ denote the homotopy category consisting of all acyclic complexes, where $* \in \{blank, -, +, b\}$.

Remark 2.2. (1) If $\mathscr{X} = \mathscr{GP}$, then $\mathscr{X}$-acyclic is called $\mathscr{GP}$-acyclic, and $\mathscr{X}$-quasi-isomorphism is called $\mathscr{GP}$-quasi-isomorphism. If $\mathscr{X} = \mathscr{GAP}$, then $\mathscr{X}$-acyclic is called $\mathscr{GAP}$-acyclic, and $\mathscr{X}$-quasi-isomorphism is called $\mathscr{GAP}$-quasi-isomorphism. Since $\mathscr{P} \subseteq \mathscr{GAP} \subseteq \mathscr{GP}$, every $\mathscr{GP}$-acyclic complex is $\mathscr{GAP}$-acyclic, and every $\mathscr{GAP}$-acyclic complex is acyclic. Moreover, every $\mathscr{GP}$-quasi-isomorphism is a $\mathscr{GAP}$-quasi-isomorphism, and every $\mathscr{GAP}$-quasi-isomorphism is a quasi-isomorphism.

(2) By [6, Lemma 2.4], a complex X is $\mathscr{GAP}$-acyclic if and only if $\mathrm{Hom}_R(D, X)$ is acyclic for each complex $D \in \mathbf{K}^-(\mathscr{GAP})$. Moreover, it follows from [6, Proposition 2.6] that a morphism $f : X \to Y$ of R-complexes is a $\mathscr{GAP}$-quasi-isomorphism if and only if $\mathrm{Hom}_R(G, f)$ is a quasi-isomorphism for each $G \in \mathbf{K}^-(\mathscr{GAP})$.

Recall a full triangulated subcategory $\mathscr{C}$ of a triangulated category $\mathscr{D}$ is said to be thick if it satisfying the following condition: assume that a morphism $f : X \to Y$ in $\mathscr{D}$ can be factored through an object from $\mathscr{C}$, and enters a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ with Z in $\mathscr{C}$, then X, Y are objects in $\mathscr{C}$. A standard example of thick subcategory is the category of all acyclic complexes in $\mathbf{K}(R)$. There is an important characterization of thick subcategories due to Rickard, called Rickard's criterion: a full triangulated subcategory $\mathscr{C}$ of a triangulated category $\mathscr{D}$ is thick if and only if every direct summand of an object of $\mathscr{C}$ is in $\mathscr{C}$ ([15, Criterion 1.3]).

Lemma 2.3. *For* $* \in \{blank, -, +, b\}$, $\mathbf{K}^*_{\mathscr{GAP}\text{-}ac}(R)$ *are thick subcategories of* $\mathbf{K}^*(R)$.

Proof. We consider the full subcategory of $\mathbf{K}^*(R)$ as follows,

$$\{Y \in \mathbf{K}^*(R) \mid \mathrm{Hom}_{\mathbf{K}^*(R)}(X[n], Y) = 0, \ \forall X \in \mathscr{GAP}, \ \forall n \in \mathbf{Z}\}.$$

Clearly, it is a triangulated subcategory of $\mathbf{K}^*(R)$ closed under direct summands, and hence is thick by Rickard's criterion. By the definition of $X \in \mathbf{K}^*_{\mathscr{GAP}\text{-}ac}(R)$, we have the following equality

$$0 = \mathrm{H}^n \mathrm{Hom}_R(G, X) = \mathrm{Hom}_{\mathbf{K}^*(R)}(G[-n], X), \quad \forall G \in \mathscr{GAP}, \ \forall n \in \mathbf{Z}.$$

It follows that $\mathbf{K}^*_{\mathscr{GAP}\text{-}ac}(R)$ are thick subcategories of $\mathbf{K}^*(R)$. $\square$

It is well known that the derived category is a Verdier quotient of the homotopy category with respect to the thick triangulated subcategory of acyclic complexes. In general, given a triangulated subcategory $\mathscr{B}$ of a triangulated category $\mathscr{K}$, in the Verdier quotient $\mathscr{K}/\mathscr{B} = S^{-1}\mathscr{K}$, where S is the compatible multiplicative system determined by $\mathscr{B}$, each morphism $f : X \to Y$ is given by an equivalent class of right fractions a/s presented by $X \overset{s}{\Leftarrow} Z \xrightarrow{a} Y$.

Note that a morphism of complexes $f : X \to Y$ is a $\mathscr{GAP}$-quasi-isomorphism if and only if its mapping cone $\mathrm{Cone}(f)$ is $\mathscr{GAP}$-acyclic. The collection of all $\mathscr{GAP}$-quasi-isomorphisms in $\mathbf{K}^*(R)$, denoted by $S_{\mathscr{GAP}}$, is a saturated compatible multiplicative system corresponding to the subcategory $\mathbf{K}^*_{\mathscr{GAP}\text{-}ac}(R)$ (see [18, chapter 3]).

DEFINITION 2.4. The derived category $\mathbf{D}^*_{\mathscr{GAP}}(R)$ with respect to Gorenstein AC-projective modules is defined to be the Verdier quotient of $\mathbf{K}^*(R)$, that is,

$$\mathbf{D}^*_{\mathscr{GAP}}(R) := \mathbf{K}^*(R)/\mathbf{K}^*_{\mathscr{GAP}\text{-}ac}(R) = S^{-1}_{\mathscr{GAP}}\mathbf{K}^*(R),$$

which is called the Gorenstein AC derived category.

Remark 2.5. (1) In fact, $\mathbf{D}^*_{\mathscr{GAP}}(R)$ is the derived category of the exact categories $(R\text{-Mod}, \mathscr{E}_{\mathscr{GAP}})$, in sense of Neeman [15], where $\mathscr{E}_{\mathscr{GAP}}$ is the collection of all the short $\mathscr{GAP}$-acyclic sequences in the category of R-modules.

(2) It follows from [7] that if R is coherent, then Gorenstein-AC projective modules coincide with Ding projective modules, and hence the corresponding Gorenstein AC derived category coincides with Ding derived category introduced in [16].

(3) If every level module has finite projective dimension, then $\mathscr{GAP} = \mathscr{GP}$ by [3], and then Gorenstein AC derived category $\mathbf{D}^*_{\mathscr{GAP}}(R)$ is precisely the Gorenstein derived category $\mathbf{D}^*_{\mathscr{GP}}(R)$ in [9].

(4) If R is a ring such that every R-module has finite projective dimension, then $\mathscr{GP} = \mathscr{GAP} = \mathscr{P}$. In this case, $\mathbf{D}^*_{\mathscr{GP}}(R)$, $\mathbf{D}^*_{\mathscr{GAP}}(R)$ and $\mathbf{D}^*(R)$ coincide.

In general, the relations between $\mathbf{D}^*_{\mathscr{GAP}}(R)$, $\mathbf{D}^*_{\mathscr{GP}}(R)$ and $\mathbf{D}^*(R)$ are as follows.

PROPOSITION 2.6. *For $* \in \{blank, -, +, b\}$, there are triangle equivalences*

$$\mathbf{D}^*(R) \cong \mathbf{D}^*_{\mathscr{GAP}}(R)/(\mathbf{K}^*_{ac}(R)/\mathbf{K}^*_{\mathscr{GAP}\text{-}ac}(R)),$$
$$\mathbf{D}^*_{\mathscr{GAP}}(R) \cong \mathbf{D}^*_{\mathscr{GP}}(R)/(\mathbf{K}^*_{\mathscr{GAP}\text{-}ac}(R)/\mathbf{K}^*_{\mathscr{GP}\text{-}ac}(R)).$$

Proof. It follows immediately from [9, Lemma 2.4] or [17, Corollary 4.3]. □

COROLLARY 2.7. *The following statements are equivalent for $* \in \{blank, -, b\}$.*
(1) $\mathbf{D}^*(R) \cong \mathbf{D}^*_{\mathscr{GAP}}(R)$;
(2) $\mathbf{K}^*_{ac}(R) \cong \mathbf{K}^*_{\mathscr{GAP}\text{-}ac}(R)$;
(3) *Any quasi-isomorphism is a $\mathscr{GAP}$-quasi-isomorphism*;
(4) *Any Gorenstein AC-projective module is a projective module.*

Proof. (1) $\Leftrightarrow$ (2), (2) $\Leftrightarrow$ (3) and (4) $\Rightarrow$ (3) are immediate follow by Proposition 2.6. It remains to prove that (3) $\Rightarrow$ (4).

Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an arbitrary short exact sequence. Considered as a map between complexes, the morphism induced by g is a quasi-isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

By assumption it is a $\mathcal{GAP}$-quasi-isomorphism. Thus $0 \to \mathrm{Hom}_R(G, X) \to \mathrm{Hom}_R(G, Y) \to \mathrm{Hom}_R(G, Z) \to 0$ is exact for any $G \in \mathcal{GAP}$. This implies that G is a projective module, and the assertion follows. $\square$

Let $F : R\text{-Mod} \to \mathbf{D}^b_{\mathcal{GAP}}(R)$ be the composition of the embedding $R\text{-Mod} \to \mathbf{K}^b(R)$ and the localization functor $\mathbf{K}^b(R) \to \mathbf{D}^b_{\mathcal{GAP}}(R)$. We get the following result.

Proposition 2.8. *The functor $F : R\text{-Mod} \to \mathbf{D}^b_{\mathcal{GAP}}(R)$ is fully faithful.*

Proof. For any $X, Y \in R$-Mod and $f \in \mathrm{Hom}_R(X, Y)$, if $F(f) = 0$ then there exists a $\mathcal{GAP}$-quasi-isomorphism $s : Z \to X$ such that $fs \sim 0$, so $\mathrm{H}^0(f)\mathrm{H}^0(s) = 0$. Since $\mathrm{H}^0(s)$ is an isomorphism, we clearly get $f = 0$. On the other hand, let $\frac{a}{s}$ be a morphism in $\mathrm{Hom}_{\mathbf{D}_{\mathcal{GAP}}(R)}(X, Y)$. Then we have a diagram $X \overset{s}{\Leftarrow} Z \overset{a}{\to} Y$, where s is a $\mathcal{GAP}$-quasi-isomorphism, and hence a quasi-isomorphism. So $\mathrm{H}^0(s) \in \mathrm{Hom}_R(\mathrm{H}^0(Z), X)$ is an isomorphism in R-Mod. Put $f = \mathrm{H}^0(a)\mathrm{H}^0(s)^{-1} \in \mathrm{Hom}_R(X, Y)$. Consider the truncation $U = \cdots \to Z^{-2} \to Z^{-1} \to \mathrm{Ker}\, d^0_Z \to 0$ of Z and the canonical map $i : U \to Z$. Since i is a $\mathcal{GAP}$-quasi-isomorphism, so is si. From the commutative diagram

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d] & Z \arrow[d, "s"] \\ \mathrm{H}^0(Z) \arrow[r, "\mathrm{H}^0(s)"] & X \end{tikzcd}$$

we get $fsi = \mathrm{H}^0(a)\mathrm{H}^0(s)^{-1}si = ai$. So the following diagram is commutative:

$$\begin{tikzcd} & Z \arrow[dl, Rightarrow, "s"'] \arrow[dr, "a"] & \\ X & U \arrow[l, Rightarrow, "si"'] \arrow[u, "i"'] \arrow[r, "ai"] \arrow[d, "si"] & Y \\ & X \arrow[ul, Rightarrow, "id"] \arrow[ur, "f"'] & \end{tikzcd}$$

It yields that $F(f) = \dfrac{f}{id_X} = \dfrac{a}{s}$, as desired. $\square$

For any R-modules M and N, it is well known that $\mathrm{Ext}^n_R(M, N) = \mathrm{Hom}_{\mathbf{D}^b(R)}(M, N[n])$. Let M be an R-module admitting a $\mathcal{GP}$-resolution $G^\bullet \to M \to 0$, i.e. a complex with each G^i Gorenstein projective which is exact by applying $\mathrm{Hom}_R(G, -)$ for any Gorenstein projective module G. For an arbitrary module N, $\mathrm{Ext}^n_{\mathcal{GP}}(M, N)$ is defined in [12] as $\mathrm{H}^n\, \mathrm{Hom}_R(G^\bullet, N)$. By [9, Theorem

3.12], $\mathrm{Ext}^n_{\mathscr{GP}}(M,N) = \mathrm{Hom}_{\mathbf{D}^b_{\mathscr{GP}}(R)}(M, N[n])$. In the rest of this section, we show that corresponding result also holds in Gorenstein AC derived category.

The following result makes the morphisms in $\mathbf{D}_{\mathscr{GAP}}(R)$ easier to understand.

LEMMA 2.9. *Let $D \in \mathbf{K}^-(\mathscr{GAP})$, $X \in \mathbf{K}(R)$. Then $\varphi : f \to f/id_D$ gives an isomorphism of abelian groups* $\mathrm{Hom}_{\mathbf{K}(R)}(D,X) \cong \mathrm{Hom}_{\mathbf{D}_{\mathscr{GAP}}(R)}(D,X)$.

Proof. If $f/id_D = 0$, then by the calculus of right fractions there is a $\mathscr{GAP}$-quasi-isomorphism $s : Y \to D$ for some complex Y such that $fs \sim 0$. It follows from [6, Proposition 2.6] that there is a morphism $g : D \to Y$ such that $sg \sim id_D$. Thus $f \sim fsg \sim 0$. Moreover, for each $f/s \in \mathrm{Hom}_{\mathbf{D}_{\mathscr{GAP}}(R)}(D,X)$ presented by $D \overset{s}{\Leftarrow} Y \overset{f}{\to} X$, there is a morphism $g : D \to Y$ such that $sg \sim id_D$. This implies that $f/s = fg/id_D = \varphi(fg)$. Hence φ is an isomorphism, as desired. □

LEMMA 2.10. *$\mathbf{D}^-_{\mathscr{GAP}}(R)$ is a full triangulated subcategory of $\mathbf{D}_{\mathscr{GAP}}(R)$; $\mathbf{D}^b_{\mathscr{GAP}}(R)$ is a full triangulated subcategory of $\mathbf{D}^-_{\mathscr{GAP}}(R)$, and hence of $\mathbf{D}_{\mathscr{GAP}}(R)$.*

Proof. Let $S = S_{\mathscr{GAP}}$. Then $\mathbf{D}_{\mathscr{GAP}}(R) = S^{-1}\mathbf{K}(R)$, $\mathbf{D}^-_{\mathscr{GAP}}(R) = (S \cap \mathbf{K}^-(R))^{-1}\mathbf{K}^-(R)$. By [10, Proposition (III) 2.10], it suffices to prove that for any $\mathscr{GAP}$-quasi-isomorphism $f : X \to Y$ with $Y \in \mathbf{K}^-(R)$, there is a morphism $g : X' \to X$ with $X' \in \mathbf{K}^-(R)$, such that fg is a $\mathscr{GAP}$-quasi-isomorphism. Then the canonical functor $(S \cap \mathbf{K}^-(R))^{-1}\mathbf{K}^-(R) \to S^{-1}\mathbf{K}(R)$ is fully faithful, and hence $\mathbf{D}^-_{\mathscr{DAP}}(R)$ is a full triangulated subcategory of $\mathbf{D}_{\mathscr{GAP}}(R)$.

Suppose that there is an integer i such that $Y^k = 0$ for any $k > i$. Let X' be the soft truncation $X^{<i}$ of X. Then there is a commutative diagram

$$\begin{array}{ccccccccccccc}
X^{<i} & \cdots & \longrightarrow & X^{i-2} & \longrightarrow & X^{i-1} & \longrightarrow & \mathrm{Ker}\, d^i & \longrightarrow & 0 & \longrightarrow & \cdots \\
{\scriptstyle g}\big\downarrow & & & \big\| & & \big\| & & \big\downarrow\cap & & \big\downarrow & & \\
X & \cdots & \longrightarrow & X^{i-2} & \longrightarrow & X^{i-1} & \longrightarrow & X^i & \longrightarrow & X^{i+1} & \longrightarrow & \cdots \\
{\scriptstyle f}\big\downarrow & & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
Y & \cdots & \longrightarrow & Y^{i-2} & \longrightarrow & Y^{i-1} & \longrightarrow & Y^i & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Since f is a $\mathscr{GAP}$-quasi-isomorphism, it is easy to see that g is also a $\mathscr{GAP}$-quasi-isomorphism, and so is fg. The second one can be proved similarly. This completes the proof. □

By [3, Theorem 8.5], we know that each R-module M has a $\mathscr{GAP}$-resolution $G^\bullet \to M \to 0$, i.e. a complex with each G^i Gorenstein AC-projective which is exact by applying $\mathrm{Hom}_R(G,-)$ for any Gorenstein AC-projective module G. Note that by a version of Comparison Theorem, the $\mathscr{GAP}$-resolution is unique up to homotopy. For an arbitrary R-module N, it is easily seen that $\mathrm{Ext}^n_{\mathscr{GAP}}(M,N) = \mathrm{H}^n\,\mathrm{Hom}_R(G^\bullet, N)$ is well defined.

THEOREM 2.11. *Let M and N be R-modules. Then we have*

$$\mathrm{Ext}^n_{\mathcal{GAP}}(M,N) \cong \mathrm{Hom}_{\mathbf{D}^b_{\mathcal{GAP}}(R)}(M, N[n]).$$

Proof. Let $G^\bullet \to M \to 0$ be a $\mathcal{GAP}$-resolution of M. Then $G^\bullet \to M$ is a $\mathcal{GAP}$-quasi-isomorphism, and so $G^\bullet \cong M$ in $\mathbf{D}^-_{\mathcal{GAP}}(R)$. Hence we have

$$\begin{aligned}\mathrm{Ext}^n_{\mathcal{GAP}}(M,N) &= \mathrm{H}^n\,\mathrm{Hom}_R(G^\bullet, N)\\ &= \mathrm{Hom}_{\mathbf{K}(R)}(G^\bullet, N[n])\\ &\cong \mathrm{Hom}_{\mathbf{D}_{\mathcal{GAP}}(R)}(G^\bullet, N[n])\\ &\cong \mathrm{Hom}_{\mathbf{D}^b_{\mathcal{GAP}}(R)}(M, N[n]),\end{aligned}$$

where the first isomorphism follows from Lemma 2.9 and the second isomorphism follows from Lemma 2.10. □

3. Bounded Gorenstein AC-derived categories

In this section, we give a description of the bounded Gorenstein AC derived category and obtain some triangle equivalences.

Define $\mathbf{K}^{-,gapb}(\mathcal{GAP})$ to be the full subcategory of $\mathbf{K}^-(\mathcal{GAP})$ by

$$\mathbf{K}^{-,gapb}(\mathcal{GAP}) := \left\{ X \in \mathbf{K}^-(\mathcal{GAP}) \;\middle|\; \begin{array}{c}\text{there exists } n = n(X) \in \mathbf{Z}, \text{ such that}\\ \mathrm{H}^i(\mathrm{Hom}(G,X)) = 0,\ \forall i \leqslant n,\ \forall G \in \mathcal{GAP}\end{array} \right\}.$$

LEMMA 3.1. *There exist a functor $F : \mathbf{K}^b(R) \to \mathbf{K}^{-,gapb}(\mathcal{GAP})$ and a $\mathcal{GAP}$-quasi-isomorphism $\varphi_X : F(X) \to X$ for any $X \in \mathbf{K}^b(R)$, which is functorial in X.*

Proof. We need to show that for each $X \in \mathbf{K}^b(R)$, there exists a $\mathcal{GAP}$-quasi-isomorphism $F(X) \to X$ with $F(X) \in \mathbf{K}^{-,gapb}(R)$. We proceed by induction on the cardinal of the finite set $\{i \in \mathbf{Z} \mid X^i \neq 0\}$, called the width of X and denoted by $\mathcal{W}(X)$.

We always identify a module with the complex concentrated in degree zero. Let $\mathcal{W}(X) = 1$. By [3, Theorem 8.5], for R-module X we have a $\mathcal{GAP}$-resolution $G^\bullet \to X \to 0$, which induces the desired $\mathcal{GAP}$-quasi-isomorphism of complexes $\varphi_X : F(X) = G^\bullet \to X$.

Now assume $\mathcal{W}(X) \geqslant 2$ with $X^j \neq 0$ and $X^i = 0$ for any $i < j$. Then we have a distinguished triangle $X_1 \xrightarrow{u} X_2 \to X \to X_1[1]$ in $\mathbf{K}^b(R)$, where $X_1 = X^j[-j-1]$ and $X_2 = X^{>j}$. By the induction there exist a $\mathcal{GAP}$-quasi-isomorphism $\varphi_{X_1} : F(X_1) \to X_1$ and $\varphi_{X_2} : F(X_2) \to X_2$ with $F(X_1), F(X_2) \in \mathbf{K}^{-,gapb}(\mathcal{GAP})$. Then by [6, Proposition 2.6], there is an isomorphism induced by φ_{X_2}

$$\mathrm{Hom}_{\mathbf{K}(R)}(F(X_1), F(X_2)) \cong \mathrm{Hom}_{\mathbf{K}(R)}(F(X_1), X_2).$$

So there exists a morphism $\varphi : F(X_1) \to F(X_2)$, which is unique up to homotopy, such that $\varphi_{X_2}\varphi = u\varphi_{X_1}$. We have a distinguished triangle

$$F(X_1) \xrightarrow{\varphi} F(X_2) \to \mathrm{Cone}(\varphi) \to F(X_1)[1]$$

in $\mathbf{K}^{-,gapb}(\mathscr{GAP})$. By the axiom of a triangulated category, there is $\varphi_X : \mathrm{Cone}(\varphi) \to X$ such that the following diagram commutes in $\mathbf{K}(R)$

$$\begin{tikzcd}
F(X_1) \arrow[r, "\varphi"] \arrow[d, "\varphi_{X_1}"'] & F(X_2) \arrow[r] \arrow[d, "\varphi_{X_2}"'] & \mathrm{Cone}(\varphi) \arrow[r] \arrow[d, "\varphi_X"'] & F(X_1)[1] \arrow[d, "\varphi_{X_1}[1]"'] \\
X_1 \arrow[r, "u"] & X_2 \arrow[r] & X \arrow[r] & X_1[1]
\end{tikzcd}$$

For each $P \in \mathscr{GAP}$, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_R(P, F(X_1)) \arrow[r] \arrow[d, "(\varphi_{X_1})_*"'] & \mathrm{Hom}_R(P, F(X_2)) \arrow[r] \arrow[d, "(\varphi_{X_2})_*"'] & \mathrm{Hom}_R(P, \mathrm{Cone}(\varphi)) \arrow[r] \arrow[d, "(\varphi_X)_*"'] & \mathrm{Hom}_R(P, F(X_1)[1]) \arrow[d, "(\varphi_{X_1}[1])_*"'] \\
\mathrm{Hom}_R(P, X_1) \arrow[r] & \mathrm{Hom}_R(P, X_2) \arrow[r] & \mathrm{Hom}_R(P, X) \arrow[r] & \mathrm{Hom}_R(P, X_1[1])
\end{tikzcd}$$

Since both φ_{X_1} and φ_{X_2} are $\mathscr{GAP}$-quasi-isomorphism, $(\varphi_{X_1})_*$ and $(\varphi_{X_2})_*$ are both quasi-isomorphism, and hence $(\varphi_X)_*$ is a quasi-isomorphism, that is, φ_X is a $\mathscr{GAP}$-quasi-isomorphism. Put $F(X) = \mathrm{Cone}(\varphi)$. This shows that F is a functor, and also that $\varphi_X : F(X) \to X$ is functorial in $X \in \mathbf{K}^b(R)$. $\square$

THEOREM 3.2. *For a ring R, there is a triangle equivalence $\mathbf{D}^b_{\mathscr{GAP}}(R) \cong \mathbf{K}^{-,gapb}(\mathscr{GAP})$.*

Proof. We denote the composition of the embedding $\mathbf{K}^{-,gapb}(\mathscr{GAP}) \to \mathbf{K}^{-}(R)$ and the localization functor $\mathbf{K}^{-}(R) \to \mathbf{D}^{-}_{\mathscr{GAP}}(R)$ by $F : \mathbf{K}^{-,gapb}(\mathscr{GAP}) \to \mathbf{D}^{-}_{\mathscr{GAP}}(R)$. For any $X \in \mathbf{K}^{-,gapb}(\mathscr{GAP})$, there exists $n \in \mathbf{Z}$ such that $\mathrm{H}^i(\mathrm{Hom}(G, X)) = 0$ for $i \leqslant n-1$ and any $G \in \mathscr{GAP}$. We have the following commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[r] \arrow[d, equal] & X^{n+2} \arrow[r] \arrow[d, equal] & \cdots \\
X^{>n} & \cdots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}\, d^{n-1} \arrow[r] & X^{n+1} \arrow[r] & X^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

where the morphism f is $\mathscr{GAP}$-quasi-isomorphism. It is clear that $F(X) \cong X^{>n}$ in $\mathbf{D}_{\mathscr{GAP}}(R)$ and $F(X) \in \mathbf{D}^b_{\mathscr{GAP}}(R)$. By Lemma 2.9, $F : \mathbf{K}^{-,gapb}(\mathscr{GAP}) \to \mathbf{D}^b_{\mathscr{GAP}}(R)$ is full faithful and F is dense by Lemma 3.1. So $F : \mathbf{K}^{-,gapb}(\mathscr{GAP}) \to \mathbf{D}^b_{\mathscr{GAP}}(R)$ is a triangle equivalence. This completes the proof. $\square$

PROPOSITION 3.3. *For a ring R, $\mathbf{K}^{-,gapb}(\mathscr{GAP})$ is a thick subcategory of $\mathbf{K}^{-}(\mathscr{GAP})$.*

Proof. It is easy to see that $\mathbf{K}^{-,gapb}(\mathscr{GAP})$ is an additive full subcategory of $\mathbf{K}^{-}(\mathscr{GAP})$. Let $X \in \mathbf{K}^{-,gapb}(\mathscr{GAP})$. Then there is an $n(X) \in \mathbf{Z}$, such that $\mathrm{H}^i(\mathrm{Hom}_R(D,X)) = 0$ for any $i \leqslant n(X)$ and $D \in \mathscr{GAP}$. For any $m \in \mathbf{Z}$, since

$$\mathrm{H}^j(\mathrm{Hom}_R(D,X[m])) = \mathrm{Hom}_{\mathbf{K}(R)}(D,X[j+m]) = \mathrm{H}^{j+m}(\mathrm{Hom}_R(D,X)),$$

$\mathrm{H}^j(\mathrm{Hom}_R(D,X[m])) = 0$ when $j \leqslant n(X) - m$, $X[m] \in \mathbf{K}^{-,gapb}(\mathscr{GAP})$. Let $X \to Y \to Z \to X[1]$ be a distinguished triangle in $\mathbf{K}^{-}(\mathscr{GAP})$ with $X, Y \in \mathbf{K}^{-,gapb}(\mathscr{GAP})$. By the definition of $\mathbf{K}^{-,gapb}(\mathscr{GAP})$, there are $n(X), n(Y) \in \mathbf{Z}$. Let $m = \min\{n(X), n(Y)\}$. Then for any $D \in \mathscr{GAP}$, we get the following exact sequence

$$\cdots \to \mathrm{H}^{n-1}(\mathrm{Hom}_R(D,Y)) \to \mathrm{H}^{n-1}(\mathrm{Hom}_R(D,Z)) \to \mathrm{H}^{n}(\mathrm{Hom}_R(D,X)) \to \cdots$$

Then $\mathrm{H}^l(\mathrm{Hom}(D,Z)) = 0$ for any $l \leqslant m-1$. Put $n(Z) = m-1$. Thus $Z \in \mathbf{K}^{-,gapb}(\mathscr{GAP})$. So $\mathbf{K}^{-,gapb}(\mathscr{GAP})$ is a triangulated subcategory of $\mathbf{K}^{-}(\mathscr{GAP})$. It is clear that $\mathbf{K}^{-,gapb}(\mathscr{GAP})$ is closed under direct summands. Thus $\mathbf{K}^{-,gapb}(\mathscr{GAP})$ is a thick subcategory of $\mathbf{K}^{-}(\mathscr{GAP})$. □

Corollary 3.4. *Let R and S be ring. Then triangle equivalence $F : \mathbf{K}^{-}(\mathscr{GAP}(R)) \cong \mathbf{K}^{-}(\mathscr{GAP}(S))$ induces a triangle equivalence $\mathbf{D}^b_{\mathscr{GAP}}(R) \cong \mathbf{D}^b_{\mathscr{GAP}}(S)$.*

Proof. Clearly, the restriction of $F : \mathbf{K}^{-}(\mathscr{GAP}(R)) \cong \mathbf{K}^{-}(\mathscr{GAP}(S))$ to $\mathbf{K}^{-,gapb}(\mathscr{GAP}(R))$ gives $\mathbf{K}^{-,gapb}(\mathscr{GAP}(R)) \cong \mathbf{K}^{-,gapb}(\mathscr{GAP}(S))$, and then the result follows from Theorem 3.2. □

4. Right recollement and stable t-structure

Let $\mathscr{D}$ be a triangulated category. A pair $(\mathscr{U}, \mathscr{V})$ of full subcategories of $\mathscr{D}$ is called a stable t-structure in $\mathscr{D}$ [14] provide that $\mathscr{U} = \mathscr{U}[1]$ and $\mathscr{V} = \mathscr{V}[1]$; $\mathrm{Hom}_{\mathscr{D}}(\mathscr{U}, \mathscr{V}) = 0$; for each $X \in \mathscr{D}$, there exists a triangle $U \to X \to V \to U[1]$ with $U \in \mathscr{U}$ and $V \in \mathscr{V}$.

A right recollement of triangulated categories is a diagram of triangulated categories and triangle functors $\mathscr{D}' \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathscr{D} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathscr{D}''$, satisfying the following conditions:

(1) $(i_*, i^!)$ and (j^*, j_*) are adjoint pairs,
(2) $j^* i_* = 0$, i_* and j_* are full embedding,
(3) each object X in $\mathscr{D}$ determines a distinguished triangle

$$i_* i^! X \to X \to j_* j^* X \to i_* i^! X[1].$$

Lemma 4.1 ([18] Theorem 11.5.3). *Let $\mathscr{C}$, $\mathscr{D}$ be triangulated categories, such that canonical embedding $i : \mathscr{C} \to \mathscr{D}$ has a right adjoint $\tau : \mathscr{D} \to \mathscr{C}$. Then there is a right recollement*

$$\mathscr{C} \underset{\tau}{\overset{i}{\rightleftarrows}} \mathscr{D} \rightleftarrows \mathrm{Ker}\,\tau.$$

We define that $\mathcal{GAP}$-resolution dimension $\mathcal{GAP}$-res.dim M of any module M is to be the minimal integer $n \geqslant 0$ such that there is an $\mathcal{GAP}$-resolution $0 \to X^{-n} \to \cdots \to X^0 \to M \to 0$, if there is no such an integer, we set $\mathcal{GAP}$-res.dim $M = \infty$. Define the global $\mathcal{GAP}$-resolution dimension of a ring R, denote $\mathcal{GAP}$-res.dim R, to be the supreme of the $\mathcal{GAP}$-resolution dimensions of all modules.

Lemma 4.2. *If $\mathcal{GAP}$-res.dim $R < \infty$, then the canonical embedding $i : \mathbf{K}(\mathcal{GAP}) \to \mathbf{K}(R)$ has a right adjoint $\tau : \mathbf{K}(R) \to \mathbf{K}(\mathcal{GAP})$. Moreover, the natural composition functor $\mathbf{K}(\mathcal{GAP}) \to \mathbf{K}(R) \to \mathbf{D}_{\mathcal{GAP}}(R)$ is a triangle equivalence.*

Proof. Put: $\mathcal{X} = \mathcal{GAP}$ in [5, Proposition 3.5]. □

Theorem 4.3. *If $\mathcal{GAP}$-res.dim $R < \infty$, then there is a right recollement*

$$\mathbf{K}(\mathcal{GAP}) \underset{\tau}{\overset{i}{\rightleftarrows}} \mathbf{K}(R) \rightleftarrows \mathbf{K}_{\mathcal{GAP}\text{-}ac}(R)$$

In this case, $(\mathbf{K}(\mathcal{GAP}), \mathbf{K}(\mathcal{GAP})^{\perp})$ is a stable t-structure in $\mathbf{K}(R)$, where $\mathbf{K}(\mathcal{GAP})^{\perp} = \{X \in \mathbf{K}(R) \mid \mathrm{Hom}_{\mathbf{K}(R)}(Y, X) = 0$ for each $Y \in \mathbf{K}(\mathcal{GAP})\}$.

Proof. By Lemma 4.2 the canonical embedding $i : \mathbf{K}(\mathcal{GAP}) \to \mathbf{K}(R)$ has a right adjoint $\tau : \mathbf{K}(R) \to \mathbf{K}(\mathcal{GAP})$. We get a right recollement from Lemma 4.1

$$\mathbf{K}(\mathcal{GAP}) \underset{\tau}{\overset{i}{\rightleftarrows}} \mathbf{K}(R) \underset{j}{\overset{\pi}{\rightleftarrows}} \mathbf{K}_{\mathcal{GAP}\text{-}ac}(R).$$

In this case, we have $\tau j = 0$. It is well known that $\mathbf{K}(\mathcal{GAP})$ and $\mathbf{K}(\mathcal{GAP})^{\perp}$ are triangulated subcategories of $\mathrm{K}(R)$. Moreover, each object X in $\mathbf{K}(R)$ determines a distinguished triangle $i\tau X \to X \to j\pi X \to i\tau X[1]$. Since i is full embedding, we get that $i\tau X \in \mathbf{K}(\mathcal{GAP})$. For any $G \in \mathbf{K}(\mathcal{GAP})$, we have

$$\mathrm{Hom}_{\mathbf{K}(R)}(G, j\pi X) \cong \mathrm{Hom}_{\mathbf{K}(R)}(iG, j\pi X) \cong \mathrm{Hom}_{\mathbf{K}(R)}(G, \tau j\pi X) = 0,$$

so $j\pi X \in \mathbf{K}(\mathcal{GAP})^{\perp}$. Therefore $(\mathbf{K}(\mathcal{GAP}), \mathbf{K}(\mathcal{GAP})^{\perp})$ is a stable t-structure in $\mathbf{K}(R)$. □

Acknowledgements. This research was partially supported by NSFC 11761060 and NWNU-LKQN-16-13. We wish to thank the referee for the very helpful suggestions which have been incorporated herein.

References

[1] J. Asadollahi, R. Hafezi and R. Vahed, Gorenstein derived equivalences and their invariants, J. Pure Appl. Algebra **218** (2014), 888–903.

[2] L. L. Avramov and A. Martsinkovsky, Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension, Proc. London Math. Soc. (3) **85** (2002), 393–440.
[3] D. Bravo, J. Gillespie and M. Hovey, The stable module category of general ring, arXiv:1405.5768[math.RA], 2014.
[4] R.-O. Buchweitz, Matrix Cohen-Macaulay modules and Tate cohomology over Gorenstein rings, Hamburg, 1987, unpublished manuscript.
[5] X. W. Chen, Homotopy equivalences induced by balanced pairs, J. Algebra **324** (2010), 2718–2731.
[6] L. W. Christensen, A. Frankild and H. Holm, On Gorenstein projective, injective and flat dimensions—A functorial description with applications, J. Algebra **302** (2006), 231–279.
[7] N. Q. Ding, Y. L. Li and L. X. Mao, Strongly Gorenstein flat modules, J. Aust. Math. Soc. **66** (2009), 323–338.
[8] E. E. Enochs and O. M. G. Jenda, Relative homological algebra, de Gruyter expositions in math. **30**, Walter de Gruyter, Berlin, 2000.
[9] N. Gao and P. Zhang, Gorenstein derived categories, J. Algebra **323** (2010), 2041–2057.
[10] S. I. Gelfand and Y. I. Manin, Methods of Homological Algebra, Springer monographs in mathematics, Second edition, Spring-Verlag, Berlin, 2003.
[11] D. Happel, On Gorenstein algebras, Representatin theory of finite groups and finite-dimensional algebras, Progress in math. **95**, Birkhäuser, Basel, 1991, 389–404.
[12] H. Holm, Gorenstein derived functors, Proc. Amer. Math. Soc. **132** (2004), 1913–1923.
[13] M. Hovey, Cotorsion pairs, model category structures, and representation theory, Math. Z. **241** (2002), 553–592.
[14] J. Miyachi, Localiztion of triangulate categories and derived categories, J. Algebra **141** (1991), 463–483.
[15] A. Neeman, The derived category of an exact category, J. Algebra **135** (1990), 388–394.
[16] W. Ren, Z. K. Liu and G. Yang, Derived categories with respect to Ding modules, J. Algebra Appl. **12**, no. 6 (2013).
[17] J. L. Verdier, Des catégories dérivées abéliennes, Asterisque **239** (1996), xii+253 pp. (1997).
[18] P. Zhang, Triangulated category and derived category (in Chinese), Science Press, Beijing, 2015.
[19] P. Zhang, Categorical resolution of a class of derived categories, Sci. China Math. (2017), https://doi.org/10.1007/s11425-016-9117-0.
[20] R. M. Zhu, Z. K. Liu and Z. P. Wang, Gorenstein homological dimensions of modules over triangular matrix rings, Turk. J. Math. **40** (2016), 146–160.

Tianya Cao
Department of Mathematics
Northwest Normal University
Lanzhou 730070
P.R. China
E-mail: caotianya1979@126.com

Zhongkui Liu
Department of Mathematics
Northwest Normal University
Lanzhou 730070
P.R. China
E-mail: liuzk@nwnu.edu.cn

Xiaoyan Yang
Department of Mathematics
Northwest Normal University
Lanzhou 730070
P.R. China
E-mail: yangxy@nwnu.edu.cn