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UNIT TANGENT SPHERE BUNDLES WITH THE REEB FLOW INVARIANT RICCI OPERATOR

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Abstract

In this paper, we study unit tangent sphere bundles T_1M whose Ricci operator $\bar{S}$ is Reeb flow invariant, that is, $L_\xi \bar{S} = 0$. We prove that for a 3-dimensional Riemannian manifold M, T_1M satisfies $L_\xi \bar{S} = 0$ if and only if M is of constant curvature 1. Also, we prove that for a 4-dimensional Riemannian manifold M, T_1M satisfies $L_\xi \bar{S} = 0$ and $\ell \bar{S}\xi = 0$ if and only if M is of constant curvature 1 or 2, where $\ell = \bar{R}(\cdot, \xi)\xi$ is the characteristic Jacobi operator.

1. Introduction

In a contact manifold $(\overline{M}, \eta)$, we have a fundamental property that the Reeb vector field ξ generates a contact diffeomorphism, that is, $L_\xi \eta = 0$. For an associated Riemannian metric $\bar{g}$, if ξ generates an isometric flow, that is, $\overline{M}$ satisfies $L_\xi \bar{g} = 0$, then $\overline{M}$ is said to be K-contact. Recently, Perrone ([11]) introduced the so-called *H-contact manifolds*, which include K-contact manifolds. It means that the Reeb vector field ξ is a harmonic vector field. In the same paper, it was shown that the Reeb vector field of an H-contact manifold is the eigenvector of the Ricci operator $\bar{S}$.

It is very intriguing to study the interplay between Riemannian manifolds (M, g) and their unit tangent sphere bundles T_1M with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. In particular, the geodesic flow generated by the Reeb vector field ξ has a crucial role on the geometry of Riemannian manifold (M, g). As a classical result, Y. Tashiro ([14]) proved that $(T_1M, \eta, \bar{g})$ is a K-contact manifold if and only if (M, g) has constant sectional curvature 1.

In this paper, we study unit tangent sphere bundles T_1M whose Ricci operator $\bar{S}$ is Reeb flow invariant, that is, $L_\xi \bar{S} = 0$. In Section 3, we prove that

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for a 3-dimensional Riemannian manifold M, T_1M satisfies $L_\xi \bar{S} = 0$ if and only if M is of constant curvature 1 (Theorem 2). In Section 4, we investigate the relationship between the condition $L_\xi \bar{S} = 0$ and H-contact condition. Then we prove that a contact metric manifold $\bar{M}$ satisfying $L_\xi \bar{S} = 0$ is H-contact if and only if $\bar{M}$ satisfies $\ell \bar{S}\xi = 0$, where ℓ is the characteristic Jacobi operator (Theorem 4). Moreover, for a 2-dimensional Riemannian manifold M, T_1M satisfies $L_\xi \bar{S} = 0$ if and only if M is of constant curvature 0 or 1 (Proposition 7). For a 4-dimensional Riemannian manifold M, we prove that T_1M satisfies $L_\xi \bar{S} = 0$ and $\ell \bar{S}\xi = 0$ if and only if M is of constant curvature 1 or 2 (Theorem 9).

2. The unit tangent sphere bundle

First, we review some fundamental facts on contact metric manifolds. We refer to [1] for more details. All manifolds are assumed to be connected and of class C^∞. A $(2n-1)$-dimensional manifold $\bar{M}$ is said to be an *almost contact manifold* if its structure group of the linear frame bundle is reducible to $U(n-1) \times \{1\}$. This is equivalent to the existence of a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1 \quad \text{and} \quad \phi^2 = -\mathrm{id} + \eta \otimes \xi. \tag{2.1}$$

Here (ϕ, ξ, η) is called an *almost contact structure*. Then one can always find a compatible Riemannian metric $\bar{g}$:

$$\bar{g}(\phi\bar{X}, \phi\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}) \tag{2.2}$$

for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$. Such a metric is called an *associated metric* and $(\bar{M}, \phi, \xi, \eta, \bar{g})$ is said to be an *almost contact metric manifold*. The *fundamental 2-form* Φ is defined by $\Phi(\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \phi\bar{Y})$. If $\bar{M}$ satisfies in addition $d\eta = \Phi$, then $\bar{M}$ is called a *contact metric manifold*, where d is the exterior differential operator. We call the structure vector field ξ the *Reeb vector field* or the *characteristic vector field*. From (2.1) and (2.2) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\bar{X}) = \bar{g}(\bar{X}, \xi).$$

Given a contact metric manifold $\bar{M}$, we define the *structural operator* h by $h = \frac{1}{2} L_\xi \phi$, where L_ξ denotes Lie differentiation for ξ. Then we may observe that h is self-adjoint and satisfies

$$h\xi = 0 \quad \text{and} \quad h\phi = -\phi h, \tag{2.3}$$

$$\bar{\nabla}_{\bar{X}}\xi = -\phi\bar{X} - \phi h\bar{X}, \tag{2.4}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. From (2.3) and (2.4) we see that each trajectory of ξ is a geodesic. We denote by $\bar{R}$ the Riemannian curvature tensor defined by

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \bar{\nabla}_{\bar{X}}(\bar{\nabla}_{\bar{Y}}\bar{Z}) - \bar{\nabla}_{\bar{Y}}(\bar{\nabla}_{\bar{X}}\bar{Z}) - \bar{\nabla}_{[\bar{X}, \bar{Y}]}\bar{Z}$$

for all vector fields $\bar{X}$, $\bar{Y}$ and $\bar{Z}$. Along a trajectory of ξ, the Jacobi operator $\ell = \bar{R}(\cdot,\xi)\xi$ is a symmetric $(1,1)$-tensor field. We call it *the characteristic Jacobi operator*. We have

$$\ell = \phi\ell\phi - 2(h^2 + \phi^2), \tag{2.5}$$

$$\bar{\nabla}_\xi h = \phi - \phi\ell - \phi h^2. \tag{2.6}$$

A contact metric manifold for which ξ is Killing is called a K-contact manifold. It is easy to see that a contact metric manifold is K-contact if and only if $h = 0$ or, equivalently, $\ell = I - \eta \otimes \xi$. It is well-known that a unit vector field V on a Riemannian manifold M determines a map between M and T_1M. Then V is said to be *harmonic* if it is a critical point of the energy functional restricted to $\mathfrak{X}_1(M)$, the set of all sections of T_1M. In particular, a contact metric manifold $\bar{M}$ is said to be an *H-contact manifold* if its Reeb vector field is harmonic in above sense. In [11] it was proved that a contact metric manifold $\bar{M}$ is H-contact if and only if ξ is an eigenvector field of the Ricci operator $\bar{S}$ on $\bar{M}$. From this, it follows that any K-contact manifold is an H-contact manifold.

Let (M,g) be an n-dimensional Riemannian manifold and ∇ the associated Levi-Civita connection. Its Riemann curvature tensor R is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ for all vector fields X, Y and Z on M. The tangent bundle over (M,g) is denoted by TM and consists of pairs (p,u), where p is a point in M and u a tangent vector to M at p. The mapping $\pi : TM \to M$, $\pi(p,u) = p$, is the natural projection from TM onto M. For a vector field X on M, its *vertical lift* X^v on TM is the vector field defined by $X^v\omega = \omega(X) \circ \pi$, where ω is a 1-form on M. For the Levi-Civita connection ∇ on M, the *horizontal lift* X^h of X is defined by $X^h\omega = \nabla_X\omega$. The tangent bundle TM can be endowed in a natural way with a Riemannian metric $\tilde{g}$, the so-called *Sasaki metric*, depending only on the Riemannian metric g on M. It is determined by

$$\tilde{g}(X^h, Y^h) = \tilde{g}(X^v, Y^v) = g(X,Y) \circ \pi, \quad \tilde{g}(X^h, Y^v) = 0$$

for all vector fields X and Y on M. Also, TM admits an almost complex structure tensor J defined by $JX^h = X^v$ and $JX^v = -X^h$. Then $\tilde{g}$ is a Hermitian metric for the almost complex structure J.

The unit tangent sphere bundle $\bar{\pi} : T_1M \to M$ is a hypersurface of TM given by $g_p(u,u) = 1$. Note that $\bar{\pi} = \pi \circ i$, where i is the immersion of T_1M into TM. A unit normal vector field $N = u^v$ to T_1M is given by the vertical lift of u for (p,u). The horizontal lift of a vector is tangent to T_1M, but the vertical lift of a vector is not tangent to T_1M in general. So, we define the *tangential lift* of X to $(p,u) \in T_1M$ by

$$X^t_{(p,u)} = (X - g(X,u)u)^v.$$

Clearly, the tangent space $T_{(p,u)}T_1M$ is spanned by vectors of the form X^h and X^t, where $X \in T_pM$.

We now define the standard contact metric structure of the unit tangent sphere bundle T_1M over a Riemannian manifold (M, g). The metric g' on T_1M is induced from the Sasaki metric $\tilde{g}$ on TM. Using the almost complex structure J on TM, we define a unit vector field ξ', a 1-form η' and a (1,1)-tensor field ϕ' on T_1M by

$$\xi' = -JN, \quad \phi' = J - \eta' \otimes N.$$

Since $g'(\overline{X}, \phi'\overline{Y}) = 2\,d\eta'(\overline{X}, \overline{Y})$, (η', g', ϕ', ξ') is not a contact metric structure. If we rescale this structure by

$$\xi = 2\xi', \quad \eta = \frac{1}{2}\eta', \quad \phi = \phi', \quad \bar{g} = \frac{1}{4}g',$$

we get the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Here the tensor ϕ is explicitly given by

$$\phi X^t = -X^h + \frac{1}{2}g(X, u)\xi, \quad \phi X^h = X^t, \tag{2.7}$$

where X and Y are vector fields on M. From now on, we consider $T_1M = (T_1M, \eta, \bar{g})$ with the standard contact metric structure.

The Levi-Civita connection $\overline{\nabla}$ of T_1M is described by

$$\begin{aligned} \overline{\nabla}_{X^t} Y^t &= -g(Y, u)X^t, \\ \overline{\nabla}_{X^t} Y^h &= \frac{1}{2}(R(u, X)Y)^h, \\ \overline{\nabla}_{X^h} Y^t &= (\nabla_X Y)^t + \frac{1}{2}(R(u, Y)X)^h, \\ \overline{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2}(R(X, Y)u)^t \end{aligned} \tag{2.8}$$

for all vector fields X and Y on M.

Also the Riemann curvature tensor $\overline{R}$ of T_1M is given by

$$\begin{aligned} \overline{R}(X^t, Y^t)Z^t &= -(g(X, Z) - g(X, u)g(Z, u))Y^t \\ &\quad + (g(Y, Z) - g(Y, u)g(Z, u))X^t, \\ \overline{R}(X^t, Y^t)Z^h &= \{R(X - g(X, u)u, Y - g(Y, u)u)Z\}^h \\ &\quad + \frac{1}{4}\{[R(u, X), R(u, Y)]Z\}^h, \\ \overline{R}(X^h, Y^t)Z^t &= -\frac{1}{2}\{R(Y - g(Y, u)u, Z - g(Z, u)u)X\}^h \\ &\quad - \frac{1}{4}\{R(u, Y)R(u, Z)X\}^h, \end{aligned} \tag{2.9}$$

$$
\begin{aligned}
\bar{R}(X^h, Y^t)Z^h &= \frac{1}{2}\{R(X,Z)(Y - g(Y,u)u)\}^t - \frac{1}{4}\{R(X, R(u, Y)Z)u\}^t \\
&\quad + \frac{1}{2}\{(\nabla_X R)(u, Y)Z\}^h, \\
\bar{R}(X^h, Y^h)Z^t &= \{R(X, Y)(Z - g(Z,u)u)\}^t \\
&\quad + \frac{1}{4}\{R(Y, R(u,Z)X)u - R(X, R(u,Z)Y)u\}^t \\
&\quad + \frac{1}{2}\{(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X\}^h, \\
\bar{R}(X^h, Y^h)Z^h &= (R(X, Y)Z)^h + \frac{1}{2}\{R(u, R(X, Y)u)Z\}^h \\
&\quad - \frac{1}{4}\{R(u, R(Y,Z)u)X - R(u, R(X,Z)u)Y\}^h \\
&\quad + \frac{1}{2}\{(\nabla_Z R)(X, Y)u\}^t
\end{aligned} \tag{2.9}
$$

for all vector fields X, Y and Z on M.

Next, to calculate the Ricci curvature tensor $\bar{\rho}$ of T_1M at the point $(p,u) \in T_1M$, let $e_1, \ldots, e_n = u$ be an orthonormal basis of T_pM. Then $\bar{\rho}$ is given by

$$
\begin{aligned}
\bar{\rho}(X^t, Y^t) &= (n-2)(g(X, Y) - g(X,u)g(Y,u)) \\
&\quad + \frac{1}{4}\sum_{i=1}^{n} g(R(u, X)e_i, R(u, Y)e_i), \\
\bar{\rho}(X^t, Y^h) &= \frac{1}{2}((\nabla_u \rho)(X, Y) - (\nabla_X \rho)(u, Y)), \\
\bar{\rho}(X^h, Y^h) &= \rho(X, Y) - \frac{1}{2}\sum_{i=1}^{n} g(R(u, e_i)X, R(u, e_i)Y),
\end{aligned} \tag{2.10}
$$

where ρ denotes the Ricci curvature tensor of M. We can refer to [4, 9] for formulas (2.8)–(2.10).

From $\xi = 2u^h$ and (2.8), it follows

$$
\bar{\nabla}_{X^t}\xi = -2\phi X^t - (R_u X)^h, \quad \bar{\nabla}_{X^h}\xi = -(R_u X)^t \tag{2.11}
$$

where $R_u = R(\cdot, u)u$ is the Jacobi operator associated with the unit vector u. From (2.4) and (2.11), it follows that

$$
\begin{aligned}
hX^t &= X^t - (R_u X)^t, \\
hX^h &= -X^h + \frac{1}{2}g(X,u)\xi + (R_u X)^h.
\end{aligned} \tag{2.12}
$$

The above formulae are also found in [2, 3, 8].

3. Reeb flow invariant Ricci operators

Suppose that the contact metric manifold $\overline{M}$ satisfies the condition $L_\xi \overline{S} = 0$ for the Ricci operator $\overline{S}$ and the Reeb vector field ξ on $\overline{M}$. Then from the definition of Lie differentiation and (2.4) we have

$$\begin{aligned}0 &= (L_\xi \overline{S})\overline{X} \\ &= L_\xi \overline{S}\overline{X} - \overline{S}(L_\xi \overline{X}) \\ &= (\overline{\nabla}_\xi \overline{S})\overline{X} - \overline{\nabla}_{\overline{S}\overline{X}}\xi + \overline{S}(\overline{\nabla}_{\overline{X}}\xi) \\ &= (\overline{\nabla}_\xi \overline{S})\overline{X} + \phi\overline{S}\overline{X} - \overline{S}\phi\overline{X} + \phi h\overline{S}\overline{X} - \overline{S}\phi h\overline{X}\end{aligned} \tag{3.1}$$

for any vector field $\overline{X}$ on $\overline{M}$. In (3.1), since $\overline{\nabla}_\xi \overline{S} + \phi\overline{S} - \overline{S}\phi$ is a symmetric operator and $\phi h\overline{S} - \overline{S}\phi h$ is a skew-symmetric operator, $\overline{M}$ satisfies the condition $L_\xi \overline{S} = 0$ if and only if it satisfies

$$\overline{\nabla}_\xi \overline{S} = \overline{S}\phi - \phi\overline{S} \tag{3.2}$$

and

$$\phi h\overline{S} = \overline{S}\phi h. \tag{3.3}$$

Now, we consider the unit tangent sphere bundle T_1M over an n-dimensional Riemannian manifold M satisfying the condition $L_\xi \overline{S} = 0$. From (2.7), (2.10) and (2.12), we can calculate

$$\begin{aligned}0 &= \bar{g}((\overline{\nabla}_\xi \overline{S})X^t - \overline{S}\phi X^t + \phi\overline{S}X^t, Y^t) \\ &= (\overline{\nabla}_\xi \bar{\rho})(X^t, Y^t) - \bar{\rho}(\phi X^t, Y^t) - \bar{\rho}(X^t, \phi Y^t) \\ &= \frac{1}{2}\sum_{i=1}^{n}\{g((\nabla_u R)(u, X)e_i, R(u, Y)e_i) + g(R(u, X)e_i, (\nabla_u R)(u, Y)e_i)\} \\ &\quad + \frac{1}{2}\{(\nabla_u \rho)(R_u X, Y) + (\nabla_u \rho)(X, R_u Y) \\ &\qquad - (\nabla_X \rho)(u, R_u Y) - (\nabla_Y \rho)(u, R_u X)\} \\ &\quad + \frac{1}{2}\{2(\nabla_u \rho)(X, Y) - (\nabla_X \rho)(u, Y) - (\nabla_Y \rho)(u, X)\} \\ &\quad - \frac{1}{2}\{g(X, u)((\nabla_u \rho)(u, Y) - (\nabla_Y \rho)(u, u)) \\ &\qquad + g(Y, u)((\nabla_u \rho)(u, X) - (\nabla_X \rho)(u, u))\},\end{aligned} \tag{3.4}$$

$$\begin{aligned}0 &= \bar{g}((\overline{\nabla}_\xi \overline{S})X^t - \overline{S}\phi X^t + \phi\overline{S}X^t, Y^h) \\ &= (\overline{\nabla}_\xi \bar{\rho})(X^t, Y^h) - \bar{\rho}(\phi X^t, Y^h) - \bar{\rho}(X^t, \phi Y^h) \\ &= (\nabla^2_{uu}\rho)(X, Y) - (\nabla^2_{uX}\rho)(u, Y) - (n-2)g(X, R_u Y) + \rho(R_u X, Y)\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&- \frac{1}{2}\sum_{i=1}^{n} g(R(u,e_i)R_uX, R(u,e_i)Y) - \frac{1}{4}\sum_{i=1}^{n} g(R(u,X)e_i, R(u,R_uY)e_i) \\
&+ \rho(X,Y) - \frac{1}{2}\sum_{i=1}^{n} g(R(u,e_i)X, R(u,e_i)Y) \\
&- g(X,u)\left\{\rho(Y,u) - \frac{1}{2}\sum_{i=1}^{n} g(R(u,e_i)u, R(u,e_i)Y)\right\} \\
&- (n-2)(g(X,Y) - g(X,u)g(Y,u)) - \frac{1}{4}\sum_{i=1}^{n} g(R(u,X)e_i, R(u,Y)e_i),
\end{aligned}
$$

$$
\begin{aligned}
(3.6)\qquad 0 &= \bar{g}((\overline{\nabla}_\xi \overline{S})X^h - \overline{S}\phi X^h + \phi\overline{S}X^h, Y^h) \\
&= (\overline{\nabla}_\xi\bar{\rho})(X^h, Y^h) - \bar{\rho}(\phi X^h, Y^h) - \bar{\rho}(X^h, \phi Y^h) \\
&= -\sum_{i=1}^{n}\{g((\nabla_u R)(u,e_i)X, R(u,e_i)Y) + g(R(u,e_i)X, (\nabla_u R)(u,e_i)Y)\} \\
&\quad - \frac{1}{2}\{(\nabla_u\rho)(R_uX, Y) + (\nabla_u\rho)(X, R_uY) \\
&\qquad - (\nabla_X\rho)(u, R_uY) - (\nabla_Y\rho)(u, R_uX)\} \\
&\quad + \frac{1}{2}\{2(\nabla_u\rho)(X,Y) + (\nabla_X\rho)(u,Y) + (\nabla_Y\rho)(u,X)\},
\end{aligned}
$$

$$
\begin{aligned}
(3.7)\qquad 0 &= \bar{g}(\overline{S}\phi hX^t - \phi h\overline{S}X^t, Y^t) \\
&= \bar{\rho}(\phi hX^t, Y^t) + \bar{\rho}(X^t, h\phi Y^t) \\
&= \frac{1}{2}\{(\nabla_Y\rho)(u,X) - (\nabla_X\rho)(u,Y) - (\nabla_u\rho)(X, R_uY) \\
&\qquad + (\nabla_u\rho)(R_uX, Y) + (\nabla_X\rho)(u, R_uY) - (\nabla_Y\rho)(u, R_uX) \\
&\qquad + g(X,u)((\nabla_u\rho)(u,Y) - (\nabla_Y\rho)(u,u)) \\
&\qquad - g(Y,u)((\nabla_u\rho)(u,X) - (\nabla_X\rho)(u,u))\},
\end{aligned}
$$

$$
\begin{aligned}
(3.8)\qquad 0 &= \bar{g}(\overline{S}\phi hX^t - \phi h\overline{S}X^t, Y^h) \\
&= \bar{\rho}(\phi hX^t, Y^h) + \bar{\rho}(X^t, h\phi Y^h) \\
&= (n-2)(g(X,Y) - g(X,u)g(Y,u)) + \frac{1}{4}\sum_{i=1}^{n} g(R(u,X)e_i, R(u,Y)e_i) \\
&\quad - (n-2)g(X, R_uY) - \frac{1}{4}\sum_{i=1}^{n} g(R(u,X)e_i, R(u,R_uY)e_i)
\end{aligned}
$$

$$
\begin{aligned}
&- \rho(X, Y) + \frac{1}{2}\sum_{i=1}^{n} g(R(u, e_i)X, R(u, e_i)Y) \\
&+ g(X, u)\left\{ \rho(Y, u) - \frac{1}{2}\sum_{i=1}^{n} g(R(u, e_i)u, R(u, e_i)Y) \right\} \\
&+ \rho(R_u X, Y) - \frac{1}{2}\sum_{i=1}^{n} g(R(u, e_i)R_u X, R(u, e_i)Y),
\end{aligned}
$$

$$
\begin{aligned}
0 &= \bar{g}(\bar{S}\phi h X^h - \phi h \bar{S} X^h, Y^h) \\
&= \bar{\rho}(\phi h X^h, Y^h) + \bar{\rho}(X^h, h\phi Y^h) \\
&= \frac{1}{2}\{(\nabla_X \rho)(u, Y) - (\nabla_Y \rho)(u, X) + (\nabla_u \rho)(R_u X, Y) \\
&\qquad - (\nabla_u \rho)(X, R_u Y) - (\nabla_{R_u X}\rho)(u, Y) + (\nabla_{R_u Y}\rho)(u, X)\}.
\end{aligned} \tag{3.9}
$$

Therefore T_1M satisfies $L_\xi \bar{S} = 0$ if and only if M satisfies (3.4)–(3.9).

THEOREM 1. *Let $M = (M, g)$ be an n-dimensional Riemannian manifold of constant curvature c and let T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ over M. Then T_1M satisfies $L_\xi \bar{S} = 0$ if and only if M is of constant curvature 1 or $n-2$.*

Proof. Suppose that M is a space of constant curvature c and T_1M satisfies $L_\xi \bar{S} = 0$. Then from (3.5) and (3.8), we obtain two equations;

$$c^3 - (n-2)c^2 - c + (n-2) = 0, \tag{3.10}$$

$$c^3 - nc^2 + (2n-3)c - (n-2) = 0. \tag{3.11}$$

Therefore we see that T_1M satisfies $L_\xi \bar{S} = 0$ if and only if $c = 1$ or $c = n-2$. □

Now, we study the case of 3-dimensional base manifold. Then we have

THEOREM 2. *Let $M = (M, g)$ be a 3-dimensional Riemannian manifold and let T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ over M. Then T_1M satisfies $L_\xi \bar{S} = 0$ if and only if M is of constant curvature 1.*

Proof. Suppose that M is a 3-dimensional Riemannian manifold and let $\{e_i\}_{i=1}^3$ be an orthonormal basis of eigenvectors of the Ricci operator S_p at point $p \in M$, that is,

$$Se_i = \alpha_i e_i, \quad i = 1, 2, 3.$$

It is well-known that the curvature tensor R of 3-dimensional Riemannian manifold (M,g) is of the following form

$$R(X,Y)Z = \rho(Y,Z)X - \rho(X,Z)Y + g(Y,Z)SX - g(X,Z)SY - \frac{\tau}{2}\{g(Y,Z)X - g(X,Z)Y\}, \tag{3.12}$$

where τ denotes the scalar curvature on M. If we put $u = e_1$, $X = Y = e_2$ in (3.8), then using (3.12), we have

$$\left(1 - \alpha_1 - \alpha_2 + \frac{\tau}{2}\right)\left(\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 - \tau\alpha_1 - \tau\alpha_2 - \alpha_2 + 1 + \frac{\tau^2}{4}\right) = 0. \tag{3.13}$$

Similarly, putting $u = e_2$, $X = Y = e_1$ in (3.8), we have

$$\left(1 - \alpha_1 - \alpha_2 + \frac{\tau}{2}\right)\left(\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 - \tau\alpha_1 - \tau\alpha_2 - \alpha_1 + 1 + \frac{\tau^2}{4}\right) = 0. \tag{3.14}$$

This time, we put $u = e_1$, $X = Y = e_3$ in (3.8), then we have

$$\left(1 - \alpha_1 - \alpha_3 + \frac{\tau}{2}\right)\left(\alpha_1^2 + \alpha_3^2 + 2\alpha_1\alpha_3 - \tau\alpha_1 - \tau\alpha_3 - \alpha_3 + 1 + \frac{\tau^2}{4}\right) = 0. \tag{3.15}$$

Similarly, putting $u = e_3$, $X = Y = e_1$ in (3.8), we have

$$\left(1 - \alpha_1 - \alpha_3 + \frac{\tau}{2}\right)\left(\alpha_1^2 + \alpha_3^2 + 2\alpha_1\alpha_3 - \tau\alpha_1 - \tau\alpha_3 - \alpha_1 + 1 + \frac{\tau^2}{4}\right) = 0. \tag{3.16}$$

In addition, put $u = e_2$, $X = Y = e_3$ in (3.8) to have

$$\left(1 - \alpha_2 - \alpha_3 + \frac{\tau}{2}\right)\left(\alpha_2^2 + \alpha_3^2 + 2\alpha_2\alpha_3 - \tau\alpha_2 - \tau\alpha_3 - \alpha_3 + 1 + \frac{\tau^2}{4}\right) = 0. \tag{3.17}$$

Similarly, put $u = e_3$, $X = Y = e_2$ in (3.8) to obtain

$$\left(1 - \alpha_2 - \alpha_3 + \frac{\tau}{2}\right)\left(\alpha_2^2 + \alpha_3^2 + 2\alpha_2\alpha_3 - \tau\alpha_2 - \tau\alpha_3 - \alpha_2 + 1 + \frac{\tau^2}{4}\right) = 0. \tag{3.18}$$

From (3.13) and (3.14), we obtain either $1 - \alpha_1 - \alpha_2 + \frac{\tau}{2} = 0$ or $\alpha_1 = \alpha_2$. Also, from (3.15) and (3.16), we obtain either $1 - \alpha_1 - \alpha_3 + \frac{\tau}{2} = 0$ or $\alpha_1 = \alpha_3$. We deduce from (3.17) and (3.18) that either $1 - \alpha_2 - \alpha_3 + \frac{\tau}{2} = 0$ or $\alpha_2 = \alpha_3$ holds. Therefore we may consider the following eight cases.

(I) $1 - \alpha_1 - \alpha_2 + \frac{\tau}{2} = 0$ and $1 - \alpha_1 - \alpha_3 + \frac{\tau}{2} = 0$ and $1 - \alpha_2 - \alpha_3 + \frac{\tau}{2} = 0$,

(II) $1 - \alpha_1 - \alpha_2 + \frac{\tau}{2} = 0$ and $1 - \alpha_1 - \alpha_3 + \frac{\tau}{2} = 0$ and $\alpha_2 = \alpha_3$,

(III) $1-\alpha_1-\alpha_2+\frac{\tau}{2}=0$ and $\alpha_1=\alpha_3$ and $1-\alpha_2-\alpha_3+\frac{\tau}{2}=0$,
(IV) $\alpha_1=\alpha_2$ and $1-\alpha_1-\alpha_3+\frac{\tau}{2}=0$ and $1-\alpha_2-\alpha_3+\frac{\tau}{2}=0$,
(V) $1-\alpha_1-\alpha_2+\frac{\tau}{2}=0$ and $\alpha_1=\alpha_3$ and $\alpha_2=\alpha_3$,
(VI) $\alpha_1=\alpha_2$ and $1-\alpha_1-\alpha_3+\frac{\tau}{2}=0$ and $\alpha_2=\alpha_3$,
(VII) $\alpha_1=\alpha_2$ and $\alpha_1=\alpha_3$ and $1-\alpha_2-\alpha_3+\frac{\tau}{2}=0$,
(VIII) $\alpha_1=\alpha_2$ and $\alpha_1=\alpha_3$ and $\alpha_2=\alpha_3$.

For cases (I), (V), (VI), (VII), we immediately see that each case gives a contradiction. In case (II), since $\tau=\alpha_1+\alpha_2+\alpha_3$ and $\alpha_2=\alpha_3$, we have $\alpha_1=2$. Also, from (3.17) we obtain

$$\alpha_2^2-3\alpha_2+2=0,$$

that is, $\alpha_2=1$ or $\alpha_2=2$. Thus, since $\alpha_1\neq\alpha_2$, we have $\alpha_1=2$ and $\alpha_2=\alpha_3=1$.

On the other hand, if we set $u=\frac{1}{\sqrt{2}}(e_1+e_2)$ and $X=Y=e_3$ in (3.8), then by the direct calculation we have

$$\left\{1-\alpha_3+\frac{1}{2}\left(\alpha_1+\alpha_3-\frac{\tau}{2}\right)^2+\frac{1}{2}\left(\alpha_2+\alpha_3-\frac{\tau}{2}\right)^2\right\} \times\left\{1-\frac{1}{2}\left(\alpha_1+\alpha_3-\frac{\tau}{2}\right)-\frac{1}{2}\left(\alpha_2+\alpha_3-\frac{\tau}{2}\right)\right\}=0. \tag{3.19}$$

But, for $\alpha_1=2$ and $\alpha_2=\alpha_3=1$, (3.19) does not hold. By similar arguments to those for case (II), we see that the cases (III) and (IV) cannot occur. Lastly, in case (VIII), we immediately see that M is Einstein and hence M is of constant curvature. Due to Theorem 1, M is of constant curvature 1 and the converse is evident. □

Together with Y. Tashiro's result, we have

Corollary 3. *Let (M,g) be a 3-dimensional Riemannian manifold. Then the unit tangent sphere bundle T_1M satisfies $L_\xi\bar{S}=0$ if and only if ξ is a Killing vector field.*

4. The case of 4-dimensional base manifolds

First, we investigate the relationship between the condition $L_\xi\bar{S}=0$ and H-contact condition on contact metric manifold. Let $\bar{M}$ be a contact metric manifold whose Ricci operator $\bar{S}$ is Reeb flow invariant. Then from (3.3), we have

$$h\bar{S}\xi=0. \tag{4.1}$$

Differentiating (4.1) with respect to ξ and using (3.2), we have

$$0 = (\overline{\nabla}_{\xi} h)\overline{S}\xi + h(\overline{S}\phi - \phi\overline{S})\xi. \tag{4.2}$$

From (2.3), (4.1) and (4.2), we see that $\overline{M}$ satisfies $(\overline{\nabla}_{\xi} h)\overline{S}\xi = 0$. We obtain from (2.6) that

$$\begin{aligned} 0 &= (\overline{\nabla}_{\xi} h)\overline{S}\xi \\ &= (\phi - \phi h^2 - \phi\ell)\overline{S}\xi \\ &= \phi\overline{S}\xi - \phi\ell\overline{S}\xi. \end{aligned} \tag{4.3}$$

Applying ϕ to (4.3), we obtain

$$-\overline{S}\xi + \eta(\overline{S}\xi)\xi + \ell\overline{S}\xi = 0, \tag{4.4}$$

and hence from (4.4), we have

Theorem 4. *Let $\overline{M}$ be a contact metric manifold and assume that $\overline{M}$ satisfies $L_{\xi}\overline{S} = 0$. Then $\overline{M}$ is H-contact if and only if $\overline{M}$ satisfies $\ell\overline{S}\xi = 0$.*

Also, from the above theorem we can easily obtain

Corollary 5. *If a contact metric manifold $\overline{M}$ satisfies $L_{\xi}\overline{S} = 0$ and $\ell\overline{S} = \overline{S}\ell$, then $\overline{M}$ is H-contact.*

In [7] the first named author classified $\overline{M}$ satisfying $L_{\xi}\overline{S} = 0$ for the dimension 3. Indeed, in the proof of Main Theorem in [7], we have

Proposition 6. *Let $\overline{M}$ be a 3-dimensional contact metric manifold. If $\overline{M}$ satisfies $L_{\xi}\overline{S} = 0$, then $\overline{M}$ is H-contact.*

Boeckx and Vanhecke ([5]) proved that the unit tangent sphere bundle of a 2- or 3-dimensional Riemannian manifold is H-contact if and only if the base manifold is of constant curvature. Calvaruso and Perrone ([6]) obtained the same result in the case of an $n(\geq 4)$-dimensional conformally flat manifold. Thus, from the result of Boeckx and Vanhecke, Proposition 6 and Theorem 1, we have

Proposition 7. *Let $M = (M, g)$ be a 2-dimensional Riemannian manifold and let T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ over M. Then T_1M satisfies $L_{\xi}\overline{S} = 0$ if and only if M is of constant curvature 0 or 1.*

Also, we have

Proposition 8. *Let $M = (M, g)$ be an $n(\geq 4)$-dimensional conformally flat manifold and let T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ over M. If T_1M satisfies $L_{\xi}\overline{S} = 0$ and $\ell\overline{S}\xi = 0$, then M is of constant curvature 1 or $n - 2$.*

Now we concentrate on the case of dim $M = 4$. Then, we have

THEOREM 9. *Let $M = (M, g)$ be a 4-dimensional Riemannian manifold and let T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ over M. Then T_1M satisfies $L_\xi \bar{S} = 0$ and $\ell \bar{S}\xi = 0$ if and only if M is of constant curvature 1 or 2.*

Proof. Suppose that the unit tangent sphere bundle T_1M over an n-dimensional Riemannian manifold M satisfies the condition $L_\xi \bar{S} = 0$ for the Ricci operator $\bar{S}$ on T_1M. Then T_1M satisfies $L_\xi \bar{S} = 0$ if and only if M satisfies (3.4)–(3.9). In (3.8) we put $X = e_a$, $Y = e_b$, $u = e_c$. Then we have

$$(4.5)\quad (n-2)(\delta_{ab} - \delta_{ac}\delta_{bc}) + \frac{1}{4}\sum_{i,j=1}^{n} R_{caij}R_{cbij} - (n-2)R_{accb} - \frac{1}{4}\sum_{i,j,k=1}^{n} R_{caij}R_{bcck}R_{ckij}$$
$$- \rho_{ab} + \frac{1}{2}\sum_{i,j=1}^{n} R_{ciaj}R_{cibj} + \delta_{ac}\left(\rho_{bc} - \frac{1}{2}\sum_{i,j=1}^{n} R_{cicj}R_{cibj}\right)$$
$$+ \sum_{k=1}^{n} R_{acck}\rho_{kb} - \frac{1}{2}\sum_{i,j,k=1}^{n} R_{acck}R_{cikj}R_{cibj} = 0,$$

where δ_{ab} denotes the Kronecker's delta, $R_{abcd} = g(R(e_a, e_b)e_c, e_d)$ and $\rho_{ab} = \rho(e_a, e_b)$. For $a = b \neq c$ in (4.5), we get

$$(4.6)\quad 4(n-2) + \sum_{i,j=1}^{n} R_{caij}^2 - 4(n-2)R_{acca} - \sum_{i,j,k=1}^{n} R_{caij}R_{acck}R_{ckij}$$
$$- 4\rho_{aa} + 2\sum_{i,j=1}^{n} R_{ciaj}^2 + 4\sum_{k=1}^{n} R_{acck}\rho_{ka} - 2\sum_{i,j,k=1}^{n} R_{acck}R_{cikj}R_{ciaj} = 0.$$

From Theorem 4, we see that T_1M satisfying $L_\xi \bar{S} = 0$ and $\ell \bar{S}\xi = 0$ has an H-contact structure. We suppose that $n = 4$. Then, owing to a result in [10], M is 2-stein, that is, an Einstein manifold satisfying $\sum_{i,j}^{n}(R_{uiuj})^2 = \mu(p)|u|^2$ for all $u \in T_pM$, $p \in M$, where $R_{uiuj} = g(R(u, e_i)u, e_j)$, $|u|^2 = g(u, u)$ and μ is a real-valued function on M. Now, since M is Einstein i.e., $\rho = \gamma g$ (γ is a constant on M), we may choose an orthonormal basis $\{e_i\}_{i=1}^4$ (known as the Singer-Thorpe basis) at each point $p \in M$ such that

$$(4.7)\quad \begin{cases} R_{1212} = R_{3434} = \lambda_1, \quad R_{1313} = R_{2424} = \lambda_2, \quad R_{1414} = R_{2323} = \lambda_3, \\ R_{1234} = \mu_1, \quad R_{1342} = \mu_2, \quad R_{1423} = \mu_3, \\ R_{ijkl} = 0 \quad \text{whenever just three of the indices } i, j, k, l \\ \qquad \text{are distinct (cf. [13]).} \end{cases}$$

Note that

$$(4.8)\quad \mu_1 + \mu_2 + \mu_3 = 0$$

by the first Bianchi identity and

$$\lambda_1 + \lambda_2 + \lambda_3 = -\frac{\tau}{4}, \tag{4.9}$$

where τ is the scalar curvature of M.

It is also known that a 4-dimensional Einstein manifold M is 2-stein if and only if

$$\mu_1 = \lambda_1 + \frac{\tau}{12}, \quad \mu_2 = \lambda_2 + \frac{\tau}{12}, \quad \mu_3 = \lambda_3 + \frac{\tau}{12} \tag{4.10}$$

or

$$-\mu_1 = \lambda_1 + \frac{\tau}{12}, \quad -\mu_2 = \lambda_2 + \frac{\tau}{12}, \quad -\mu_3 = \lambda_3 + \frac{\tau}{12}$$

holds for any Singer-Thorpe basis $\{e_i\}_{i=1}^4$ at each point $p \in M$ (cf. [12]).

On the other hand, if we put $a = b = 1$, $c = 2$ and $a = b = 3$, $c = 4$ in (4.6), then, using (4.7), we have

$$(1 + \lambda_1)(2\gamma - 4 - 2\lambda_1^2 - \mu_1^2 - \mu_2^2 - \mu_3^2) = 0. \tag{4.11}$$

Similarly, put $a = b = 1$, $c = 3$ and $a = b = 2$, $c = 4$ in (4.6) to have

$$(1 + \lambda_2)(2\gamma - 4 - 2\lambda_2^2 - \mu_1^2 - \mu_2^2 - \mu_3^2) = 0. \tag{4.12}$$

For $a = b = 1$, $c = 4$ and $a = b = 2$, $c = 3$ in (4.6), we have

$$(1 + \lambda_3)(2\gamma - 4 - 2\lambda_3^2 - \mu_1^2 - \mu_2^2 - \mu_3^2) = 0. \tag{4.13}$$

From (4.11)–(4.13), we get the following cases.

(i) $\lambda_1 = \lambda_2 = \lambda_3 = -1$,
(ii) $\lambda_1 = \lambda_2 = -1$ and $2\gamma = 4 + 2\lambda_3^2 + \mu_1^2 + \mu_2^2 + \mu_3^2$,
(iii) $\lambda_1 = \lambda_3 = -1$ and $2\gamma = 4 + 2\lambda_2^2 + \mu_1^2 + \mu_2^2 + \mu_3^2$,
(iv) $\lambda_2 = \lambda_3 = -1$ and $2\gamma = 4 + 2\lambda_1^2 + \mu_1^2 + \mu_2^2 + \mu_3^2$,
(v) $\lambda_1 = -1$ and $\lambda_2^2 = \lambda_3^2$,
(vi) $\lambda_2 = -1$ and $\lambda_1^2 = \lambda_3^2$,
(vii) $\lambda_3 = -1$ and $\lambda_1^2 = \lambda_2^2$,
(viii) $\lambda_1^2 = \lambda_2^2 = \lambda_3^2$.

From case (i), we see that M is of constant curvature 1. In case (ii), we get from (4.9) and (4.10)

$$\lambda_3 = 2 - \frac{\tau}{4}, \quad \mu_1 = \mu_2 = -1 + \frac{\tau}{12}, \quad \mu_3 = 2 - \frac{\tau}{6}. \tag{4.14}$$

Applying (4.14) in case (ii), we have

$$(\tau - 12)(\tau - 9) = 0. \tag{4.15}$$

Similarly, in cases (iii) and (iv), we get (4.15). But, the case $\tau = 12$ yields again that M is of constant curvature 1. For the case $\tau = 9$, from (4.14) we get $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = -\frac{1}{4}$, $\mu_1 = \mu_2 = -\frac{1}{4}$ and $\mu_3 = \frac{1}{2}$. Use (4.7) to check (3.5), a necessary equation for T_1M to satisfy $L_\xi \overline{S} = 0$. Indeed, the right hand side of (3.5) for $u = e_1$, $X = Y = e_2$, for example, becomes $-4 + \frac{\tau}{2} - 2\lambda_1^2 - \mu_1^2 - \mu_2^2 - \mu_3^2$. It gives a contradiction. In case (v), we consider two cases $\lambda_2 = \lambda_3$ or $\lambda_2 = -\lambda_3$. If $\lambda_2 = \lambda_3$, from (4.9) and (4.10) we get

$$\lambda_2 = \lambda_3 = \frac{1}{2} - \frac{\tau}{8}, \quad \mu_1 = -1 + \frac{\tau}{12}, \quad \mu_2 = \mu_3 = \frac{1}{2} - \frac{\tau}{24}. \tag{4.16}$$

From (4.12) and (4.16), we obtain

$$(\tau - 12)^2 = 0,$$

that is, $\lambda_2 = \lambda_3 = -1$, which yields that this is a contradiction. If $\lambda_2 = -\lambda_3$, from (4.9) and (4.10) we get

$$\tau = 4, \quad \mu_1 = -\frac{2}{3}, \quad \mu_2 = \lambda_2 + \frac{1}{3}, \quad \mu_3 = \lambda_3 + \frac{1}{3}. \tag{4.17}$$

From (4.12) and (4.17), we obtain

$$3\lambda_2^2 + 2 = 0,$$

which can not occur. Similarly, the cases (vi) and (vii) can not hold.

Lastly, we consider the case (viii);

$$\lambda_1^2 = \lambda_2^2 = \lambda_3^2. \tag{4.18}$$

Then, from (4.8), (4.9), (4.10) and (4.18) we obtain the following four cases.

(a) $\lambda_1 = \lambda_2 = \lambda_3 = -\frac{\tau}{12}$ and $\mu_1 = \mu_2 = \mu_3 = 0$,

(b) $\lambda_1 = \lambda_2 = -\frac{\tau}{4}$, $\lambda_3 = \frac{\tau}{4}$ and $\mu_1 = \mu_2 = -\frac{\tau}{6}$, $\mu_3 = \frac{\tau}{3}$,

(c) $\lambda_1 = \lambda_3 = -\frac{\tau}{4}$, $\lambda_2 = \frac{\tau}{4}$ and $\mu_1 = \mu_3 = -\frac{\tau}{6}$, $\mu_2 = \frac{\tau}{3}$,

(d) $\lambda_2 = \lambda_3 = -\frac{\tau}{4}$, $\lambda_1 = \frac{\tau}{4}$ and $\mu_2 = \mu_3 = -\frac{\tau}{6}$, $\mu_1 = \frac{\tau}{3}$.

In cases (b)–(d), we get from (4.12)

$$7\tau^2 - 12\tau + 96 = 0,$$

which can not occur. In case (a), we get from (4.12)

$$(\tau - 12)(\tau - 24) = 0.$$

Therefore M is of constant sectional curvature 1 or 2. Since the unit tangent sphere bundle of a space of constant curvature is H-contact ([5]), the converse follows from Theorem 1. □

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