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THE EULER PRODUCT FOR THE RIEMANN ZETA-FUNCTION IN THE CRITICAL STRIP

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Abstract

In this paper we study a pointwise asymptotic behavior of the partial Euler product for the Riemann zeta-function on the right half of the critical strip. We discuss relations among the behavior of the partial Euler product, the distribution of the prime numbers and the distribution of nontrivial zeros of the Riemann zeta-function.

1. Introduction

The Riemann zeta-function $\zeta(s)$ has the Euler product expression:

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}, \tag{1.1}$$

where p runs over the prime numbers. The product (1.1) converges absolutely in $\mathrm{Re}(s) > 1$. We consider the partial Euler product

$$\prod_{p \le x} (1 - p^{-s_0})^{-1} \tag{1.2}$$

as $x \to \infty$ for fixed $s_0 \in \mathbf{C}$ satisfying $1/2 \le \mathrm{Re}(s_0) \le 1$. When $\mathrm{Re}(s_0) = 1$, we know that

$$\prod_{p \le x} (1 - p^{-1})^{-1} \sim e^{c_E} \log x,$$

$$\prod_{p \le x} (1 - p^{-1-it_0})^{-1} \to \zeta(1 + it_0)$$

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as $x \to \infty$ for any $t_0 \in \mathbf{R} \setminus \{0\}$, where c_E is the Euler constant. The former formula was obtained by Mertens in [12, p. 53]. See also [17, §3.15] for the Euler product on $\mathrm{Re}(s_0) = 1$.

In this paper we investigate a behavior of the partial Euler product (1.2) at $s_0 \in \mathbf{C}$ satisfying $1/2 \le \mathrm{Re}(s_0) < 1$. There is corresponding research for L-functions of elliptic curves over the rational number field: see Goldfeld [5], Kuo–Murty [10] and Conrad [2]. Their motivation comes from the initial form of the Birch and Swinnerton-Dyer conjecture, which predicts a behavior of partial Euler products attached to elliptic curves at the central point. We note that Conrad has treated a certain class of L-functions, which includes, for example, Dirichlet L-functions of nonprincipal Dirichlet characters as well as L-functions of elliptic curves. However L-functions belonging to his class are holomorphic in the right of a critical line. Thus his results do not apply to the Riemann zeta-function because of the pole at $s = 1$. Moreover, when we consider a behavior of the partial Euler product as $x \to \infty$, we cannot ignore a contribution of the pole as explained below.

We state our results. Let $\Lambda(n)$ be the von Mangoldt function and we put $\psi(x) := \sum_{n \le x} \Lambda(n)$. When $\mathrm{Re}(s_0) = 1/2$, a behavior of the partial Euler product can be characterized as follows:

Theorem 1. *The following conditions* (i)–(iii) *are equivalent*:

(i) $\psi(x) = x + o(x^{1/2} \log x)$ *as* $x \to \infty$.

(ii) *There exists* $t_0 \in \mathbf{R}$ *such that*

$$(1.3) \qquad (\log x)^m \prod_{p \le x} (1 - p^{-s_0})^{-1} \Big/ \exp\left[\lim_{\varepsilon \downarrow 0}\left(\int_{1+\varepsilon}^{x} \frac{du}{u^{s_0} \log u} - \log \frac{1}{\varepsilon}\right)\right]$$

has a nonzero limit as $x \to \infty$, *where* $s_0 := \frac{1}{2} + it_0$ *and* m *is the multiplicity of the zero of* $\zeta(s)$ *at* $s = s_0$.

(iii) *The quantity* (1.3) *has a nonzero limit as* $x \to \infty$ *for any* $t_0 \in \mathbf{R}$.

If the above conditions are valid, *then the Riemann hypothesis holds and* (1.3) *converges to*

$$(1.4) \qquad e^{(1-m)c_E}(s_0 - 1)\frac{\zeta^{(m)}(s_0)}{m!} \times \begin{cases} \sqrt{2} & \text{if } t_0 = 0, \\ 1 & \text{otherwise} \end{cases}$$

as $x \to \infty$, *where* $\zeta^{(m)}(s)$ *is the* m*-th derivative of* $\zeta(s)$.

Under the Riemann hypothesis, the current best estimate for $\psi(x)$ is $\psi(x) = x + O(x^{1/2}(\log x)^2)$. Therefore we cannot reach the first condition in Theorem 1 under the Riemann hypothesis at present. On the other hand, Montgomery [13, p. 16] predicts that

$$(1.5) \quad \limsup_{x \to \infty} \frac{\psi(x) - x}{x^{1/2}(\log\log\log x)^2} \overset{?}{=} \frac{1}{2\pi}, \quad \liminf_{x \to \infty} \frac{\psi(x) - x}{x^{1/2}(\log\log\log x)^2} \overset{?}{=} -\frac{1}{2\pi}.$$

If both equations are true, then the first condition in Theorem 1 holds. We also remark that Cramér [3, pp. 24–25] has essentially obtained

$$\int_1^X (\psi(x) - x)^2 \frac{dx}{x} \ll X \tag{1.6}$$

under the Riemann hypothesis (see also [4]). In particular (1.6) implies that

$$\frac{1}{X} \operatorname{meas}\{x \in [X, 2X] : |\psi(x) - x| > \varepsilon X^{1/2} \log X\} \ll \frac{1}{\varepsilon^2 (\log X)^2}$$

holds for $X \geq 2$ and $\varepsilon > 0$, where meas is the Lebesgue measure on $\mathbf{R}$. Thus the first condition in Theorem 1 is reasonable from a statistical standpoint.

When $1/2 < \operatorname{Re}(s_0) < 1$, a behavior of the partial Euler product (1.2) is characterized as follows:

Theorem 2. *Let $\sigma_0 \in (1/2, 1)$ be fixed. Then the following conditions* (i)–(iv) *are equivalent.*

(i) $\psi(x) = x + O(x^{\sigma_0})$.

(ii) *There exists $t_0 \in \mathbf{R}$ such that*

$$(\log x)^m \prod_{p \leq x} (1 - p^{-s_0})^{-1} \Big/ \exp\left[\lim_{\varepsilon \downarrow 0} \left(\int_{1+\varepsilon}^{x} \frac{du}{u^{s_0} \log u} - \log \frac{1}{\varepsilon} \right)\right] \tag{1.7}$$

has a nonzero limit as $x \to \infty$, where $s_0 := \sigma_0 + it_0$ and m is the multiplicity of the zero of $\zeta(s)$ at $s = s_0$.

(iii) *The quantity* (1.7) *has a nonzero limit as $x \to \infty$ for any $t_0 \in \mathbf{R}$.*

(iv) $\zeta(s) \neq 0$ *in* $\operatorname{Re}(s) > \sigma_0$.

If the above conditions are valid, then (1.7) *converges to*

$$e^{(1-m)c_E}(s_0 - 1) \frac{\zeta^{(m)}(s_0)}{m!} \tag{1.8}$$

as $x \to \infty$.

In Theorem 2 the asymptotic behavior of the partial Euler product is equivalent that $\zeta(s)$ is zero-free in $\operatorname{Re}(s) > \sigma_0$, which is different from Theorem 1. This difference comes from a zero density theorem: see Remark 4.3 below.

Theorems 1 and 2 should be compared with Conrad's results [2, Theorems 3.3, 5.11, 6.3]. We see that the denominator of (1.3), (1.7) and $e^{c_E}(s_0 - 1)$ on (1.4), (1.8) are regarded as the contribution of the pole at $s = 1$.

Theorems 1 and 2 follow from Theorem 3 below. We put $\Theta := \sup\{\operatorname{Re}(\rho) : \rho \in \mathbf{C}, \zeta(\rho) = 0\}$. We know that $1/2 \leq \Theta \leq 1$ holds unconditionally and that the Riemann hypothesis is equivalent to $\Theta = 1/2$. With this notation we have

Theorem 3. *We assume $\Theta < 1$. Let $s_0 = \sigma_0 + it_0$ with $\Theta \leq \sigma_0 < 1$ and $t_0 \in \mathbf{R}$. The nonnegative integer m denotes the multiplicity of the zero of $\zeta(s)$ at $s = s_0$. Then we have*

$$(1.9)\qquad \sum_{2\le n\le x}\frac{\Lambda(n)}{n^{s_0}\log n}-\lim_{\varepsilon\downarrow 0}\left(\int_{1+\varepsilon}^{x}\frac{du}{u^{s_0}\log u}-\log\frac{1}{\varepsilon}\right)+m\log\log x$$
$$=\lim_{\substack{s=\sigma+it_0\\ \sigma\downarrow\sigma_0}}(\log\zeta(s)-m\log(s-s_0)+\log(s-1))+(1-m)c_E$$
$$+\frac{\psi(x)-x}{x^{s_0}\log x}+O\left(\frac{1}{x^{\sigma_0-\Theta}\log x}\right),$$

where the implied constant depends only on s_0. *Here* $\log\zeta(s)-m\log(s-s_0)+\log(s-1)$ *is determined such that the following conditions are satisfied*:

- $\log\zeta(s)=\sum_{n=2}^{\infty}\dfrac{\Lambda(n)}{n^s\log n}$ *in* $\mathrm{Re}(s)>1$,
- $\arg(s-1)\in(-\pi/2,\pi/2)$ *and* $\arg(s-s_0)\in(-\pi/2,\pi/2)$ *in* $\mathrm{Re}(s)>1$,
- $\log\zeta(s)-m\log(s-s_0)+\log(s-1)$ *is holomorphic in* $\mathrm{Re}(s)>\Theta$.

The partial Dirichlet series for $\log\zeta(s)$, i.e., the first term on (1.9), is more tractable than the logarithm of the partial Euler product (1.2). Therefore we state Theorem 3 in terms of the partial Dirichlet series instead of the partial Euler product. We will discuss a relation between the partial Euler product and the partial Dirichlet series in Lemma 2.1.

Roughly speaking, our main tools to derive Theorem 3 are the following two facts. The first one is Abel's theorem for Dirichlet series, which says that convergence of Dirichlet series at one point implies uniform convergence on a certain sector. The second one is Weierstrass's theorem in complex analysis, which asserts that uniform convergence for a sequence of holomorphic functions guarantees holomorphy for the limit function.

When we restrict our attention to the case of real s_0, Theorems 1–3 can be described in terms of the logarithmic integral. See Corollaries 4.4, 4.5 and 3.6.

After completing the first version of this paper, the author noticed Ramanujan's work [15, §68], which was unpublished until the late twentieth century. Roughly speaking, Ramanujan discovered more precise formulas than (1.9) in the case $s_0\in[1/2,1)$, probably assuming the Riemann hypothesis. However it seems that he did not give a convincing proof. We will compare our results with Ramanujan's formulas briefly in §5.

In a trial to justify Ramanujan's formulas, the author found another approach to Theorem 3 based on a classical method using Perron's formula. This approach has an advantage to clarify a contribution of the nontrivial zeros more explicitly at least in the case that s_0 is real. However we adopt the original approach in this paper because it is sufficient for our purpose and the method is of some interest. The other approach will be included in the forthcoming paper, in which we will discuss the part of [15] on maximal orders of the divisor functions $\sigma_{-s}(n):=\sum_{d|n}d^{-s}$ for $s>0$.

This paper is organized as follows. In §2 we investigate a relation between the partial Euler product and the partial Dirichlet series for $\log\zeta(s)$. In §3 we prove Theorem 3. In §4 we show Theorems 1 and 2, using Theorem 3. In §5 we give some remarks related to our results. In Appendix we give some numerical computations regarding the partial Euler product.

2. The partial Euler product and the partial Dirichlet series

In this section we relate the partial Dirichlet series $\sum_{2\le n\le x}\frac{\Lambda(n)}{n^{s_0}\log n}$ to the logarithm of the partial Euler product, whose branch is determined by

$$\log\left(\prod_{p\le x}(1-p^{-s_0})^{-1}\right):=\sum_{p\le x}\sum_{k=1}^{\infty}\frac{p^{-ks_0}}{k},$$

for fixed $s_0\in\mathbf{C}$ satisfying $1/2\le\mathrm{Re}(s_0)<1$. First of all we give an explicit relation between them.

Lemma 2.1. *Let $s_0=\sigma_0+it_0$ with $\sigma_0\in[1/2,1]$ and $t_0\in\mathbf{R}$. Then for $x\ge 2$ we have*

$$\sum_{2\le n\le x}\frac{\Lambda(n)}{n^{s_0}\log n}=\sum_{p\le x}\sum_{k=1}^{\infty}\frac{1}{kp^{ks_0}}+\begin{cases}-\frac{1}{2}\log 2+O((\log x)^{-1}) & \text{if } s_0=1/2,\\ O(x^{-(\sigma_0-1/2)}(\log x)^{-1}) & \text{if } s_0\ne 1/2,\end{cases}$$

where the implied constant depends only on s_0.

Proof. We have

$$\sum_{p\le x}\sum_{k=1}^{\infty}\frac{1}{kp^{ks_0}}-\sum_{2\le n\le x}\frac{\Lambda(n)}{n^{s_0}\log n}=\sum_{p\le x}\sum_{\substack{k\ge 1\\ p^k>x}}\frac{1}{kp^{ks_0}}.$$

For $y\in[2,x]$ we divide the sum as follows:

$$=\frac{1}{2}\sum_{\sqrt{x}<p\le x}\frac{1}{p^{2s_0}}+\sum_{p\le y}\sum_{\substack{k\ge 3\\ p^k>x}}\frac{1}{kp^{ks_0}}+\sum_{y<p\le x}\sum_{\substack{k\ge 3\\ p^k>x}}\frac{1}{kp^{ks_0}}. \tag{2.1}$$

By the prime number theorem the second and last sums are

$$\left|\sum_{p\le y}\sum_{\substack{k\ge 3\\ p^k>x}}\frac{1}{kp^{ks_0}}\right|\le\sum_{p\le y}\sum_{k>\log x/\log p}\frac{1}{p^{k\sigma_0}}\ll x^{-\sigma_0}\sum_{p\le y}1\ll\frac{x^{-\sigma_0}y}{\log y},$$

$$\left|\sum_{y<p\le x}\sum_{\substack{k\ge 3\\ p^k>x}}\frac{1}{kp^{ks_0}}\right|\le\sum_{y<p\le x}\sum_{k=3}^{\infty}\frac{1}{p^{k\sigma_0}}\ll\frac{1}{y^{3\sigma_0-1}\log y},$$

respectively. Inserting these into (2.1) and taking $y = x^{1/3}$, we obtain

$$\sum_{p \le x} \sum_{k=1}^{\infty} \frac{1}{kp^{ks_0}} - \sum_{2 \le n \le x} \frac{\Lambda(n)}{n^{s_0} \log n} = \frac{1}{2} \sum_{\sqrt{x} < p \le x} \frac{1}{p^{2s_0}} + O\left(\frac{1}{x^{\sigma_0 - 1/3} \log x}\right). \tag{2.2}$$

When $1/2 < \sigma_0 \le 1$, we easily see $\sum_{\sqrt{x} < p \le x} p^{-2s_0} = O(x^{-(\sigma_0 - 1/2)} (\log x)^{-1})$ by the prime number theorem, so that we reach the desired result. We concentrate on the case $\sigma_0 = 1/2$. First of all we show the following:

$$\sum_{p \le X} \frac{1}{p^{1+2it_0}} = \begin{cases} \log \log X + c(0) + O((\log X)^{-1}) & \text{if } t_0 = 0, \\ c(t_0) + O((\log X)^{-1}) & \text{if } t_0 \ne 0, \end{cases} \tag{2.3}$$

where $c(t_0)$ is a constant determined by t_0 and the implied constants depend only on t_0. The formula (2.3) is well known in the case of $t_0 = 0$: see [14, Theorem 2.7 (d)] for example. We consider the case $t_0 \ne 0$. Let $X \le Y$. Then by integration by parts we have

$$\begin{aligned} \sum_{X < p \le Y} \frac{1}{p^{1+2it_0}} &= \int_X^Y \frac{d\pi(u)}{u^{1+2it_0}} = \frac{\pi(Y)}{Y^{1+2it_0}} - \frac{\pi(X)}{X^{1+2it_0}} + (1 + 2it_0) \int_X^Y \frac{\pi(u)}{u^{2+2it_0}} \, du \\ &= (1 + 2it_0) \int_X^Y \frac{du}{u^{1+2it_0} \log u} + O\left(\frac{1}{\log X}\right), \end{aligned} \tag{2.4}$$

where $\pi(x) := |\{p \le x : \text{prime numbers}\}|$. Here in the last equality we applied $\pi(x) = \dfrac{x}{\log x} + O\left(\dfrac{x}{(\log x)^2}\right)$. We easily see from integration by parts that the last integral on (2.4) is $O((\log X)^{-1})$. Consequently we obtain

$$\sum_{X < p \le Y} \frac{1}{p^{1+2it_0}} = O\left(\frac{1}{\log X}\right). \tag{2.5}$$

This implies that $\sum_{p \le X} p^{-1-2it_0}$ converges as $X \to \infty$. Taking the limit $Y \to \infty$ on (2.5), we obtain (2.3) in the case of $t_0 \ne 0$.

By (2.3) we have

$$\frac{1}{2} \sum_{\sqrt{x} < p \le x} \frac{1}{p^{1+2it_0}} = \begin{cases} \frac{1}{2} \log 2 + O((\log x)^{-1}) & \text{if } t_0 = 0, \\ O((\log x)^{-1}) & \text{if } t_0 \ne 0. \end{cases}$$

Applying this to (2.2), we obtain the stated result in the case of $\sigma_0 = 1/2$. The proof is completed. □

Using Lemma 2.1 together with standard treatments of a logarithmic branch (see [1, §2.2 of Chapter 5] for example), we show the following:

LEMMA 2.2. *Let $s_0 = \sigma_0 + it_0$ with $\sigma_0 \in [1/2, 1)$ and $t_0 \in \mathbf{R}$. Let m denote the multiplicity of the zero of $\zeta(s)$ at $s = s_0$. Then the following conditions* (i) *and* (ii) *are equivalent*:

(i) *As* $x \to \infty$,

$$(\log x)^m \prod_{p \le x} (1 - p^{-s_0})^{-1} \Big/ \exp\left[\lim_{\varepsilon \downarrow 0}\left(\int_{1+\varepsilon}^{x} \frac{du}{u^{s_0} \log u} - \log \frac{1}{\varepsilon}\right)\right] \tag{2.6}$$

has a nonzero limit.

(ii) *As* $x \to \infty$,

$$\sum_{2 \le n \le x} \frac{\Lambda(n)}{n^{s_0} \log n} - \lim_{\varepsilon \downarrow 0}\left(\int_{1+\varepsilon}^{x} \frac{du}{u^{s_0} \log u} - \log \frac{1}{\varepsilon}\right) + m \log\log x$$

converges.

Proof. According to Lemma 2.1, (ii) is equivalent that

$$-\sum_{p \le x} \mathrm{Log}(1 - p^{-s_0}) - \lim_{\varepsilon \downarrow 0}\left(\int_{1+\varepsilon}^{x} \frac{du}{u^{s_0} \log u} - \log \frac{1}{\varepsilon}\right) + m \log\log x \tag{2.7}$$

converges as $x \to \infty$, where $\mathrm{Log}\, z$ is the principal branch of $\log z$. Thus we easily see that (ii) implies (i) by taking the exponential on (2.7). Next we show that (2.7) converges as $x \to \infty$ under (i). We denote (2.6) by $P(s_0; x)$ and its limit by $P(s_0)$. Then we have $\mathrm{Log}(P(s_0; x)/P(s_0)) \to 0$ as $x \to \infty$. For any $x \ge 2$ there exists $h(x) \in \mathbf{Z}$ such that

$$\begin{aligned} \mathrm{Log} \frac{P(s_0; x)}{P(s_0)} &= m \log\log x - \sum_{p \le x} \mathrm{Log}(1 - p^{-s_0}) \\ &\quad - \lim_{\varepsilon \downarrow 0}\left(\int_{1+\varepsilon}^{x} \frac{du}{u^{s_0} \log u} - \log \frac{1}{\varepsilon}\right) - \mathrm{Log}(P(s_0)) + 2\pi i h(x). \end{aligned} \tag{2.8}$$

For $\eta \in (0, 1]$ we replace x with $x + \eta$ on (2.8) and subtract (2.8) from it. Consequently we have

$$\begin{aligned} \mathrm{Log} \frac{P(s_0; x+\eta)}{P(s_0)} - \mathrm{Log} \frac{P(s_0; x)}{P(s_0)} &= m \int_x^{x+\eta} \frac{du}{u \log u} - \sum_{x < p \le x+\eta} \mathrm{Log}(1 - p^{-s_0}) \\ &\quad - \int_x^{x+\eta} \frac{du}{u^{s_0} \log u} + 2\pi i (h(x+\eta) - h(x)). \end{aligned}$$

Estimating the sum and the integrals trivially, we obtain

$$2\pi i (h(x+\eta) - h(x)) = \mathrm{Log} \frac{P(s_0; x+\eta)}{P(s_0)} - \mathrm{Log} \frac{P(s_0; x)}{P(s_0)} + O(x^{-\sigma_0})$$

as $x \to \infty$ uniformly for $\eta \in (0,1]$. Thus we obtain $h(x+\eta) - h(x) \to 0$ as $x \to \infty$ uniformly for $\eta \in (0,1]$. This together with $h(x) \in \mathbf{Z}$ for any $x \geq 2$ yields that there exists $X \geq 2$ and $h \in \mathbf{Z}$ such that $h(x) = h$ for any $x \geq X$. Inserting this into (2.8) and taking the limit $x \to \infty$, we see that (2.7) converges to $\mathrm{Log}(P(s_0)) - 2\pi i h$. This completes the proof. □

3. Proof of Theorem 3

In this section we prove Theorem 3. Throughout this section we assume $\Theta < 1$ and fix $s_0 = \sigma_0 + it_0$ with $\Theta \leq \sigma_0 < 1$ and $t_0 \in \mathbf{R}$ as in Theorem 3.

Lemma 3.1. *Keep the assumption and the notation. Put*

$$A(x) := \sum_{2 \leq n \leq x} \frac{\Lambda(n)}{n^{s_0} \log n} - \lim_{\varepsilon \downarrow 0}\left(\int_{1+\varepsilon}^{x} \frac{du}{u^{s_0} \log u} - \log \frac{1}{\varepsilon}\right) - \frac{\psi(x) - x}{x^{s_0} \log x} + m \log \log x. \tag{3.1}$$

Then for $2 \leq x \leq y$ we have

$$A(y) - A(x) = O\left(\frac{1}{x^{\sigma_0 - \Theta} \log x}\right), \tag{3.2}$$

where the implied constant depends only on s_0. In particular, the function $A(x)$ converges as $x \to \infty$.

Proof. Let $2 \leq x \leq y$. Then integration by parts gives

$$\begin{aligned} &\sum_{x < n \leq y} \frac{\Lambda(n)}{n^{s_0} \log n} - \int_x^y \frac{du}{u^{s_0} \log u} \\ &= \int_x^y \frac{d(\psi(u) - u)}{u^{s_0} \log u} = \frac{\psi(y) - y}{y^{s_0} \log y} - \frac{\psi(x) - x}{x^{s_0} \log x} \\ &\quad + s_0 \int_x^y \frac{\psi(u) - u}{u^{s_0+1} \log u}\, du + \int_x^y \frac{\psi(u) - u}{u^{s_0+1} (\log u)^2}\, du. \end{aligned} \tag{3.3}$$

We treat the last two terms. Here we recall the following formula for $u \geq 2$ and $T \geq 2$ (see [14, Theorem 12.5]):

$$\psi(u) = u - \sum_{\substack{\rho = \beta + i\gamma \\ -T \leq \gamma \leq T}} \frac{u^\rho}{\rho} + O(\log u) + O\left(\frac{u (\log(uT))^2}{T}\right),$$

where $\rho = \beta + i\gamma$ runs over the nontrivial zeros of $\zeta(s)$ with $-T \leq \gamma \leq T$ counted with multiplicity and the implied constant is absolute. Thus for $\alpha \in \{1, 2\}$ and

$T > |t_0|$ we have

$$\int_x^y \frac{\psi(u) - u}{u^{s_0+1}(\log u)^\alpha}\, du = -\sum_{\substack{\rho=\beta+i\gamma \\ -T\le\gamma\le T}} \frac{1}{\rho}\int_x^y \frac{du}{u^{s_0-\rho+1}(\log u)^\alpha} + O\left(\int_x^y \frac{du}{u^{\sigma_0+1}(\log u)^{\alpha-1}}\right) + O\left(\frac{1}{T}\int_x^y \frac{(\log(uT))^2}{u^{\sigma_0}(\log u)^\alpha}\, du\right). \tag{3.4}$$

We easily see that

$$\int_x^y \frac{du}{u^{\sigma_0+1}(\log u)^{\alpha-1}} \ll \frac{1}{x^{\sigma_0}(\log x)^{\alpha-1}},$$

$$\frac{1}{T}\int_x^y \frac{(\log(uT))^2}{u^{\sigma_0}(\log u)^\alpha}\, du \ll \frac{1}{T}\left(y^{1-\sigma_0}(\log y)^{2-\alpha} + \frac{y^{1-\sigma_0}(\log T)^2}{(\log y)^\alpha}\right).$$

We treat the sum on (3.4). We divide it into $\rho = s_0$ and $\rho \neq s_0$. When $\rho \neq s_0$, integration by parts gives

$$\int_x^y \frac{du}{u^{s_0-\rho+1}(\log u)^\alpha} = \frac{1}{\rho - s_0}\frac{y^{\rho-s_0}}{(\log y)^\alpha} - \frac{1}{\rho - s_0}\frac{x^{\rho-s_0}}{(\log x)^\alpha} + \frac{\alpha}{\rho - s_0}\int_x^y \frac{du}{u^{s_0-\rho+1}(\log u)^{\alpha+1}}.$$

Applying these to (3.4) and taking the limit $T \to \infty$, we obtain

$$\begin{aligned}\int_x^y &\frac{\psi(u) - u}{u^{s_0+1}(\log u)^\alpha}\, du \\ &= -\frac{m}{s_0}\int_x^y \frac{du}{u(\log u)^\alpha} - \frac{1}{(\log y)^\alpha}\sum_{\rho\neq s_0}\frac{y^{\rho-s_0}}{\rho(\rho - s_0)} + \frac{1}{(\log x)^\alpha}\sum_{\rho\neq s_0}\frac{x^{\rho-s_0}}{\rho(\rho - s_0)} \\ &\quad - \alpha\sum_{\rho\neq s_0}\frac{1}{\rho(\rho - s_0)}\int_x^y \frac{du}{u^{s_0-\rho+1}(\log u)^{\alpha+1}} + O\left(\frac{1}{x^{\sigma_0}(\log x)^{\alpha-1}}\right).\end{aligned}$$

We estimate the sums over the nontrivial zeros by using $\sum_\rho |\rho|^{-2} < \infty$, so that

$$\int_x^y \frac{\psi(u) - u}{u^{s_0+1}(\log u)^\alpha} = -\frac{m}{s_0}\int_x^y \frac{du}{u(\log u)^\alpha} + O\left(\frac{1}{x^{\sigma_0-\Theta}(\log x)^\alpha}\right).$$

Applying this to (3.3) and noting that $m = 0$ if $\sigma_0 > \Theta$, we obtain the desired result. □

Next we compute the limit of $A(x)$.

Lemma 3.2. *Keep the assumption and the notation as in Theorem* 3 *and Lemma* 3.1. *Then we have*

$$\lim_{x\to\infty} A(x) = \lim_{\substack{s=\sigma+it_0\\ \sigma\downarrow\sigma_0}} (\log\zeta(s) - m\log(s-s_0) + \log(s-1)) + (1-m)c_E.$$

where $\log\zeta(s) - m\log(s-s_0) + \log(s-1)$ *is determined in the same as Theorem* 3.

We put

$$F(s;x) := \int_{\uparrow 2}^{x} u^{s_0-s}\, dA(u). \tag{3.5}$$

We show Lemma 3.2 by calculating $\lim_{\sigma\downarrow\sigma_0}\lim_{x\to\infty} F(\sigma+it_0;x)$ by two ways. Firstly we investigate convergence of $F(s;x)$ as $x\to\infty$.

Lemma 3.3. *Keep the assumption and the notation as in Theorem* 3 *and Lemma* 3.1. *Then* $F(s;x)$ *converges as* $x\to\infty$ *uniformly on* $s\in\mathscr{S}_H := \{\sigma+it : |t-t_0| \le H(\sigma-\sigma_0)\}$ *for each* $H>0$. *In particular*, $\lim_{x\to\infty} F(s;x)$ *is continuous on* $\mathscr{S}_H$ *and holomorphic in the interior* $\mathscr{S}_H^\circ$ *of* $\mathscr{S}_H$ *for each* $H>0$.

We remark that the prototype of Lemma 3.3 is Abel's theorem for Dirichlet series, which can be found in [14, Theorem 1.1] for instance.

Proof of Lemma 3.3. Let $H>0$ be fixed. Thanks to Lemma 3.1, the function $A(x)$ has a limit, say A. Let $\varepsilon>0$ be arbitrary. Then there exists $X\ge 2$, which depends only on s_0, H and ε, such that $|A(x)-A| \le \varepsilon/(3+H)$ $(=: \varepsilon')$ holds for any $x\ge X$. We take $s=\sigma+it\in\mathscr{S}_H$ and $(X\le)x\le y$ and treat $|F(s;y)-F(s;x)|$. If $s=s_0$, then $|F(s_0;y)-F(s_0;x)| = |A(y)-A(x)| \le 2\varepsilon' \le \varepsilon$. Next we consider the case $s\ne s_0$. In this case $\sigma>\sigma_0$ holds. Integration by parts gives

$$\begin{aligned} F(s;y)-F(s;x) &= \int_x^y u^{s_0-s} d(A(u)-A) \\ &= y^{s_0-s}(A(y)-A) - x^{s_0-s}(A(x)-A) \\ &\quad - (s_0-s)\int_x^y u^{s_0-s-1}(A(u)-A)\, du. \end{aligned}$$

Estimating each term trivially, we obtain

$$|F(s;y)-F(s;x)| \le 2\varepsilon' + |s_0-s|\varepsilon' \int_x^y u^{\sigma_0-\sigma-1}\, du \le \left(2+\frac{|s_0-s|}{\sigma-\sigma_0}\right)\varepsilon'.$$

Since $|s_0-s| \le (\sigma-\sigma_0)+|t-t_0| \le (1+H)(\sigma-\sigma_0)$, this is bounded above by $\le (3+H)\varepsilon' = \varepsilon$. Thus $F(s;x)$ converges as $x\to\infty$ uniformly on $s\in\mathscr{S}_H$ as desired. In consequence we see that $\lim_{x\to\infty} F(s;x)$ is continuous on $\mathscr{S}_H$, and is holomorphic in $\mathscr{S}_H^\circ$ by Weierstrass's theorem (see [1, §1.1 of Chapter 5]). We complete the proof. □

Next we calculate $\lim_{x\to\infty} F(s;x)$ when $\mathrm{Re}(s) > 1$. Inserting (3.1) into (3.5) and using $\log \zeta(s) = \sum_{n=2}^{\infty} \dfrac{\Lambda(n)}{n^s \log n}$ in $\mathrm{Re}(s) > 1$, we easily see that

$$\lim_{x\to\infty} F(s;x) = \log \zeta(s) - \int_2^\infty \frac{du}{u^s \log u} + m \int_2^\infty \frac{du}{u^{s-s_0+1} \log u} - \int_{\uparrow 2}^\infty u^{s_0-s} d\left(\frac{\psi(u) - u}{u^{s_0} \log u}\right) \tag{3.6}$$

holds for $\mathrm{Re}(s) > 1$. To calculate the first two integrals, we show the following formula:

LEMMA 3.4. *In* $\arg(z-1) \in (-\pi/2, \pi/2)$ *we have*

$$\lim_{\varepsilon \downarrow 0} \left(\int_{1+\varepsilon}^\infty \frac{du}{u^z \log u} - \log \frac{1}{\varepsilon} \right) = -c_E - \log(z-1). \tag{3.7}$$

Proof. We restrict z to $z > 1$. Changing the variable u by $u = e^v$ and integrating by parts, we have

$$\int_{1+\varepsilon}^\infty \frac{du}{u^z \log u} = -(1+\varepsilon)^{-(z-1)} \log\log(1+\varepsilon) + (z-1) \int_{\log(1+\varepsilon)}^\infty e^{-(z-1)v} \log v \, dv.$$

Since $-(1+\varepsilon)^{-(z-1)} \log\log(1+\varepsilon) - \log(1/\varepsilon)$ tends to zero as $\varepsilon \downarrow 0$, we have

$$\lim_{\varepsilon \downarrow 0} \left(\int_{1+\varepsilon}^\infty \frac{du}{u^z \log u} - \log \frac{1}{\varepsilon} \right) = (z-1) \int_0^\infty e^{-(z-1)v} \log v \, dv.$$

Changing the variable v by $w = (z-1)v$ and applying $\Gamma(1) = 1$ and $\Gamma'(1) = -c_E$, where $\Gamma(z)$ is the gamma function, we obtain (3.7) for $z > 1$. Since both sides on (3.7) are holomorphic in $\arg(z-1) \in (-\pi/2, \pi/2)$, the identity (3.7) holds for $\arg(z-1) \in (-\pi/2, \pi/2)$ by the identity theorem. This completes the proof. □

Next we investigate analytic properties of the last term on (3.6).

LEMMA 3.5. *Keep the assumption and the notation as in Theorem* 3 *and Lemma* 3.1. *Then for* $s = \sigma + it$ *with* $\sigma > 1$ *and* $t \in \mathbf{R}$ *we have*

$$\int_{\uparrow 2}^\infty u^{s_0-s} d\left(\frac{\psi(u) - u}{u^{s_0} \log u}\right) = \frac{2^{1-s}}{\log 2} - (s_0 - s) \int_s^{+\infty+it} \left(-\frac{1}{z} \frac{\zeta'}{\zeta}(z) + \frac{2^{1-z}}{1-z} \right) dz. \tag{3.8}$$

Each term on the right-hand side is holomorphic in $\mathrm{Re}(s) > \Theta$. *On* $t = t_0$ *the last term has the following asymptotic behavior as* $\sigma \downarrow \sigma_0$:

$$\int_{\sigma+it_0}^{+\infty+it_0} \left(-\frac{1}{z} \frac{\zeta'}{\zeta}(z) + \frac{2^{1-z}}{1-z} \right) dz = -\frac{m}{s_0} \log \frac{1}{\sigma - \sigma_0} + O(1). \tag{3.9}$$

Proof. First of all we prove (3.8) for $\mathrm{Re}(s) > 1$. Integrating by parts, we have

$$\int_{\uparrow 2}^{\infty} u^{s_0 - s} d\left(\frac{\psi(u) - u}{u^{s_0} \log u} \right) = \frac{2^{1-s}}{\log 2} - (s_0 - s) \int_2^{\infty} \frac{\psi(u) - u}{u^{s+1} \log u} du. \tag{3.10}$$

We treat the last term on (3.10). We note that

$$\int_s^{+\infty+it} \frac{1}{u^{z+1}} dz = \frac{1}{u^{s+1} \log u}$$

holds for $u \geq 2$. Applying this and Fubini's theorem, we have

$$\int_2^{\infty} \frac{\psi(u) - u}{u^{s+1} \log u} du = \int_s^{+\infty+it} \int_2^{\infty} \frac{\psi(u) - u}{u^{z+1}} dudz.$$

We note that

$$\frac{\zeta'}{\zeta}(z) = -z \int_2^{\infty} \frac{\psi(u)}{u^{z+1}} du \tag{3.11}$$

holds for $\mathrm{Re}(z) > 1$. In fact, expressing $(\zeta'/\zeta)(z) = -\sum_{n=1}^{\infty} \Lambda(n) n^{-z}$ in terms of the Stieltjes integral and integrating by parts, we can plainly see (3.11). Thus we obtain

$$\int_2^{\infty} \frac{\psi(u) - u}{u^{s+1} \log u} du = \int_s^{+\infty+it} \left(-\frac{1}{z} \frac{\zeta'}{\zeta}(z) + \frac{2^{1-z}}{1-z} \right) dz. \tag{3.12}$$

Inserting (3.12) into (3.10), we reach (3.8). Since the integrand $-\frac{1}{z}\frac{\zeta'}{\zeta}(z) + \frac{2^{1-z}}{1-z}$ is holomorphic in $\mathrm{Re}(z) > \Theta$, the right-hand side of (3.12) is holomorphic in $\mathrm{Re}(s) > \Theta$.

Finally we show (3.9). We write

$$\int_{\sigma+it_0}^{+\infty+it_0} \left(-\frac{1}{z} \frac{\zeta'}{\zeta}(z) + \frac{2^{1-z}}{1-z} \right) dz$$
$$= -\int_{\sigma+it_0}^{+\infty+it_0} \frac{m}{z(z-s_0)} dz + \int_{\sigma+it_0}^{+\infty+it_0} \left(-\frac{1}{z} \frac{\zeta'}{\zeta}(z) + \frac{m}{z(z-s_0)} + \frac{2^{1-z}}{1-z} \right) dz.$$

Since the integrand of the last term is holomorphic at $z = s_0$, the last term is $O(1)$ as $\sigma \downarrow \sigma_0$. The first term is calculated as follows:

$$\begin{aligned} -\int_{\sigma+it_0}^{+\infty+it_0} \frac{m}{z(z-s_0)}\,dz &= -\int_{\sigma+it_0}^{2+it_0} \frac{m}{z(z-s_0)}\,dz + O(1) \\ &= -\frac{m}{s_0}\int_{\sigma+it_0}^{2+it_0}\left(\frac{1}{z-s_0} - \frac{1}{z}\right) dz + O(1) \\ &= -\frac{m}{s_0}\log\frac{1}{\sigma-\sigma_0} + O(1). \end{aligned}$$

Combining these, we obtain (3.9). □

Proof of Lemma 3.2. Inserting (3.7) and (3.8) into (3.6), we see that

$$\begin{aligned} \lim_{x\to\infty} F(s;x) &= (\log\zeta(s) - m\log(s-s_0) + \log(s-1)) + (1-m)c_E \\ &\quad + \lim_{\varepsilon\downarrow 0}\left(\int_{1+\varepsilon}^{2} \frac{du}{u^s\log u} - \log\frac{1}{\varepsilon}\right) \\ &\quad - m\lim_{\varepsilon\downarrow 0}\left(\int_{1+\varepsilon}^{2} \frac{du}{u^{s-s_0+1}\log u} - \log\frac{1}{\varepsilon}\right) \\ &\quad - \frac{2^{1-s}}{\log 2} + (s_0 - s)\int_s^{+\infty+it}\left(-\frac{1}{z}\frac{\zeta'}{\zeta}(z) + \frac{2^{1-z}}{1-z}\right) dz \end{aligned} \tag{3.13}$$

holds for $s = \sigma + it$ with $\sigma > 1$ and $t \in \mathbf{R}$. It follows from Lemmas 3.3 and 3.5 that both sides of (3.13) are holomorphic in $s \in \mathscr{S}_H^\circ$ for each $H > 0$. Thus (3.13) holds for $s \in \mathscr{S}_H^\circ$ by the identity theorem. We put $s = \sigma + it_0$ and take the limit $\sigma \downarrow \sigma_0$. Then by Lemma 3.3 again we have

$$\lim_{\substack{s=\sigma+it_0 \\ \sigma\downarrow\sigma_0}} \lim_{x\to\infty} F(s;x) = \lim_{x\to\infty} F(s_0;x) = \lim_{x\to\infty} A(x) - A(2-). \tag{3.14}$$

By the definition of $A(x)$ we have

$$A(2-) = -\lim_{\varepsilon\downarrow 0}\left(\int_{1+\varepsilon}^{2} \frac{du}{u^{s_0}\log u} - \log\frac{1}{\varepsilon}\right) + m\log\log 2 + \frac{2^{1-s_0}}{\log 2}. \tag{3.15}$$

It follows from (3.9) that

$$\lim_{\substack{s=\sigma+it_0 \\ \sigma\downarrow\sigma_0}} (s_0 - s)\int_s^{+\infty+it_0}\left(-\frac{1}{z}\frac{\zeta'}{\zeta}(z) + \frac{2^{1-z}}{1-z}\right) dz = 0. \tag{3.16}$$

Combining (3.13)–(3.16), we obtain the desired result. □

Proof of Theorem 3. Taking the limit $y \to \infty$ on (3.2) and applying Lemma 3.2, we complete the proof. □

Finally in this section we discuss Theorem 3 when s_0 is real. In this case we can write the second term on (1.9) in terms of the logarithmic integral $\mathrm{Li}(x)$, which is defined by

$$\mathrm{Li}(x) := \lim_{\varepsilon\downarrow 0}\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x}\right)\frac{du}{\log u}.$$

In fact, Theorem 3 turns to

Corollary 3.6. *We assume* $\Theta < 1$. *Then for each* $\alpha \in [\Theta, 1)$ *we have*

$$\sum_{2\le n\le x}\frac{\Lambda(n)}{n^{\alpha}\log n} - \mathrm{Li}(x^{1-\alpha}) = \log(-\zeta(\alpha)) + \frac{\psi(x)-x}{x^{\alpha}\log x} + O\left(\frac{1}{x^{\alpha-\Theta}\log x}\right),$$

where $\log(-\zeta(\alpha)) \in \mathbf{R}$.

Here we keep in mind that $\zeta(\alpha) < 0$ for $\alpha \in (0,1)$, which follows from [17, (2.1.5)] for example. In order to show Corollary 3.6, we need the following formula:

Lemma 3.7. *Let* $\alpha \in (0,1)$ *and* $x > 1$. *Then we have*

$$\lim_{\varepsilon\downarrow 0}\left(\int_{1+\varepsilon}^{x}\frac{du}{u^{\alpha}\log u} - \log\frac{1}{\varepsilon}\right) = \mathrm{Li}(x^{1-\alpha}) - \log(1-\alpha) - c_E.$$

Proof. Changing the variable u by $u^{1/(1-\alpha)}$, we have

$$\int_{1+\varepsilon}^{x}\frac{du}{u^{\alpha}\log u} = \int_{(1+\varepsilon)^{1-\alpha}}^{x^{1-\alpha}}\frac{du}{\log u} \tag{3.17}$$
$$= \int_{(1+\varepsilon)^{1-\alpha}}^{1+\varepsilon}\frac{du}{\log u} + \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x^{1-\alpha}}\right)\frac{du}{\log u} - \int_0^{1-\varepsilon}\frac{du}{\log u}.$$

We write the first term on the right as

$$\int_{(1+\varepsilon)^{1-\alpha}}^{1+\varepsilon}\frac{du}{\log u} = \int_{(1+\varepsilon)^{1-\alpha}}^{1+\varepsilon}\left(\frac{1}{\log u} - \frac{1}{u-1}\right)du + \int_{(1+\varepsilon)^{1-\alpha}}^{1+\varepsilon}\frac{du}{u-1}. \tag{3.18}$$

We easily see that the first integral on the right is $o(1)$ and the last integral is $-\log(1-\alpha) + o(1)$ as $\varepsilon \downarrow 0$. Thus (3.18) equals $-\log(1-\alpha) + o(1)$ as $\varepsilon \downarrow 0$. Inserting this into (3.17), we have

$$\lim_{\varepsilon\downarrow 0}\left(\int_{1+\varepsilon}^{x}\frac{du}{u^{\alpha}\log u} - \log\frac{1}{\varepsilon}\right) = -\log(1-\alpha) + \mathrm{Li}(x^{1-\alpha}) - \lim_{\varepsilon\downarrow 0}\left(\int_0^{1-\varepsilon}\frac{du}{\log u} + \log\frac{1}{\varepsilon}\right).$$

We easily see that the last term on the right is

$$\lim_{\varepsilon\downarrow 0}\left(\int_0^{1-\varepsilon}\frac{du}{\log u}+\log\frac{1}{\varepsilon}\right)=c_E$$

by changing the variable u by $u=e^{-v}$ and integrating by parts in the same manner as the proof of Lemma 3.4. This completes the proof. □

Proof of Corollary 3.6. Since $\zeta(\alpha)\neq 0$, the multiplicity m equals 0 in Theorem 3 with $s_0=\alpha$. Applying Lemma 3.7 to Theorem 3 with $s_0=\alpha$, we obtain the stated result. □

4. Proof of Theorems 1 and 2

In this section we prove Theorems 1 and 2. First of all we show the following:

PROPOSITION 4.1. *Let* $s_0:=\sigma_0+it_0$ *with* $1/2\leq\sigma_0<1$ *and* $t_0\in\mathbf{R}$. *We assume that*

$$\sum_{2\leq n\leq x}\frac{\Lambda(n)}{n^{s_0}\log n}-\lim_{\varepsilon\downarrow 0}\left(\int_{1+\varepsilon}^{x}\frac{du}{u^{s_0}\log u}-\log\frac{1}{\varepsilon}\right)+m\log\log x \tag{4.1}$$

converges as $x\to\infty$, *where* m *is the multiplicity of the zero of* $\zeta(s)$ *at* $s=s_0$. *Then the following assertions hold*:

(1) $\psi(x)=x+o(x^{\sigma_0}\log x)$ *as* $x\to\infty$.

(2) $\zeta(s)\neq 0$ *in* $\mathrm{Re}(s)>\sigma_0$.

(3) *The quantity* (4.1) *converges to*

$$\lim_{\substack{s=\sigma+it_0\\ \sigma\downarrow\sigma_0}}(\log\zeta(s)-m\log(s-s_0)+\log(s-1))+(1-m)c_E$$

as $x\to\infty$, *where* $\log\zeta(s)-m\log(s-s_0)+\log(s-1)$ *is determined as in Theorem* 3.

Proof. We show (1). From the assumption, the quantity (4.1) has a limit, which is denoted by c. We put

$$\Delta(x):=\sum_{2\leq n\leq x}\frac{\Lambda(n)}{n^{s_0}\log n}-\lim_{\varepsilon\downarrow 0}\left(\int_{1+\varepsilon}^{x}\frac{du}{u^{s_0}\log u}-\log\frac{1}{\varepsilon}\right)+m\log\log x-c.$$

Then $\Delta(x)=o(1)$ as $x\to\infty$. We have

$$\begin{aligned}\psi(x)&=\int_{\uparrow 2}^{x}u^{s_0}\log u\,d\left(\sum_{2\leq n\leq u}\frac{\Lambda(n)}{n^{s_0}\log n}\right)\\&=\int_{\uparrow 2}^{x}u^{s_0}\log u\,d\Delta(u)+x+O(x^{\sigma_0}).\end{aligned} \tag{4.2}$$

Integration by parts gives

$$(4.3)\qquad \int_{\uparrow 2}^{x} u^{s_0} \log u d\Delta(u)$$

$$= x^{s_0}\Delta(x)\log x - s_0 \int_2^x u^{s_0-1}\Delta(u)\log u\, du - \int_2^x u^{s_0-1}\Delta(u)\, du + O(1).$$

The first term is $o(x^{\sigma_0}\log x)$. In order to treat the second term, we divide the interval $2 \le u \le x$ into $2 \le u \le x^{1/2}$ and $x^{1/2} < u \le x$. We see from $\Delta(u) = o(1)$ that the integral on $2 \le u \le x^{1/2}$ is $O(x^{\sigma_0/2}\log x)$ and the integral on $x^{1/2} < u \le x$ is $o(x^{\sigma_0}\log x)$. Thus the second term on the right-hand side of (4.3) is $o(x^{\sigma_0}\log x)$. In the same manner the third term is $o(x^{\sigma_0})$. In total (4.3) is $o(x^{\sigma_0}\log x)$ as $x \to \infty$. Applying this to (4.2), we obtain (1).

It is well known that (1) implies (2). In fact, we note that the following equation holds in $\mathrm{Re}(s) > 1$ (see (3.11)):

$$(4.4)\qquad \frac{\zeta'}{\zeta}(s) = -s\int_2^\infty \frac{\psi(u) - u}{u^{s+1}}\,du - \frac{s\cdot 2^{1-s}}{s-1}.$$

We see from (1) that the integral converges absolutely and locally uniformly in $\mathrm{Re}(s) > \sigma_0$. Thus $(\zeta'/\zeta)(s)$ is holomorphic in $\{s \in \mathbf{C} : \mathrm{Re}(s) > \sigma_0,\ s \ne 1\}$, which is nothing but (2).

Lastly we show (3). The assertion (2) says $\sigma_0 \ge \Theta$. Inserting (1) into (1.9), we obtain (3). □

Proof of Theorem 1. We prove the former part of Theorem 1 by showing that (iii) implies (ii), that (ii) implies (i) and that (i) implies (iii). It is clear that (iii) implies (ii). We show that (ii) implies (i). By Lemma 2.2 (ii) implies that (4.1) converges as $x \to \infty$ for some $s_0 = \frac{1}{2} + it_0$. Thus by Proposition 4.1 (1), we obtain (i). Next we show that (i) implies (iii). Let $s_0 = \frac{1}{2} + it_0$ with $t_0 \in \mathbf{R}$ be arbitrary. We note that (i) implies the Riemann hypothesis (see (4.4)). Applying the assumption $\psi(x) = x + o(x^{1/2}\log x)$ to Theorem 3 with $\sigma_0 = 1/2$, we see that (4.1) converges as $x \to \infty$. Thanks to Lemma 2.2, we obtain (iii). We complete the proof of the former part of Theorem 1.

We show the latter part. In the proof of the former part we have already mentioned that (i) implies the Riemann hypothesis. Inserting Lemma 2.1 and (i) into Theorem 3 with $\sigma_0 = 1/2$ and taking the exponential, we find that (1.3) converges to (1.4) as $x \to \infty$. This completes the proof. □

Before we prove Theorem 2, we quote the following result of Grosswald [7, Théorème 1]:

LEMMA 4.2. *If* $\Theta > 1/2$, *then we have* $\psi(x) = x + O(x^{\Theta})$.

Proof of Theorem 2. We prove the former part of Theorem 2 by showing that (iii) implies (ii), that (ii) implies (iv), that (iv) implies (i) and that (i) implies (iii). Clearly (iii) implies (ii). We assume (ii). Then we can show $\psi(x) = x + o(x^{\sigma_0} \log x)$ in the same manner as the proof of Theorem 1. Since the integral on (4.4) converges locally uniformly in $\mathrm{Re}(s) > \sigma_0$, we obtain (iv). We immediately see from Lemma 4.2 that (iv) implies (i). We can easily show that (i) implies (iii) in the same manner as the proof of Theorem 1. We complete the proof of the former part.

We can also show the latter part in the same manner as the proof of Theorem 1. We omit the detail. □

Remark 4.3. Lemma 4.2 is a typical application of a zero density theorem. Thus the slight difference between Theorems 1 and 2 comes from the zero density theorem.

At the end of this section we discuss the case that s_0 is real. Taking Lemma 3.7 into account, we immediately see that the following are weaker versions of Theorems 1 and 2, respectively.

COROLLARY 4.4. *The following conditions* (i) *and* (ii) *are equivalent*:
(i) $\psi(x) = x + o(x^{1/2} \log x)$ *holds as* $x \to \infty$.
(ii) *As* x *tends to infinity*, *we have*

$$\prod_{p \le x} (1 - p^{-1/2})^{-1} \Big/ \exp[\mathrm{Li}(x^{1/2})] \to -\sqrt{2}\zeta\left(\frac{1}{2}\right).$$

If the above conditions hold, *then the Riemann hypothesis is true.*

COROLLARY 4.5. *Let* $\alpha \in (1/2, 1)$. *Then the following conditions* (i)–(iii) *are equivalent*:
(i) $\psi(x) = x + O(x^{\alpha})$ *holds as* $x \to \infty$.
(ii) *As* x *tends to infinity*, *we have*

$$\prod_{p \le x} (1 - p^{-\alpha})^{-1} \Big/ \exp[\mathrm{Li}(x^{1-\alpha})] \to -\zeta(\alpha).$$

(iii) $\zeta(s) \neq 0$ *in* $\mathrm{Re}(s) > \alpha$.

5. Concluding remarks

In this section we give several remarks related to our results.

First of all we mention a formula of Guinand [8, Theorem 2] regarding the number $N(T)$ of the zeros ρ of $\zeta(s)$ in $0 < \mathrm{Im}(\rho) \le T$. Now we assume the Riemann hypothesis. We take $T > 0$. For simplicity suppose that $\zeta(1/2 + iT) \neq 0$. Taking the imaginary part on (1.9) with $s_0 = 1/2 + iT$ and taking the limit as $x \to \infty$, we obtain

$$\arg\zeta\left(\frac{1}{2}+iT\right) = \lim_{x\to\infty}\left(-\sum_{2\le n\le x}\frac{\Lambda(n)\sin(T\log n)}{n^{1/2}\log n}+\int_1^x\frac{\sin(T\log u)}{u^{1/2}\log u}\,du + \frac{(\psi(x)-x)\sin(T\log x)}{x^{1/2}\log x}\right) - \arg\left(-\frac{1}{2}+iT\right).$$

Here $\arg\zeta\big(\frac{1}{2}+iT\big)$ and $\arg\big(-\frac{1}{2}+iT\big)$ are determined such that $\arg\zeta(\sigma+iT)$ and $\arg(\sigma-1+iT)$ are continuous on $\sigma\ge 1/2$ and they tend to 0 as $\sigma\to+\infty$. Applying this to the following formula (see [14, Theorem 14.1])

$$N(T) = \frac{1}{\pi}\arg\Gamma\left(\frac{1}{4}+\frac{iT}{2}\right) - \frac{T}{2\pi}\log\pi + \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right)+1,$$

we obtain a formula for $N(T)$. This essentially agrees with [8, Theorem 2]. Guinand used Hankel transforms and his method is different from ours.

In this paper we treat asymptotic behaviors of $\sum_{2\le n\le x}\frac{\Lambda(n)}{n^{s_0}\log n}$. On the other hand, Gonek [6] considered behaviors of

$$\sum_{n=2}^{\infty}\frac{\Lambda(n)v(n;x)}{n^{s_0}\log n}, \tag{5.1}$$

where $v(\cdot;x)$ is a weight function. If we choose $v(\cdot;x)$ appropriately, then a behavior of (5.1) can be obtained under a milder assumption than the case of our unweighted sum. Compare the displayed formula just before Theorem 9.1 in [6], which is a consequence of the Riemann hypothesis, with Theorem 1.

Lastly we compare Theorem 3 with Ramanujan's formulas [15, (359) and (361)]. For simplicity we assume the Riemann hypothesis. When $s_0\in[1/2,1)$, we can rewrite Theorem 3 as follows:

Proposition 5.1. *Assume the Riemann hypothesis. Then,*
(1) *We have*

$$\prod_{p\le x}(1-p^{-1/2})^{-1} = -\sqrt{2}\zeta\left(\frac{1}{2}\right)\exp\left[\mathrm{Li}(\vartheta(x)^{1/2}) + O\left(\frac{1}{\log x}\right)\right], \tag{5.2}$$

where $\vartheta(x) := \sum_{p\le x}\log p$.
(2) *For* $\alpha\in(1/2,1)$ *we have*

$$\prod_{p\le x}(1-p^{-\alpha})^{-1} = -\zeta(\alpha)\exp\left[\mathrm{Li}(\vartheta(x)^{1-\alpha}) + O\left(\frac{1}{x^{\alpha-1/2}\log x}\right)\right], \tag{5.3}$$

where the implied constant depends only on α.

For the proof we start with Theorem 3. Let $\alpha\in[1/2,1)$. Then under the Riemann hypothesis we have

$$\sum_{2\le n\le x}\frac{\Lambda(n)}{n^\alpha\log n}-\lim_{\varepsilon\downarrow 0}\left(\int_{1+\varepsilon}^{x}\frac{du}{u^\alpha\log u}-\log\frac{1}{\varepsilon}\right) \tag{5.4}$$
$$=\log(-(1-\alpha)\zeta(\alpha))+c_E+\frac{\psi(x)-x}{x^\alpha\log x}+O\left(\frac{1}{x^{\alpha-1/2}\log x}\right).$$

We calculate the last term on the left by using the following lemma together with Lemma 3.7.

Lemma 5.2. *Let $\alpha>0$ be fixed. Suppose $r(x)=o(x)$ as $x\to\infty$. Then it holds unconditionally that*

$$\int_x^{x+r(x)}\frac{du}{u^\alpha\log u}=\frac{r(x)}{x^\alpha\log x}+O\left(\frac{|r(x)|^2}{x^{1+\alpha}\log x}\right).$$

Proof. We write the left-hand side as

$$\int_x^{x+r(x)}\frac{du}{u^\alpha\log u}=\int_0^{r(x)}\frac{du}{(x+u)^\alpha\log(x+u)}. \tag{5.5}$$

We see from the Taylor expansions that the identities

$$(x+u)^{-\alpha}=x^{-\alpha}\left(1+\frac{u}{x}\right)^{-\alpha}=x^{-\alpha}\left(1+O\left(\frac{|u|}{x}\right)\right)$$

and

$$\log(x+u)=\log x+O\left(\frac{|u|}{x}\right)=\log x\left(1+O\left(\frac{|u|}{x\log x}\right)\right)$$

hold uniformly for $-x/2\le u\le x/2$. Thus we have

$$\frac{1}{(x+u)^\alpha\log(x+u)}=\frac{1}{x^\alpha\log x}\left(1+O\left(\frac{|u|}{x}\right)\right)$$

uniformly for $-x/2\le u\le x/2$. Inserting this into (5.5), we complete the proof. □

Proof of Proposition 5.1. For $\alpha\in[1/2,1)$ we calculate the last term on the left-hand side of (5.4). By Lemmas 3.7 and 5.2 together with the conditional estimate $\vartheta(x)=x+O(x^{1/2}(\log x)^2)$, we have

$$\lim_{\varepsilon\downarrow 0}\left(\int_{1+\varepsilon}^{x}\frac{du}{u^\alpha\log u}-\log\frac{1}{\varepsilon}\right)$$
$$=\lim_{\varepsilon\downarrow 0}\left(\int_{1+\varepsilon}^{\vartheta(x)}\frac{du}{u^\alpha\log u}-\log\frac{1}{\varepsilon}\right)-\int_x^{\vartheta(x)}\frac{du}{u^\alpha\log u}$$
$$=\mathrm{Li}(\vartheta(x)^{1-\alpha})-\log(1-\alpha)-c_E-\frac{\vartheta(x)-x}{x^\alpha\log x}+O\left(\frac{(\log x)^3}{x^\alpha}\right).$$

Applying this and $\vartheta(x) = \psi(x) + O(x^{1/2})$ (see [14, Corollary 2.5]) to (5.4), we obtain

$$\sum_{2 \le n \le x} \frac{\Lambda(n)}{n^\alpha \log n} = \mathrm{Li}(\vartheta(x)^{1-\alpha}) + \log(-\zeta(\alpha)) + O\left(\frac{1}{x^{\alpha-1/2}\log x}\right).$$

Inserting Lemma 2.1 and taking the exponential, we reach the desired result. □

We give some comments on Proposition 5.1. According to Proposition 5.1,

$$\prod_{p \le x} (1 - p^{-1/2})^{-1} \sim -\sqrt{2}\zeta\left(\frac{1}{2}\right) \exp[\mathrm{Li}(\vartheta(x)^{1/2})]$$

holds as $x \to \infty$ only assuming the Riemann hypothesis. On the other hand, in view of Corollary 4.4,

$$\prod_{p \le x} (1 - p^{-1/2})^{-1} \sim -\sqrt{2}\zeta\left(\frac{1}{2}\right) \exp[\mathrm{Li}(x^{1/2})]$$

is not achieved under the Riemann hypothesis at present.

In [15, (359) and (361)] Ramanujan gives more precise formulas than (5.2) and (5.3) without a convincing proof. For example, Ramanujan asserts that $O((\log x)^{-1})$ on (5.2) can be replaced by

$$\frac{1}{\log x}\left(1 - \frac{1}{2}\sum_\rho \frac{x^{\rho-1/2}}{\rho(\rho - \frac{1}{2})}\right) + O\left(\frac{1}{(\log x)^2}\right).$$

We will discuss Ramanujan's formulas in the forthcoming paper.

Appendix A. Numerical calculations

In this appendix we give numerical data for

$$E(\alpha; x) := \prod_{p \le x} (1 - p^{-\alpha})^{-1} \Big/ \exp[\mathrm{Li}(x^{1-\alpha})].$$

In view of Corollaries 4.4 and 4.5, $E(\alpha; x)$ is expected to converge to

$$E(\alpha) := -\zeta(\alpha) \times \begin{cases} \sqrt{2} & \text{if } \alpha = 1/2, \\ 1 & \text{if } 1/2 < \alpha < 1 \end{cases}$$

as $x \to \infty$ for $\alpha \in [1/2, 1)$.

Table 1 presents numerical values for $E(\alpha; x)$ and $E(\alpha)$. This calculation was done by PARI/GP. Our expectation $E(\alpha; x) \to E(\alpha)$ is not far from the truth in view of the numerical data. In order to give some more observations on the numerical data, we look at the ratio $E(\alpha; x)/E(\alpha)$ for $\alpha \in [1/2, 1)$. We

Table 1. $E(\alpha; x)$ and $E(\alpha)$.

$E(\alpha; x)$	$\alpha = 1/2$	$\alpha = 5/8$	$\alpha = 3/4$	$\alpha = 7/8$
$x = 10$	2.336986	2.798311	4.074990	8.246520
$x = 10^2$	2.188085	2.499169	3.690993	7.660053
$x = 10^3$	2.172603	2.376645	3.566187	7.518212
$x = 10^4$	2.182170	2.300975	3.507474	7.467063
$x = 10^5$	2.136782	2.234054	3.471481	7.443684
$x = 10^6$	2.028938	2.178329	3.451221	7.434380
$x = 10^7$	2.022247	2.158042	3.446000	7.432788
$x = 10^8$	2.053075	2.148405	3.444133	7.432372
$x = 10^9$	2.074954	2.140566	3.442940	7.432150
$x = 10^{10}$	2.106856	2.135434	3.442312	7.432055
$E(\alpha)$	2.065253	2.117418	3.441285	7.431961

assume the Riemann hypothesis. Then, inserting Lemma 2.1 into Corollary 3.6 and taking the exponential, we obtain

$$E(\alpha; x) = E(\alpha) \exp\left[\frac{\psi(x) - x}{x^{\alpha} \log x} + O\left(\frac{1}{x^{\alpha-1/2} \log x}\right)\right]. \tag{A.1}$$

Thus, the conjectural bound $\psi(x) = x + O(x^{1/2}(\log\log\log x)^2)$, which is a consequence of Montgomery's conjecture (1.5), implies

$$\frac{E(\alpha; x)}{E(\alpha)} = 1 + O\left(\frac{(\log\log\log x)^2}{x^{\alpha-1/2} \log x}\right). \tag{A.2}$$

On the other hand, numerical calculation gives

$$\frac{E(1/2; 10^9)}{E(1/2)} = 1.0046\cdots, \quad \frac{E(1/2; 10^{10})}{E(1/2)} = 1.0201\cdots. \tag{A.3}$$

To some extent, the conjectural estimate (A.2) seems reasonable from a standpoint of (A.3) because $(\log\log\log x)^2/\log x$ approximately equals 0.0593 for $x = 10^9$ and 0.0567 for $x = 10^{10}$.

From Table 1 we may wonder if $E(\alpha; x) > E(\alpha)$ for any $\alpha > 1/2$ and $x \geq 10$. However we cannot expect it because of the following result:

Proposition A.1. *We assume the Riemann hypothesis. Then for $\alpha \in (1/2, 1)$ we have*

$$E(\alpha; x) = E(\alpha)\left(1 + \Omega_{\pm}\left(\frac{\log\log\log x}{x^{\alpha-1/2} \log x}\right)\right).$$

In particular under the Riemann hypothesis there exists $\{x_n\}_{n=1}^{\infty}$ such that x_n tends to infinity and $E(\alpha; x_n) - E(\alpha)$ changes sign on $n \in \mathbf{Z}_{\geq 1}$.

Proof. We assume the Riemann hypothesis. It follows from (A.1) and the conditional estimate $\psi(x) = x + O(x^{1/2}(\log x)^2)$ that

$$E(\alpha; x) = E(\alpha)\left(1 + \frac{\psi(x) - x}{x^\alpha \log x} + O\left(\frac{1}{x^{\alpha-1/2}\log x}\right)\right).$$

Inserting Littlewood's omega result $\psi(x) = x + \Omega_\pm(x^{1/2}\log\log\log x)$ (see [11], [9, Theorem 5.8], [14, Theorem 15.11] for example), we complete the proof. □

Remark A.2. In [16, Théorème 2] Robin has shown unconditionally that there exists $\{x_n\}_{n=1}^\infty$ such that x_n tends to infinity and

$$\prod_{p \le x_n} (1 - p^{-1})^{-1} - e^{c_E} \log x_n$$

changes sign on $n \in \mathbf{Z}_{\ge 1}$.

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