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KODAI MATH. J.
34 (2011), 191–201

GENERALIZE SOME NORM INEQUALITIES OF SAITOH

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Abstract

In this paper we present various norm inequalities in framework Sobolev spaces by using Hölder's inequality. In particular, we imply some of the corresponding results of Saitoh whose proofs were based on Aronszajn's theory of reproducing kernels.

1. Introduction and results

In a series of papers ([19], [20], [22], [23], [24], [25], [26], [28]) S. Saitoh deduced various norm inequalities when he studied some concepts in sums, products and other operators of reproducing kernels through transforms of the Hilbert spaces by using Aronszajn's theory of reproducing kernels ([2], [27]). Some of these norm inequalities were generalized and reproved using various technics. J. Burbea ([3], [4]) considered Saitoh's norm inequalities for entire functions and functions holomorphic in the unit disk while M. Cwikel and R. Kerman ([5]) and K. F. Andersen ([1]) generalized some convolution inequalities which were also studied later by S. Saitoh, V. K. Tuan and M. Yamamoto ([29], [30], [31], [32]) and the authors ([6], [7], [16], [17], [18]). In this paper, based on Hölder's inequality, we would like to present some more generalizations of Saitoh's inequalities.

First, we examine the Sobolev space $H^p_{\alpha,\beta}$, $p > 1$, on $\mathbf{R}$ consisting of all real-valued and absolutely continuous functions $f(x)$ with finite norm

$$(1.1)\qquad \|f\|_{H^p_{\alpha,\beta}} = \left\{\int_{-\infty}^{\infty} \{\alpha^p|f'(x)|^p + \beta^p|f(x)|^p\}\,dx\right\}^{1/p} < \infty \quad (\alpha,\beta > 0).$$

In the case of $p = 2$, the Sobolev Hilbert space $H^2_{\alpha,\beta}$ has been examined extensively by many authors in view of Aronszajn's theory of reproducing

2000 *Mathematics Subject Classification*. 26D10; 26A46; 46E35.

Key words and phrases. Inequality; absolutely continuous function; Sobolev spaces; weight.

This research was supported by Vietnam's National Foundation for Science and Technology Development (NAFOSTED), 101.01.45.09.

Received October 5, 2010; revised October 27, 2010.

kernels. In [8], M. Hegland and J. T. Marti obtained the best constant of the well-known Sobolev's inequality

$$\left(\sup_{x\in\mathbf{R}}|f(x)|\right)^2 \le C\|f\|^2_{H^2_{1,1}}. \tag{1.2}$$

Some extensions of the above result may be found in, for instance, [15], or in [9, 10, 11, 12], and the references there in. In [25], S. Saitoh found that if $f \in H^2_{\alpha_1,\beta_1}$ and $g \in H^2_{\alpha_2,\beta_2}$ then $fg \in H^2_{\alpha,\beta}$, where $\alpha = \alpha_1\alpha_2$, $\beta = \alpha_1\beta_2 + \alpha_2\beta_1$, and moreover,

$$\|fg\|^2_{H^2_{\alpha,\beta}} \le \frac{1}{2}\left(\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}\right)\|f\|^2_{H^2_{\alpha_1,\beta_1}}\|g\|^2_{H^2_{\alpha_2,\beta_2}}. \tag{1.3}$$

More generally, we have

THEOREM 1.1. *Let* $\alpha_j > 0$, $\beta_j > 0$ *for* $j = 1,2,\dots,m$, *and*

$$\alpha = \prod_{j=1}^m \alpha_j, \quad \beta = \sum_{j=1}^m \beta_j \prod_{i\ne j}\alpha_i.$$

If $f_j \in H^p_{\alpha_j,\beta_j}$ *for all* $j = 1,2,\dots,m$, *then* $\prod_{j=1}^m f_j \in H^p_{\alpha,\beta}$, *and moreover,*

$$\left\|\prod_{j=1}^m f_j\right\|^p_{H^p_{\alpha,\beta}} \le C(p)\prod_{j=1}^m \|f_j\|^p_{H^p_{\alpha_j,\beta_j}}, \tag{1.4}$$

where

$$C(p) = \begin{cases} \left(\dfrac{2p-2}{p}\right)^{(p-1)(m-1)}\left(\sum_{j=1}^m \prod_{i\ne j}\dfrac{\alpha_i}{\beta_i}\right)^{p-1} & \text{if } 1 < p \ne 2, \\ \left(\dfrac{1}{2}\right)^{m-1}\sum_{j=1}^m \prod_{i\ne j}\dfrac{\alpha_i}{\beta_i} & \text{if } p = 2. \end{cases}$$

Remark 1.2. (1) The case $m = 1$ in inequality (1.4) is trivial.

(2) For $p = 2$ and $m \ne 1$, unless $f_j = 0$ a.e. for some j, equatity holds in (1.4) if and only if

$$f_j(x) = C_j e^{-(\beta_j/\alpha_j)|x-y|}$$

for some real constants C_j, $j = 1,2,\dots,m$, and some point $y \in \mathbf{R}$ which is independent of j.

(3) Since $\lim_{p\to 2} C(p) = 2^{m-1}C(2)$ it follows that the above constant $C(p)$, for $p \ne 2$ and $m \ne 1$, is not the best.

CONJECTURE 1.3. *The best constant in the inequality* (1.4) *is*

$$C(p) = \left(\frac{1}{2}\right)^{m-1}\left(\frac{2p-2}{p}\right)^{(p-1)(m-1)}\left(\sum_{j=1}^m \prod_{i\ne j}\frac{\alpha_i}{\beta_i}\right)^{p-1}, \quad p > 1.$$

Note that ([25])

$$G_{\alpha,\beta}(x, y) = \frac{1}{2\alpha\beta} e^{-(\beta/\alpha)|x-y|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\alpha^2\xi^2 + \beta^2} \, d\xi \tag{1.5}$$

is the reproducing kernel for the Sobolev Hilbert space $H^2_{\alpha,\beta}$. Hence, any member $f \in H^2_{\alpha,\beta}$ is expressible in the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(\xi)}{\alpha^2\xi^2 + \beta^2} e^{i\xi x} \, d\xi \tag{1.6}$$

for a complex-valued function F satisfying

$$\int_{-\infty}^{\infty} \frac{|F(\xi)|^2}{\alpha^2\xi^2 + \beta^2} \, d\xi < \infty, \tag{1.7}$$

and furthermore we have the isometrical identity

$$\|f\|^2_{H^2_{\alpha,\beta}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|F(\xi)|^2}{\alpha^2\xi^2 + \beta^2} \, d\xi. \tag{1.8}$$

Therefore, in the case of $p = 2$, Theorem 1.1 can be transformed in the following form by means of Fourier's integral.

Corollary 1.4. *For any complex-valued functions*

$$F_j \in L_2(\mathbf{R}, (\alpha_j^2\xi^2 + \beta_j^2)^{-1} \, d\xi), \quad j = 1, 2, \ldots, m,$$

and for the iterated convolution $\prod_{j=1}^m *$, *we have the inequality*

$$\begin{aligned} \frac{1}{(2\pi)^{m-1}} \int_{-\infty}^{\infty} \left| \left(\prod_{j=1}^{m} * \frac{F_j(\xi)}{\alpha_j^2\xi^2 + \beta_j^2} \right)(\xi) \right|^2 (\alpha^2\xi^2 + \beta^2) \, d\xi \\ \le C(2) \prod_{j=1}^{m} \int_{-\infty}^{\infty} \frac{|F_j(\xi)|^2}{\alpha_j^2\xi^2 + \beta_j^2} \, d\xi. \end{aligned} \tag{1.9}$$

Unless $m = 1$ or $F_j = 0$ a.e. for some j, equality holds here if and only if

$$F_j(\xi) = D_j e^{i\xi y} \tag{1.10}$$

for some complex numbers D_j, $j = 1, 2, \ldots, m$, and some point $y \in \mathbf{R}$ which is independent of j.

Remark 1.5. Some generalized versions of the inequality (1.9) can be found in [1] and [17].

Next, let ω be a weight, which means a positive continuous function on $(a, b) \subset \mathbf{R}$ satisfying $\omega \in L_1(a, b)$, and $W^p_\omega(a, b)$, $p > 1$, be a weighted Sobolev space of a function f, real-valued absolutely continuous on (a,b), $\lim_{x\to a} f(x) = 0$ and whose norm is

$$\|f\|_{\omega} = \left(\int_a^b \frac{|f'(x)|^p}{\omega(x)^{p-1}}\, dx\right)^{1/p} < \infty. \tag{1.11}$$

In the case of $\omega \equiv 1$, this space is denoted by $W^p(a,b)$.

We note that $W_{\omega}^2(a,b)$ is a weighted Sobolev space admitting the reproducing kernel

$$K(x,s) = \int_a^{\min(x,s)} \omega(t)\, dt,$$

which was used in various contexts. Y. Sawano, H. Fujiwara and S. Saitoh [33] investigated compactness of linear operators associated with the real inversion formulas of the Laplace transform, coming with the space $W_{\omega}^2(0,\infty)$ while A. Yamada [34] considered an elementary integral inequality which extends a norm inequality of Saitoh and yields well-known Opial-type inequalities.

Our main theorem in this direction is:

THEOREM 1.6. *For some weights ω_j, $j = 1,2,\dots,m$, let us consider a new weight*

$$\rho(x) = \left(\prod_{j=1}^m \int_a^x \omega_j(t)\, dt\right)', \quad x \in (a,b).$$

Then, for $f_j \in W_{\omega_j}^p(a,b)$, $j = 1,2,\dots,m$, we have $\prod_{j=1}^m f_j \in W_{\rho}^p(a,b)$, and moreover,

$$\left\|\prod_{j=1}^m f_j\right\|_{\rho} \le \prod_{j=1}^m \|f_j\|_{\omega_j}. \tag{1.12}$$

Unless $m = 1$ or $f_j = 0$ a.e. for some j, equality holds in (1.12) for $f_j \in W_{\omega_j}^p(a,b)$ if and only if

$$f_j(x) = C_j \int_a^{\min(x,s)} \omega_j(t)\, dt \tag{1.13}$$

for some real constants $C_j \neq 0$, $j = 1,2,\dots,m$, and some point $s \in (a,b)$ which is independent of j.

Remark 1.7. (1) The above result is known in the case $p = 2$ and $\omega_j \equiv \omega$, $j = 1,2,\dots,m$, where it was proved by Saitoh [23].
(2) From Theorem 1.6 we can obtain many interesting norm inequalities by investigating various weights. We refer to [33, Section 3] for some typical weights.

In the simple case of Theorem 1.6, when $f_j \equiv f$ and $\omega_j \equiv 1$ on (a,b) for $j = 1,2,\dots,m$, we obtain the following corollary.

Corollary 1.8. *For $f \in W^p(a,b)$, we have*

$$\int_a^b \frac{|\{f(x)^m\}'|^p}{(x-a)^{(m-1)(p-1)}}\,dx \le m^{p-1}\left[\int_a^b |f'(x)|^p\,dx\right]^m. \tag{1.14}$$

The constant m^{p-1} is the best possible. Moreover, the extremal function is of the form $f(x) = C[\min(x,s) - a]$ a.e., where C is a constant and $s \in (a,b)$.

In connection with Fourier sine and Fourier cosine transforms, let us consider the integral transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{F(t)\sin xt}{t^2}\,dt, \quad x > 0 \tag{1.15}$$

for real-valued functions $F(t)$ satisfying

$$\int_0^\infty \frac{|F(t)|^2}{t^2}\,dt < \infty. \tag{1.16}$$

Then (see [21]), $f(x)$ is absolutely continuous, $\lim_{x\to 0} f(x) = 0$, $f'(x) \in L_2(0,\infty)$ and

$$\int_0^\infty |f'(x)|^2\,dx = \frac{2}{\pi}\int_0^\infty \frac{|F(t)|^2}{t^2}\,dt. \tag{1.17}$$

In view of Corollary 1.8, we have

Corollary 1.9. *For any real-valued functions $F(t)$ satisfying* (1.16), *we have the inequality*

$$\frac{8}{\pi^2}\int_0^\infty \frac{1}{x}\left|\int_0^\infty \frac{F(t)\sin xt}{t^2}\,dt \int_0^\infty \frac{F(t)\cos xt}{t}\,dt\right|^2 dx \le \left[\int_0^\infty \frac{|F(t)|^2}{t^2}\,dt\right]^2. \tag{1.18}$$

The equality holds here if and only if

$$F(t) = C\sin st \quad \text{on } (0,\infty), \tag{1.19}$$

where C is a constant and $s \in (0,\infty)$.

Remark 1.10. The equality statement in Corollary 1.9 is proved as follows. The equatity holds in (1.18) if and only if

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{F(t)\sin xt}{t^2}\,dt = C\min(x,s) \quad \text{on } (0,\infty),$$

where C is a constant and $s \in (0,\infty)$. Thus we have

$$\frac{2}{\pi}\int_0^\infty \frac{F(t)\cos xt}{t}\,dt = C\chi(x;(0,s])$$

and so, by the inverse Fourier cosine transform,

$$F(t) = Ct \int_0^\infty \chi(x; (0, s]) \cos xt \, dx = C \sin st.$$

Conversely, it is easy to see that for $F(t) = C \sin st$, equality holds in (1.18).

2. Proof of Theorem 1.1

We begin with the following lemma:

LEMMA 2.1. *For all* $f_j \in H^p_{\alpha_j, \beta_j}$, $j = 1, 2, \ldots, m$ *and* $x \in \mathbf{R}$ *we have*

$$\alpha^p \left| \left\{ \prod_{j=1}^m f_j(x) \right\}' \right|^p + \beta^p \left| \prod_{j=1}^m f_j(x) \right|^p \leq \left(\sum_{j=1}^m \prod_{i \neq j} \frac{\alpha_i}{\beta_i} \right)^{p-1} \left(\sum_{j=1}^m [\alpha_j^p |f_j'(x)|^p + \beta_j^p |f_j(x)|^p] \prod_{i \neq j} \alpha_i \beta_i^{p-1} |f_i(x)|^p \right), \tag{2.1}$$

and so

$$\alpha^2 \left| \left\{ \prod_{j=1}^m f_j(x) \right\}' \right|^2 + \beta^2 \left| \prod_{j=1}^m f_j(x) \right|^2 \leq \left(\sum_{j=1}^m \prod_{i \neq j} \frac{\alpha_i}{\beta_i} \right) \left(\sum_{j=1}^m |\alpha_j f_j'(x) + \beta_j f_j(x)|^2 \prod_{i \neq j} \alpha_i \beta_i |f_i(x)|^2 \right) - \alpha\beta \left(\prod_{j=1}^m |f_j(x)|^2 \right)'. \tag{2.2}$$

Proof. By the Leibniz rule we have

$$\alpha^p \left| \left\{ \prod_{j=1}^m f_j(x) \right\}' \right|^p = \left| \sum_{j=1}^m [\alpha_j f_j'(x)] \prod_{i \neq j} \alpha_i f_i(x) \right|^p = \left| \sum_{j=1}^m [\alpha_j f_j'(x)] \prod_{i \neq j} [\alpha_i^{1/p} \beta_i^{1-1/p} f_i(x)] \frac{\alpha_i^{1-1/p}}{\beta_i^{1-1/p}} \right|^p.$$

On applying Hölder's inequality we get

$$\alpha^p \left| \left\{ \prod_{j=1}^m f_j(x) \right\}' \right|^p \leq \left(\sum_{j=1}^m \prod_{i \neq j} \frac{\alpha_i}{\beta_i} \right)^{p-1} \left(\sum_{j=1}^m \alpha_j^p |f_j'(x)|^p \prod_{i \neq j} \alpha_i \beta_i^{p-1} |f_i(x)|^p \right). \tag{2.3}$$

Since

$$\beta^p = \left(\sum_{j=1}^m \beta_j \prod_{i\neq j} \alpha_i\right)^p = \left(\sum_{j=1}^m \prod_{i\neq j} \frac{\alpha_i}{\beta_i}\right)^{p-1} \sum_{j=1}^m \beta_j^p \prod_{i\neq j} \alpha_i \beta_i^{p-1}$$

it follows that

$$\beta^p \left|\prod_{j=1}^m f_j(x)\right|^p = \left(\sum_{j=1}^m \prod_{i\neq j} \frac{\alpha_i}{\beta_i}\right)^{p-1} \sum_{j=1}^m \beta_j^p |f_j(x)|^p \prod_{i\neq j} \alpha_i \beta_i^{p-1} |f_i(x)|^p,$$

which together with (2.3) yield (2.1).

In the case of $p = 2$, we have

$$\begin{aligned}
&\alpha^2 \left|\left\{\prod_{j=1}^m f_j(x)\right\}'\right|^2 + \beta^2 \left|\prod_{j=1}^m f_j(x)\right|^2 \\
&\leq \left(\sum_{j=1}^m \prod_{i\neq j} \frac{\alpha_i}{\beta_i}\right)\left(\sum_{j=1}^m [\alpha_j^2 |f_j'(x)|^2 + \beta_j^2 |f_j(x)|^2] \prod_{i\neq j} \alpha_i\beta_i |f_i(x)|^2\right) \\
&= \left(\sum_{j=1}^m \prod_{i\neq j} \frac{\alpha_i}{\beta_i}\right)\left(\sum_{j=1}^m [|\alpha_j f_j'(x) + \beta_j f_j(x)|^2 - \alpha_j\beta_j (|f_j(x)|^2)'] \prod_{i\neq j} \alpha_i\beta_i |f_i(x)|^2\right) \\
&= \left(\sum_{j=1}^m \prod_{i\neq j} \frac{\alpha_i}{\beta_i}\right)\left(\sum_{j=1}^m |\alpha_j f_j'(x) + \beta_j f_j(x)|^2 \prod_{i\neq j} \alpha_i\beta_i |f_i(x)|^2\right) - \alpha\beta\left(\prod_{j=1}^m |f_j(x)|^2\right)'.
\end{aligned}$$

This completes the proof of Lemma 2.1. □

LEMMA 2.2. *For all* $f_j \in H^p_{\alpha_j,\beta_j}$, $j = 1, 2, \ldots, m$, *and* $x \in \mathbf{R}$ *we have*

$$\alpha_j \beta_j^{p-1} |f_j(x)|^p \leq \left(\frac{p-1}{p}\right)^{p-1} \int_{-\infty}^x |\alpha_j f_j'(y) + \beta_j f_j(y)|^p \, dy. \tag{2.4}$$

Proof. The proof immediate from the following

$$f_j(x) = \frac{1}{\alpha_j} \int_{-\infty}^x [\alpha_j f_j'(y) + \beta_j f_j(y)] e^{(\beta_j/\alpha_j)(y-x)} \, dy$$

and Hölder's inequality. □

Now, from Lemmas 2.1 and 2.2 we have

LEMMA 2.3. *For all* $f_j \in H^p_{\alpha_j,\beta_j}$, $j = 1, 2, \dots, m$, *and* $x \in \mathbf{R}$ *we have*

$$\alpha^p \left| \left\{ \prod_{j=1}^m f_j(x) \right\}' \right|^p + \beta^p \left| \prod_{j=1}^m f_j(x) \right|^p \le C(p) \left(\prod_{j=1}^m \int_{-\infty}^x [\alpha_j^p |f_j'(y)|^p + \beta_j^p |f_j(y)|^p] \, dy \right)' \tag{2.5}$$

and

$$\alpha^2 \left| \left\{ \prod_{j=1}^m f_j(x) \right\}' \right|^2 + \beta^2 \left| \prod_{j=1}^m f_j(x) \right|^2 \le C(2) \left(\prod_{j=1}^m \int_{-\infty}^x |\alpha_j f_j'(y) + \beta_j f_j(y)|^2 \, dy \right)' - \alpha\beta \left(\prod_{j=1}^m |f_j(x)|^2 \right)'. \tag{2.6}$$

The theorem now follows from Lemma 2.3 and the following

$$\int_{-\infty}^{\infty} \left(\prod_{j=1}^m |f_j(x)|^2 \right)' dx = 0 \tag{2.7}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} |\alpha_j f_j(x) + \beta_j f_j'(x)|^2 \, dx &= \int_{-\infty}^{\infty} (\alpha_j^2 |f_j(x)|^2 + \beta_j^2 |f_j'(x)|^2) \, dx + \int_{-\infty}^{\infty} \alpha_j \beta_j (|f_j(x)|^2)' \, dx \\ &= \int_{-\infty}^{\infty} (\alpha_j^2 |f_j(x)|^2 + \beta_j^2 |f_j'(x)|^2) \, dx \end{aligned} \tag{2.8}$$

for all $j = 1, 2, \dots, m$.

3. Proof of Theorem 1.6

Since f_j is absolutely continuous with $\lim_{x \to a} f_j(x) = 0$, it follows that

$$f_j(x) = \int_a^x f_j'(s) \, ds, \quad x \in (a, b), \ j = 1, 2, \dots, m.$$

Then, for $f_j \in W^p_{\omega_j}(a, b)$, $j = 1, 2, \dots, m$, by Hölder's inequality, we obtain

$$\begin{aligned} |f_j(x)| = \left| \int_a^x f_j'(s) \, ds \right| &\le \int_a^x |f_j'(s)| \, ds \\ &\le \left(\int_a^x \frac{|f_j'(s)|^p}{\omega_j(s)^{p-1}} \, ds \right)^{1/p} \left(\int_a^x \omega_j(s) \, ds \right)^{(p-1)/p} \end{aligned} \tag{3.1}$$

and thus,

$$\left|\left\{\prod_{j=1}^{m} f_j(x)\right\}'\right|^p \le \left[\sum_{k=1}^{m} |f_k'(x)| \prod_{j\neq k}\left\{\left(\int_a^x \frac{|f_j'(s)|^p}{\omega_j(s)^{p-1}}\,ds\right)^{1/p}\left(\int_a^x \omega_j(s)\,ds\right)^{(p-1)/p}\right\}\right]^p$$

$$\le \left[\sum_{k=1}^{m} |f_k'(x)|\left\{\prod_{j\neq k}\int_a^x \frac{|f_j'(s)|^p}{\omega_j(s)^{p-1}}\,ds\right\}^{1/p}\left\{\prod_{j\neq k}\int_a^x \omega_j(s)\,ds\right\}^{(p-1)/p}\right]^p$$

$$\le \left\{\sum_{k=1}^{m} \frac{|f_k'(x)|^p}{\omega_k(x)^{p-1}}\prod_{j\neq k}\int_a^x \frac{|f_j'(s)|^p}{\omega_j(s)^{p-1}}\,ds\right\}\rho(x)^{p-1}.$$

Hence,

$$\int_a^b \frac{|\{\prod_{j=1}^m f_j(x)\}'|^p}{\rho(x)^{p-1}}\,dx \le \int_a^b\left\{\sum_{k=1}^{m} \frac{|f_k'(x)|^p}{\omega_k(x)^{p-1}}\prod_{j\neq k}\int_a^x \frac{|f_j'(s)|^p}{\omega_j(s)^{p-1}}\,ds\right\}dx$$

$$= \int_a^b\left\{\prod_{j=1}^{m}\int_a^x \frac{|f_j'(s)|^p}{\omega_j(s)^{p-1}}\,ds\right\}' dx$$

$$= \prod_{j=1}^{m}\int_a^b \frac{|f_j'(x)|^p}{\omega_j(x)^{p-1}}\,dx,$$

which gives (1.12).

Next we determine under what conditions equality can hold in (1.12). Equality in (1.12) implies that equality holds in (3.1) for each $x \in (a,b)$ and $j = 1,2,\ldots,m$. This happens only if for each $x \in (a,b)$ and $j = 1,2,\ldots,m$,

$$\left|\int_a^x f_j'(s)\,ds\right| = \int_a^x |f_j'(s)|\,ds \tag{3.2}$$

and

$$A_j\frac{|f_j'(s)|^p}{\omega_j(s)^{p-1}} = B_j\omega_j(s) \tag{3.3}$$

almost everywhere on $(a,x]$, where A_j and B_j are real constants.

By continuity of ω_j on (a,b), the conditions (3.2) and (3.3) imply that there exist some real constants C_j, $j = 1,2,\ldots,m$, such that

$$f_j'(s) = C_j\omega_j(s)\chi(s;(a,x]) \quad \text{for each } x \in (a,b),$$

which establishes the formula (1.13).

Conversely, we see directly that for f_j, $j = 1,2,\ldots,m$, satisfying (1.13), equality holds in (1.12).

Acknowledgment. The authors wish to express our deep thanks for the anonymous referees for their valuable comments and suggestions on the earlier version of this paper.

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