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FORMULAS OF F-THRESHOLDS AND F-JUMPING COEFFICIENTS ON TORIC RINGS*

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Abstract

Mustaţă, Takagi and Watanabe define F-thresholds, which are invariants of a pair of ideals in a ring of characteristic $p > 0$. In their paper, it is proved that F-thresholds are equal to jumping numbers of test ideals on regular local rings. In this note, we give formulas of F-thresholds and F-jumping coefficients on toric rings. By these formulas, we prove that there exists an inequality between F-jumping coefficients and F-thresholds. In particular, we observe a difference between F-pure thresholds and F-thresholds on certain rings. As applications, we give a characterization of regularity for toric rings defined by simplicial cones, and we prove the rationality of F-thresholds on certain rings.

1. Introduction

Let R be a commutative Noetherian ring of characteristic $p > 0$. Suppose $\mathfrak{a}$ is an ideal of R and c is a positive real number. In [HY], Hara and Yoshida defined a generalized test ideal $\tau(\mathfrak{a}^c)$ of $\mathfrak{a}$ with exponent c. This is a generalization of the test ideal $\tau(R)$, which appeared in the theory of tight closure (cf. [HH]). On the other hand, this ideal is a characteristic p analogue of a multiplier ideal (cf. [Laz]). Similarly, one can define a characteristic p analogue of a jumping coefficient of a multiplier ideal, which is called the F-jumping coefficient. In other words, a positive real number c is an F-jumping coefficient of an ideal $\mathfrak{a}$ of R if $\tau(\mathfrak{a}^c) \neq \tau(\mathfrak{a}^{c-\varepsilon})$ for all positive real numbers ε.

Mustaţă, Takagi and Watanabe studied F-jumping coefficients. In [MTW], they defined an another invariant of singularities, which is called the F-threshold. They proved that an F-threshold coincides with an F-jumping coefficient on a regular local ring of characteristic $p > 0$. Using this relation, they proved basic properties of F-jumping coefficients. Blickle, Mustaţă and Smith studied F-jumping coefficients or F-thresholds on F-finite regular rings. In particular, they proved the rationality and discreteness of F-thresholds for F-finite regular rings under some assumptions (cf. [BMS1] and [BMS2] for details), which partially solves an open problem in [MTW].

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However, if rings have singularities, F-thresholds may not coincide with F-jumping coefficients. In [HMTW], Huneke, Mustaţă, Takagi and Watanabe studied various topics of F-thresholds. For example, they defined a new invariant called the F-threshold of a module, which coincides with an F-jumping coefficient for F-finite and F-regular local normal $\mathbf{Q}$-Gorenstein rings. As a corollary, they proved an inequality between the F-threshold and the F-pure threshold, which is the smallest F-jumping coefficient for a fixed ideal. They also gave examples of non-regular rings and ideals whose F-thresholds coincide with their F-pure thresholds.

In this paper, we consider F-thresholds and F-jumping coefficients of monomial ideals for toric rings, which are not necessarily regular. We give the explicit formula of F-thresholds in section 3, which is written in terms of cones corresponding to toric rings and Newton polyhedrons corresponding to monomial ideals. Using this formula, we attempt a comparison between F-thresholds and F-jumping coefficients in section 4. As applications, we give a characterization of regularity of toric rings defined by simplicial cones in Theorem 5.3. We also prove the rationality of F-thresholds of monomial ideals for toric rings defined by simplicial cones in Theorem 5.5.

2. The definition of F-thresholds

Throughout this paper, we assume that every ring R is reduced and contains a perfect field k whose characteristic is $p > 0$. Let $F : R \to R$ be the Frobenius map which sends an element x of R to x^p. For a positive integer e, the ring R viewed as an R-module via the e-times iterated Frobenius map is denoted by ${}^eR$. We assume that a ring R is F-finite, that is, ${}^1R$ is a finitely generated R-module. We also assume that a ring R is F-pure, that is, the Frobenius map F is pure. For an ideal J and a positive integer e, $J^{[p^e]}$ is the ideal generated by p^e-th power elements of J. We recall the definition and some remarks of F-thresholds which are defined by Mustaţă, Takagi and Watanabe in [MTW]. These are invariants of a pair of ideals.

Definition 2.1 (F-threshold, cf. [MTW, §1]). Let $\mathfrak{a}$ and J be nonzero proper ideals of a ring R such that $\mathfrak{a} \subseteq \sqrt{J}$. The p^e-th threshold $\nu_{\mathfrak{a}}^J(p^e)$ of $\mathfrak{a}$ with respect to J is defined as

$$\nu_{\mathfrak{a}}^J(p^e) := \max\{r \in \mathbf{N} \mid \mathfrak{a}^r \not\subseteq J^{[p^e]}\}.$$

Then we define the F-threshold $\mathrm{c}^J(\mathfrak{a})$ of $\mathfrak{a}$ with respect to J as

$$\mathrm{c}^J(\mathfrak{a}) := \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}.$$

Remark. Since R is F-pure, if $u \notin J^{[p^e]}$, then $u^p \notin J^{[p^{e+1}]}$. This implies that $\nu_{\mathfrak{a}}^J(p^e)/p^e \le \nu_{\mathfrak{a}}^J(p^{e+1})/p^{e+1}$, and hence $\mathrm{c}^J(\mathfrak{a})$ exists under our assumption. Furthermore, if $\mathfrak{a} \subseteq \sqrt{J}$, then $\mathrm{c}^J(\mathfrak{a})$ is a finite number. However, in general, the existence of this limit has not proved. In [HMTW], Huneke, Mustaţă, Takagi and Watanabe defined $\mathrm{c}_{-}^{J}(\mathfrak{a})$ and $\mathrm{c}_{+}^{J}(\mathfrak{a})$ as

$$\mathrm{c}_{-}^{J}(\mathfrak{a}) := \liminf \frac{\nu_{\mathfrak{a}}^{J}(p^e)}{p^e}, \quad \mathrm{c}_{+}^{J}(\mathfrak{a}) := \limsup \frac{\nu_{\mathfrak{a}}^{J}(p^e)}{p^e},$$

for ideals $\mathfrak{a}$ and J such that $\mathfrak{a} \subseteq \sqrt{J}$. When $\mathrm{c}_{-}^{J}(\mathfrak{a}) = \mathrm{c}_{+}^{J}(\mathfrak{a})$, they call it the F-threshold of $\mathfrak{a}$ with respect to J, which is denoted by $\mathrm{c}^{J}(\mathfrak{a})$. They give a sufficient condition when $\mathrm{c}^{J}(\mathfrak{a})$ exists (cf. [HMTW, Lemma 2.3]).

Let R° be the set of elements of R which are not contained in any minimal prime ideals of R. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^{\circ} \neq \emptyset$, and let c be a positive real number. For an R-module D, we define the $\mathfrak{a}^c$-tight closure of the zero submodule in D as the following. We denote it by $0_D^{*\mathfrak{a}^c}$. For an element z of D, an element z is contained in $0_D^{*\mathfrak{a}^c}$ if there exists an element x of R° such that

$$x\mathfrak{a}^{\lceil cp^e \rceil}(1 \otimes z) = 0 \in {}^{e}R \otimes D,$$

where e runs all sufficiently large positive integers.

Definition 2.2 (test ideal). Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^{\circ} \neq \emptyset$, and c a positive real number. We define the R-module E as $\bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$, where $\mathfrak{m}$ runs all maximal ideals of R and $E_R(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$. The test ideal $\tau(\mathfrak{a}^c)$ of $\mathfrak{a}$ with exponent c is defined as

$$\tau(\mathfrak{a}^c) := \bigcap_{D \subseteq E} \mathrm{Ann}_R\, 0_D^{*\mathfrak{a}^c},$$

where D runs all finitely generated R-submodules of E.

In [MTW], Mustaţă, Takagi and Watanabe also proved the connection between F-thresholds and test ideals on regular local rings. Moreover, in [BMS2], Blickle, Mustaţă and generalized it on regular rings.

Theorem 2.3 ([MTW, Proposition 2.7] and [BMS2, Proposition 2.23]). *Let $\mathfrak{a}$ and J be proper ideals of a regular ring R such that $\mathfrak{a} \subseteq \sqrt{J}$. Then*

$$\tau(\mathfrak{a}^{\mathrm{c}^{J}(\mathfrak{a})}) \subseteq J.$$

On the other hand, for a positive real number c, the ideal $\mathfrak{a}$ is included in $\sqrt{\tau(\mathfrak{a}^c)}$, and also

$$\mathrm{c}^{\tau(\mathfrak{a}^c)}(\mathfrak{a}) \leq c.$$

In addition, there exists a map from the set of F-thresholds of $\mathfrak{a}$ to the set of test ideals of $\mathfrak{a}$ which sends the test ideal J to $\mathrm{c}^{J}(\mathfrak{a})$. Moreover, this map is bijective. The inverse map sends an F-threshold c of $\mathfrak{a}$ to $\tau(\mathfrak{a}^c)$.

By the two inequalities in Theorem 2.3, F-thresholds on a regular ring are equal to F-jumping coefficients. They are analogues of jumping coefficients of a multiplier ideal.

COROLLARY 2.4. *For a fixed nonzero proper ideal* $\mathfrak{a}$ *of a regular ring* R, *the set of F-thresholds of* $\mathfrak{a}$ *is equal to the set of F-jumping coefficients of* $\mathfrak{a}$.

3. A formula of F-thresholds on toric rings

Let us begin with fixing the notation about toric geometries. Let N be the lattice of rank d, and M the dual lattice of N. We recall that M is isomorphic to $\mathbf{Z}^d$. We denote $N \otimes_{\mathbf{Z}} \mathbf{R}$ and $M \otimes_{\mathbf{Z}} \mathbf{R}$ by $M_{\mathbf{R}}$ and $N_{\mathbf{R}}$ respectively. The duality pairing of $M_{\mathbf{R}}$ and $N_{\mathbf{R}}$ is denoted by

$$\langle\, , \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \to \mathbf{R}.$$

For a strongly convex rational polyhedral cone σ in $N_{\mathbf{R}}$, we define the dual cone $\sigma^{\vee}$ of σ as

$$\sigma^{\vee} := \{u \in M_{\mathbf{R}} \mid \langle u, v\rangle \geq 0, \forall v \in \sigma\}.$$

Let R be a toric ring defined by σ. In other words, R is the subalgebra of Laurent polynomial $k[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ generated by sets $\{X^u \mid u \in \sigma^{\vee} \cap M\}$, where X^u expresses $X_1^{u_1} \cdots X_d^{u_d}$ for a lattice point $u = (u_1, \dots, u_d)$ of M. Since we always assume that k is a perfect field, a toric ring is F-finite under our assumption. A proper ideal $\mathfrak{a}$ of R is said to be a monomial ideal if $\mathfrak{a}$ is generated by monomials. For a monomial ideal $\mathfrak{a}$, we define two types of sets in $\sigma^{\vee}$.

DEFINITION 3.1. The Newton polyhedron $\mathrm{P}(\mathfrak{a})$ of $\mathfrak{a}$ is defined as

$$\mathrm{P}(\mathfrak{a}) := \mathrm{conv}\{u \in M \mid X^u \in \mathfrak{a}\},$$

and $\mathrm{Q}(\mathfrak{a})$ is defined as

$$\mathrm{Q}(\mathfrak{a}) := \bigcup_{X^u \in \mathfrak{a}} u + \sigma^{\vee}.$$

Suppose λ is a positive real number. The sets $\lambda\mathrm{P}(\mathfrak{a})$ is defined as

$$\lambda\mathrm{P}(\mathfrak{a}) := \{\lambda u \in M_{\mathbf{R}} \mid u \in \mathrm{P}(\mathfrak{a})\}.$$

We define $\lambda\mathrm{Q}(\mathfrak{a})$ by the same way.

The following proposition is basic properties of $\mathrm{Q}(\mathfrak{a})$ and $\mathrm{P}(\mathfrak{a})$, which follows immediately.

PROPOSITION 3.2. *Let* $\mathfrak{a}$ *be a monomial ideal of a toric ring* R *defined by a cone* σ *in* $N_{\mathbf{R}}$.

(i) *For* $e \in \mathbf{Z}_{>0}$, *it holds that* $\mathrm{Q}(\mathfrak{a}) = (1/p^e)\mathrm{Q}(\mathfrak{a}^{[p^e]})$.
(ii) $\mathrm{P}(\mathfrak{a}) + \sigma^\vee \subseteq \mathrm{P}(\mathfrak{a})$.
(iii) *If* $\mathfrak{a} = (X^{\mathbf{a}_1}, \dots, X^{\mathbf{a}_s})$, *then* $\mathrm{P}(\mathfrak{a}) = \mathrm{conv}\{\mathbf{a}_1, \dots, \mathbf{a}_s\} + \sigma^\vee$.

Using this notation, we give a computation of F-thresholds. This formula is a generalization of [HMTW, Example 2.7]. Let R be a toric ring defined by a cone σ in $N_{\mathbf{R}}$. Let $\mathfrak{a}$ be a monomial ideal of R. For an element u of $\sigma^\vee$, we define $\lambda_{\mathfrak{a}}(u)$ as

$$\lambda_{\mathfrak{a}}(u) := \sup\{\lambda \in \mathbf{R}_{>0} \mid u \in \lambda \mathrm{P}(\mathfrak{a})\}.$$

If u is not contained in $\lambda\mathrm{P}(\mathfrak{a})$ for all positive real numbers λ, then we set $\lambda_{\mathfrak{a}}(u) := 0$ by convention.

THEOREM 3.3. *Let R be a toric ring defined by σ, and also let $\mathfrak{a}$ and J be monomial ideals of R such that $\mathfrak{a} \subseteq \sqrt{J}$. Then*

$$\mathrm{c}^J(\mathfrak{a}) = \sup_{u \in \sigma^\vee \setminus \mathrm{Q}(J)} \lambda_{\mathfrak{a}}(u).$$

Proof. We assume that $\mathfrak{a} = (X^{\mathbf{a}_1}, \dots, X^{\mathbf{a}_s})$ where $\mathbf{a}_i$ are lattice points of M for $i = 1, \dots, s$. To prove the theorem, we need the following two claims.

CLAIM 1. For all positive integers e, there exists an element u of $\sigma^\vee \setminus \mathrm{Q}(J)$ such that $\nu_{\mathfrak{a}}^J(p^e)/p^e \le \lambda_{\mathfrak{a}}(u)$.

CLAIM 2. For every element u of $\sigma^\vee \setminus \mathrm{Q}(J)$, there exists a positive integer e such that $\nu_{\mathfrak{a}}^J(p^e)/p^e \ge \lambda_{\mathfrak{a}}(u)$.

Claim 1 implies that

$$\nu_{\mathfrak{a}}^J(p^e)/p^e \le \sup_{u \in \sigma^\vee \setminus \mathrm{Q}(J)} \lambda_{\mathfrak{a}}(u).$$

Thus $\mathrm{c}^J(\mathfrak{a}) \le \sup \lambda_{\mathfrak{a}}(u)$ by the definition of F-thresholds. By the similar argument, Claim 2 implies $\mathrm{c}^J(\mathfrak{a}) \ge \sup \lambda_{\mathfrak{a}}(u)$.

Proof of Claim 1. We fix a positive integer e. Since the definition of the p^e-th threshold, there are nonnegative integers r_i with $\sum r_i = \nu_{\mathfrak{a}}^J(p^e)$ such that $X^{\sum r_i \mathbf{a}_i}$ is not contained in $J^{[p^e]}$. In particular, $\sum r_i \mathbf{a}_i \notin \mathrm{Q}(J^{[p^e]})$. This is equivalent to the condition that $(1/p^e)\sum r_i \mathbf{a}_i$ is not contained in $(1/p^e)\mathrm{Q}(J^{[p^e]})$. By Proposition 3.2 (i), we have $(1/p^e)\sum r_i \mathbf{a}_i \notin \mathrm{Q}(J)$. Hence

$$\frac{1}{p^e}\sum r_i \mathbf{a}_i = \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}\sum \frac{r_i}{\nu_{\mathfrak{a}}^J(p^e)}\mathbf{a}_i,$$

which is an element of $(\nu_{\mathfrak{a}}^J(p^e)/p^e)\mathrm{P}(\mathfrak{a})$. Thus $\nu_{\mathfrak{a}}^J(p^e)/p^e \le \lambda_{\mathfrak{a}}((1/p^e)\sum r_i \mathbf{a}_i)$. □

Proof of Claim 2. We fix u an element of $\sigma^\vee \backslash \mathrm{Q}(J)$, such that $\lambda_{\mathfrak{a}}(u) \neq 0$. We find an integer e which satisfies the assertion of Claim 2 by three steps.

Step 1. We prove that there exists an element u' of the boundary $(\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil / p^e)\mathrm{P}(\mathfrak{a})$ such that $u' \notin \mathrm{Q}(J)$ for sufficiently large e. The following sequence of real numbers

$$\lambda_{\mathfrak{a}}(u) \leq \cdots \leq \frac{\lceil p^{e+1} \lambda_{\mathfrak{a}}(u) \rceil}{p^{e+1}} \leq \frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil}{p^e} \leq \cdots \leq \frac{\lceil p \lambda_{\mathfrak{a}}(u) \rceil}{p}$$

induces the sequence of Newton polyhedrons

$$\frac{\lceil p \lambda_{\mathfrak{a}}(u) \rceil}{p} \mathrm{P}(\mathfrak{a}) \subseteq \cdots \subseteq \frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil}{p^e} \mathrm{P}(\mathfrak{a}) \subseteq \frac{\lceil p^{e+1} \lambda_{\mathfrak{a}}(u) \rceil}{p^{e+1}} \mathrm{P}(\mathfrak{a}) \subseteq \cdots \subseteq \lambda_{\mathfrak{a}}(u) \mathrm{P}(\mathfrak{a}).$$

In particular, the above sequences are strict if $\lambda_{\mathfrak{a}}(u) \notin (1/p^e)\mathbf{Z}$ for all integers e. Since $u \notin \mathrm{Q}(J)$, we can find such u' by taking e sufficiently large.

Step 2. We prove that there exist nonnegative integers r_i such that $\sum r_i/p^e \geq \lambda_{\mathfrak{a}}(u)$ and $\sum r_i \mathbf{a}_i / p^e$ is not contained in $\mathrm{Q}(J)$. We denote $\sum r_i \mathbf{a}_i / p^e$ by u''. Since u' is contained in $(\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil / p^e)\mathrm{P}(\mathfrak{a})$, u' can be written as

$$\frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil}{p^e} \left(\sum c_i \mathbf{a}_i + \omega \right),$$

where c_i are nonnegative real numbers with $\sum c_i = 1$ and $\omega \in \sigma^\vee$ by Proposition 3.2 (iii). Let

$$r_i := \lceil \lceil p^e \lambda_{\mathfrak{a}}(u) \rceil c_i \rceil.$$

Then

$$\sum \frac{r_i}{p^e} \geq \frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil}{p^e} \sum c_i \geq \lambda_{\mathfrak{a}}(u).$$

Moreover,

$$\left| u'' + \frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil}{p^e} \omega - u' \right| \leq \sum \left| \frac{\lceil \lceil p^e \lambda_{\mathfrak{a}}(u) \rceil c_i \rceil}{p^e} - \frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil c_i}{p^e} \right| \cdot |\mathbf{a}_i| < \frac{1}{p^e} \sum |\mathbf{a}_i|.$$

Since $u' \notin \mathrm{Q}(J)$, an element $u'' + (\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil / p^e)\omega$ is not contained in $\mathrm{Q}(J)$ if we choose e sufficiently large. Hence u'' is not contained in $\mathrm{Q}(J)$.

Step 3. Since $u'' \notin \mathrm{Q}(J)$,

$$p^e u'' \notin p^e \mathrm{Q}(J) = \mathrm{Q}(J^{[p^e]}).$$

Therefore $X^{p^e u''}$ is not contained in $J^{[p^e]}$. On the other hand, $X^{p^e u''} \in \mathfrak{a}^{\sum r_i}$ by the construction of u''. Therefore $\sum r_i \leq \nu_{\mathfrak{a}}^J(p^e)$. This implies $\lambda_{\mathfrak{a}}(u) \leq \nu_{\mathfrak{a}}^J(p^e)/p^e$. □

We complete the proof of Theorem 3.3. □

4. A comparison between F-jumping coefficients and F-thresholds

In [TW], Takagi and Watanabe defined the F-pure threshold $\mathrm{c}(\mathfrak{a})$ of an ideal $\mathfrak{a}$ of a ring R as

$$\mathrm{c}(\mathfrak{a}) := \sup\{c \in \mathbf{R}_{\geq 0} \mid (R, \mathfrak{a}^c) \text{ is F-pure}\}.$$

See [TW, Definition 1.3, Definition 2.1] for the details. They also proved that if a ring R is strongly F-regular, then F-pure thresholds are described as in Definition 4.1. Since F-finite toric rings are strongly F-regular, we define F-pure thresholds as follows.

Definition 4.1 (F-pure thresholds). Let R be a toric ring, and $\mathfrak{a}$ a monomial ideal. The F-pure threshold $\mathrm{c}(\mathfrak{a})$ of $\mathfrak{a}$ is defined as

$$\mathrm{c}(\mathfrak{a}) := \sup\{c \in \mathbf{R}_{\geq 0} \mid \tau(\mathfrak{a}^c) = R\}.$$

Hence the F-pure threshold of $\mathfrak{a}$ is the smallest F-jumping coefficient of $\mathfrak{a}$. In [HMTW], the inequality between an F-pure threshold and an F-threshold on a local ring was given in terms of the F-threshold of a module ([HMTW, Section 4.]). In this section, we consider the inequality on toric rings, by a combinatorial method. Furthermore, we consider the connection between arbitrary F-jumping coefficients and F-thresholds. To compute F-pure thresholds and F-jumping coefficients of monomial ideals, we introduce the following theorem given by Blickle.

Theorem 4.2 ([B, Theorem 3]). *We set $\{v_j\}$ are the set of primitive lattice points of N. We consider a cone σ generated by $\{v_j\}$. Let R be the toric ring defined by σ, and $\mathfrak{a}$ a monomial ideal of R. Then for a positive real number c, the test ideal $\tau(\mathfrak{a}^c)$ of $\mathfrak{a}$ with exponent c is also a monomial ideal. Moreover, $X^u \in \tau(\mathfrak{a}^c)$ for a lattice point u of M if and only if there exists an element ω of $M_{\mathbf{R}}$ such that*

$$\langle \omega, v_j \rangle \leq 1 \quad (j = 1, \dots, n),$$

and

$$u + \omega \in \mathrm{Int}(c\mathrm{P}(\mathfrak{a})).$$

By this theorem, the F-pure threshold of a monomial ideal of a toric ring can be described as in the following corollary.

Corollary 4.3. *Let R and $\mathfrak{a}$ be as in Theorem 4.2. Then the F-pure threshold $\mathrm{c}(\mathfrak{a})$ of $\mathfrak{a}$ is described as*

$$\mathrm{c}(\mathfrak{a}) = \sup_{u \in \sigma^\vee \setminus \mathrm{O}} \lambda_{\mathfrak{a}}(u),$$

where

$$\mathrm{O} := \{u \in \sigma^\vee \mid \exists j, \langle u, v_j \rangle \geq 1\}.$$

Proof. First, we assume that $\mathrm{c}(\mathfrak{a}) < \sup \lambda_{\mathfrak{a}}(u)$. Then there exists a positive real number α such that

$$\mathrm{c}(\mathfrak{a}) < \alpha < \sup \lambda_{\mathfrak{a}}(u).$$

By the definition of F-pure thresholds, $\tau(\mathfrak{a}^{\alpha})$ is a proper ideal of R. Then there exists a positive real number β such that

$$\alpha < \beta < \sup \lambda_{\mathfrak{a}}(u)$$

and $\beta = \lambda_{\mathfrak{a}}(u')$ for an element u' of $\sigma^{\vee} \backslash \mathrm{O}$. This implies that $u' \in \beta \mathrm{P}(\mathfrak{a})$. In particular, u' is an element of $\mathrm{Int}(\alpha \mathrm{P}(\mathfrak{a}))$. In addition, $\langle u', v_j \rangle < 1$ for all j. By Theorem 4.2, it contradicts that $\tau(\mathfrak{a}^{\alpha}) \subsetneq R$. Therefore $\mathrm{c}(\mathfrak{a}) \geq \sup \lambda_{\mathfrak{a}}(u)$. Second, we assume $\mathrm{c}(\mathfrak{a}) > \sup \lambda_{\mathfrak{a}}(u)$. There exists a positive number α such that

$$\sup \lambda_{\mathfrak{a}}(u) < \alpha < \mathrm{c}(\mathfrak{a})$$

and $\tau(\mathfrak{a}^{\alpha}) = R$. This implies that there exists an element ω of $\sigma^{\vee}$ such that $\langle \omega, v_j \rangle \leq 1$ for all j and

$$\omega \in \mathrm{Int}(\alpha \mathrm{P}(\mathfrak{a})).$$

If $1 > \varepsilon > 0$, then $\langle (1-\varepsilon)\omega, v_j \rangle < 1$ for all j. Thus $(1-\varepsilon)\omega$ is contained in $\sigma^{\vee} \backslash \mathrm{O}$. On the other hand, since $\omega \in \mathrm{Int}(\alpha \mathrm{P}(\mathfrak{a}))$, it holds that

$$(1-\varepsilon')\omega \in \alpha \mathrm{P}(\mathfrak{a}),$$

for sufficiently small ε'. Therefore

$$\sup_{u \in \sigma^{\vee} \backslash \mathrm{O}} \lambda_{\mathfrak{a}}(u) < \lambda_{\mathfrak{a}}((1-\varepsilon')\omega),$$

which is a contradiction. Thus $\mathrm{c}(\mathfrak{a}) \geq \sup \lambda_{\mathfrak{a}}(u)$, which completes the proof of the corollary. $\square$

Using this presentation, we give an inequality between an F-pure threshold and an F-threshold with respect to the maximal monomial ideal on a toric ring.

PROPOSITION 4.4. *Let R, σ and $\mathfrak{a}$ be as in Theorem 4.2, and $\mathfrak{m}$ the maximal monomial ideal of R. Then*

$$\mathrm{c}(\mathfrak{a}) \leq \mathrm{c}^{\mathfrak{m}}(\mathfrak{a}).$$

Proof. By the definitions, it is enough to show that $\mathrm{Q}(\mathfrak{m}) \subseteq \mathrm{O}$. In particular, it is enough to show $\mathrm{Q}(\mathfrak{m}) \cap M \subseteq \mathrm{O}$. It follows immediately. $\square$

Remark. In general, for an ideal $\mathfrak{a}$, we have $\mathrm{c}^{J'}(\mathfrak{a}) \leq \mathrm{c}^{J}(\mathfrak{a})$, where J and J' are ideals such that $J \subseteq J'$ and $\mathfrak{a} \subseteq \sqrt{J}$. Therefore the F-pure threshold of $\mathfrak{a}$ is less than or equal to all F-thresholds of $\mathfrak{a}$.

Now we give a generalization of this comparison.

PROPOSITION 4.5. *Let R, σ and $\mathfrak{a}$ be as in Theorem* 4.2. *For a lattice point u of $\sigma^\vee$, we define the nonnegative number $\mu_{\mathfrak{a}}(u)$ as*

$$\mu_{\mathfrak{a}}(u) := \sup_{\omega \in \sigma^\vee \setminus O} \lambda_{\mathfrak{a}}(u + \omega),$$

and the nonnegative number $\mathrm{c}^i(\mathfrak{a})$ as

$$\mathrm{c}^i(\mathfrak{a}) = \inf_{X^u \in \tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})})} \mu_{\mathfrak{a}}(u),$$

where $\mathrm{c}^0(\mathfrak{a}) := 0$. Then $\mathrm{c}^i(\mathfrak{a})$ is the i-th F-jumping coefficient of $\mathfrak{a}$.

LEMMA 4.6. *Let R, σ and $\mathfrak{a}$ be as in Theorem* 4.2. *Suppose ω and ω' are elements of $\sigma^\vee$. For all $j = 1, \ldots, n$, we assume that*

$$\langle \omega, v_j \rangle \le \langle \omega', v_j \rangle.$$

Then $\lambda_{\mathfrak{a}}(\omega) \le \lambda_{\mathfrak{a}}(\omega')$.

Proof. If $\lambda_{\mathfrak{a}}(\omega) = 0$, it is trivial. We prove this lemma in the case $\lambda_{\mathfrak{a}}(\omega) \neq 0$. By the assumption, there exists an element ω'' of $\sigma^\vee$ such that $\omega' = \omega + \omega''$. Let $\lambda := \lambda_{\mathfrak{a}}(\omega)$. Since $\omega/\lambda \in \mathrm{P}(\mathfrak{a})$ and $\omega''/\lambda \in \sigma^\vee$,

$$\frac{\omega'}{\lambda} \in \mathrm{P}(\mathfrak{a}) + \sigma^\vee.$$

By Proposition 3.2 (ii), we have $\omega'/\lambda \in \mathrm{P}(\mathfrak{a})$. Hence $\lambda \le \lambda_{\mathfrak{a}}(\omega')$. □

Proof of Proposition 4.5. We show that $\mathrm{c}^i(\mathfrak{a})$ is a jumping number of the test ideal. We assume that

$$\tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})}) = (X^{\mathbf{b}_1}, \ldots, X^{\mathbf{b}_t}).$$

By Lemma 4.6,

$$\mathrm{c}^i(\mathfrak{a}) = \inf_{j=1,\ldots,t} \mu_{\mathfrak{a}}(\mathbf{b}_j).$$

Since $\{\mathbf{b}_j\}$ is a finite set, there exists j' such that $\mathrm{c}^i(\mathfrak{a}) = \mu_{\mathfrak{a}}(\mathbf{b}_{j'})$. By the definition of $\mathrm{c}^i(\mathfrak{a})$, for all elements ω of $\sigma^\vee \setminus O$,

$$\mathbf{b}_{j'} + \omega \notin \mathrm{Int}(c^i(\mathfrak{a})\mathrm{P}(\mathfrak{a})).$$

This implies that $X^{\mathbf{b}_{j'}} \notin \tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})})$ by Theorem 4.2. On the other hand, there exists an element ω' of $\sigma^\vee \setminus O$ such that

$$\mathbf{b}_{j'} + \omega' \in \mathrm{Int}((\mathrm{c}^i(\mathfrak{a}) - \varepsilon)\mathrm{P}(\mathfrak{a})),$$

for all positive real numbers ε. This also implies that $X^{\mathbf{b}_{j'}} \in \tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a}) - \varepsilon})$. Therefore $\tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})}) \subsetneq \tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a}) - \varepsilon})$ and hence $\mathrm{c}^i(\mathfrak{a})$ is a jumping number.

We show that $\mathrm{c}^i(\mathfrak{a})$ is the i-th F-jumping coefficient of $\mathfrak{a}$. In other words, $\tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a}) - \varepsilon}) = \tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})})$ for all positive numbers ε such that $\mathrm{c}^{i-1}(\mathfrak{a}) \le \mathrm{c}^i(\mathfrak{a}) - \varepsilon$.

The inclusion $\tau(\mathfrak{a}^{c^i(\mathfrak{a})-\varepsilon}) \subseteq \tau(\mathfrak{a}^{c^{i-1}(\mathfrak{a})})$ follows immediately from Theorem 4.2. The opposite inclusion follows from the definition of $c^i(\mathfrak{a})$. In fact, if $X^u \in \tau(\mathfrak{a}^{c^{i-1}(\mathfrak{a})})$, then $c^i(\mathfrak{a}) \le \mu_{\mathfrak{a}}(u)$ by definition of $c^i(\mathfrak{a})$. Hence there exists an element ω of $\sigma^\vee \backslash \mathrm{O}$ such that

$$u + \omega \in \mathrm{Int}((c^i(\mathfrak{a}) - \varepsilon)\mathrm{P}(\mathfrak{a})).$$

This implies that $X^u \in \tau(\mathfrak{a}^{c^i(\mathfrak{a})-\varepsilon})$ by Theorem 4.2. We complete the proof of the proposition. □

Proposition 4.7. *We have the following inequality*:

$$c^i(\mathfrak{a}) \le c^{\tau(\mathfrak{a}^{c^i(\mathfrak{a})})}(\mathfrak{a}).$$

Proof. Since $\tau(\mathfrak{a}^{c^i(\mathfrak{a})}) \subsetneq \tau(\mathfrak{a}^{c^{i-1}(\mathfrak{a})})$, there exists a lattice point u in $\sigma^\vee$ such that $X^u \in \tau(\mathfrak{a}^{c^{i-1}(\mathfrak{a})})$ and $X^u \notin \tau(\mathfrak{a}^{c^i(\mathfrak{a})})$. By Proposition 4.5,

$$c^i(\mathfrak{a}) \le \mu_{\mathfrak{a}}(u). \tag{1}$$

We claim that for all elements ω of $\sigma^\vee \backslash \mathrm{O}$,

$$\omega + u \in \sigma^\vee \backslash \mathrm{Q}(\tau(\mathfrak{a}^{c^i(\mathfrak{a})})). \tag{2}$$

By Theorem 3.3, this claim implies that

$$\mu_{\mathfrak{a}}(u) \le c^{\tau(\mathfrak{a}^{c^i(\mathfrak{a})})}(\mathfrak{a}). \tag{3}$$

The proof of the proposition is completed from inequalities (1) and (3). Now we prove the claim (2). We assume that there exists an element ω of $\sigma^\vee \backslash \mathrm{O}$ such that $u + \omega \in \mathrm{Q}(\tau(\mathfrak{a}^{c^i(\mathfrak{a})}))$. There exist a lattice point u' of M and an element ω' of $\sigma^\vee$ such that $X^{u'} \in \tau(\mathfrak{a}^{c^i(\mathfrak{a})})$ and $u + \omega = u' + \omega'$. Thus $u - u'$ and $\omega' - \omega$ are lattice points. On the other hand, since u is a lattice point, $u = u' + \omega' - \omega$ and $X^u \notin \tau(\mathfrak{a}^{c^i(\mathfrak{a})})$, we have $\omega' - \omega \notin \sigma^\vee$. That is, there exists j such that $\langle (\omega' - \omega), v_j \rangle < 0$. Therefore

$$0 \le \langle \omega', v_j \rangle < \langle \omega, v_j \rangle < 1.$$

It contradicts that $\omega' - \omega \in M$. Hence we have the claim, and then we complete the proof of the proposition. □

Remark. Since an F-finite toric ring is strongly F-regular, $\mathfrak{a} \subseteq \tau(\mathfrak{a}^{c^i(\mathfrak{a})})$. Hence $c^{\tau(\mathfrak{a}^{c^i(\mathfrak{a})})}(\mathfrak{a})$ exists and is a finite number.

5. Applications

Let us give some applications of the results of the previous sections. As we see in Corollary 2.4, for an arbitrary ideal $\mathfrak{a}$, the set of the F-thresholds of $\mathfrak{a}$ is equal to the set of the F-jumping coefficients of $\mathfrak{a}$ on regular rings. By Theorem 3.3, if R is a toric ring which has at most Gorenstein singularities, then there exists a monomial ideal $\mathfrak{a}$ of R such that $c(\mathfrak{a}) = c^{\mathfrak{m}}(\mathfrak{a})$.

PROPOSITION 5.1. *Let R be a Gorenstein toric ring defined by a cone σ in $N_{\mathbf{R}}$ and $\mathfrak{m}$ the maximal monomial ideal. There exists a monomial ideal $\mathfrak{a}$ of R such that* $\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$.

Proof. We assume that σ is generated by primitive lattice points $v_1, \dots, v_n$ of N. For a Gorenstein toric ring R, there exists a lattice point ω of $\sigma^\vee$ such that $\langle \omega, v_j \rangle = 1$ for all $j = 1, \dots, n$. By Lemma 4.6, for a monomial ideal $\mathfrak{a}$ of R, we have

$$\mathrm{c}(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\omega).$$

Let $\mathfrak{a}$ be a monomial ideal generated by X^ω. We have $\mathrm{P}(\mathfrak{a}) = \omega + \sigma^\vee$, and clearly $\mathrm{c}(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\omega) = 1$. Since ω is a nonzero lattice point of M, we have $\omega \in \mathrm{Q}(\mathfrak{m})$. Hence $\mathrm{P}(\mathfrak{a}) \subseteq \mathrm{Q}(\mathfrak{m})$. By Theorem 3.3, that implies $\mathrm{c}^{\mathfrak{m}}(\mathfrak{a}) \leq 1$. On the other hand, the inequality $\mathrm{c}(\mathfrak{a}) \leq \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$ follows by Proposition 4.4. We complete the proof of the proposition. □

For 2-dimensional toric rings, the opposite assertion of Proposition 5.1 holds. However, it is false in general toric rings whose dimension are greater than 3.

PROPOSITION 5.2. *Let R be a 2-dimensional toric ring, and $\mathfrak{m}$ the maximal monomial ideal of R. If there exists a monomial ideal $\mathfrak{a}$ of R such that* $\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$, *then R has at most Gorenstein singularities.*

Proof. Suppose that R is defined by a cone σ. By taking a suitable change of coordinates, it suffices to consider cones generated by $(1,0)$ and (a,b) such that $b > 0$ and the greatest common divisor of a and b is 1. The following three cases are trivial: If $a = 0$, then R is the polynomial ring. If $a = 1$ and $b = 1$, then $R = k[X_1, X_1^{-1}X_2]$, which is a regular ring. If $a = 1$ and $b > 1$, then $R = k[X_1, X_2, X_1^b X_2^{-1}] \cong k[x,y,z]/(xz - y^b)$. We recall that $\operatorname{Spec} R$ has an A_{b-1} singularity. Hence R is a Gorenstein ring. In the following, we assume that $a > 1$. The dual cone $\sigma^\vee$ is generated by $(0,1)$ and $(b,-a)$. We set the point $\omega = (1, (1-a)/b)$, which satisfies

$$\langle \omega, (1,0) \rangle = \langle \omega, (a,b) \rangle = 1.$$

If $\omega \notin \mathrm{Q}(\mathfrak{m})$, then for all monomial ideals $\mathfrak{a}$, we have $\mathrm{c}(\mathfrak{a}) < \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$. In fact, by taking $\varepsilon > 0$ with $(1+\varepsilon)\omega \notin \mathrm{Q}(\mathfrak{m})$, we have a strict inequality;

$$\mathrm{c}(\mathfrak{a}) < \lambda_{\mathfrak{a}}((1+\varepsilon)) \leq \mathrm{c}^{\mathfrak{m}}(\mathfrak{a}).$$

By the assumption of the proposition, $\omega \in \mathrm{Q}(\mathfrak{m})$. Thus it is enough to prove that $\omega \in M$ under the assumption $\omega \in \mathrm{Q}(\mathfrak{m})$. By the definition of $\mathrm{Q}(\mathfrak{m})$, if $\omega \in \mathrm{Q}(\mathfrak{m})$, then there exists a nonzero lattice point u of $\sigma^\vee$ such that $\omega - u \in \sigma^\vee$. Since $u \in \sigma^\vee$, the lattice point u is written as $u = \lambda_1(0,1) + \lambda_2(b,-a)$, where λ_1 and λ_2 are positive. Since $\omega - u \in \sigma^\vee$, we have $(1/b) - \lambda_1 \geq 0$ and $(1/b) - \lambda_2 \geq 0$. Since u is a nonzero lattice point and b is a positive integer, we have $\lambda_2 = 1/b$.

Hence $u = (1, \lambda_1 - (a/b))$. Since u is a lattice point, there exists an integer l such that $l = \lambda_1 - (a/b)$ and

$$-\frac{a}{b} \le l \le \frac{1-a}{b}.$$

Since a and b are integers and the greatest common divisor of a and b is 1, we have $bl = 1 - a$. Thus $1 - a$ is divisible by b. This implies that $\omega \in M$. The remaining cases are $a < 0$. They follow by the same argument. We complete the proof of the proposition. □

Example 1. Suppose $N = \mathbf{Z}^3$. We define generators $\{v_i\}$ of a cone σ in $N_{\mathbf{R}}$ as

$$v_1 := (1, 0, 0), \quad v_2 := (1, 1, 0), \quad v_3 := (0, 1, r).$$

We also define an element ω of $\sigma^\vee$ as $(1, 0, 1/r)$. Since $\langle \omega, v_i \rangle = 1$ for all i, the toric ring R defined by σ has an r-Gorenstein singularity. A set of generators $\{u_i\}$ of $\sigma^\vee$ is written as

$$u_1 := (r, -r, 1), \quad u_2 := (0, r, -1), \quad u_3 := (0, 0, 1).$$

Then

$$\omega = \frac{1}{r} u_1 + \frac{1}{r} u_2 + \frac{1}{r} u_3.$$

Since $\omega - (1/r)u_3$ is a lattice point of $\sigma^\vee$, we have $\omega \in \mathrm{Q}(\mathfrak{m})$, where $\mathfrak{m}$ is the maximal monomial ideal of R. Let $\mathfrak{a}$ be a monomial ideal generated by $X^{r\omega}$. Then $(1/r)\mathrm{P}(\mathfrak{a}) = \omega + \sigma^\vee$. Hence $(1/r)\mathrm{P}(\mathfrak{a}) \subseteq \mathrm{Q}(\mathfrak{m})$. The same argument in the proof of Proposition 5.1 implies $\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a}) = 1/r$.

Example 2. Suppose $N = \mathbf{Z}^d$, where $d > 3$. We consider the cone σ generated by

$$\begin{aligned} v_1 &:= (1, 0, 0, 0, \ldots, 0) \\ v_2 &:= (1, 1, 0, 0, \ldots, 0) \\ v_3 &:= (0, 1, r, 0, \ldots, 0) \\ v_i &:= (0, 0, 0, 0, \ldots, 0, \overset{\overset{i}{\smile}}{1}, 0, \ldots, 0), \quad 3 < i \le d. \end{aligned}$$

Let R be a toric ring defined by σ, then R is a d-dimensional r-Gorenstein ring. By the same argument in Example 1, there exists a monomial ideal $\mathfrak{a}$ of R such that $\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$.

Using F-thresholds and F-pure thresholds, we give a criterion of regularities of a toric ring defined by a simplicial cone.

Theorem 5.3. *Let R be a toric ring defined by a simplicial cone σ, and $\mathfrak{m}$ the maximal monomial ideal. If there exists a monomial ideal $\mathfrak{a}$ such that $\sqrt{\mathfrak{a}} = \mathfrak{m}$ and*

$$\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a}),$$

then R is a regular ring.

Proof. Since σ is simplicial, we may assume that

$$\sigma = \mathbf{R}_{\geq 0}v_1 + \cdots \mathbf{R}_{\geq 0}v_d,$$

where $v_j \in N$ and $\{v_1, \ldots, v_d\}$ are $\mathbf{R}$-linearly independent. Hence there exist lattice points u_i of M and positive integers l_i such that

$$\sigma^\vee = \mathbf{R}_{\geq 0}u_1 + \cdots + \mathbf{R}_{\geq 0}u_d,$$

and $\langle u_i, v_j \rangle = l_i\delta_{ij}$. Moreover, for all $i, j = 1, \ldots, d$, we assume that v_j and u_i are primitive. Since σ is simplicial, R is $\mathbf{Q}$-Gorenstein. Hence there exists a rational point ω of $M_{\mathbf{R}}$ such that

$$\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\omega).$$

By Theorem 3.3,

$$\lambda_{\mathfrak{a}}(\omega)\mathrm{P}(\mathfrak{a}) \subseteq \mathrm{Q}(\mathfrak{m}). \tag{4}$$

To prove the theorem, it is enough to show that $l_i = 1$ for every $i = 1, \ldots, d$. We derive a contradiction assuming $l_i > 1$ for some i. Since $\sqrt{\mathfrak{a}} = \mathfrak{m}$, for a sufficiently large nonnegative integer l, we have $X^{lu_i} \in \mathfrak{a}$. In particular, $\lambda_{\mathfrak{a}}(\omega)lu_i \in \lambda_{\mathfrak{a}}(\omega)\mathrm{P}(\mathfrak{a})$. If we choose sufficiently large l, then we have

$$0 < \frac{l_i - 1}{\lambda_{\mathfrak{a}}(\omega)ll_i - 1} < 1.$$

Let α be a positive real number such that $0 < \alpha < (l_i - 1)/(\lambda_{\mathfrak{a}}(\omega)ll_i - 1)$. By the definition of $\mathrm{P}(\mathfrak{a})$ and (4),

$$\alpha\lambda_{\mathfrak{a}}(\omega)lu_i + (1 - \alpha)\omega \in \mathrm{Q}(\mathfrak{m}).$$

On the other hand, for all j,

$$\langle \alpha\lambda_{\mathfrak{a}}(\omega)lu_i + (1 - \alpha)\omega, v_j \rangle = \begin{cases} 1 - \alpha < 1 & (j \neq i), \\ \alpha\lambda_{\mathfrak{a}}(\omega)ll_i + 1 - \alpha < l_i & (j = i). \end{cases}$$

By the definition of $\mathrm{Q}(\mathfrak{m})$, there exist a positive integer l_i', a lattice point u of $\mathrm{Q}(\mathfrak{m})$ and an element u' of $\sigma^\vee$ such that

$$\langle u, v_j \rangle = \begin{cases} 0 & (j \neq i) \\ l_i' < l_i & (j = i), \end{cases}$$

and

$$\alpha\lambda_{\mathfrak{a}}(\omega)lu_i + (1 - \alpha)\omega = u + u'.$$

However, the existence of u contradicts the primitiveness of u_i. Thus $l_i = 1$. Eventually, for every $i = 1, \dots, d$, we have $l_i = 1$. Therefore we complete the proof of the theorem. $\square$

On the other hand, there exist a toric ring R defined by a non-simplicial cone with a maximal ideal $\mathfrak{m}$ such that $\mathrm{c}(\mathfrak{m}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{m})$.

Example 3 ([HMTW, Remark 2.5]). If $R = k[X_1X_3, X_2X_3, X_3, X_1X_2X_3]$ and $\mathfrak{m} = (X_1X_3, X_2X_3, X_3, X_1X_2X_3)$, then R is a toric ring whose defining cone is

$$\sigma = \mathbf{R}_{\geq 0}(1,0,0) + \mathbf{R}_{\geq 0}(0,1,0) + \mathbf{R}_{\geq 0}(-1,0,1) + \mathbf{R}_{\geq 0}(0,-1,1).$$

We denote by ω the element $(1,1,2)$ of $\sigma^\vee$. Then

$$\langle \omega, (1,0,0) \rangle = \langle \omega, (0,1,0) \rangle = \langle \omega, (-1,0,1) \rangle = \langle \omega, (0,-1,1) \rangle = 1.$$

By Corollary 4.3 and Lemma 4.6, for every monomial ideal $\mathfrak{a}$, we have $\mathrm{c}(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\omega)$. Hence $\mathrm{c}(\mathfrak{m}) = 2$. On the other hand, $\mathrm{c}^{\mathfrak{m}}(\mathfrak{m}) = 2$.

Finally, we discuss the rationality of F-thresholds. This was given as an open problem in [MTW]. For some regular rings, Blickle, Mustaţă and Smith give the affirmative answer. In [BMS2], they prove the rationality of F-thresholds of all proper ideals $\mathfrak{a}$ with respect to ideals J which entail $\mathfrak{a} \subseteq \sqrt{J}$ on an F-finite regular ring essentially of finite type over k ([BMS2, Theorem 3.1]). In addition, they also prove in cases that $\mathfrak{a}$ is a principal ideal on an F-finite regular ring ([BMS1, Theorem 1.2]). On the other hand, Katzman, Lyubeznik and Zhang prove it in cases that $\mathfrak{a}$ is a principal ideal on an excellent regular local ring, that is not necessarily F-finite ([KLZ]). We will prove the rationality of an F-threshold of a monomial ideal $\mathfrak{a}$ with respect to an $\mathfrak{m}$-primary monomial ideal J on a toric ring. For an element v of $N_{\mathbf{R}}$ and a real number λ, we define the affine half space $\mathrm{H}^+(v;\lambda)$ as

$$\mathrm{H}^+(v;\lambda) := \{u \in M_{\mathbf{R}} \mid \langle u, v \rangle \geq \lambda\}.$$

We also define the hyperplane $\partial\mathrm{H}^+(v;\lambda)$ as

$$\partial\mathrm{H}^+(v;\lambda) := \{u \in M_{\mathbf{R}} \mid \langle u, v \rangle = \lambda\}.$$

Assume that $\mathfrak{a}$ is a monomial ideal of a toric ring. Since $\mathrm{P}(\mathfrak{a})$ is a convex polyhedral set, it is written as an intersection of finite affine half spaces. We observe a form of $\mathrm{P}(\mathfrak{a})$.

LEMMA 5.4. *Let R be a toric ring defined by a cone σ in $N_{\mathbf{R}}$, and $\mathfrak{a}$ a monomial ideal of R. Then there exist rational points v'_l of $N_{\mathbf{R}}$ and rational numbers λ'_l for $l = 1, \dots, t$ such that* $\mathrm{P}(\mathfrak{a}) = \bigcap_{l=1}^{t} \mathrm{H}^+(v'_l; \lambda'_l)$.

Proof. Since σ is a rational polyhedral cone, so is $\sigma^\vee$. Hence there exist lattice points u_i of M such that

$$\sigma^\vee = \mathbf{R}_{\geq 0}u_1 + \cdots + \mathbf{R}_{\geq 0}u_m.$$

We assume that $\mathfrak{a} = (X^{\mathbf{a}_1}, \ldots, X^{\mathbf{a}_s})$. We define the rational polyhedral cone τ of $M_{\mathbf{R}} \times \mathbf{R}$ as

$$\tau := \mathbf{R}_{\geq 0}(\mathbf{a}_1, 1) + \cdots + \mathbf{R}_{\geq 0}(\mathbf{a}_s, 1) + \mathbf{R}_{\geq 0}(u_1, 0) + \cdots + \mathbf{R}_{\geq 0}(u_m, 0).$$

For such τ and $\mathrm{P}(\mathfrak{a})$,

$$\tau \cap (M_{\mathbf{R}} \times \{1\}) = \mathrm{P}(\mathfrak{a}) \times \{1\}. \tag{5}$$

In fact, let $(u, 1)$ be an element of the left-hand side. Then

$$(u, 1) = \sum_{i=1}^{s} a_i(\mathbf{a}_i, 1) + \sum_{j=1}^{m} b_j(u_j, 0),$$

where a_i and b_j are nonnegative numbers. By the definition, $\sum a_i = 1$. By Proposition 3.2 (iii), $u \in \mathrm{P}(\mathfrak{a})$. The similar argument implies the opposite inclusion. Since τ is the rational polyhedral convex cone, for $l = 1, \ldots, t$, there exist rational points (v_l', μ_l) of $N_{\mathbf{R}}$ such that

$$\tau = \bigcap_{l=1}^{t} \mathrm{H}^+((v_l', \mu_l); 0), \tag{6}$$

where $\mathrm{H}^+((v_l', \mu_l); 0)$ is the affine half space of $M_{\mathbf{R}} \times \mathbf{R}$. The duality pairing of $M_{\mathbf{R}} \times \mathbf{R}$ and $N_{\mathbf{R}} \times \mathbf{R}$ is defined as

$$\langle (u, \lambda), (v, \mu) \rangle := \langle u, v \rangle + \lambda\mu,$$

for all elements (u, λ) of $M_{\mathbf{R}} \times \mathbf{R}$ and all elements (v, μ) of $N_{\mathbf{R}} \times \mathbf{R}$. Under this duality,

$$\mathrm{H}^+((v, \mu); 0) \cap (M_{\mathbf{R}} \times \{1\}) = \mathrm{H}^+(v; -\mu) \times \{1\}.$$

Therefore if we set $\lambda_l' := -u_l$ for each $l = 1, \ldots, t$, the assertion of the lemma follows by (5) and (6). $\square$

THEOREM 5.5. *Let R, σ and $\mathfrak{a}$ be as in Lemma 5.4. Furthermore, we assume that σ is a d-dimensional simplicial cone. Let J be an $\mathfrak{m}$-primary monomial ideal, where $\mathfrak{m}$ is the maximal monomial ideal of R. Then the F-threshold $\mathrm{c}^J(\mathfrak{a})$ of $\mathfrak{a}$ with respect to J is a rational number.*

Proof. We denote by $\partial\mathrm{Q}(J)$ the boundary of $\mathrm{Q}(J)$ in $\sigma^\vee$, and also denote by $M_{\mathbf{Q}}$ the set of the rational points of $M_{\mathbf{R}}$. By Lemma 5.4, if there exists a finite set B of $M_{\mathbf{Q}} \cap \partial\mathrm{Q}(J)$ such that

$$\mathrm{c}^J(\mathfrak{a}) = \max_{\omega \in B} \lambda_{\mathfrak{a}}(\omega),$$

then $\mathrm{c}^J(\mathfrak{a})$ is a rational number.

First, we prove that

$$\mathrm{c}^J(\mathfrak{a}) = \sup_{\omega \in \partial\mathrm{Q}(J)} \lambda_{\mathfrak{a}}(\omega).$$

By Theorem 3.3, if there exists an element ω of $\sigma^\vee$ such that $c^J(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\omega)$, then ω is an element of $\partial Q(J)$. In fact, if such ω is contained in $\sigma^\vee \backslash Q(J)$, there exists a positive real number ε such that $(1+\varepsilon)\omega \in \sigma^\vee \backslash Q(J)$. This implies that $c^J(\mathfrak{a}) \geq (1+\varepsilon)\lambda_{\mathfrak{a}}(\omega)$. It is a contradiction.

Second, we prove the existence of B. We assume that $\sigma = \mathbf{R}_{\geq 0} v_1 + \cdots + \mathbf{R}_{\geq 0} v_d$, where v_j are primitive lattice points. Since σ is simplicial, for every j, there exists an element u_j of $M_{\mathbf{Q}}$ such that

$$\langle u_j, v_l \rangle = \delta_{jl}, \quad l \in \{1, \ldots, d\}.$$

Since J is $\mathfrak{m}$-primary, there exist nonnegative integers r_j such that $r_j u_j \in Q(J)$. That implies $\partial Q(J)$ is bounded. The order $\leq_\sigma$ over $\partial Q(J)$ is defined by $u \leq_\sigma u'$ if

$$\langle u, v_j \rangle \leq \langle u', v_j \rangle, \quad \forall j = 1, \ldots, d.$$

Then $\partial Q(J)$ has maximal elements with respect to this order. Let B be the set of maximal elements of $\partial Q(J)$ with respect to the order $\leq_\sigma$. By Lemma 4.6, we conclude

$$c^J(\mathfrak{a}) = \sup_{\omega \in \partial Q(J)} \lambda_{\mathfrak{a}}(\omega) = \sup_{\omega \in B} \lambda_{\mathfrak{a}}(\omega).$$

To show that B is a finite set of $M_{\mathbf{Q}}$, we prove the following claim.

Claim. Let J be the ideal of R generated by elements $X^{\mathbf{b}_1}, \ldots, X^{\mathbf{b}_t}$. We assume that $u \in B$, that is,

(i) $u \in \partial Q(J)$,

(ii) u is a maximal element with respect to the order $\leq_\sigma$ in $\partial Q(J)$.

Then for every $j = 1, \ldots, d$, there exists integer i_j such that

$$u \in \bigcap_{j=1}^{d} (\mathbf{b}_{i_j} + (\partial H^+(v_j; 0) \cap \sigma^\vee)). \tag{7}$$

In particular, B is a finite set and $u \in M_{\mathbf{Q}}$.

Proof of Claim. We suppose that u does not satisfy (7). Then there exists j' in $\{1, \ldots, d\}$ such that

$$u \notin \mathbf{b}_i + (\partial H^+(v_{j'}; 0) \cap \sigma^\vee), \tag{8}$$

for all $i = 1, \ldots, t$. We choose an element u' of $\sigma^\vee$ such that

$$\langle u', v_j \rangle = \langle u, v_j \rangle, \quad (j \neq j'),$$
$$\langle u', v_{j'} \rangle = \lfloor \langle u, v_{j'} \rangle \rfloor + 1.$$

Since σ is simplicial, u' uniquely exists. We will show that the existence of u' contradicts the assumption (ii). By the construction of u', we have $u' \in Q(J)$. To see $u' \notin \operatorname{Int} Q(J)$, we paraphrase the assumption (i). Since $u \notin \operatorname{Int} Q(J)$, we have $u \notin \mathbf{b}_i + \mathrm{Int}(\sigma^\vee)$ for all $i = 1, \dots, t$. Furthermore, this is equivalent to the existence of l_i such that

$$\langle u, v_{l_i} \rangle \leq \langle \mathbf{b}_i, v_{l_i} \rangle, \tag{9}$$

for each $i = 1, \dots, t$. If l_i is not j', we have directly

$$\langle u', v_{l_i} \rangle = \langle u, v_{l_i} \rangle \leq \langle \mathbf{b}_i, v_{l_i} \rangle,$$

by the construction of u' and the relation (9). On the other hand, if l_i is j', then the relations (9) and (8) imply

$$\lfloor \langle u, v_{j'} \rangle \rfloor \leq \langle \mathbf{b}_i, v_{j'} \rangle - 1,$$

because $\mathbf{b}_i$ is a lattice point of M. Hence $\langle u', v_{l_i} \rangle \leq \langle \mathbf{b}_i, v_{l_i} \rangle$. Eventually, in both cases, $u' \notin \mathrm{Int}\, \mathrm{Q}(J)$. Therefore $u' \in \partial \mathrm{Q}(J)$. By the construction of u', the element u is not a maximal element in $\partial \mathrm{Q}(J)$. It contradicts the assumption (ii). We complete the proof of Claim. □

We complete the proof of the theorem. □

Now we consider the rationality of F-jumping coefficients on **Q**-Gorenstein toric rings. The rationality of F-jumping coefficients is the consequence of the fact that test ideals are equal to multiplier ideals ([HY, Theorem 4.8] and [B, Theorem 1]). However, we also give its proof by a combinatorial method.

PROPOSITION 5.6. *Let R, σ and $\mathfrak{a}$ be as in Lemma 5.4. Moreover, we assume R is an r-Gorenstein toric ring. Then for all i, the i-th F-jumping coefficient $\mathrm{c}^i(\mathfrak{a})$ of $\mathfrak{a}$ is a rational number.*

Proof. In the proof of Proposition 4.5, we have seen that there exists a lattice point $\mathbf{b}$ of M such that $\mathrm{c}^i(\mathfrak{a}) = \mu_{\mathfrak{a}}(\mathbf{b})$, where $X^{\mathbf{b}}$ is one of generators of $\tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})})$. By the similar argument in the proof of Proposition 5.1, there exists an element ω of $\sigma^\vee$ such that $\mathrm{c}^i(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\mathbf{b} + \omega/r)$. Let ω_R be the canonical module of R. Since ω corresponds to the generator of $\omega_R^{(r)}$, we see $\omega \in M$. Hence $\mathbf{b} + \omega/r$ is in $M_{\mathbf{Q}}$. Therefore $\mathrm{c}^i(\mathfrak{a})$ is a rational number. □

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