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ON RIEMANNIAN MANIFOLDS WITH SASAKIAN 3-STRUCTURE OF CONSTANT HORIZONTAL SECTIONAL CURVATURE

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§ 0. Introduction.

In 1970 Kuo [4] introduced differentiable manifolds with normal contact (Sasakian) 3-structure and discussed about its associated metric, dimension, structure group of the tangent bundle and so on. Moreover, these differentiable manifolds have been studied from various viewpoints by Ishihara, Kashiwada, Tachibana, Tanno, Yu, one of the authors and others (See, for example [1], [2], [5], [7], [8]).

In 1964, Ogiue [6] has studied Riemannian manifolds with normal contact (Sasakian) structure (ϕ, ξ, g), when the sectional curvature of the two plane orthogonal to ξ (so-called, C-holomorphic sectional curvature) does not depend on the C-holomorphic section, and given the form of the curvature tensor and obtained results on the problems of admitting the axiom of planes and the free mobility.

In the present note, we consider the condition on the sectional curvature of a Riemannian manifold with Sasakian 3-structure, corresponding to [6].

That is we shall prove

THEOREM. *Let M be an n-dimensional Riemannian manifold with Sasakian 3-structure* ($n \geqq 7$). *Assume that M is of constant horizontal sectional curvature, then M is necessarily of constant curvature.*

In § 1, we give definitions of horizontal sections, horizontal sectional curvatures, and some formulas on the curvature tensors. § 2 will be devoted to the proof of the theorem above.

§ 1. Preliminaries.

Let (M, g) be a Riemannian manifold with Sasakian 3-structure (See, Ishihara and Konishi [1]). That is, $\xi_{(1)}$, $\xi_{(2)}$ and $\xi_{(3)}$ are mutually orthogonal unit Killing

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vectors satisfying

$$K(X, \xi_{(A)})Y=g(\xi_{(A)}, Y)X-g(X, Y)\xi_{(A)} \qquad (A=1, 2, 3)\,, \tag{1.1}$$

where K denotes the Riemannian curvature tensor. Denoting by ∇ the Riemannian connection formed with g and defining a (1,1)-tensor field $\phi_{(A)}$ by $\phi_{(A)}=\nabla\xi_{(A)}$, we have

$$\phi_{(A)}\xi_{(A)}=0\,, \tag{1.2}$$

$$\phi_{(A)}\phi_{(A)}X=-X+g(\xi_{(A)}, X)\xi_{(A)}\,, \tag{1.3}$$

$$g(\phi_{(A)}X, \phi_{(A)}Y)=g(X, Y)-g(\xi_{(A)}, X)g(\xi_{(A)}, Y)\,, \tag{1.4}$$

for $A=1,2,3$, and

$$\xi_{(A)}=\phi_{(C)}\xi_{(B)}=-\phi_{(B)}\xi_{(C)}\,, \tag{1.5}$$

$$\phi_{(A)}X=\phi_{(C)}\phi_{(B)}X-g(\xi_{(B)}, X)\xi_{(C)}=-\phi_{(B)}\phi_{(C)}X+g(\xi_{(C)}, X)\xi_{(B)} \tag{1.6}$$

for any even permutation $(A. B, C)$ of $(1, 2, 3)$.

We denote by $T_p(M)$ the tangent space at p in M, and by $T_p{}^H(M)$ the subspace of all tangent vectors which are orthogonal to $\xi_{(1)}$, $\xi_{(2)}$ and $\xi_{(3)}$ at p. Each element of $T_p{}^H(M)$ is called a *horizontal vector* at p. For any horizontal vector X at p, we consider the 4-dimensional subspace $H(X)$, which are spanned by X, $\phi_{(1)}X$, $\phi_{(2)}X$ and $\phi_{(3)}X$, i. e.,

$$H(X)=\{Y \mid Y=aX+b\phi_{(1)}X+c\phi_{(2)}X+d\phi_{(3)}X,\ a, b, c, d\in R\}\,.$$

We call $H(X)$ the *horizontal section* determined by X. If the sectional curvature for each two vectors belonging to $H(X)$ is a constant $k(X)$ depending only upon the horizontal vector X at p, then $k(X)$ is said to be the *horizontal sectional curvature* with respect to X at p. Especially, if each horizontal section $H(X)$ at each point p in M, X being an arbitrary horizontal vector at p, has a horizontal sectional curvature whose value $k(X)$ is independent of X, we say that the manifold M is of *constant horizontal sectional curvature* at p. It is easily seen that M is of constant horizontal sectional curvature, if and only if the sectional curvature σ satisfies

$$\sigma(X, \phi_{(A)}X)=k(p) \qquad (A=1, 2, 3)$$

for any horizontal vector X, at each point p in M, $k(p)$ being independent of X.

Remark. If for a fixed A, $\sigma(X, \phi_{(A)}X)=k(p)$ for any tangent vector X orthogonal to $\xi_{(A)}$ is constant, then M is of constant curvature (Kashiwada [2]). This situation is a little different from ours.

Now we define (1, 1)-tensor fields $\phi_{(A)}{}^H$ $(A=1, 2, 3)$ with

$$\phi_{(A)}{}^H X=\phi_{(A)}X-g(\xi_{(C)}, X)\xi_{(B)}+g(\xi_{(B)}, X)\xi_{(C)}\,, \tag{1.7}$$

X being arbitrary tangent vector, where (A, B, C) is an even permutation of $(1, 2, 3)$. Then each vector $\phi_{(A)}{}^{H}X$ is horizontal and satisfies

$$g(\phi_{(A)}{}^{H}X, Y)+g(X, \phi_{(A)}{}^{H}Y)=0, \qquad (A=1, 2, 3)$$

for any tangent vector X and Y. Particularly, we can get

$$\phi_{(A)}{}^{H}\xi_{(B)}=0, \qquad (A, B=1, 2, 3). \tag{1.8}$$

Let $\{U, x^{h}\}$ be a local chart of M with local coordinates $\{x^{h}\}$(*). Denoting by $\phi=\phi_{(1)}$ in $\{U, x^{h}\}$ and taking account of Ogiue [6], we have

$$\begin{aligned} K_{bjai}\phi^{H}{}_{k}{}^{b}\phi^{H}{}_{h}{}^{a}-K_{bjah}\phi^{H}{}_{k}{}^{b}\phi^{H}{}_{i}{}^{a} \\ =g^{H}{}_{ki}g_{jh}-g^{H}{}_{kh}g_{ji}-\phi^{H}{}_{ki}\phi^{H}{}_{jh}+\phi^{H}{}_{kh}\phi^{H}{}_{ji}, \end{aligned} \tag{1.9}$$

$$\begin{aligned} K_{baih}\phi^{H}{}_{k}{}^{b}\phi^{H}{}_{j}{}^{a}=K_{biah}\phi^{H}{}_{k}{}^{b}\phi^{H}{}_{j}{}^{a}-K_{biah}\phi^{H}{}_{j}{}^{b}\phi^{H}{}_{k}{}^{a} \\ =K_{kjih}+g_{ki}g_{jh}-g_{kh}g_{ji}-\phi^{H}{}_{ki}\phi^{H}{}_{jh}+\phi^{H}{}_{kh}\phi^{H}{}_{ji}. \end{aligned} \tag{1.10}$$

where we have put

$$g^{H}{}_{ji}=g_{ji}-\sum_{A=1}^{3}\xi_{(A)j}\xi_{(A)i}, \qquad \phi^{H}{}_{ji}=g^{H}{}_{it}\phi^{H}{}_{j}{}^{t}. \tag{1.11}$$

Besides, we obtain

$$\phi^{H}{}_{j}{}^{t}\phi^{H}{}_{t}{}^{i}=-\delta_{j}{}^{i}+\sum_{A=1}^{3}\xi_{(A)j}\xi_{(A)}{}^{i}, \tag{1.12}$$

$$\phi^{H}{}_{j}{}^{t}\phi^{H}{}_{i}{}^{s}g^{H}{}_{ts}=g^{H}{}_{ji}. \tag{1.13}$$

§ 2. Proof of Theorem.

Suppose that the Riemannian manifold M is of constant horizontal sectional curvature $k=k(p)$ at a point p in M. The sectional curvature for the horizontal section determined by $\phi^{H}X$ and $\phi(\phi^{H}X)$ is given by

$$k=-\frac{K_{dcba}(\phi^{H}X)^{d}(\phi(\phi^{H}X))^{c}(\phi^{H}X)^{b}(\phi(\phi^{H}X))^{a}}{g_{db}(\phi^{H}X)^{d}(\phi^{H}X)^{b}g_{ca}(\phi(\phi^{H}X))^{c}(\phi(\phi^{H}X))^{a}} \tag{1.2}$$

for any tangent vector X at p. Taking account of (1.1), (1.3), (1.11) and (1.13), we have from (2.1)

$$[K_{djbh}\phi^{H}{}_{k}{}^{d}\phi^{H}{}_{i}{}^{b}+g^{H}{}_{ki}g_{jh}+(k-1)g^{H}{}_{ki}g^{H}{}_{jh}]X^{k}X^{j}X^{i}X^{h}=0.$$

Since X being arbitrary at p, we have

(*) The Latin indices $a, b, c \cdots, h, i, j, \cdots$ run over the range $\{1, 2, \cdots, n\}$ and, the components of geometrical objects and the summation convention will be used with respect to this system of indices throughout this paper.

$$
\begin{aligned}
&K_{tjsh}\phi^{H}{}_{k}{}^{t}\phi^{H}{}_{i}{}^{s}+K_{tjsi}\phi^{H}{}_{k}{}^{t}\phi^{H}{}_{h}{}^{s}+K_{thsk}\phi^{H}{}_{i}{}^{t}\phi^{H}{}_{j}{}^{s}+K_{tksi}\phi^{H}{}_{j}{}^{t}\phi^{H}{}_{h}{}^{s}\\
&\quad+K_{tjsi}\phi^{H}{}_{h}{}^{t}\phi^{H}{}_{k}{}^{s}+K_{tjsk}\phi^{H}{}_{h}{}^{t}\phi^{H}{}_{i}{}^{s}+K_{thsi}\phi^{H}{}_{j}{}^{t}\phi^{H}{}_{k}{}^{s}+K_{thsk}\phi^{H}{}_{j}{}^{t}\phi^{H}{}_{i}{}^{s}\\
&\quad+K_{thsi}\phi^{H}{}_{k}{}^{t}\phi^{H}{}_{j}{}^{s}+K_{tjsh}\phi^{H}{}_{i}{}^{t}\phi^{H}{}_{k}{}^{s}+K_{tksi}\phi^{H}{}_{h}{}^{t}\phi^{H}{}_{j}{}^{s}+K_{tjsk}\phi^{H}{}_{i}{}^{t}\phi^{H}{}_{h}{}^{s}\\
&\quad+2(g^{H}{}_{ki}g_{jh}+g^{H}{}_{ih}g_{kj}+g^{H}{}_{kh}g_{ji}+g^{H}{}_{jh}g_{ki}+g^{H}{}_{kj}g_{ih}+g^{H}{}_{ji}g_{kh})\\
&\quad+4(k-1)(g^{H}{}_{ki}g^{H}{}_{jh}+g^{H}{}_{kj}g^{H}{}_{ih}+g^{H}{}_{kh}g^{H}{}_{ji})=0
\end{aligned}
$$

from which we get

$$
\begin{aligned}
&K_{tjsh}\phi^{H}{}_{k}{}^{t}\phi^{H}{}_{i}{}^{s}+K_{tjsi}\phi^{H}{}_{h}{}^{t}\phi^{H}{}_{k}{}^{s}+K_{thsi}\phi^{H}{}_{k}{}^{t}\phi^{H}{}_{j}{}^{s}\\
&\quad+g^{H}{}_{ki}g_{jh}+g^{H}{}_{kh}g_{ji}+g^{H}{}_{kj}g_{ih}-\phi^{H}{}_{ki}\phi^{H}{}_{jh}+\phi^{H}{}_{kh}\phi^{H}{}_{ji}+\phi^{H}{}_{kj}\phi^{H}{}_{ih}\\
&\quad+(k-1)(g^{H}{}_{ki}g^{H}{}_{jh}+g^{H}{}_{kh}g^{H}{}_{ji}+g^{H}{}_{kj}g^{H}{}_{ih})=0
\end{aligned}
$$

by virtue of (1.9). Moreover, we have from (1.9) and (1.10)

$$
\begin{aligned}
&3K_{tjsh}\phi^{H}{}_{m}{}^{t}\phi^{H}{}_{l}{}^{s}-K_{mhjl}-K_{mljh}\\
&\qquad+g_{mh}g_{jl}+g_{jh}g_{ml}-2g_{mj}g_{lh}+3g^{H}{}_{ml}g_{jh}-3\phi^{H}{}_{mh}\phi^{H}{}_{lj}\\
&\qquad+(k-1)(g^{H}{}_{ml}g^{H}{}_{jh}+g^{H}{}_{mh}g^{H}{}_{jl}+g^{H}{}_{mj}g^{H}{}_{lh})=0\,.
\end{aligned}
$$

where we have used the indices m and l instead of k and i respectively. Transvecting this with $\phi^{H}{}_{i}{}^{m}\phi^{H}{}_{k}{}^{l}$ and using (1.1), (1.8), (1.9), (1.11), (1.12) and (1.13), we find

$$
\begin{aligned}
K_{mhlj}\phi^{H}{}_{i}{}^{m}\phi^{H}{}_{k}{}^{l}=&-3K_{khij}-K_{kijh}+g_{kh}g_{ji}-g_{kj}g_{ih}\\
&-3g_{ki}g_{jh}-g^{H}{}_{ki}g_{jh}-3g^{H}{}_{kj}g_{ih}+\phi^{H}{}_{kh}\phi^{H}{}_{ij}\\
&-(k-1)(g^{H}{}_{ki}g^{H}{}_{jh}+\phi^{H}{}_{kh}\phi^{H}{}_{ij}+\phi^{H}{}_{kj}\phi^{H}{}_{ih})\,.
\end{aligned}
$$

Taking the skew-symmetric part of this equation with respect to k and j, and using the first Bianchi identities, we obtain

$$
\begin{aligned}
(2.2)\qquad 4K_{kjih}=&4(g_{kh}g_{ji}-g_{ki}g_{jh})\\
&+(k-1)(g^{H}{}_{kh}g^{H}{}_{ji}-g^{H}{}_{ki}g^{H}{}_{jh})\\
&+(k-1)(\phi^{H}{}_{kh}\phi^{H}{}_{ji}-\phi^{H}{}_{ki}\phi^{H}{}_{jh}-2\phi^{H}{}_{kj}\phi^{H}{}_{ih})
\end{aligned}
$$

by virtue of (1.9).

As for another structure $\psi=\phi_{(2)}$, taking account of (1.7), we can get, by similar devices,

$$
\begin{aligned}
(2.3)\qquad 4K_{kjih}=&4(g_{kh}g_{ji}-g_{ki}g_{jh})\\
&+(k-1)(g^{H}{}_{kh}g^{H}{}_{ji}-g^{H}{}_{ki}g^{H}{}_{jh})\\
&+(k-1)(\psi^{H}{}_{kh}\psi^{H}{}_{ji}-\psi^{H}{}_{ki}\psi^{H}{}_{jh}-2\psi^{H}{}_{kj}\psi^{H}{}_{ih})\,.
\end{aligned}
$$

Compare with (2.2) and (2.3), we have $k=1$ because of independency of ϕ^H and ψ^H in the tensor algebra over M. This means that M is of constant curvature 1. Consequently our theorem is completely proved.

Since the distribution D spanned by ξ_1, ξ_2 and ξ_3 is involutive, for a sufficiently small neighborhood V, we have a local fibering $\pi; V \rightarrow V/D$, where V/D is a quaternion Kähler manifold. Each horizontal section is projected by π to a Q-section, and hence the condition in our theorem is equivalent to that V/D is of constant Q-sectional curvature (See Konishi [3] and Tanno [8]). A 4-dimensional quaternion Kähler manifold of constant Q-sectional curvature is necessarily of constant curvature. So, if we adopt the above fibering, the case of dim $M=7$ is specially treated.

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