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METRIC POLYNOMIAL STRUCTURES

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0. The paper is devoted to the study of metric polynomial structures, i. e. polynomial structures f for which there exists a positive definite Riemannian metric g such that there is $g(f(X), f(Y))=g(X, Y)$. In the first paragraph we divide the metric polynomial structures into four groups, restricting then ourselves to the first group only. In the second paragraph we are concerned with the integrability conditions of the distributions naturally arising in the study of these structures. In the last third paragraph we establish the existence of special connections associated with the metric polynomial structures.

1. Let M be a differentiable manifold of class C^∞. By a *polynomial structure* on M we mean a C^∞-tensor field of type (1, 1) on M satisfying a polynomial equation

$$P(f)=f^n+a_1f^{n-1}+\cdots+a_{n-1}f+a_nI=0$$

where $a_1, \cdots, a_n$ are real numbers, at every point of M. Moreover we shall suppose that the polynomial P is the minimal polynomial of f_x at every point $x\in M$.

Example: If f satisfies a polynomial equation $P(f)=0$ then P need not be necessarily the minimal polynomial of f_x at every point $x\in M$, even if we suppose that f has a constant rank on M. Let us take for example $M=\boldsymbol{R}^4$ with cartesian coordinates (X_1, X_2, X_3, X_4) and let us define f by

$$f=\begin{pmatrix} 0 & x_1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that f satisfies on $\boldsymbol{R}^4$ the equation $P(\xi)=\xi^3=0$. Its minimal polynomial at a point with $x_1\neq 0$ is ξ^3 whereas it is ξ^2 at a point with $x_1=0$. Clearly rank $f=2$ on M.

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By this time the most intensively studied polynomial structures have been the almost complex structures and the polynomial structures satisfying $f^3+f=0$. (see [1]-[5]). The latter ones are closely connected with the almost contact structures. Two other types of polynomial structures can be found in [6] and [7].

We consider now a polynomial structure f on M satisfying $P(f)=0$. Let us write a decomposition of P into the prime factors, i. e.

$$P(\xi)=\tilde{P}_1'(\xi)\cdots\tilde{P}_r'(\xi)\cdot\tilde{P}_1''(\xi)\cdots\tilde{P}_s''(\xi)$$

where $\tilde{P}_i'(\xi)=(\xi-c_i)^{k_i}$, $\tilde{P}_j''(\xi)=(\xi^2+2a_j\xi+b_j)^{l_j}$, $i=1,\cdots,r$ $j=1,\cdots,s$. Of course here $\xi^2+2a_j\xi+b_j$ are irreducible polynomials (over real numbers), i. e. $a_j^2<b_j$. We define $r+s$ distributions D_i' and D_j'' on M by setting

$$D_i'=\operatorname{Ker}\tilde{P}_i'(f)\,,\qquad D_j''=\operatorname{Ker}\tilde{P}_j''(f)$$

D_i' and D_j'' are obviously invariant under f. Further $\tilde{P}_i'(\tilde{P}_j'')$ is the minimal polynomial of the restriction of f to D_i' (D_j'').

PROPOSITION 1. *There exist uniquely determined polynomials* P_i', P_j'' *such that*

(i) $P_i'(f)$ $(P_j''(f))$ *is the projector onto* D_i' (D_j'').

(ii) $\deg P_i'<\deg P$, $\deg P_j''<\deg P$, *where* deg *denotes degree of a polynomial.*

Proof. For the sake of simplicity we shall prove the existence and uniqueness of P_i'. The greatest common divisor of the polynomials $\tilde{P}_1'$ and $\tilde{P}_2'\cdots\tilde{P}_r'\cdot\tilde{P}_1''\cdots\tilde{P}_s''$ is equal to 1 and thus there exist polynomials Q_1', R_1' such that

$$Q_1'\tilde{P}_1'+R_1'\tilde{P}_2'\cdots\tilde{P}_s''=1\,.$$

Here the polynomial $R_1'\tilde{P}_2'\cdots\tilde{P}_s''$ has already the property (i). Writing $R_1'=S_1'\tilde{P}_1'+T_1'$ we find easily that the polynomial $T_1'\tilde{P}_2'\cdots\tilde{P}_s''$ has both the properties (i) and (ii). The uniqueness follows immediately from the fact that P is the minimal polynomial of f at every point.

From this proposition we get easily

COROLLARY 1. $\dim D_i'$ $(\dim D_j'')$ *is constant on* M *and thus* $(D_1',\cdots,D_r',D_1'',\cdots,D_s'')$ *is a* $(r+s)$-π-*structure on* M. *We shall call it* $(r+s)$-π-*structure associated with the polynomial structure* f.

In what follows V denotes a finite dimensional vector space over the field of real numbers. We shall need the following two lemmas.

LEMMA 1. *Let* $f:V\underset{k}{\rightarrow}V$ *be an endomorphism with the minimal polynomial* $P(\xi)=(\xi-c)^k$. *Then there exists a positive definite metric* g *on* V *such that* $g(f(v),f(w))=g(v,w)$ *for any vectors* $v,w\in V$ *if and only if there is* $c=\pm1$, $k=1$.

Proof. The sufficienty of the conditions is obvious. Thus let us suppose that there exists on V an invariant metric g. $(\xi-c)^k$ is the minimal polynomial and therefore $\operatorname{Ker}(f-cI)$ is a non-trivial subspace of V which is obviously invariant under f, and this implies $c=\pm 1$. Let us take the case $c=1$. The other one can be treated in the exactly same way. Let us suppose now that $k\geqq 2$. Obviously $\operatorname{Ker}(f-I)^2\supset\operatorname{Ker}(f-I)$ and again because $(\xi-c)^k$ is minimal there is $\operatorname{Ker}(f-I)^2\neq\operatorname{Ker}(f-I)$. We choose $v\in\operatorname{Ker}(f-I)^2$, $v\notin\operatorname{Ker}(f-I)$. For any $w\in\operatorname{Ker}(f-I)$ we get

$$\begin{aligned} g((f-I)v, w) &= g(fv, w)-g(v, w) \\ &= g(fv, fw)-g(v, w) \\ &= g(v, w)-g(v, w) \\ &= 0 . \end{aligned}$$

But because $(f-I)v\in\operatorname{Ker}(f-I)$ this implies $(f-I)v=0$ which is a contradiction. So there must be $k=1$.

LEMMA 2. *Let $f: V\to V$ be an endomorphism with the minimal polynomial $P(\xi)=(\xi^2+2a\xi+b)^l$ such that $a^2-b<0$. Then there exists a positive definite metric g on V such that $g(fv, fw)=g(v, w)$ for any two vectors $v, w\in V$ if and only if there is $b=1$, $l=1$.*

Proof. If $b=1$, $l=1$ let us take on V any positive definite metric h and define g by

$$g(v, w)=h(v, w)+\frac{1}{1-a^2}h((f+aI)v, (f+aI)w) .$$

An easy calculation shows that g is an invariant metric.

The converse part proceeds similarly as in Lemma 1. $(\xi^2+2a\xi+b)^l$ is the minimal polynomial and therefore $\operatorname{Ker}(f^2+2af+bI)$ is a non-trivial subspace of V which is invariant under f. For any $v, w\in\operatorname{Ker}(f^2+2af+bI)$ we get

$$\begin{aligned} 0 &= g((f^2+2af+bI)v, f^2w) \\ &= g(f^2v, f^2w)+2ag(fv, f^2w)+bg(v, f^2w) \\ &= g(v, w)+2ag(v, fw)+bg(v, f^2w) \\ &= g(v, (bf^2+2af+I)w) \end{aligned}$$

which implies $bf^2+2af+I=0$ on $\operatorname{Ker}(f^2+2af+bI)$. But $\xi^2+2a\xi+b$ is the minimal polynomial of f restricted to $\operatorname{Ker}(f^2+2af+bI)$ and thus there is $b\xi^2+2a\xi+1=b(\xi^2+2a\xi+1)$ on $\operatorname{Ker}(f^2+2af+bI)$ from which we get immediately $b=1$. So let us suppose now f to have the minimal polynomial $(\xi^2+2a\xi+1)^l$ with $a^2<1$ and $l\geqq 2$. Again $\operatorname{Ker}(f^2+2af+I)^2\supset\operatorname{Ker}(f^2+2af+I)$ and because $(\xi^2+2af+1)^l$ is minimal there is $\operatorname{Ker}(f^2+2af+I)^2\neq\operatorname{Ker}(f^2+2af+I)$. We choose $v\in\operatorname{Ker}(f^2+2af+I)^2$, $v\notin\operatorname{Ker}(f^2+2af+I)$. For any $w\in\operatorname{Ker}(f^2+2af+I)$ we get

$$\begin{aligned} g((f^2+2af+I)v, f^2w) &= g(f^2v, f^2w)+2ag(fv, f^2w)+g(v, f^2w) \\ &= g(v, w)+2ag(v, fw)+g(v, f^2w) \\ &= g(v, (f^2+2af+I)w) \\ &= 0 \end{aligned}$$

which implies $v\in\operatorname{Ker}(f^2+2af+I)$ because $(f^2+2af+I)v\in\operatorname{Ker}(f^2+2af+I)$ and f^2 is an automorphism of $\operatorname{Ker}(f^2+2af+I)$. This contradiction shows that there must be $l=1$.

Let us notice here the obvious fact that there exists on M a positive definite Riemannian metric g such that there is $g(f(X), f(Y))=g(X, Y)$ if and only if such a metric exists on every distribution $\mathcal{D}'_i$ and $\mathcal{D}''_j$. A polynomial structure for which there exists a metric with the above properties will be called **metric polynomial structure**. The metric in question (which is not uniquely determined) we shall call **invariant metric**. The following proposition is a consequence of Lemma 1 and Lemma 2.

PROPOSITION 2. *There are exactly four types of metric polynomial structures the minimal polynomials of which are given by*

(i) $P(\xi)=(\xi^2+2a_1\xi+1)\cdots(\xi^2+2a_s\xi+1)$

(ii) $P(\xi)=(\xi-1)(\xi^2+2a_1\xi+1)\cdots(\xi^2+2a_s\xi+1)$

(iii) $P(\xi)=(\xi+1)(\xi^2+2a_1\xi+1)\cdots(\xi^2+2a_s\xi+1)$

(iv) $P(\xi)=(\xi-1)(\xi+1)(\xi^2+2a_1\xi+1)\cdots(\xi^2+2a_s\xi+1)$

where $a_i^2<1$, $a_i\neq a_j$ *for* $i\neq j$, $i, j=1, \cdots, s$.

In the next we shall restrict ourselves to the study of metric polynomial structures of the first type only.

2. Let f be a metric polynomial structure of the first type on a manifold M, and denote by $(D_1, \cdots, D_s)$ the $s-\pi$-structure associated with f. P_i will be the corresponding projectors. We are going in this paragraph to give a necessary and sufficient condition for integrability of this $s-\pi$-structure. First we need

LEMMA 3. *Let* $(D_1, \cdots, D_s)$ *be a* $s-\pi$-*structure on a manifold* M, P_i *the corresponding projectors. Let* $c_1, \cdots, c_s$ *be real numbers,* $c_i\neq c_j$ *for* $i\neq j$, *and define a tensor* $I_{(c_1,\cdots,c_s)}=\sum_{i=1}^{s} c_iP_i$. *Then* $(D_1, \cdots, D_s)$ *is integrable if and only if*

$$[I_{(c_1,\cdots,c_s)}, I_{(c_1,\cdots,c_s)}]=0$$

where bracket denotes Nijenhuis torsion tensor.

Proof. According to [8] $(D_1, \cdots, D_s)$ is integrable if and only if $[P_i, P_j]=0$ for $i, j=1, \cdots, s$. Our condition is therefore necessary.

If Q is a tensor field of type (1, 2) and C a tensor field of type (1, 1) we denote by CQ, QC, $Q\cdot C$ tensor fields of type (1, 2) defined by

$$CQ(X, Y)=C(Q(X, Y)), \quad QC(X, Y)=Q(CX, Y), \quad Q\cdot C(X, Y)=Q(X, CY)$$

where X, Y are two vectors. Now if A, B, C are tensor fields of type (1, 1) we have the following identity

$$(1) \qquad [A, BC]+[AC, B]=A[B, C]+B[A, C]+[A, B]C+[A, B]\cdot C$$

(see e. g. [9]).

Let us suppose now that there is $[I_{(c_1,\cdots,c_s)}, I_{(c_1,\cdots,c_s)}]=0$, i. e. $\sum_{i,j=1}^{s} c_i c_j [P_i, P_j]=0$. Using (1) we get

$$(2) \qquad [P_i, P_jP_k]+[P_iP_k, P_j]$$
$$=P_i[P_j, P_k]+P_j[P_i, P_k]+[P_i, P_j]P_k+[P_i, P_j]\cdot P_k$$

which implies

$$\sum_{i,j=1}^{s} c_i c_j [P_i, P_jP_k]+\sum_{i,j=1}^{s} c_i c_j [P_iP_k, P_j]$$
$$=\sum_{i,j=1}^{s} c_i c_j P_i[P_j, P_k]+\sum_{i,j=1}^{s} c_i c_j P_j[P_i, P_k].$$

Then applying P_l on both sides we have

$$(c_k-c_l)P_l[\sum_{i=1}^{s} c_i P_i, P_k]=0$$

from which for $l\neq k$ follows $P_l[\sum_{i=1}^{s} c_i P_i, P_k]=0$. But from this, by virtue of $\sum_{k=1}^{s} c_k[\sum_{i=1}^{s} c_i P_i, P_k]=0$, we get easily $[\sum_{i=1}^{s} c_i P_i, P_k]=0$, at first for all k with $ck\neq 0$, and then using $\sum_{l=1}^{s} P_l=I$ also for a possible k with $c_k=0$. Using this last relation we have from (2)

$$\sum_{i=1}^{s} c_i [P_iP_k, P_j]=\sum_{i=1}^{s} c_i P_i[P_j, P_k]$$

and again applying P_l on both sides we get

$$(c_k-c_l)P_l[P_j, P_k]=0.$$

Now as a consequence of this equality we can find, in the exactly same way as above,

$$[P_j, P_k]=0 \qquad \text{for } j, k=1, \cdots, s$$

and this finishes the proof.

PROPOSITION 3. *$s-\pi$-structure associated with a metric polynomial structure f is integrable if and only if there is*

$$[f+f^{-1}, f+f^{-1}]=0.$$

Proof is an easy application of Lemma 3. It is enough to set $c_i=2a_i$ and to notice that $I_{(2a_1,\cdots,2a_s)}=-(f+f^{-1})$. The last equality follows immediately from the fact that on D_i there is $f^2+2a_if+I=0$.

3. In this last paragraph we establish the existence of a linear connection ∇ satisfying $\nabla f=0$ and give a necessary and sufficient condition for the existence of a symmetric connection with the same property.

Let us define on M a tensor J by

$$J=\sum_{i=1}^{s}\frac{f+a_iI}{\sqrt{1-a_i^2}}P_i.$$

It is easy to check that J is an almost complex structure on M and that all the distributions D_i are J-invariant. Clearly they are also even-dimensional. From this we can conclude that a metric polynomial structure is very closely related to a couple of another structures, namely almost complex structure and $s-\pi$-structure the all distributions of which are invariant under this almost complex structure.

PROPOSITION 4. *A linear connection ∇ on M satisfies $\nabla f=0$ if and onlf if it satisfies $\nabla J=0$ and $\nabla P_i=0$, $i=1,\cdots,s$.*

Proof. $\nabla f=0$ implies $\nabla J=0$ and $\nabla P_i=0$. The second equality we get by virtue of Proposition 1 expressing P_i as a polynomial in f and the first one follows from the definition of J when using $\nabla P_i=0$. On the other hand $\nabla J=0$ and $\nabla P_i=0$ implies $\nabla f=0$ because there is $f=\sum_{i=1}^{s}(\sqrt{1-a_i^2}J-a_iI)P_i$.

PROPOSITION 5. *There exists a linear connection ∇ on M satisfying $\nabla f=0$.*

Proof. This proposition follows from a result of Wong (see [10], Theorem 1). Obviously at any point $u\in M$ we can find a frame $r(u)$ such that a matrix expression of f with respect to $r(u)$ is

$$\begin{pmatrix} H_1 & & & 0 \\ & H_2 & & \\ & & \ddots & \\ 0 & & & H_s \end{pmatrix} \quad \text{where} \quad H_i=\left(\begin{array}{c|c} -a_iI_i & -\sqrt{1-a_i^2}\,I \\ \hline \sqrt{1-a_i^2}\,I_i & -a_iI_i \end{array}\right).$$

I_i denotes here a unit matrix of dimension equal to $\frac{1}{2}\dim D_i$.

Before stating the last proposition we must have a new tensor defined. Let A, B be two tensor fields of type (1, 1) on M such that $AB=BA$. Then we define a tensor $\{A, B\}$ of type (1, 2) by

$$\{A, B\}(X, Y)=[AX, BY]+AB[X, Y]-A[X, BY]-B[AX, Y]$$

where X, Y are vector fields on M. One can easily check that this definition is good.

PROPOSITION 6. *There exists a symmetric linear connection ∇ on M with $\nabla f=0$ if and only if the following conditions are satisfied*

(i) *J is integrable*

(ii) *$(D_1, \cdots, D_s)$ is integrable*

(iii) $\{J, P_i\}=0$, $i=1, \cdots, s$.

Proof. If there exists a symmetric connection ∇ on M satisfying $\nabla f=0$ then we have also $\nabla J=0$ and $\nabla P_i=0$ by Proposition 4. But the existence of a symmetric connection ∇ satisfying $\nabla J=0$ is equivalent to the integrability of J (see e.g. [11], Chap. IX, § 3) as well as the existence of a symmetric connection ∇ satisfying $\nabla P_i=0$ is equivalent to the integrability of $(D_1, \cdots, D_s]$ (see [8], § 4). Moreover if ∇ is symmetric we have for any vector fields X, Y

$$\begin{aligned}\{J, P_i\}(X, Y)=&\nabla_{JX}(P_iY)-\nabla_{P_iY}(JX)+JP_i\nabla_XY-JP_i\nabla_YX-J\nabla_X(P_iY)\\ &+J\nabla_{P_iY}X-P_i\nabla_{JX}Y+P_i\nabla_Y(JX)\\ =&P_i\nabla_{JX}Y-J\nabla_{P_iY}X+JP_i\nabla_XY-JP_i\nabla_YX-JP_i\nabla_XY\\ &+J\nabla_{P_iY}X-P_i\nabla_{JX}Y+P_iJ\nabla_YX\\ =&0\,.\end{aligned}$$

To prove the converse part we remember the fact that if ∇' is any symmetric connection and J a complex structure then a connection ∇ defined by $\nabla_XY=\nabla'_XY-Q(X, Y)$ where

$$4Q(X, Y)=(\nabla'_{JX}J)(Y)+J((\nabla'_XJ)(Y))+2J((\nabla'_YJ)(X))$$

satiffies $\nabla J=0$ (see e.g. [11], Char. IX, § 3). Let us take for ∇' a symmetric connection satisfying $\nabla' P_i=0$, $i=1, \cdots, s$. Such a cennection surely exists because $(D_1, \cdots, D_s)$ is integrable. Now it suffices to show that the new connection ∇ has also the property $\nabla P_i=0$, $i=1, \cdots, s$. In other words we are to prove $Q(X, P_iY)-P_iQ(X, Y)=0$. We get

$$\begin{aligned}&4Q(X, P_iY)-4P_iQ(X, Y)\\ &=(\nabla'_{JX}J)(P_iY)+J((\nabla'_XJ)(P_iY))+2J((\nabla'_{P_iY}J)(X))\\ &\quad -P_i((\nabla'_{JX}J)(Y))-P_iJ((\nabla'_XJ)(Y))-2P_iJ((\nabla'_YJ)(X))\\ &=P_i\nabla'_{JX}(JY)-P_iJ\nabla'_JY+JP_i\nabla'_X(JY)-J^2P_i\nabla'_XY\\ &\quad +2J\nabla'_{P_iY}(JX)-2J^2\nabla'_{P_iY}X-P_i\nabla'_{JX}(JY)+P_iJ\nabla'_{JX}Y\\ &\quad -P_iJ\nabla'_X(JY)+P_iJ^2\nabla'_XY-2P_iJ\nabla'_Y(JX)+2P_iJ^2\nabla'_YX\\ &=2\{J\nabla'_{P_iY}(JX)+\nabla'_{P_iY}X-P_iJ\nabla'_Y(JX)-P_i\nabla'_YX\}\end{aligned}$$

$$
\begin{aligned}
&=2\{JP_i\nabla'_{JX}Y+J[P_iY, JX]\\
&\qquad +P_i\nabla'_XY+[P_iY, X]-P_iJ\nabla'_Y(JX)-P_i\nabla'_YX\}\\
&=2\{JP_i[JX, Y]-J[JX, P_iY]+P_i[X, Y]-[X, P_iY]\}\\
&=2\{J, P_i\}(JX, Y)\\
&=0
\end{aligned}
$$

and this finishes the proof.

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