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COMPLEX SUBMANIFOLDS WITH CERTAIN CONDITIONS

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§ 0. Introduction.

Complex Einstein hypersurfaces in a complex space form were classified by Smyth [8]. He showed that they are locally symmetric and used Cartan's list of irreducible Hermitian symmetric spaces. Nomizu and Smyth [3] continued their study of complex hypersurfaces in a complex space form.

On the other hand, Ogiue [4], applying a formula of Simons' type and results obtained by O'Neill [6], studied complex submanifolds of constant holomorphic sectional curvature in a complex space form.

In this paper, we shall study complex submanifolds, especially complex Einstein submanifolds, in a complex space form which satisfy certain conditions for the normal bundle. In § 1, we give basic formulas concerning complex submanifolds. In § 2, we study complex submanifolds with certain holonomy groups with respect to the induced connection in the normal bundle. In § 3, applying a formula of Simons' type, we study, in a complex projective space with Fubini-Study metric, complex Einstein submanifolds with certain curvature condition concerning the normal bundle.

§ 1. Preliminaries.

Let $\bar{M}^{n+p}$ be a complex $(n+p)$-dimensional Kaehler manifold with complex structure J and Kaehler metric g and M^n be a complex submanifold in $\bar{M}^{n+p}$ of complex dimension n. Then M^n is a Kaehler manifold with the induced complex structure and the induced metric, which will be also denoted by J and g respectively. Let $\bar{\nabla}$ (resp. ∇) be the connection with respect to the metric of $\bar{M}^{n+p}$ (resp. the induced metric of M^n). We can easily see that the connection ∇ in M^n is a Kaehler connection. If we denote by H the second fundamental form of M^n, then the equation of Gauss can be written as

$$\bar{\nabla}_X Y=\nabla_X Y+H(X, Y) \tag{1.1}$$

for any local vector fields X and Y of M^n. We note that the second fundamental form H satisfies

$$H(JX, Y)=H(X, JY)=JH(X, Y) \tag{1.2}$$

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for any vectors X and Y tangent to M^n.

Throughout this paper, X, Y and Z will be either local vector fields of M^n or vectors tangent to M^n at a point and the inner product $g(X, Y)$ of X and Y will be denoted by $\langle X, Y\rangle$.

Let $N(M^n)$ be the normal bundle of M^n in $\bar{M}^{n+p}$. Then $N(M^n)$ is a Hermitian vector bundle with the induced complex structure J^* and the induced metric g^*. The induced connection ∇^* in $N(M^n)$ is a Hermitian connection. Choosing local fields of orthonormal vectors $C_1, \cdots, C_p, JC_1, \cdots, JC_p$ normal to M^n, equations of Weingarten may be written as

$$\bar{\nabla}_X C_i = -A_i X + \nabla^*_X C_i\,, \qquad \bar{\nabla}_X JC_i = -A_{\bar{i}} X + \nabla^*_X JC_i \tag{1.3}$$

for each i where the index i runs over the range $\{1, \cdots, p\}$ and $A_1, \cdots, A_p$, $A_{\bar{1}}, \cdots, A_{\bar{p}}$ are local symmetric tensor fields of type (1, 1) on M^n satisfying

$$\langle H(X, Y), C_i\rangle = \langle A_i X, Y\rangle\,, \qquad \langle H(X, Y), JC_i\rangle = \langle A_{\bar{i}} X, Y\rangle \tag{1.4}$$

for each i. We have from (1.2) and (1.4)

$$A_{\bar{i}} = JA_i\,, \tag{1.5}$$

$$JA_i + A_i J = 0 \tag{1.6}$$

for each i and hence we see that M^n is a minimal submanifold in $\bar{M}^{n+p}$.

Next, we consider the structure equations of the submanifold M^n in $\bar{M}^{n+p}$. Let TM be the tangent bundle of M^n. If we denote by ∇' the induced connection in the bundle $TM+N(M^n)$ and denote by Proj_{TM} (resp. $\mathrm{Proj}_{N(M)}$) the projection map of vectors of the ambient manifold $\bar{M}^{n+p}$ to the tangent space of M^n (resp. normal space), then structure equations of Gauss, Codazzi and Ricci may be written as, for any X, Y and Z,

$$\begin{aligned}\mathrm{Proj}_{TM}\bar{R}(X, Y)Z = R(X, Y)Z &+ \sum_i \{\langle A_i X, Z\rangle A_i Y - \langle A_i Y, Z\rangle A_i X\} \\ &+ \sum_i \{\langle JA_i X, Z\rangle JA_i Y - \langle JA_i Y, Z\rangle JA_i X\}\,,\end{aligned} \tag{1.7}$$

$$\mathrm{Proj}_{N(M)}\bar{R}(X, Y)Z = (\nabla'_X H)(Y, Z) - (\nabla'_Y H)(X, Z)\,, \tag{1.8}$$

$$\begin{aligned}\mathrm{Proj}_{N(M)}\bar{R}(X, Y)C_i = R^*(X, Y)C_i &- \sum_j \{\langle A_i A_j X, Y\rangle - \langle A_j A_i X, Y\rangle\} C_j \\ &- \sum_j \{\langle A_i JA_j X, Y\rangle - \langle JA_j A_i X, Y\rangle\} JC_j\end{aligned} \tag{1.9}$$

respectively, where $\bar{R}$, R and R^* are the Riemann curvature tensors of $\bar{M}^{n+p}$, M^n and $N(M^n)$ respectively.

By a complex space form $\bar{M}^{n+p}(c)$, we shall mean a complex $(n+p)$-dimensional connected complete Kaehler manifold of constant holomorphic sectional curvature c. We assume that the ambient manifold $\bar{M}^{n+p}$ is a complex space form $\bar{M}^{n+p}(c)$. Then the curvature tensor $\bar{R}$ satisfies

$$\begin{aligned}(1.10)\qquad &\bar{R}(\bar{X},\bar{Y})\bar{Z}\\ &=\frac{c}{4}\{\langle\bar{Y},\bar{Z}\rangle\bar{X}-\langle\bar{X},\bar{Z}\rangle\bar{Y}+\langle J\bar{Y},\bar{Z}\rangle J\bar{X}-\langle J\bar{X},\bar{Z}\rangle J\bar{Y}-2\langle J\bar{X},\bar{Y}\rangle J\bar{Z}\}\end{aligned}$$

for any vectors $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ tangent to $\bar{M}^{n+p}(c)$. Thus we have, from (1.7), (1.8) and (1.9),

$$\begin{aligned}(1.11)\qquad R(X,Y)Z=&\frac{c}{4}\{\langle Y,Z\rangle X-\langle X,Z\rangle Y\\ &+\langle JY,Z\rangle JX-\langle JX,Z\rangle JY-2\langle JX,Y\rangle JZ\}\\ &+\sum_i\{\langle A_iY,Z\rangle A_iX-\langle A_iX,Z\rangle A_iY\}\\ &+\sum_i\{\langle JA_iY,Z\rangle JA_iX-\langle JA_iX,Z\rangle JA_iY\},\end{aligned}$$

$$(1.12)\qquad (\nabla'_XH)(Y,Z)=(\nabla'_YH)(X,Z),$$

$$\begin{aligned}(1.13)\qquad R^*(X,Y)C_i=&\sum_j\langle[A_i,A_j]X,Y\rangle C_j+\sum_j\langle[A_i,JA_j]X,Y\rangle JC_j\\ &-\frac{c}{2}\langle JX,Y\rangle JC_i\end{aligned}$$

for any X, Y and Z. We can easily show from (1.11) that the Ricci tensor S and the scalar curvature ρ satisfy

$$(1.14)\qquad S(X,Y)=\frac{1}{2}(n+1)c\langle X,Y\rangle-2\sum_i\langle A_i^2X,Y\rangle,$$

$$(1.15)\qquad \rho=\frac{1}{2}(n+1)c-\frac{1}{n}\sum_i\operatorname{tr}A_i^2$$

respectively, where $\operatorname{tr}A_i^2$ is the trace of A_i^2.

§2. Submanifolds with certain holonomy groups in the normal bundle.

Let M^n be a complex submanifold of complex dimension n in a Kaehler manifold $\bar{M}^{n+p}$ of complex dimension $n+p$. Using (1.7) and (1.9), we obtain

LEMMA 1. *If $\bar{S}$ and S are the Ricci tensors of $\bar{M}^{n+p}$ and M^n respectively, then we have*

$$(2.1)\qquad \bar{S}(X,JY)=S(X,JY)+\lambda(X,Y)$$

for any vectors X and Y tangent to M^n where λ is a globally defined two form on M^n such that

$$(2.2)\qquad \lambda(X,Y)=\sum_i\langle R^*(X,Y)C_i,JC_i\rangle=-\frac{1}{2}\operatorname{tr}J^*R^*(X,Y).$$

Proof. We note that $\bar{S}(X,JY)$ is equal to $-\frac{1}{2}\operatorname{tr}J\bar{R}(X,Y)$. Therefore, taking orthonormal basis $X_1,\cdots,X_n$, $JX_1,\cdots,JX_n$ of the tangent space of M^n

at each point, we shall compute $\operatorname{tr} J\bar{R}(X, Y)$, i.e.,

$$\sum_t \langle J\bar{R}(X, Y)X_t, X_t\rangle + \sum_t \langle J\bar{R}(X, Y)JX_t, JX_t\rangle + \sum_i \langle J\bar{R}(X, Y)C_i, C_i\rangle + \sum_i \langle J\bar{R}(X, Y)JC_i, JC_i\rangle$$

where the index t runs over the range $\{1, \cdots, n\}$. From (1.7), we easily find

$$\begin{aligned}\sum_t \langle J\bar{R}(X, Y)X_t, X_t\rangle &= \sum_t \langle JR(X, Y)X_t, X_t\rangle \\ &\quad + 2_i \sum_{i,t} \{\langle A_i X, X_t\rangle\langle JA_i Y, X_t\rangle - \langle A_i Y, X_t\rangle\langle JA_i X, X_t\rangle\}, \\ \sum_t \langle J\bar{R}(X, Y)JX_t, JX_t\rangle &= \sum_t \langle JR(X, Y)X_t, X_t\rangle \\ &\quad + 2_i \sum_{i,t} \{\langle A_i X, JX_t\rangle\langle JA_i Y, JX_t\rangle - \langle A_i Y, JX_t\rangle\langle JA_i X, JX_t\rangle\}.\end{aligned}$$

Thus, noting that $S(X, JY)$ is equal to $-\frac{1}{2} \operatorname{tr} JR(X, Y)$, we have

$$\sum_t \langle J\bar{R}(X, Y)X_t, X_t\rangle + \sum_t \langle J\bar{R}(X, Y)JX_t, JX_t\rangle = -2S(X, JY) + 4\sum_i \langle A_i X, JA_i Y\rangle .$$

On the other hand, from (1.9), we find

$$\begin{aligned}\sum_i \langle J\bar{R}(X, Y)C_i, C_i\rangle &= \sum_i \langle JR^*(X, Y)C_i, C_i\rangle + \sum_i \{\langle JA_i X, A_i Y\rangle - \langle JA_i^2 X, Y\rangle\}, \\ \sum_i \langle J\bar{R}(X, Y)JC_i, JC_i\rangle &= \sum_i \langle JR^*(X, Y)JC_i, JC_i\rangle + \sum_i \{\langle JA_i X, A_i Y\rangle - \langle JA_i^2 X, Y\rangle\}.\end{aligned}$$

Since $\sum_i \{\langle JA_i X, A_i Y\rangle - \langle JA_i^2 X, Y\rangle\}$ is equal to $-2\sum_i \langle A_i X, JA_i Y\rangle$, we have

$$\sum_i \langle J\bar{R}(X, Y)C_i, C_i\rangle + \sum_i \langle J\bar{R}(X, Y)JC_i, JC_i\rangle = \operatorname{tr} J^*R^*(X, Y) - 4\sum_i \langle A_i X, JA_i Y\rangle .$$

Therefore we obtain

$$\bar{S}(X, JY) = S(X, JY) - \frac{1}{2} \operatorname{tr} J^*R^*(X, Y) . \qquad \text{Q. E. D.}$$

Let G^* be the restricted holonomy group with respect to the induced connection in the normal bundle $N(M^n)$. Then G^* is a Lie subgroup of $U(p)$. Applying Lemma 1, we have

THEOREM 1. *Let M^n be a complex submanifold in a Kaehler manifold $\bar{M}^{n+p}$. The restricted holonomy group G^* in the normal bundle $N(M^n)$ is contained in $SU(p)$ if and only if $\bar{S}=S$ on TM.*

Proof. G^* is contained in the real representation of $SU(p)$ if and only if $\operatorname{tr} J^*R^*(X, Y)=0$ for every tangent vectors X and Y of M^n (see [2], p. 151). Therefore we see from Lemma 1 that G^* is contained in $SU(p)$ if and only if $\bar{S}=S$ on TM. Q. E. D.

In particular, if the ambient manifold $\bar{M}^{n+p}$ is a complex space from $\bar{M}^{n+p}(c)$, we have

COROLLARY. *Let M^n be a complex submanifold in a complex space form $\bar{M}^{n+p}(c)$. If the restricted holonomy group G^* in $N(M^n)$ is contained in $SU(p)$, then c must be non-positive and M^n is an Einstein manifold, and moreover if $c=0$, then M^n is a totally geodesic submanifold.*

Proof. Since $\bar{M}^{n+p}(c)$ is a complex space form, $\bar{S}$ is given by $\bar{S}=\frac{1}{2}(n+p+1)cg$. From Theorem 1, we obtain $S=\frac{1}{2}(n+p+1)cg$, and hence M^n is an Einstein manifold. We have from (1.14)

$$\sum_i \langle A_i^2 X, Y\rangle = -\frac{pc}{4}\langle X, Y\rangle \qquad \text{for any } X \text{ and } Y.$$

Thus we see that c must be non-positive and that M^n is totally geodesic if $c=0$. Q. E. D.

Let M^n be a complex submanifold in a complex space form $\bar{M}^{n+p}(c)$. If G^* is trivial, then $R^*=0$. We have

THEOREM 2. *Let M^n be a complex submanifold in a complex space from $\bar{M}^{n+p}(c)$. The restricted holonomy group G^* in $N(M^n)$ is trivial if and only if $c=0$ and M^n is a totally geodesic submanifold.*

Proof. Using (1.13) we have

$$[A_i, A_j]=[A_i, JA_j]=0 \qquad \text{for all } i, j\ (i\neq j),$$

$$[A_i, JA_i]=\frac{c}{2}J.$$

From the former equations we obtain $A_iA_j=0$ for all i, j $(i\neq j)$. Taking suitable orthonormal basis $X_1, \cdots, X_n, JX_1, \cdots, JX_n$ of M^n, we can represent A_i $(i=1, \cdots, p)$ by diagonal matrices of the form

$$(*)\qquad \begin{pmatrix} \alpha_1^i & & & & & \\ & \ddots & & & & \\ & & \alpha_n^i & & & \\ & & & -\alpha_1^i & & \\ & & & & \ddots & \\ & & & & & -\alpha_n^i \end{pmatrix}$$

such that α_t^{i}'s satisfy $\alpha_t^i\alpha_t^j=0$ ($t=1,\cdots,n$, $i\neq j$, $1,\cdots,p$). Noting $[A_i, JA_i]=-2JA_i^2$, from latter equation, we find $(\alpha_t^i)^2=-\dfrac{c}{4}$ for all i and t. Therefore we can see that $c=0$ and $\alpha_t^i=0$ for all i and t if $p\geqq 2$. When $p=1$, the theorem is proved by Nomizu and Smyth [3]. Q. E. D.

Remark. Let M^n be a complex Einstein submanifold with non-zero scalar curvature in a complex space form $\bar{M}^{n+p}(c)$ with $c\neq 0$. When G^* is abelian, by taking suitable local field of orthonormal vectors normal to M^n, we can represent every element of the Lie algebra of G^* by matrices of the form

$$\left(\begin{array}{ccc|ccc} & & & \lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \\ \hline -\lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & -\lambda_n & & & \end{array}\right).$$

Since $R^*(X, Y)$ is contained in the Lie algebra of G^* for every X and Y, we see from (1.13) that A_i's with respect to the above local normal frame field satisfy $A_iA_j=0$ ($i\neq j$). Hence we see that A_i's can be represented by diagonal matrices of the form (*) with respect to suitable orthonormal basis of M^n. Using the method in [3] and formula of Simons' type which will be given in § 3, we can show that the restricted homogeneous holonomy group of the submanifold M^n is either $U(n)$ or $SO(n)\times SO(2)$.

§ 3. Formula of Simons' type and it's application (cf. [4], [5]).

Let M^n be a complex submanifold in a complex space form $\bar{M}^{n+p}(c)$. First we compute the Laplacian of the square of the length of the second fundamental form by taking a local cordinate system of M^n. The components of the metric tensor, the complex structure etc. will be denoted as follows;

$$g=(g_{ba}),\quad J=(J_b{}^a),\quad R=(R_{dcb}{}^a),\quad S=(S_{ba}),\quad H=(h_{ba}{}^x),$$
$$g^*=(g_{yx}),\quad J^*=(J_y{}^x),\quad R^*=(R_{bay}{}^x),$$

where the indices $a, b,\cdots$ run over the range $\{1,\cdots,2n\}$ and the indices $x, y,\cdots$ over the range $\{1,\cdots,p,\tilde{1},\cdots,\tilde{p}\}$

LEMMA 2. *We obtain the formula of Simons' type;*

$$\frac{1}{2}\Delta\|H\|^2=\frac{c}{2}(n+2)\|H\|^2-8\operatorname{tr}(\sum_i A_i^2)^2 \tag{3.1}$$
$$-2\sum_{i,j}(\operatorname{tr} A_iA_j)^2-2\sum_{i,j}(\operatorname{tr} JA_iA_j)^2+\|\nabla' H\|^2,$$

where $\|H\|^2=2\sum_i \operatorname{tr} A_i^2=h_{ba}{}^x h^{ba}{}_x$.

Proof. We first note that $h_b{}^{ax}, h_{bax}, \cdots$ are defined by $h_b{}^{ax}=h_{bc}{}^x g^{ca}$, $h_{bax}=h_{ba}{}^y g_{yx}, \cdots$. Since the Laplacian $\Delta\|H\|^2$ of the square of the length of the second fundamental form is defined by

$$\Delta\|H\|^2=g^{ed}(\nabla'_e\nabla'_d\|H\|^2),$$

we have

$$\frac{1}{2}\Delta\|H\|^2=g^{ed}(\nabla'_e\nabla'_d h_{ba}{}^x)h^{ba}{}_x+\|\nabla' H\|^2.$$

We shall compute the first tirm of the right hand side. The structure equations (1.11), (1.12) and (1.13) are given in tirms of local coordinates by

$$R_{dcba}=\frac{c}{4}(g_{da}g_{cb}-g_{ca}g_{db}+J_{da}J_{cb}-J_{ca}J_{db}-2J_{dc}J_{ba}) \tag{1.11$'$}$$
$$+h_{da}{}^x h_{cbx}-h_{ca}{}^x h_{dbx},$$

$$\nabla'_c h_{ba}{}^x=\nabla'_b h_{ca}{}^x, \tag{1.12$'$}$$

$$R_{dcy}{}^x=h_{de}{}^x h_c{}^e{}_y-h_{ce}{}^x h_d{}^e{}_y-\frac{c}{2}J_{dc}J_y{}^x, \tag{1.13$'$}$$

where $R_{dcba}=R_{dcb}{}^e g_{ea}$ and $J_{dc}=J_d{}^b g_{bc}$. From (1.12)$'$ and Ricci equality, we have

$$g^{ed}(\nabla'_e\nabla'_d h_{ba}{}^x)h^{ba}{}_x=g^{ed}(\nabla'_e\nabla'_b h_{da}{}^x)h^{ba}{}_x$$
$$=g^{ed}(\nabla'_b\nabla'_e h_{da}{}^x-R_{ebd}{}^c h_{ca}{}^x-R_{eba}{}^c h_{dc}{}^x+R_{eby}{}^x h_{da}{}^y)h^{ba}{}_x.$$

Substituting (1.11)$'$ and (1.13)$'$, we obtain from minimality of M^n

$$g^{ed}(\nabla'_e\nabla'_d h_{ba}{}^x)h^{ba}{}_x$$
$$=\left\{\frac{c}{2}(n+1)\delta_b^c-h^c{}_e{}^y h_b{}^e{}_y\right\}h_{ca}{}^x h^{ba}{}_x$$
$$-\left\{\frac{c}{4}(g_{ec}g_{ba}-g_{bc}g_{ea}+J_{ec}J_{ba}-J_{bc}J_{ea}-2J_{eb}J_{ac})+h_{ec}{}^y h_{bay}-h_{bc}{}^y h_{eay}\right\}h^{ecx}h^{ba}{}_x$$
$$+\left(h_{ed}{}^x h_b{}^d{}_y-h_{bd}{}^x h_e{}^d{}_y-\frac{c}{2}J_{eb}J_y{}^x\right)h^{eay}h^b{}_{ax}$$
$$=\frac{c}{2}(n+2)\|H\|^2-h_{ec}{}^y h_{bay}h^{ecx}h^{ba}{}_x+2h_{ed}{}^x h_b{}^d{}_y h^{eay}h^b{}_{ax}-2h_{bd}{}^y h_e{}^d{}_y h^{eay}h^b{}_{ax}.$$

Since we can easily show

$$h_{ec}{}^{y}h_{bay}h^{ecx}h^{ba}{}_{x}=2\sum_{i,j}(\operatorname{tr}A_iA_j)^2+2\sum_{i,j}(\operatorname{tr}JA_iA_j)^2\,,$$

$$h_{ed}{}^{x}h_{b}{}^{d}{}_{y}h^{eay}h^{b}{}_{ax}=0\,,\qquad h_{bd}{}^{x}h_{e}{}^{d}{}_{y}h^{eay}h^{b}{}_{ax}=4\operatorname{tr}(\sum_i A_i{}^2)^2\,,$$

we obtain (3.1). Q. E. D.

Next, using (3.1), we study complex Einstein submanifold M^n satisfying the condition $\sum_t R^*(X_t, JX_t)=\mu J^*$ in a complex projective space CP^{n+p} of constant holomorphic sectional curvature $c\ (>0)$, where $X_1, \cdots, X_n, JX_1, \cdots, JX_n$ are orthonormal basis of the tangent space M^n and μ is a globally difined function on M^n. We note that the condition $\sum_t R^*(X_t, JX_t)=\mu J^*$ is always satisfied when $p=1$. We need the following Lemma.

LEMMA 3. *Let M^n be a complex Einstein submanifold* (i. e. $S=\rho g$) *satisfying the condition $\sum_t R^*(X_t, JX_t)=\mu J^*$ in a complex space form $\bar{M}^{n+p}(c)$. If M^n is not totally geodesic, then the codimension p is smaller than $\frac{1}{2}n(n+1)$ and M^n is of constant holomorphic sectional curvature if and only if $\rho=\frac{n+1}{2}c$ or $p=\frac{1}{2}n(n+1)$.*

Proof. We first prove

$$\operatorname{tr}A_i{}^2=\frac{n\alpha}{p}\,, \tag{3.2}$$

$$\operatorname{tr}A_iA_j=0 \qquad (i\neq j)\,, \tag{3.3}$$

$$\operatorname{tr}JA_iA_j=0 \tag{3.4}$$

for any i and j, where $\alpha=\frac{1}{2}\{(n+1)c-2\rho\}$. From (1.14) and (2.1), we have

$$\begin{aligned}&\sum_i\langle A_i{}^2X, Y\rangle=-\frac{pc}{4}\langle X, Y\rangle-\frac{1}{2}\lambda(X, JY)\,,\\ &\sum_i\langle A_i{}^2X, Y\rangle=\frac{\alpha}{2}\langle X, Y\rangle\end{aligned} \tag{3.5}$$

and hence we obtain

$$\lambda(X, JY)=-\Big(\frac{pc}{2}+\alpha\Big)\langle X, Y\rangle\,.$$

Thus from (2.2) and the condition $\sum_t R^*(X_t, JX_t)=\mu J^*$,

$$\mu=\frac{1}{p}\sum_{t,i}\langle R^*(X_t, JX_t)C_i, JC_i\rangle=\frac{1}{p}\sum_t\lambda(X_t, JX_t)=-\frac{nc}{2}-\frac{n}{p}\alpha\,.$$

On the other hand, using (1.13), we find immediately that

$$\mu=\sum_t \langle R^*(X_t, JX_t)C_i, JC_i\rangle=-\operatorname{tr} A_i^2-\frac{nc}{2},$$

$$\sum_t \langle R^*(X_t, JX_t)C_i, JC_j\rangle=-\operatorname{tr} A_i A_j \qquad (i\neq j),$$

$$\sum_t \langle R^*(X_t, JX_t)C_i, C_j\rangle=-\operatorname{tr} JA_i A_j$$

for all i and j. Therefore we see that the condition $\sum_t R^*(X_t, JX_t)=\mu J^*$ implies (3.2), (3.3) and (3.4).

The equation (1.11) may be written as

$$R(X, Y)Z=\frac{c}{4}R_0(X, Y)Z+D(X, Y)Z,$$

where

$$\begin{aligned} R_0(X, Y)Z=&\langle Y, Z\rangle X-\langle X, Z\rangle Y\\ &+\langle JY, Z\rangle JX-\langle JX, Z\rangle JY-2\langle JX, Y\rangle JZ,\\ D(X, Y)Z=&\sum_i\{\langle A_i Y, Z\rangle A_i X-\langle A_i X, Z\rangle A_i Y\}\\ &+\sum_i\{\langle JA_i Y, Z\rangle JA_i X-\langle JA_i X, Z\rangle JA_i Y\}. \end{aligned}$$

Next, we compute

$$\left\|D+\frac{\nu}{4}R_0\right\|^2=\|D\|^2+\frac{\nu}{2}\langle D, R_0\rangle+\frac{\nu^2}{16}\|R_0\|^2,$$

where ν is arbitrary number and $\langle\,,\rangle$ means the extended inner product on the tensor space of type (1.3). Using (3.2), (3.3) and (3.4), we can easily find

$$\|D\|^2=4\frac{n^2}{p}\alpha^2, \quad \frac{\nu}{2}\langle D, R_0\rangle=-8n\alpha\nu, \quad \frac{\nu^2}{16}\|R_0\|^2=2n(n+1)\nu^2.$$

Thus it follows that

$$\left\|D+\frac{\nu}{4}R_0\right\|^2=2n\left\{(n+1)\nu^2-4\alpha\nu+\frac{2n}{p}\alpha^2\right\}. \tag{3.6}$$

Since the left-hand side is non-negative for arbitary number ν, we obtain

$$\alpha^2\left\{p-\frac{1}{2}n(n+1)\right\}\leqq 0.$$

The left-hand side is equal to zero for some ν if and only if $\alpha=0$ or $p=\frac{1}{2}n(n+1)$. This completes the proof. Q. E. D.

THEOREM 3. *Let M^n be a connected complete complex* Einstein *submanifold* (i. e., $S=\rho g$) *satisfying the condition* $\sum_t R^*(X_t, JX_t)=\mu J^*$ *in a complex projective space* CP^{n+p} *of holomorphic sectional curvature* c. *If* $\rho\geqq\frac{1}{2}\frac{n(n+p+1)}{2p+n}c$, *then*

$\nabla' H=0$ and ρ is $\frac{n+1}{2}c$ or $\frac{1}{2}\frac{n(n+p+1)}{2p+n}c$. If $\rho=\frac{n+1}{2}c$, then M^n is a totally geodesic submanifold, i. e., *CP^n. If $\rho=\frac{1}{2}\frac{n(n+p+1)}{2p+n}c$, then M^n is the complex quadric Q^n when $p=1$ and M^n is the complex projective space of constant holomorphic sectional curvature $\frac{c}{2}$ when $p=\frac{1}{2}n(n+1)$.*

Proof. From (3.3) and (3.4), we see that (3.1) reduces to

$$\frac{1}{2}\Delta\|H\|^2=\frac{c}{2}(n+2)\|H\|^2-8\operatorname{tr}(\sum_i A_i^2)^2-2\sum_i(\operatorname{tr}A_i^2)^2+\|\nabla' H\|^2. \tag{3.7}$$

Using (3.5), we obtain

$$\|H\|^2=2\sum_i\operatorname{tr}A_i^2=2n\alpha, \tag{3.8}$$

$$\operatorname{tr}(\sum_i A_i^2)^2=\frac{n}{2}\alpha^2. \tag{3.9}$$

Substituting (3.2), (3.8) and (3.9) in (3.7), we have

$$n\alpha\left\{(n+2)c-4\alpha-2\frac{n}{p}\alpha\right\}+\|\nabla' H\|^2=0.$$

Therefore if $\alpha\leqq\frac{1}{2}\frac{p(n+2)}{2p+n}c$ $\left(\text{i. e., } \rho\geqq\frac{1}{2}\frac{n(n+p+1)}{2p+n}c\right)$, then $\nabla' H=0$ and $\rho=\frac{n+1}{2}c$ or $\rho=\frac{1}{2}\frac{n(n+p+1)}{2p+n}c$. If $\rho=\frac{n+1}{2}c$, then $H=0$, i. e., M^n is totally geodesic. If $\rho=\frac{1}{2}\frac{n(n+p+1)}{2p+n}c$, then M^n is the complex quadric Q^n when $p=1$ (cf. [8]). If $\rho=\frac{1}{2}\frac{n(n+p+1)}{2p+n}c$ and $p=\frac{1}{2}n(n+1)$, then we see that M^n is of constant holomorphic sectional curvature by Lemma 3. Substituting $\alpha=\frac{1}{2}\frac{p(n+2)}{2p+n}c$ and $p=\frac{1}{2}n(n+1)$ in (3.6), we have

$$\left\|D+\frac{\nu}{4}R_0\right\|^2=2n(n+1)\left(\nu-\frac{c}{2}\right)^2.$$

Thus, if $\rho=\frac{1}{2}\frac{n(n+p+1)}{2p+n}c$ and $p=\frac{1}{2}n(n+1)$, then we see that M^n is of constant holomorphic sectional curvature $\frac{c}{2}$ and hence from results obtained by Ogiue [5] that M^n is rigid. For the imbedding of complex projective space of holomorphic sectional curvature $\frac{c}{2}$ into complex projective space of holomorphic sectional curvature c, see O'Neill [6]. Q. E. D.

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