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ON QUATERNION KÄHLERIAN MANIFOLES ADMITTING THE AXIOM OF PLANES

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§ 0. Introduction.

A Riemannian manifold satisfies, as is well known, the axiom of planes if and only if it is of constant curvature (See Cartan [3]). In 1953, Yano and Mogi [8] proved that a Kählerian manifold is of constant holomorphic sectional curvature if and only if it admits the axiom of holomorphic planes. Thereafter Ogiue [7] proved in 1964 that a Sasakian manifold is of constant C-holomorphic sectional curvature if and only if it admits the axiom of C-holomorphic planes or C-holomorphic free mobility.

Recently, quaternion Kählerian manifolds have been studied by several authors [1], [2], [4], [5], [6] and interesting results have been obtained. In a recent paper [5], Ishihara have determined the form of the curvature tensor of a quaternion Kählerian manifold with constant Q-sectional curvature (See the formula (1.8)). The purpose of the present paper is to prove

THEOREM. *A quaternion Kählerian manifold M admits the axiom of Q-planes if and only if it is of constant Q-sectional curvature, provided that* $\dim M \geqq 8$.

COROLLARY. *A quaternion Kählerian manifold M of dimension $4m$ admits the axiom of Q-planes of order p $(1 \leqq p \leqq m)$ if and only if it is of constant Q-sectional curvature, provided* $\dim M \geqq 8$.

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§ 1. Preriminaries.

We now recall definitions and some formulas in quaternion Kählerian manifolds (See [5]). Consider a Riemannian manifold (M, g) which admits 3-dimensional vector bundle V consisting of tensors of type $(1, 1)$ over M. The triple (M, g, V) is called a *quaternion Kählerian manifold* if M, g and V satisfy the following conditions (a) and (b):

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(*a*) In any coordinate neighborhood U of M, there is a local base $\{F, G, H\}$ of the bundle V, such that F, G and H are tensor fields of type (1, 1) in U satisfying

$$F^2=G^2=H^2=-I, \tag{1.1}$$

$$GH=-HG=F, \qquad HF=-FH=G, \qquad FG=-GF=H,$$

I being the identity tensor fields of type (1, 1) in M, and each of F, G and H forms an almost Hermitian structure together with g. Such a local base $\{F, G, H\}$ of V is called a *canonical local base* of the bundle V in U.

(*b*) If φ is a cross section of the bundle V, then $\nabla_X\varphi$ is also a cross section of V for any vector field X in M, where ∇ denotes the Riemannian connection of M.

We call (g, V) a *quaternion Kählerian structure*. From now on, we denote a quaternion Kählerian manifold (M, g, V) simply by (M, g) or more simply by M, for the sake of simplicity.

Any quaternion Kählerian manifold (M, g) is of dimension $4m$ ($m\geqq 1$) and is an Einstein space, that is, the Ricci tensor has the components K_{ji} of the form

$$K_{ji}=4(m+2)ag_{ji}, \tag{1.2}$$

g_{ji} being components of g (See [5]), where the indicies h, i, j, k and l run over the range $\{1, 2, \cdots, 4m\}$ and the summation convention will be used with respect to these indices.

We assume that M is of dimension $4m\geqq 8$. We denote by $F_i{}^h$, $G_i{}^h$ and $H_i{}^h$ respectively components of F, G and H, and by $K_{kji}{}^h$ components of the curvature tensor of (M, g). In terms of these notations, the following formulas were established in [5]:

$$K_{ktsh}F^{ts}=-\frac{m}{m+2}K_{ks}F_h{}^s, \qquad K_{ktsh}G^{ts}=-\frac{m}{m+2}K_{ks}G_h{}^s, \tag{1.3}$$

$$K_{ktsh}H^{ts}=-\frac{m}{m+2}K_{ks}H_h{}^s,$$

$$K_{ts}F_k{}^tF_j{}^s=K_{ts}G_k{}^tG_j{}^s=K_{ts}H_k{}^tH_j{}^s=K_{kj}, \tag{1.4}$$

$$-K_{kjts}F_i{}^tF_h{}^s+K_{kjih}=-4a(G_{kj}G_{ih}+H_{kj}H_{ih}), \tag{1.5}$$

$$K_{tjsh}F_k{}^tF_i{}^s-K_{tjsi}F_k{}^tF_h{}^s=-4a(G_{kj}G_{ih}+H_{kj}H_{ih}), \tag{1.6}$$

a being a real constant appearing in (1.2), where we have put $F_{kj}=F_k{}^tg_{tj}$, $F^{ji}=g^{jt}F_t{}^i$ and so on.

Using the first Bianchi identity, (1.5) and (1.6), we get easily

$$\begin{aligned} &K_{tjsh}F_k{}^tF_i{}^s-K_{tish}F_k{}^tF_j{}^s \\ &\quad=K_{jikh}+4a(G_{ji}G_{kh}+G_{ki}G_{jh}+G_{kj}G_{ih}+H_{ji}H_{kh}+H_{ki}H_{jh}+H_{kj}H_{ih}). \end{aligned} \tag{1.7}$$

Taking a point x of a quaternion Kählerian manifold M, and a tangent vector

X at x, we put

$$Q(X)=\{Y \mid Y=aX+bFX+cGX+dHX\},$$

where a, b, c and d are arbitary real numbers, and we call $Q(X)$ the *Q-section* determined by X, which is a 4-dimensional subspace in the tangent space $T_x(M)$ of M at x. If the sectional curvature for any two vectors belonging to $Q(X)$ is a constant $c(X)$ depending only upon the vector X at x, then the constant $c(X)$ is called the *Q-sectional curvature* with respect to X at x. If the Q-sectional curvature $c(X)$ is a constant c independent of the choice of X and x, then the manifold M is a quaternion Kählerian manifold of *constant Q-sectional curvature c*. This definition leads us to the following result ([5]). That is, a quaternion Kählerian manifold of dimension $4m\geqq 8$ is of constant Q-sectional curvature c if and only if its curvature tensor has the components of the form

$$\begin{aligned}(1.8)\qquad K_{kjih}=\frac{c}{4}(&g_{kh}g_{ji}-g_{jh}g_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}\\ &+G_{kh}G_{ji}-G_{jh}G_{ki}-2G_{kj}G_{ih}+H_{kh}H_{ji}-H_{jh}H_{ki}-2H_{kj}H_{ih}).\end{aligned}$$

We now introduce the notion of the axiom of Q-planes and that of Q-planes of order p. Let M be a $4m$-dimensional quaternion Kählerian manifold with a quaternion Kählerian structure V. A quaternion Kählerian manifold is said to admit *the axiom of Q-planes* if there exists a 4-dimensional totally geodesic submanifold tangent to any Q-section at each point. Next we take a point x in M, and linearly independent vectors $X_{(q)}$ $(q=1, \cdots, p,\ 1\leqq q\leqq m)$ at x such that $X_{(q)}$, $FX_{(q)}$, $GX_{(q)}$, $HX_{(q)}$ $(q=1, \cdots, p)$ are also linearly independent. The $4p$-dimensional vector subspace

$$Q(X_{(1)}, \cdots, X_{(p)})=\{Y \mid Y=\sum_{q=1}^{p}(a_qX_{(q)}+b_qFX_{(q)}+c_qGX_{(q)}+d_qHX_{(q)})\},$$

a_q, b_q, c_q and d_q being arbitary real numbers, in the tangent space $T_x(M)$ is called a *Q-planes of order p* determined by $X_{(1)}, \cdots, X_{(p)}$. If a quaternion Kählerian manifold admits a $4p$-dimensional totally geodesic submanifold tangent to any Q-planes of order p at each point, we say that the manifold admits *the axiom of Q-planes of order p*. Thus a Q-plane of order 1 is nothing but a Q-section. The axiom of Q-planes of order 1 coincides with the axiom of Q-planes. As a consequence of (1.8), a quaternian Kählerian manifold of dimension $4m$ admits the axiom of Q-planes of order p $(1\leqq p\leqq m)$ if it is of constant Q-sectional curvature. Therefore, the theorem stated in § 0 implies immediately the corollary stated in § 0.

§ 2. Proof of the theorem.

We are now going to prove our theorem stated in § **0**. Assume that a quaternion Kählerian manifold (M, g) of dimension $4m\geqq 8$ admits the axiom of Q-planes. We take an arbitrary point x in M and Q-section $Q(X)$ determined by an arbitrary unit tangent vector X at x. Denote by N the totally geodesic submanifold of dimension 4 passing through x and being tangent to $Q(X)$ at x. If the submanifold N is assumed to be expressed by parametric equations $x^h = x^h(v^\lambda)$, where x^h and v^λ are local coordinates in M and N respectively. Then, since N is totally geodesic, the equations

$$\frac{\partial^2 x^h}{\partial v^\nu \partial v^\mu} + \left\{ {h \atop j\ \ i} \right\} \frac{\partial x^j}{\partial v^\nu} \frac{\partial x^i}{\partial v^\mu} - \frac{\partial x^h}{\partial v^\lambda} \left\{ {\lambda \atop \nu\ \ \mu} \right\} = 0$$

are established, $\left\{ {h \atop j\ \ i} \right\}$ and $\left\{ {\lambda \atop \nu\ \ \mu} \right\}$ being Christoffel symbols formed respectively with g_{ji} and the induced metric $g_{\nu\mu}$ of N, where the indices κ, λ, μ and ν run over the range $\{1, 2, 3, 4\}$ and the summation convention will be used with respect to these indices. Since the integrability conditions of these differential equations above are given by the equation of Gauss, we have

$$K_{\nu\mu\lambda}{}^\kappa B_\kappa{}^h = K_{kji}{}^h B_\nu{}^k B_\mu{}^j B_\lambda{}^i, \tag{2.1}$$

where $B_\nu{}^i = \partial x^i / \partial v^\nu$, and $K_{\nu\mu\lambda}{}^\kappa$ are components of the curvature tensor of N. If $\{F, G, H\}$ is a canonical local base of V around the point x of M, then each of $B_\mu{}^h$, for a fixed index μ, is at x linear combination of X^h, $F_i{}^h X^i$, $G_i{}^h X^i$ and $H_i{}^h X^i$ where X^h are components of X, because the submanifold N is tangent to the Q-section $Q(X)$. Conversely, each of X^h, $F_i{}^h X^i$, $G_i{}^h X^i$ and $H_i{}^h X^i$ is a linear combination of $B_1{}^h$, $B_2{}^h$, $B_3{}^h$ and $B_4{}^h$. Thus, taking account of (2.1), we have

$$K_{kji}{}^h F_s{}^k X^s X^j X^i = \alpha X^h + \beta F_i{}^h X^i + \gamma G_i{}^h X^i + \delta H_i{}^h X^i$$

where α, β, γ and δ are local functions in M. Since X is a vector, the equation above is equivalent to the following equation:

$$K_{kji}{}^h F_s{}^k X^s X^j X^i = (\alpha \delta_i^h g_{js} + \beta F_i{}^h g_{js} + \gamma G_i{}^h G_{js} + \delta H_i{}^h g_{js}) X^i X^j X^s. \tag{2.2}$$

Since X can be arbitrarily taken, we have from (2.2)

$$\begin{aligned} &K_{kji}{}^h F_s{}^k + K_{kis}{}^h F_j{}^k + K_{ksj}{}^h F_i{}^k + K_{kjs}{}^h F_i{}^k + K_{kij}{}^h F_s{}^k + K_{ksi}{}^h F_j{}^k \\ &\quad = 2\alpha(\delta_i^h g_{js} + \delta_s^h g_{ij} + \delta_j^h g_{si}) + 2\beta(F_i{}^h g_{js} + F_s{}^h g_{ij} + F_j{}^h g_{si}) \\ &\qquad + 2\gamma(G_i{}^h g_{js} + G_s{}^h g_{ij} + G_j{}^h g_{si}) + 2\delta(H_i{}^h g_{js} + H_s{}^h g_{ij} + H_j{}^h g_{si}). \end{aligned} \tag{2.3}$$

Transvecting the equation above with $F_t{}^s$ and taking the skew-symmetric parts of the both sides with respect to the indicies t and j, we have

$$\begin{aligned} &-K_{tji}{}^h - K_{tji}{}^h + K_{jti}{}^h + K_{jit} \\ &\quad + K_{kis}{}^h F_j{}^k F_t{}^s - K_{kis}{}^h F_t{}^k F_j{}^s + K_{ksj}{}^h F_i{}^k F_t{}^s - K_{kst}{}^h F_i{}^k F_j{}^s \\ &\quad + K_{kjs}{}^h F_i{}^k F_t{}^s - K_{kts}{}^h F_i{}^k F_j{}^s + K_{ksi}{}^h F_j{}^k F_t{}^s - K_{ksi}{}^h F_t{}^k F_j{}^s \\ &= 2\alpha(2\delta_i^h F_{tj} + F_t{}^h g_{ij} - F_j{}^h g_{it} + \delta_j^h F_{ti} - \delta_t^h F_{ji}) \end{aligned} \tag{2.4}$$

$$+2\beta(2F_i{}^hF_{tj}-\delta_t^h g_{ij}+\delta_j^h g_{it}+F_j{}^hF_{ti}-F_t{}^hF_{ji})$$
$$+2\gamma(2G_i{}^hF_{tj}-H_t{}^hg_{ij}+H_j{}^hg_{it}+G_j{}^hF_{ti}-G_t{}^hF_{ji})$$
$$+2\delta(2H_i{}^hF_{tj}+G_t{}^hg_{ij}-G_j{}^hg_{it}+H_j{}^hF_{ti}-H_t{}^hF_{ji})\,.$$

We are now going to determine the coefficients α, γ and δ. First of all, α vanishes identically. In fact, contracting (2.4) with respect to i and h and taking account of (1.3) and (1.4), we obtain

$$(8m+4)\alpha F_{tj}=0\,,$$

which implies $\alpha=0$.

Next, both of γ and δ are also vanishing identically. In fact, by transvecting (2.3) with $F_t{}^i$, we get

$$K_{kjih}F_s{}^kF_t{}^i+K_{kish}F_j{}^kF_t{}^i+K_{ksjh}F_i{}^kF_t{}^i+K_{kjsh}F_i{}^kF_t{}^i$$
$$+K_{kijh}F_s{}^kF_t{}^i+K_{ksih}F_j{}^kF_t{}^i$$
$$=2\beta(-g_{th}g_{js}+F_{sh}F_{tj}+F_{jh}F_{ts})+2\gamma(-H_{th}g_{js}+G_{sh}F_{tj}+G_{jh}F_{ts})$$
$$+2\delta(G_{th}g_{js}+H_{sh}F_{tj}+H_{jh}F_{ts})\,.$$

If we take skew-symmetric parts of the both side in the equation above with respect to the indices t and h, then we have, using (1.5) and (1.6),

$$2\gamma(2H_{ht}g_{js}+G_{sh}F_{tj}+G_{jh}F_{ts}-G_{st}F_{hj}-G_{jt}F_{hs})$$
$$+2\delta(2G_{th}g_{js}+H_{sh}F_{tj}+H_{jh}F_{ts}-H_{st}F_{hj}-H_{jt}F_{hs})=0\,,$$

from which, transvecting with g^{js},

$$2\gamma(8m+4)H_{ht}+2\delta(8m+4)G_{th}=0\,.$$

Therefore we obtain $\gamma=\delta=0$. Thus (2.4) reduces to the following equation:

$$-3K_{tjih}+K_{kish}F_j{}^kF_t{}^s-K_{kish}F_t{}^kF_j{}^s \tag{2.5}$$
$$+K_{ksjh}F_i{}^kF_t{}^s-K_{ksth}F_i{}^kF_j{}^s+K_{kjsh}F_i{}^kF_t{}^s-K_{ktsh}F_i{}^kF_j{}^s$$
$$+K_{ksih}F_j{}^kF_t{}^s-K_{ksih}F_t{}^kF_j{}^s$$
$$=2\beta(2F_{ih}F_{tj}-g_{th}g_{ij}+g_{jh}g_{it}+F_{jh}F_{ti}-F_{th}F_{ji})\,.$$

On the other hand, using the first Bianchi identity and (1.5), we have

$$K_{kish}F_j{}^kF_t{}^s-K_{kish}F_t{}^kF_j{}^s=-K_{tjih}-4a(G_{tj}G_{ih}+H_{tj}H_{ih})\,.$$

By similar divices, we have also

$$K_{ksjh}F_i{}^kF_t{}^s-K_{ksth}F_i{}^kF_j{}^s$$
$$=-K_{tjih}+4a(G_{jh}G_{it}+H_{jh}H_{it})-4a(G_{th}G_{ij}+H_{th}H_{ij})\,.$$

Substituting these two equations into (2.5) and using (1.5) and (1.7), we have

$$K_{tjih}=\frac{\beta}{4}(g_{th}g_{ji}-g_{jh}g_{ti}+F_{th}F_{ji}-F_{jh}F_{ti}-2F_{tj}F_{ih})$$
$$+a(G_{th}G_{ji}-G_{jh}G_{ti}-2G_{tj}G_{ih}+H_{th}H_{ji}-H_{jh}H_{ti}-2H_{tj}H_{ih}). \tag{2.6}$$

Next, transvecting (2.6) with g^{th}, we have $\beta=4a$ as a consequence of (1.2). Thus the Riemannian manifold (M, g) has the curvature tensor of the form (1.8) with a function c. However, in such a case, the function c is necessarily a constant (See [5]). Therefore, (M, g, V) is a quaternion Kählerian manifold of constant Q-sectional curvature. Thus our theorem has been completely proved.

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