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EXISTENCE OF MAXIMAL ANALYTIC FUNCTIONS ON RIEMANN SURFACES

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1. Introduction. Given a Riemann surface W (open or closed), let $\mathcal{A}(W)$ and $\mathcal{M}(W)$ denote respectively the set of all single-valued analytic functions on W and the set of all single-valued meromorphic functions on W.

For any function $f \in \mathcal{A}(W)$ (or $g \in \mathcal{M}(W)$), take an arbitrary $\varphi \in \mathcal{A}(f(W))$ (or $\psi \in \mathcal{M}(f(W))$) and the composite function $\varphi \circ f$ (or $\psi \circ g$) still belongs to $\mathcal{A}(W)$ (or $\mathcal{M}(W)$). This fact suggests us to consider the indecomposable functions (i.e. impossible to be represented in the above-mentioned composite form) as, in a sense, fundamental for the surface W. In the present paper we shall be concerned with the existence theorem of such functions. Naturally, if φ is a one-to-one conformal map of $f(W)$ onto itself, then we have $f = \varphi \circ (\varphi^{-1} \circ f)$. So we have to speak of the indecomposability, always up to such trivial decompositions.

2. The reasoning being completely parallel for the case of meromorphic functions, in what follows we shall mostly confine ourselves to analytic functions on W.

DEFINITION. For two functions $f, g \in \mathcal{A}(W)$, if we have

$$g = \varphi \circ f \qquad \text{where} \qquad \varphi \in \mathcal{A}(f(W))$$

($f(W)$ being the image of W by f in the complex plane), we shall say: 'f *majorizes* g' and we shall express this fact by $g \prec f$.

Obviously, '$f \prec g$ and $g \prec f$' is equivalent to '$f = \varphi \circ g$ and φ is a one-to-one conformal map of $g(W)$ onto itself'. So we may define an equivalence relation on $\mathcal{A}(W)$ by

$$f \sim g \Longleftrightarrow f \prec g \quad \text{and} \quad g \prec f \tag{2.1}$$

and the relation $\prec$ induces an order relation on the set of quivalence classes of $\mathcal{A}(W)$. Our problem is reduced to assure the existence of maximal equivalence classes with respect to this order relation.

DEFINITION. A function $f \in \mathcal{A}(W)$ is said to be *maximal* if f belongs to a maximal equivalence class.

Our final result is:

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THEOREM A. *For any* $f \in \mathcal{A}(W)$, *there exists a maximal analytic function* f_0 *such that* $f \prec f_0$.

Proof. According to Zorn's lemma, it suffices to show that the set of equivalence classes of $\mathcal{A}(W)$ is inductively ordered i.e. every totally ordered subset has an upper bound (cf. Bourbaki [1]). We shall do it in the next sections.

3. Let $\mathfrak{F}$ be a totally ordered subset of the equivalence classes of $\mathcal{A}(W)$. We choose a representative element from each class in $\mathfrak{F}$ and denote them by $\{f_i\}_{i\in I}$, where $f_i \in \mathcal{A}(W)$ and I is an index set isomorphically ordered to $\mathfrak{F}$:

$$(3.1) \qquad i, j \in I \text{ and } i<j \text{ imply } f_i=\varphi_{ij}\circ f_j \quad (\varphi_{ij}\in\mathcal{A}(f_j(W))).$$

We shall call $\{f_i\}_{i\in I}$ *a totally ordered set of analytic functions on* W.

We first remark some elementary properties of a totally ordered set $\{f_i\}_{i\in I}$ of analytic functions which will be useful later.

LEMMA. I. *For two points* p, q *of* W, *if* $f_{i_0}(p)\neq f_{i_0}(q)$ *then* $f_j(p)\neq f_j(q)$ *for all* j *such that* $i_0<j$.

II. *When we fix a point* p *of* W, *the multiplicity of* f_i *at* p *denoted* $n(p; f_i)$ *is a decreasing function of* i *and if* $i<j$ *then* $n(p; f_i)$ *is a multiple of* $n(p; f_j)$.

III. *If* f_{i_0} *has the minimum multiplicity at* p *then there exists a neighbourhood* V (*independent of* j) *of* p *such that*

$$(3.2) \qquad f_j=\chi_{ji_0}\circ f_{i_0} \qquad (\chi_{ji_0}\in\mathcal{A}(f_{i_0}(V))$$

holds in V *for all* $j>i_0$.

Proof. I and II are direct consequences of (3.1). III. Clearly, it is sufficient to prove the relation (3.2) for $f_{i_0}^*$ and f_j^* (where $f_i^*(q)\equiv f_i(q)-f_i(p)$) instead of f_{i_0} and f_j and so we may assume without loss of generality that $f_{i_0}(p)=f_j(p)=0$. Let τ denote a local uniformizer of W having as its domain a disc $\Delta=\{z||z|<1\}$ and satisfying $\tau(0)=p, f_{i_0}\circ\tau=z^m$ ($m=n(p; f_{i_0})$). Put $\tau(\Delta)=V$. Then every $f_j\circ\tau$ can be expanded in Δ in power series in z. And it suffices to prove that this power series contains only terms in powers of z^m. Suppose that it were not the case and let $mk+l$ $(0<l<m)$ be the lowest power in the expansion of f_j which is not a multiple of m, i.e.

$$f_j\circ\tau=\sum_{\nu=1}^{k} c_\nu z^{m\nu}+dz^{mk+l}+\cdots.$$

Consider the function

$$g=\left(f_j-\sum_{\nu=1}^{k} c_\nu (f_{i_0})^\nu\right)/(f_{i_0})^k.$$

g is an analytic function in V whose power series begins by the term z^l. On the other hand, because of the relation $f_{i_0}=\varphi_{i_0j}\circ f_j$, g can be represented in the form

$$\phi \circ f_j \quad \text{where} \quad \phi \in \mathcal{A}(f_j(V))$$

and its expansion must begin with a term which is a multiple of z^m and so we have a contradiction.

PROPOSITION. *Let W be a Riemann surface on which there exists a family $\{f_i\}_{i\in I}$ such that*

i) *$\{f_i\}_{i\in I}$ is a totally ordered set of analytic functions on W,*

ii) *$\{f_i\}_{i\in I}$ separates W, i.e. if $p, q\in W$ and $p\neq q$, there is an f_i which assumes different values at p and q.*

Then W is a planar surface.

Proof. It is well-known that if every relatively compact subdomain G of W is planar then W itself is planar. So we shall show that for every G there exists an f_i which is univalent on G. Suppose this were not true, then we can find a relatively compact subdomain G_0 of W and a sequence of pairs of points of G_0:

$$\{(p_i, q_i)\}_{i\in I} \quad \text{such that} \quad f_i(p_i)=f_i(q_i).$$

Put

$$P_i=\overline{\bigcup_{k\geqq i}\{p_k\}}, \qquad Q_i=\overline{\bigcup_{k\geqq i}\{q_k\}} \qquad (i\in I)$$

(the bar means the closure in W) and

$$P=\bigcap_i P_i, \qquad Q=\bigcap_i Q_i.$$

Then P and Q, being intersections of decreasing sequences of compact sets, are not empty and we have the following two possibilities.

First, if we can take two different points $p\in P$ and $q\in Q$, according to the separating property ii) and the continuity of f_i, we have an f_{i_0} and disjoint neighbourhoods V_1 and V_2 of p and q such that $f_{i_0}(V_1)\cap f_{i_0}(V_2)=\phi$. Then property I of the preceding lemma implies

$$f_j(V_1)\cap f_j(V_2)=\phi \qquad \text{for all} \quad j>i_0,$$

which contradicts the definitions of p and q.

Secondly, if P and Q reduce to the same single point $\{p\}$, then as the consequence of the separating property ii) and the property III of the preceding lemma there is an f_{i_0} such that $n(p; f_{i_0})=1$. Then f_{i_0} is univalent in a neighbourhood V of p and also all f_j with $j>i_0$ are univalent in V, which is also a contradiction.

4. Now we recall the following Theorem C of Heins [2], with some changes of notations adapted to ours:

THEOREM. *If K_1 is an arbitrary subfield of $\mathcal{M}(W)$ containing the complex constants and functions other than constants, then there exists a conformal map ϕ of W onto a Riemann surface W_1 and a separating subfield K_2 of $\mathcal{M}(W_1)$ such* *that $g \to g \circ \psi$ maps K_2 onto K_1. The representation of K_1 so given is determined up to a conformal equivalence.*

When we have a totally ordered set of anlytic functions $\{f_i\}_{i \in I}$ on a Riemann surface W, consider the sets

$$M_i = \{\varphi \circ f_i | \varphi \in \mathcal{M}(f_i(W))\} \qquad (i \in I).$$

M_i $(i \in I)$ are clearly subfields of $\mathcal{M}(W)$ and if we put $K_1 = \cup_{i \in I} M_i$, K_1 is also a subfield of $\mathcal{M}(W)$ as it is the union of increasing sequence of subfields. Then the above-mentioned Heins' theorem applies to W and K_1 and show the existence of a second Riemann surface W_1, a conformal map ψ: $W \to W_1$ and $g_1 \in \mathcal{A}(W_1)$ $(i \in I)$ such that $f_i = g_i \circ \psi$. And even more, $\{g_i\}_{i \in I}$ is a totally ordered set of analytic functions on W_1 separating W_1.

Thus we have proved:

THEOREM B. *Let $\{f_i\}_{i \in I}$ be a totally ordered set of analytic functions on W, then there exist a Riemann surface W_1, a conformal map ψ of W onto W_1 and $g_i \in \mathcal{A}(W_1)$ $(i \in I)$ such that*

$$f_i = g_i \circ \psi \tag{4. 1}$$

and $\{g_i\}_{i \in I}$ is separationg on W_1.

The existence of a separating $\{g_i\}_{i \in I}$ implies, according to our Proposition, that W_1 is actually a planar surface and so ψ an analytic function on W. And the relations (4. 1) show that it is an upper bound for the family $\{f_i\}_{i \in I}$.

COROLLARY. *The set of equivalence classes of $\mathcal{A}(W)$ is inductively ordered.*

Thus the proof of our Theorem A is complete.

5. We conclude this paper by citing some remarks.

I. If W is a planar surface, then the set of univalent functions on W is the unique maximal (and so maximum) equivalence class.

II. If W is a closed surface, our Theorem A still applies to $\mathcal{M}(W)$ and meromorphic functions of minimum order are, for example, maximal.

III. We may prove Theorem B directly, without making use of the fields of meromorphic functions, following the line of proof of Heins. W_1 is the quotient set of W by the equivalence relation '$p \sim q$ if and only if $f_i(p) = f_i(q)$ for all $i \in I$'. So ψ is not only an upper bound of $\{f_i\}_{i \in I}$ but the least upper bound.

REFERENCES

[1] BOURBAKI, N. Théorie des ensembles, Chap. III, (Act. Sci. Ind. 1243), Hermann (1956).

[2] HEINS, M. Algebraic structure and conformal mapping. Trans. Amer. Math. Soc. **89** (1958), 267–276.

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