LENGTH OF THE SINGULAR SET OF SCHOTTKY GROUP

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1. Let B_0 be an infinite domain on the z-plane, whose boundary consists of $2p$ ($p \geqq 2$) mutually disjoint circles H_i, H_i' ($i=1, 2, \cdots, p$). These circles are equivalent in pairs (H_i, H_i') ($i=1, 2, \cdots, p$); the outside of H_i is mapped onto the inside of H_i' by hyperbolic or loxodromic transformations

$$S_i: \quad z'=\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i} \qquad (\alpha_i\delta_i-\beta_i\gamma_i=1).$$

The transformations S_i ($i=1, 2, \cdots, p$) generate a Schottky group G with the fundamental domain B_0.

2. Now let us define the grade of a transformation $S \in G$. Any element S of G is represented as the product of generators S_i ($i=1, 2, \cdots, p$) in the form

$$S=S_{i_1}^{\lambda_1}S_{i_2}^{\lambda_2}\cdots S_{i_k}^{\lambda_k},$$

where the exponents λ_j are integers. We call the sum

$$m=\sum_{j=1}^{k} |\lambda_j|$$

the grade of S and that of the image $S(B_0)$ of B_0. In particular, the identical transformation and B_0 have the grade 0, and any generator S_i ($i=1, 2, \cdots, p$) together with its inverse S_i^{-1} and the image $S_i(B_0)$ of B_0 have the grade 1.

Consider an infinite set of circles which are obtained from p pairs of circles H_i, H_i' ($i=1, 2, \cdots, p$) of B_0 by all the transformations of G. We say that a circle of the set is of grade m, if it is surrounded by m circles of the set. The total number of circles of grade m is obviously equal to $2p(2p-1)^m$. If we perform a transformation of grade m (>0) on B_0, we obtain a domain of grade m whose outer boundary is a circle of grade $m-1$ and inner boundaries are $2p-1$ circles of grade m.

Denote by D_m a domain bounded by the whole circles of grade m. Then D_m ($m=0, 1, 2, \cdots$) are a monotone increasing sequence of domains, so that D_μ ($\mu<m$) is contained in D_m as a subdomain. Further, denote by D_m^c the complement of D_m with respect to the extended z-plane. Then D_m^c consists of $2p(2p-1)^m$ closed disks which are mutually disjoint. For $m\to\infty$ D_m^c converges to a perfect non-dense set E. We call E the singular set of G. G is properly discontinuous in the complement of E.

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It is well known that, in the case $p \geqq 2$, the logarithmic capacity of E is positive (Myrberg [1]) and the 2-dimensional measure of E is equal to zero (Sario [2]).

3. Denote by $r_{i,j}$ $(i=1, 2, \cdots, 2p(2p-1)^j)$ the radii of circles of grade j and put

$$L_m = \sum_{j=0}^{m} \sum_{i=1}^{2p(2p-1)^j} r_{i,j}. \tag{1}$$

Schottky [3] proved the following: Suppose that the $2p-3$ circles $K_1, K_2, \cdots, K_{2p-3}$ can be described so that each K_j is disjoint from each other, K_1 contains two circles of $\{H_i, H_i'\}_{i=1}^{p}$, K_1 and K_2 surround a domain together with a circle of $\{H_i, H_i'\}_{i=1}^{p}$ and so on, and finally there are two circles of $\{H_i, H_i'\}_{i=1}^{p}$ outside K_{2p-3}. Then $\lim_{m\to\infty} L_m$ is finite.

The Schottky theorem implies that, under the same assumption as in the theorem, the 1-dimensional measure of E is equal to zero. But it seems still open whether L_m is convergent or not in the general case where the above condition for B_0 is not satisfied. The condition of Schottky is geometric and we shall give other quantitative conditions.

4. Let R and r be radii of two circles in the z-plane, and d the distance between their centers. Then

$$\frac{(R^2+r^2-d^2)^2}{4R^2r^2} - 1 = K \tag{2}$$

is invariant under any linear transformation of z. There are three cases: (i) K is 0, if they are tangent, (ii) negative, if they intersect themselves, (iii) positive, otherwise. In the third case we obtain

$$R^2+r^2-d^2 = \pm 2Rr\sqrt{1+K} \tag{3}$$

where plus sign is used in the case where a circle is contained in the inside of the other, and minus sign in the other case.

To make our discussion clear, we treat the case $p=2$ in B_0, in which G is generated by two transformations. Take a domain B_m of grade m. Then, B_m is bounded by one outer circle $C^{(m-1)}$ of grade $m-1$ with radius $R^{(m-1)}$ and three inner circles $C_1^{(m)}, C_2^{(m)}, C_3^{(m)}$ of grade m with radii $r_1^{(m)}, r_2^{(m)}, r_3^{(m)}$, respectively. For $C^{(m-1)}$ and $C_i^{(m)}$ $(i=1, 2, 3)$, we always have

$$R^{(m-1)} - d_i^{(m)} > r_i^{(m)} \quad (i=1, 2, 3), \tag{4}$$

where $d_i^{(m)}$ denotes the distance between the centers of $C^{(m-1)}$ and $C_i^{(m)}$. From (3) and (4) we obtain

$$R^{(m-1)} > r_i^{(m)}\sqrt{1+k} + d_i^{(m)}, \tag{5}$$

where k is the minimum of the K's which are six in number. Hence we have

$$3R^{(m-1)} > \sqrt{1+k}\ (r_1^{(m)}+r_2^{(m)}+r_3^{(m)}) + d_1^{(m)}+d_2^{(m)}+d_3^{(m)}. \tag{6}$$

Denote by $'d_1^{(m)}$, $'d_2^{(m)}$ and $'d_3^{(m)}$ the distances between the centers of $C_1^{(m)}$ and $C_2^{(m)}$, of $C_2^{(m)}$ and $C_3^{(m)}$, and of $C_3^{(m)}$ and $C_1^{(m)}$, respectively. Then, it is easily seen that

$$(7) \qquad 2(d_1^{(m)}+d_2^{(m)}+d_3^{(m)})>'d_1^{(m)}+'d_2^{(m)}+'d_3^{(m)}.$$

As to $'d_i^{(m)}$ $(i=1, 2, 3)$, for example $'d_1^{(m)}$, from (3) with minus sign, it follows

$$r_1^{(m)2}+r_2^{(m)2}-'d_1^{(m)2}=-2r_1^{(m)}r_2^{(m)}\sqrt{1+k}.$$

Therefore we obtain

$$(8) \qquad 'd_1^{(m)}=(r_1^{(m)}+r_2^{(m)})\sqrt{1+\frac{4r_1^{(m)}r_2^{(m)}}{(r_1^{(m)}+r_2^{(m)})^2}\frac{\sqrt{1+k}-1}{2}}.$$

Denote by $G_{12}(m)$ and $A_{12}(m)$ the geometric and arithmetical means of the radii $r_1^{(m)}$ and $r_2^{(m)}$ of grade m. Obviously

$$1\geqq\frac{G_{12}(m)}{A_{12}(m)}>0$$

for any m and $i\neq j$, and

$$1\geqq M=\inf_{i,j,m}\left(\frac{G_{ij}(m)}{A_{ij}(m)}\right)^2\geqq 0.$$

Hence M is independent of i, j and m. Therefore we obtain from (8)

$$(9) \qquad \begin{cases} 'd_1^{(m)}=(r_1^{(m)}+r_2^{(m)})\sqrt{1+M^*(k)}, \\ 'd_2^{(m)}=(r_2^{(m)}+r_3^{(m)})\sqrt{1+M^*(k)}, \\ 'd_3^{(m)}=(r_3^{(m)}+r_1^{(m)})\sqrt{1+M^*(k)}, \end{cases}$$

where

$$M^*(k)=M\frac{\sqrt{1+k}-1}{2}.$$

From (6), (7) and (9), it follows

$$r_1^{(m)}+r_2^{(m)}+r_3^{(m)}<R^{(m-1)}\cdot\rho,$$

where

$$\rho=\frac{3}{\sqrt{1+k}+\sqrt{1+M^*(k)}}.$$

If we take $\rho<1$, we obtain

$$\lim_{m\to\infty}L_m<R_1^{(0)}+R_2^{(0)}+R_3^{(0)}+R_4^{(0)}+(R_1^{(0)}\cdot\rho+R_2^{(0)}\cdot\rho+R_3^{(0)}\cdot\rho+R_4^{(0)}\cdot\rho)$$
$$+\cdots=4R(1+\rho+\rho^2+\cdots)=\frac{4R}{1-\rho},$$

where

$$R=\max_i R_i^{(0)} \quad \text{and} \quad R_i^{(0)}=r_{i0} \qquad (i=1, 2, 3, 4)$$

are the radii of circles B_0, that is, the circles of grade 0. Hence it is seen that

$$\sqrt{1+k}+\sqrt{1+M^*(k)}>3 \tag{10}$$

is a sufficient condition for L_m to be convergent in the case $p=2$.

Since the above discussion remains true in the general case $p>2$, we obtain the following

THEOREM 1. *If*

$$\sqrt{1+k}+\sqrt{1+M^*(k)}>2p-1 \qquad (p\geqq 2), \tag{11}$$

then $\lim_{m\to\infty}L_m<\infty$. *In particular, the* 1*-dimensional measure of the singular set E of G is zero.*

Since

$$\sqrt{1+k}+\sqrt{1+M^*(k)}\geqq 1+\sqrt{1+k},$$

we have

THEOREM 2. *If*

$$k>(2p-2)^2-1, \tag{12}$$

the same conclusion as in Theorem 1 *holds.*

5. Let us compare our condition with the condition of Schottky.

If $p=2$, we obtain from (12) $k>3$. If we assume that B_0 is bounded by four circles with unit radius, the mutual distances between any two circles are greater than 0.449··· by (3). In such a domain the condition of Schottky is always satisfied. But in the case of four circles with unequal radii, there are many examples which satisfy our condition but do not satisfy the Schottky's.

Consider two pairs of circles C_1, C_1' and C_2, C_2' with radii 1 and 1/10, respectively. We take the mutual distance between C_1 and C_1' is slightly greater than 0.449···. We see that in general the mutual distance between C and C^* with radii 1 and $1/N$, respectively, is greater than $k/2(N+1)$ by (3). For $N=10$, it is greater than 0.136···. Draw two common tangents L_1 and L_2 between C_1 and C_1', and let the point of intersection be the origin and further draw C_2 and C_2' near enough the origin such that they intersect L_1 and L_2, and the distance of C_1 from C_1' is slightly greater than 0.136···. Obviously such a domain B_0 does not satisfy the condition of Schottky.

6. REMARK.

(i) In the case of $p=3$, even if we assume that B_0 is bounded by six circles with unit radius, there are many examples which satisfy our condition but do not satisfy the Schottky's.

(ii) Our theorem is not necessarily an extension of the Schottky theorem. Because it is easy to get the fundamental domains, which do not satisfy our conditions but do the Schottky's.

References

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