ON BOREL'S DIRECTIONS OF MEROMORPHIC FUNCTIONS OF FINITE ORDER, II.

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I. Main Theorems

In Part I,[1] I have proved the following two theorems.

Theorem 1. Let $w(z)$ be meromorphic in $|z|<1$ and the number of zero points of $\prod_{i=1}^{3}(w(z)-a_i)$ in $|z|\leqq 1$ be $\leqq n$. Then

$$S(r)\leqq n+\frac{A}{1-r} \quad (0\leqq r<1),$$

where A is a constant, which depends on a_1, a_2, a_3, only and

$$S(r)=\frac{1}{\pi}\iint_{|z|\leqq r}\left(\frac{|w'(z)|}{1+|w(z)|^2}\right)^2 t\,dt\,d\theta \quad (z=te^{i\theta}).$$

Theorem 2. Let $w(z)$ be meromorphic in an angular domain Δ_0:

$$|\arg z|\leqq\alpha_0, \quad 0\leqq|z|<\infty$$

and Δ:

$$|\arg z|\leqq\alpha<\alpha_0, \quad 0\leqq|z|<\infty.$$

Then for any $\lambda>1$,

$$T(r;\Delta)\leqq 3\sum_{i=1}^{3}N(\lambda r,a_i;\Delta_0)+A(\log r)^2 \quad (r\geqq 1),$$

where A is a constant, which depends on a_1, a_2, a_3, α_0, α, λ only and

$$T(r;\Delta)=\int_1^r\frac{S(t;\Delta)}{t}dt,$$

$$S(r;\Delta)=\frac{1}{\pi}\int_1^r\int_{-\alpha}^{\alpha}\left(\frac{|w'(te^{i\theta})|}{1+|w(te^{i\theta})|^2}\right)^2 t\,dt\,d\theta,$$

$$N(r,a;\Delta_0)=\int_1^r\frac{n(t,a;\Delta_0)}{t}dt,$$

$n(r,a;\Delta_0)$ being the number of zero points of $w(z)-a$ in $|\arg z|\leqq\alpha_0$, $0\leqq|z|\leqq r$.

We see easily that the constant k in the proof of Theorem 1 is

$$k\leqq B\left(\frac{1}{[a_1,a_2]}+\frac{1}{[a_2,a_3]}+\frac{1}{[a_3,a_1]}\right),$$

where B is a numerical constant and

$$[a,b]=\frac{|a-b|}{\sqrt{(1+|a|^2)(1+|b|^2)}},$$

so that A in Theorem 1 is

$$A\leqq C\left(\frac{1}{[a_1,a_2]^2}+\frac{1}{[a_2,a_3]^2}+\frac{1}{[a_3,a_1]^2}\right),$$

where C is a numerical constant. Hence Theorem 1 becomes

$$S(r)\leqq n+\frac{C}{1-r}\left(\frac{1}{[a_1,a_2]^2}+\frac{1}{[a_2,a_3]^2}+\frac{1}{[a_3,a_1]^2}\right).$$

By means of this, we can easily prove the existence of the "cercles de remplissage" of Valiron and Milloux. We will prove the following theorem.

Theorem 3. Let $w(z)$ be meromorphic in Δ_0: $|\arg z|\leqq\alpha_0$, $0\leqq|z|<1$ and Δ: $|\arg z|\leqq\alpha<\alpha_0$, $0\leqq|z|<1$. Then

$$T(r;\Delta)\leqq 63\sum_{i=1}^{3}N\left(\frac{r+3}{4},a_i;\Delta_0\right)+A\log\frac{1}{1-r}, \quad \left(\tfrac{1}{2}\leqq r<1\right),$$

where A is a constant, which depends on a_1, a_2, a_3, α_0, α only and

$$T(r;\Delta)=\int_{\frac{1}{2}}^{r}\frac{S(t;\Delta)}{t}dt,$$

$$S(r;\Delta)=\frac{1}{\pi}\int_{\frac{1}{2}}^{r}\int_{-\alpha}^{\alpha}\left(\frac{|w'(te^{i\theta})|}{1+|w(te^{i\theta})|^2}\right)^2 t\,dt\,d\theta,$$

$$N(r,a;\Delta_0)=\int_{\frac{1}{2}}^{r}\frac{n(t,a;\Delta_0)}{t}dt,$$

$n(r,a;\Delta_0)$ being the number of zero points of $w(z)-a$ in $|\arg z|\leqq\alpha_0$, $0\leqq|z|\leqq r<1$.

Proof. Let

$$r_\nu=1-\frac{1}{2^\nu}, \quad r_{\nu+1}=\frac{1+r_\nu}{2}, \quad r_\nu-r_{\nu-1}=\frac{1}{2^\nu}, \quad (\nu=1,2,\cdots) \tag{1}$$

$$\left.\begin{aligned}
&Q_\nu:\ r_{\nu-1}\leqq|z|\leqq r_\nu,\ |\arg z|\leqq\alpha;\\
&Q_\nu^{(k)}:\ r_{\nu-1}\leqq|z|\leqq r_\nu,\ \frac{k-1}{2^\nu}\leqq\arg z\leqq\frac{k}{2^\nu};\\
&Q_\nu^{0}:\ r_{\nu-2}\leqq|z|\leqq r_{\nu+1},\ |\arg z|\leqq\alpha_0;\\
&Q_\nu^{0(k)}:\ r_{\nu-2}\leqq|z|\leqq r_{\nu+1},\\
&\qquad\frac{k-2}{2^\nu}\leqq\arg z\leqq\frac{k+1}{2^\nu};\\
&(\nu\geqq 2,\ k=0,\pm1,\pm2,\cdots),
\end{aligned}\right\} \tag{2}$$

where k varies in the range, such that $Q_\nu^{0(k)}$ is contained in Δ_0. If we take ν_0 so large that $2/2^\nu$

$<\alpha_0-\alpha$, then $Q_\nu^{o(k)}$ $(\nu \geqq \nu_0)$ is contained in Δ_0, if $Q_\nu^{(k)}$ has a common point with Δ.

$$\left.\begin{aligned} S_\nu &= \frac{1}{\pi}\iint_{Q_\nu}\left(\frac{|w'(z)|}{1+|w(z)|^2}\right)^2 r\,dr\,d\theta, \\ S_\nu^{(k)} &= \frac{1}{\pi}\iint_{Q_\nu^{(k)}}\left(\frac{|w'(z)|}{1+|w(z)|^2}\right)^2 r\,dr\,d\theta \\ &\qquad (z=re^{i\theta}) \end{aligned}\right\} \quad (3)$$

and n_ν^o, $n_\nu^{o(k)}$ be the number of zero points of $\prod_{i=1}^3 (w(z)-a_i)$ in Q_ν^o, $Q_\nu^{o(k)}$ respectively.

In the following, we denote constants, which depend on $a_1, a_2, a_3, \alpha_0, \alpha$, only by the same letter A. Since $Q_\nu^{o(k)}$ $(\nu \geqq \nu_0)$ is contained in Δ_0 with $Q_\nu^{(k)}$, we have by Theorem 1.

$$S_\nu^{(k)} \leqq n_\nu^{o(k)} + A, \quad (\nu \geqq \nu_0). \quad (4)$$

Since $Q_\nu^{o(k)}$ overlap twice and the number of $Q_\nu^{(k)}$, which have common points with Δ is $\leqq A\cdot 2^\nu$, we have for $\nu \geqq \nu_0$,

$$S_\nu = \sum_k S_\nu^{(k)} \leqq \sum_k n_\nu^{o(k)} + A\cdot 2^\nu \leqq 3\cdot n_\nu^o + A\cdot 2^\nu, \quad (5)$$

For $1 \leqq \nu \leqq \nu_0$, we apply Theorem 1 to Q_ν, Q_ν^o, then we have $S_\nu \leqq n_\nu^o + A$, so that (5) holds for $\nu = 1, 2, \cdots$ with a suitable A. Let $\frac{1}{2} \leqq r < 1$. We choose N, such that $r_{N-1} < r \leqq r_N$, then

$$2^N \leqq \frac{2}{1-r}. \quad (6)$$

From (5), we have

$$S(r;\Delta) \leqq S(r_N;\Delta) = S_1 + S_2 + \cdots + S_N \leqq$$

$$\leqq 3(n_1^o + n_2^o + \cdots + n_N^o) + A\cdot 2^N.$$

We put

$$n(r;\Delta_0) = \sum_{i=1}^3 n(r, a_i; \Delta_0), \quad (7)$$

then since Q_ν^o overlap twice, $n_1^o + n_2^o + \cdots + n_N^o \leqq 3n(r_{N+1};\Delta_0)$, so that by (6),

$$\begin{aligned} S(r;\Delta) &\leqq 9n(r_{N+1};\Delta_0) + A\cdot 2^N \\ &\leqq 9n(r_{N+1};\Delta_0) + \frac{A}{1-r}. \end{aligned}$$

Since

$$r_{N+1} = \frac{1+r_N}{2}, \quad r_N = \frac{1+r_{N-1}}{2} \leqq \frac{1+r}{2},$$

we have $r_{N+1} \leqq \frac{r+3}{4}$, so that

$$S(r;\Delta) \leqq 9n\left(\frac{r+3}{4};\Delta_0\right) + \frac{A}{1-r},$$

$$T(r;\Delta) = \int_{\frac{1}{2}}^r \frac{S(r;\Delta)}{r}\,dr$$

$$\leqq 9\int_{\frac{1}{2}}^r \frac{n\left(\frac{r+3}{4};\Delta_0\right)}{r}\,dr + A\cdot\log\frac{1}{1-r}$$

$$= 36\int_{\frac{7}{8}}^{\frac{r+3}{4}} \frac{n(t;\Delta_0)}{4t-3}\,dt + A\cdot\log\frac{1}{1-r}.$$

Since $4t-3 \geqq \frac{4}{7}t$ for $t \geqq \frac{7}{8}$, we have

$$T(r;\Delta) \leqq 63\int_{\frac{1}{2}}^{\frac{r+3}{4}} \frac{n(t,\Delta_0)}{t}\,dt + A\cdot\log\frac{1}{1-r}$$

$$= 63\sum_{i=1}^3 N\left(\frac{r+3}{4}, a_i;\Delta_0\right) + A\cdot\log\frac{1}{1-r},$$

q.e.d.

II. Meromorphic functions in a unit circle.

By means of Theorem 3, we can prove simply the following Valiron's theorems[2] (4-6). In the following $T(r)$ denotes the Nevanlinna's characteristic function.

<u>Theorem 4</u>. Let $w(z)$ be meromorphic in $|z|<1$ and

$$\int_0^1 T(r)(1-r)^{\lambda-1}\,dr = \infty \quad (\lambda > 0),$$

Then there exists a direction J, such that

$$\sum_\nu (1-|z_\nu(a;\Delta)|)^{\lambda+1} = \infty$$

for any a, with two possible exceptions, where $z_\nu(a;\Delta)$ are zero points of $w(z)-a$ in any angular domain Δ, which contains J.

<u>Proof</u>. By dividing $(0, 2\pi)$ into 2^n equal parts, we obtain angular domains Δ_n of magnitude $2\pi/2^n$, such that $\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_n \supset \cdots$

$$\int_0^1 T(r;\Delta_n)(1-r)^{\lambda-1}\,dr = \infty$$

Let Δ_n converge to an direction J: $\arg z = \alpha$, then for any angular domain Δ, which contains J,

$$\int_0^1 T(r;\Delta)(1-r)^{\lambda-1}\,dr = \infty.$$

Let for any $\delta > 0$,

$$\Delta: \ |\arg z - \alpha| \leqq \delta, \ 0 \leqq |z| < 1;$$

$$\Delta_0: \ |\arg z - \alpha| \leqq 2\delta, \ 0 \leqq |z| < 1.$$

Then by Theorem 3,

$$\int_0^r T(r;\Delta)(1-r)^{\lambda-1}\,dr$$

$$\leqq 63\sum_{i=1}^3 N\left(\frac{r+3}{4}, a_i;\Delta_0\right)(1-r)^{\lambda-1}\,dr + O(1)$$

so that by (1),

$$\int^1 N(r, a; \Delta_0)(1-r)^{\lambda-1} dr = \infty,$$

or

$$\sum_\nu (1 - |z_\nu(a; \Delta_0)|)^{\lambda+1} = \infty .$$

for any a, with two possible exceptions.

Theorem 5. Let $w(z)$ be meromorphic in $|z|<1$, such that

$$\overline{\lim_{r\to1}} \frac{T(r)}{\log(1/(1-r))} = \infty .$$

Then there exists a direction J, such that in any angular domain Δ, which contains J, $w(z)$ takes any value infinitely often with two possible exceptions. More precisely,

$$\overline{\lim_{r\to1}} \frac{N(r, a; \Delta_0)}{\log(1/(1-r))} = \infty,$$

with two possible exceptions.

Proof. As before there exists a direction J; $\arg z = \alpha$, such that for any angular domain Δ, which contains J,

$$\overline{\lim_{r\to1}} \frac{T(r, \Delta)}{\log(1/(1-r))} = \infty \qquad (1)$$

Let for any $\delta > 0$,

$$\Delta : |\arg z - \alpha| \leqq \delta, \quad 0 \leqq |z| < 1 \ ;$$

$$\Delta_0 : |\arg z - \alpha| \leqq 2\delta, \quad 0 \leqq |z| < 1 \ .$$

Then from (1) and Theorem 3, we see that

$$\overline{\lim_{r\to1}} \frac{N(r, a; \Delta_0)}{\log(1/(1-r))} = \infty ,$$

for any a, with two possible exceptions.

Theorem 6. Let $w(z)$ be meromorphic in $|z|<1$, such that

$$\lim_{r\to1} T(r) = \infty .$$

Then there exists a direction J, such that in any angular domain Δ, which contains J, $w(z)$ takes any value infinitely often, except a set of logarithmic capacity zero.

Proof. As before, there exists a direction J : $\arg z = \alpha$, such that for any angular domain Δ, which contains J,

$$\lim_{r\to1} T(r; \Delta) = \infty . \qquad (1)$$

We suppose that $\alpha = 0$ and let for any $\delta > 0$,

$$\left.\begin{aligned} &\Delta : |\arg z| \leqq \delta, \quad 0 \leqq |z| < 1 , \\ &\Delta_0 : |\arg z| \leqq 2\delta, \quad 0 \leqq |z| < 1 . \end{aligned}\right\} (2)$$

We map Δ_0 on D_0 : $|\zeta|<1, |\arg \zeta| \leqq \frac{\pi}{2}$ by $\zeta = z^{\pi/4\delta}$ and put

$$|\zeta| = R, \quad |z| = r , \quad R = r^{\pi/4\delta}$$

Then Δ is mapped on $|\arg \zeta| \leqq \frac{\pi}{4}, |\zeta|<1$. We map D_0 on $|x|<1$ by

$$x = -\frac{\zeta^2 - 1 + 2\zeta}{\zeta^2 - 1 - 2\zeta} , \quad \text{or} \quad \frac{x-i}{x+i} = i\left(\frac{\zeta - i}{\zeta + i}\right)^2 ,$$

such that $\zeta = i, 0, -i$ correspond to $x = i, -1, -i$. We put $w(z) = v(x)$. Let for $0 \leqq R < 1$,

$$A(R) = \operatorname*{Max}_{|\zeta|\leqq R, |\arg\zeta|\leqq\frac{\pi}{4}} \left| \frac{\zeta^2 - 1 + 2\zeta}{\zeta^2 - 1 - 2\zeta} \right| ,$$

then the image of $R_1 \leqq |\zeta| \leqq R$, $|\arg \zeta| \leqq \frac{\pi}{4}$ is contained in $|x| = \rho = A(R)$, when R_1 is suitably chosen. Hence if we put

$$S(\rho, v) = \frac{1}{\pi} \iint_{|x|\leqq\rho} \left(\frac{|v'(x)|}{1+|v(x)|^2}\right)^2 t\,dt\,d\theta$$

$$(x = te^{i\theta}) ,$$

then $S(r, w; \Delta) \leqq S(\rho, v) + O(1)$

Since there exists a constant A such that $A d\rho \geqq dr$, we have

$$T(r, w; \Delta) = \int_0^r \frac{S(r, w; \Delta)}{r} dr$$

$$\leqq A_1 \int_0^\rho \frac{S(\rho, v)}{\rho} d\rho + O(1) = A_1 T(\rho, v) + O(1)$$

$$(A_1 = \text{const.}),$$

where $T(\rho, v)$ is the characteistic function of $v(x)$. From (1), we have $\lim_{\rho\to1} T(\rho, v) = \infty$, so that by Frostman's theorem[3], $v(x)$ takes any value infinitely often in $|x|<1$, except a set of logarithmic capacity zero. Returning to the z-plane, $w(z)$ takes any value infinitely often in Δ_0, except a set of logarithmic capacity zero.

III. Ahlfors' directions

1. Let K be a sphere of diameter 1 and F be a finite covering surface of K. We denote its area by $|F|$ and the length of its boundary by L and put $S = |F|/\pi$. Let D be a simply connected domain on K and $F(D)$ be the part of F, which lies above D and put

$$S(D) = \frac{|F(D)|}{|D|} ,$$

where $|D|$ denotes the area of D and $|F(D)|$ is that of $F(D)$. Then by Ahlfors' covering theorem[4],

$$|S(D) - S| \leqq kL, \qquad (1)$$

where k is a constant, which depends on D only. $F(D)$ consists of a finite number of connected parts, which are called islands (Inseln) and peninsulas (Zungen). Let $\{D^i\}$ be islands in $F(D)$ and $\rho(D^i)$ be its Euler's characteristic. We put

$$p(D^i) = -\rho(D^i), \quad p(D) = \sum_i p(D^i).$$

Let $D_1, \ldots, D_q$ $(q \geqq 3)$ be q simply connected domains on K, which have no common points each other. Then Ahlfors[5] proved that

$$\sum_{i=1}^{q} p(D_i) \geqq (q-2)S - kL, \qquad (2)$$

where k is a constant, which depends on $D_1, \ldots, D_q$ only. Since $p(D^i) = 1$, if D^i is simply connected and $p(D^i) \leqq 0$ otherwise, if we denote the number of simply connected islands in $F(D)$ by $n_1(D_i)$, then by (2),

$$\sum_{i=1}^{q} n_1(D_i) \geqq (q-2)S - kL. \qquad (3)$$

Let $n_0(D)$ be the number of schlicht islands in $F(D)$ and $n_1'(D)$ be that of non-schlicht simply connected islands in $F(D)$, then $n_1(D) = n_0(D) + n_1'(D)$. Since $n_1'(D) \leqq \frac{1}{2}S(D)$, we have by (1),

$$n_1(D) = n_0(D) + n_1'(D) \leqq n_0(D) + \frac{1}{2}S(D)$$
$$\leqq n_0(D) + \frac{1}{2}S + kL,$$

so that from (3),

$$\sum_{i=1}^{q} n_0(D_i) \geqq \frac{(q-4)}{2}S - kL. \qquad (4)$$

If we take $q = 3$ in (3) and $q = 5$ in (4), then we have

$$\sum_{i=1}^{3} n_1(D_i) \geqq S - kL, \qquad (5)$$

$$2\sum_{i=1}^{5} n_0(D_i) \geqq S - kL. \qquad (6)$$

From (5), (6), we can prove similarly as Theorem 1, the following theorem.

Theorem 7. Let $w(z)$ be meromorphic in $|z|<1$ and F be its Riemann surface spread over the w-sphere K.

(i) Let D_1, D_2, D_3 be three simply connected domains on K. If the total number of simply connected islands of F above D_1, D_2, D_3 be $\leqq n_1$; then

$$S(r) \leqq n_1 + \frac{A}{1-r}, \quad (0 \leqq r < 1),$$

where A is a constant, which depends on D_1, D_2, D_3 only.

(ii) Let $D_1, \ldots, D_5$ be five simply connected domains on K. If the total number of schlicht islands of F above $D_1, \ldots, D_5$ be $\leqq n_0$, then

$$S(r) \leqq 2n_0 + \frac{A}{1-r}, \quad (0 \leqq r < 1),$$

where A is a constant, which depends on $D_1, \ldots, D_5$ only.

2. By this theorem, we can prove an anlogous theorem as Theorem 2 and by means of which we can prove the following theorem.

Theorem 8. Let $w = f(z)$ be a meromorphic function of finite order $\rho > 0$, such that[6]

$$\int^{\infty} \frac{T(r)}{r^{k+1}} dr = \infty, \quad (k > 0).$$

Then there exists a direction $J: \arg z = \alpha$, which has the following property. Let $\Delta: |\arg z - \alpha| \leqq \delta$, $0 \leqq |z| < \infty$ be any-angular domain, which contains J and $\Delta_r: |\arg z - \alpha| \leqq \delta, 0 \leqq |z| \leqq r$ and $F_r(\Delta)$ be the Riemann surface generated by $w = f(z)$ on the w-sphere K, when z varies in Δ_r.

Let D be a simply connected domain on K and $n_1(r, D; \Delta)$ be the number of simply connected islands in $F_r(\Delta)$ above D and $n_0(r, D; \Delta)$ be that of schlicht islands in $F_r(\Delta)$. Then for any $\delta > 0$,

$$\text{(i)} \int^{\infty} \frac{n_1(r, D; \Delta)}{r^{k+1}} dr = 0$$

holds for a certain one of any three simply connected domains on K and

$$\text{(ii)} \int^{\infty} \frac{n_0(r, D; \Delta)}{r^{k+1}} dr = \infty$$

holds for a certain one of any five simply connected domains on K.

We may call J an Ahlfors' direction of $f(z)$. We remark that, as Dinghas[7] proved,

$$\int^{\infty} \frac{n(r, D)}{r^{\rho+\varepsilon+1}} dr < \infty$$

for any simply connected domain D and $\varepsilon > 0$, where $n(r, D)$ is the number of (not necessarily simply connected) islands in F_r above D, F_r being the Riemann surface generated by $w = f(z)$ on K, when z varies in $|z| \leqq r$.

(i) is evidently a generalization of Valiron's theorem on Borel's directions.

We can prove analogous theorems which correspond to Theorem 5 and 6 for meromorphic functions in a unit circle.

Similarly as Theorem 4 of Part I, we can extend Theorem 8 as follows.

Theorem 9. Let $w=f(z)$ be a meromorphic function of finite order $\rho>0$, such that

$$\int^{\infty}\frac{T(r)}{r^{k+1}}dr=\infty \qquad (k>0),$$

C; $z=z(t)$ $(0\leqq t<\infty)$ be a simple curve, which connects $z=0$ with $z=\infty$ and for any $\delta>0$, Δ be the set of points, which is covered by all discs : $|z-z(t)|\leqq\delta|z(t)|$ $(0\leqq t<\infty)$, $\Delta(\theta)$ be the set obtained from Δ by rotating an angle θ, $\Delta_r(\theta)$ be its part, which is contained in $|z|\leqq r$ and $F_r(\Delta(\theta))$ be the Riemann surface generated by $w=f(z)$ on the w-sphere K, when z varies in $\Delta_r(\theta)$.

Let D be a simply connected domain on K and $n_1(r,D;\Delta(\theta))$ be the number of simply connected islands in $F_r(\Delta(\theta))$ above D and $n_0(r,D;\Delta(\theta))$ be that of schlicht islands in $F_r(\Delta(\theta))$ above D. Then there exists a certain θ_0, such that for any $\delta>0$,

$$\text{(i)}\quad \int^{\infty}\frac{n_1(r,D;\Delta(\theta_0))}{r^{k+1}}dr=\infty$$

holds for a certain one of any three simply connected domains on K and

$$\text{(ii)}\quad \int^{\infty}\frac{n_0(r,D;\Delta(\theta_0))}{r^{k+1}}dr=\infty$$

holds for a certain one of any five simply connected domains on K.

3. We will prove a theorem, which is analogous to the theorem on the "cercles de remplisage". Let $w=f(z)$ be a meromorphic function of finite order $\rho>0$. Then we can prove easily that for any $\varepsilon>0$, $\delta>0$, there exists a sequence of discs $\Delta_n: |z-z_n|\leqq\delta|z_n|$ $(n=1,2,\cdots)$, such that

$$S_n\geqq r_n^{\rho-\varepsilon}, \quad (|z_n|=r_n),$$

where

$$S_n=\frac{1}{\pi}\iint_{\Delta_n}\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 r\,dr\,d\theta, \quad (z=re^{i\theta}).$$

If we take $\varepsilon_1>\varepsilon_2>\cdots>\varepsilon_n\searrow 0$ for ε and $\delta_1>\delta_2>\cdots>\delta_n\searrow 0$ for δ, then we see that there exists a sequence of points z_n, which is independent of ε and δ, such that (1) holds for any $\varepsilon>0$, $\delta>0$.

Let $\Delta_n^0: |z-z_n|\leqq 2\delta|z_n|$ and we apply Theorem 7 on Δ_n and Δ_n^0 then we have the following theorem.

Theorem 10. Let $w=f(z)$ be a meromorphic function of finite order $\rho>0$. Then there exists a sequence of points z_n, which is independent of ε and δ and satisfies the following condition.

Let for any $\delta>0$, $\Delta_n: |z-z_n|\leqq\delta|z_n|$ $(|z_n|=r_n)$ and F_n be the Riemann surface generated by $w=f(z)$ on the w-sphere K, when z varies in Δ_n. Let D be a simply connected domain on K and $n_1(D,\Delta_n)$ be the number of simply connected islands in F_n above D and $n_0(D,\Delta_n)$ be that of schlicht islands in F_n above D. Then for any $\varepsilon>0$, $\delta>0$,

$$\text{(i)}\quad \sum_{i=1}^{3} n_1(D_i;\Delta_n)\geqq r_n^{\rho-\varepsilon}$$

for any three simply connected domains D_1, D_2, D_3 on K, if $n\geqq n(D_1,D_2,D_3,\varepsilon,\delta)$,

$$\text{(ii)}\quad \sum_{i=1}^{5} n_0(D_i;\Delta_n)\geqq r_n^{\rho-\varepsilon}$$

for any five simply connected domains $D_1,\cdots,D_5$ on K, if $n\geqq n(D_1,\cdots,D_5,\varepsilon,\delta)$.

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