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SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH TURNING POINTS AND SINGULARITIES I

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§ 1. Introduction.

Differential equations containing a positive small parameter ε

$$\varepsilon^2 \frac{d^2y}{dx^2} - p(x)y = 0 \tag{1.1}$$

are considered. The independent variable x is complex. The coefficient $p(x)$ is a rational function of the form $p(x)=r(x)/q(x)$, where $r(x)$ and $q(x)$ have no common factors. Zeros of $r(x)$ are called turning points and zeros of $q(x)$ (and possibily the point at infinity) are singular points of the differential equation (1.1).

When the parameter ε tends to zero, asymptotic solutions of (1.1) are valid only in some domain in the x-plane. The principal parts of the formal solutions are of the form

$$y(x, \varepsilon) \sim p(x)^{-1/4} \exp\left[\pm\frac{1}{\varepsilon}\int p(x)^{1/2}dx\right] \qquad (\varepsilon\to 0), \tag{1.2}$$

which are called WKB approximations. In this paper we consider several cases of $p(x)$ and show how to construct unbounded domains called canonical regions. The validity of WKB approximations in the canonical region can be established as in Evgrafov-Fedoryuk [1] or Nakano [3].

In § 3 and § 5 the case $r(x)=(x-1)^2$, $q(x)=x$ is treated and in § 4 and § 6 the case $r(x)=-(x-1)^2$, $q(x)=x$ is treated. These two cases are very similar but some difference appears between them. The two cases have similar property in the small but different property in the large. In § 2 some common property between them is treated. In § 7 canonical paths are treated. In these cases $x=1$ is a turning point of order two, the origin is a regular singular point and the point at infinity is an irregular singular point of (1.1). In § 8 the case $r(x)=(x-1)^2$, $q(x)=x^3$ is treated, which is the simplest case of (1.1) having a turning point at $x=1$ and two irregular singular points at $x=0$ and point at infinity. In the part 2 we shall consider cases containing a logarithmic term in (2.1) below and a matching method.

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§ 2. Level curves defined by the integral.

The regions or domains in which the WKB approximations of (1.1) are valid are determined by the integral

$$\xi(1, x)=\int_1^x p(x)^{1/2}dx, \qquad p(x)=\frac{(x-1)^2}{x}, \tag{2.1}$$

where $x=1$ is the turning point of (1.1). Since we must consider two WKB approximations on two sheets of complex planes it is sufficient for our purpose to choose one of two square roots of the integrand of the above integral. By choosing one branch in (2.1) such that it takes positive values for $x>1$, we get

$$\xi(1, x)=\frac{2}{3}x^{3/2}-2x^{1/2}+\frac{4}{3},$$

where the point x moves in the one sheet of complex planes with a branch cut along the negative real axis. For the local property we have

$$\xi(1, x)\sim\begin{cases} \dfrac{2}{3}x^{3/2} & (\text{near } x=\infty) \\ \dfrac{4}{3} & (\text{near } x=0)\,. \end{cases}$$

If we put

$$\nu x=+\iota\mu, \qquad \nu=r\cos\theta, \qquad \mu=r\sin\theta$$

$$a=\sqrt{\frac{\sqrt{\nu^2+\mu^2}+\nu}{2}}, \qquad b=\sqrt{\frac{\sqrt{\nu^2+\mu^2}-\nu}{2}},$$

the integral (2.1) is expressed by its real and imaginary parts:

$$\xi(1, x)=\left[\frac{2}{3}(a^3-3ab^2)-2a+\frac{4}{3}\right]+i\left[\frac{2}{3}(3a^2b-b^3)-2b\right]. \tag{2.2}$$

Stokes curves for (1.1) are obtained by the equations $\operatorname{Re}\xi(1, x)=\text{const.}$ or $\operatorname{Im}\xi(1, x)=\text{const.}$ First we consider curves defined by the real part of $\xi(1, x)$. The following equation defines Stokes curves $l_1, \cdots l_4$

$$\operatorname{Re}\xi(1, x)=0, \text{ i.e., } a^3-3ab^2-3a+2=0 \text{ or } a^2-3b^2-3=-\frac{2}{a},$$

and it becomes in the polar form

$$r-2r\cos\theta+3=\frac{2\sqrt{2}}{\sqrt{r}}\frac{1}{\sqrt{1+\cos\theta}}.$$

We consider curves defined by the equation

$$\operatorname{Re}\xi(1,x)=\frac{4}{3},\qquad \text{i.e., } a^3-3ab^2-3a=0.$$

This equation represents curves (l_5, l_6) defined by the relations

$$\mu=0,\qquad \nu\leqq 0$$

and

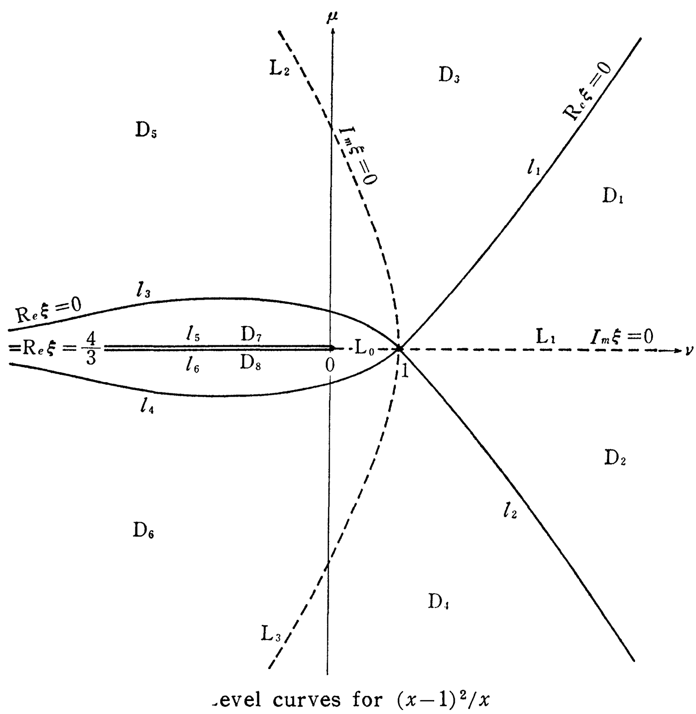
$$3\nu^2-\mu^2-12\nu+9=0,\qquad \nu\geqq\frac{3}{2}.$$

For the imaginary part, we consider the equation

$$\operatorname{Im}\xi(1,x)=0,$$

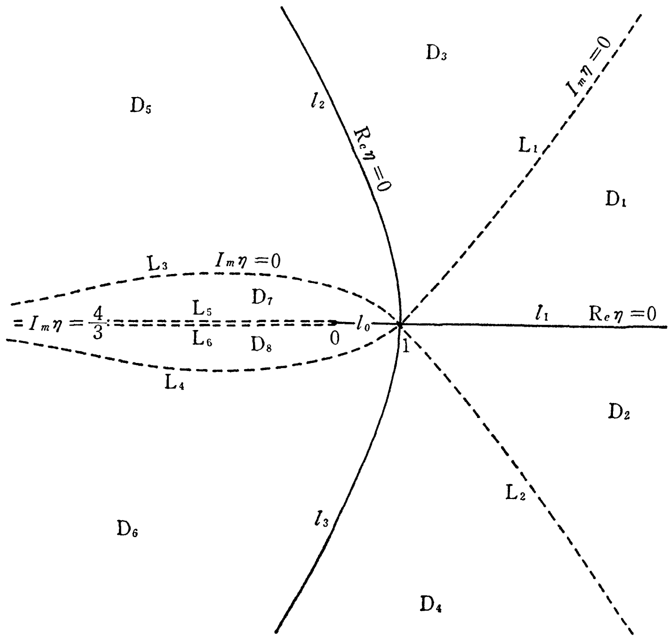
for which we get two equations, the first of which represents

$$L_0 \text{ and } L_1:$$



Level curves for $(x-1)^2/x$

Fig. 1.



Level curves for $-(x-1)^2/x$

Fig. 2.

$$b=0, 3a^2-b^2-3=0\,.$$

From the second equation we get the relation, representing curves L_2, L_3

$$3\nu^2-\mu^2-12\nu+9=0\,, \qquad \nu\leqq\frac{3}{2}\,.$$

Other level curves defined by the equations

$$\operatorname{Re}\xi(1,x)=\text{const.} \quad \text{and} \quad \operatorname{Im}\xi(1,x)=\text{const.}$$

are obtained from the curves obtained above.

Thus we could get curves which play the important role to determine the canonical regions for (1.1). These level curves are represented in Fig. 1.

§ 3. Canonical regions for the case $p(x)=(x-1)^2/x$.

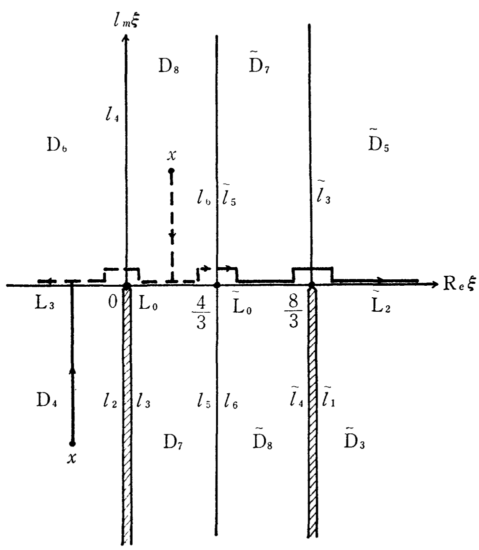
In the preceding section we considered the level curves defined by the integral (2.1). In this section we analyze what region surrounded by particular level curves is mapped one-to-one onto the region in the ξ-plane with perpendicular coordinates $(\operatorname{Re}\xi, \operatorname{Im}\xi)$ if the integral (2.1) is considered as the mapping defined on the x-plane into the ξ-plane. In the figure 1 D_i denotes regions bounded by curves L_j, L_k.

Since $\mathrm{Re}\,\xi=0$ and $\mathrm{Im}\,\xi$ takes positive values on the line l_1 proceeding from the point $x=1, l_1$ is mapped onto the positive imaginary axis of the ξ-plane. Since on the lines l_3 and l_2 $\mathrm{Re}\,\xi=0$ and $\mathrm{Im}\,\xi$ takes negative values they are mapped onto the negative imaginary axis. We assume that the line l_3 is mapped onto the left side of the nagative imaginary axis,for in the domain D_5 $\mathrm{Re}\,\xi$ takes negative values. Similarly we assume that the line l_2 is mapped onto the right side of the negative imaginary axis.

Since $\mathrm{Im}\,\xi=0$ and $\mathrm{Re}\,\xi$ takes positive values on the line L_1, the line L_1 is mapped onto the positive real axis. By similar consideration the line L_2 is mapped onto the negative real axis. Thus the domain $D_5\cup L_2\cup D_3\cup l_1\cup D_1\cup L_1 \cup D_2$ is mapped one-to-one onto the whole ξ-plane except for the negative imaginary axis. The mapping ξ is conformal except for the turning point because $d\xi/dx\neq 0$ for $x\neq 1$.

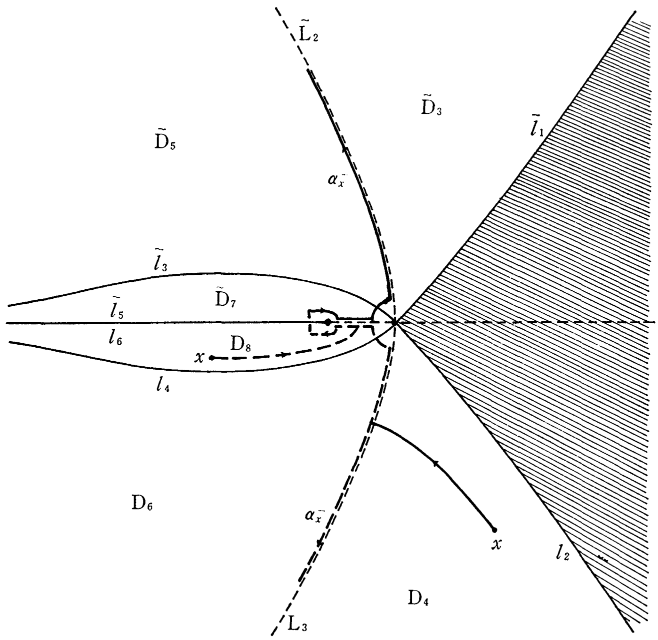
By analyzing in the similar way the domain $D_6\cup L_3\cup D_4\cup l_2\cup D_2\cup L_1\cup D_1$ is mapped one-to-one conformaly onto the whole ξ-plane except for the positive imaginary axis.

In the domain D_7 $\mathrm{Im}\,\xi$ takes negative values and $\mathrm{Re}\,\xi$ changes from 0 to 4/3, and in the domain D_8 $\mathrm{Im}\,\xi$ takes positive values and $\mathrm{Re}\,\xi$ changes from 0 to 4/3. The lines l_5 and l_6 are the very same lines, i.e., they are the negative real axis. But we assume that the line l_5 is the upper side of the negative real axis and line l_6 is the lower side of it. The line l_4



The image of $\mathscr{D}_5$

Fig. 3.



The canonical region $\mathscr{D}_5$ and paths of integration

Fig. 4.

onto the positive imaginary axis, for on it $\mathrm{Re}\,\xi$ takes value zero only and $\mathrm{Im}\,\xi$ is positive. On the line L_0 $\mathrm{Im}\,\xi$ vanishes and $\mathrm{Re}\,\xi$ changes from 0 to 4/3. Thus the domain $D_7 \cup L_0 \cup D_8$ is mapped onto the strip $6 \leqq \mathrm{Re}\,\xi \leqq \frac{4}{3}$ shown in Fig. 3. The domain $D_3 \cup L_2 \cup D_5 \cup l_3 \cup D_7 \cup L_0 \cup D_8$ is mapped onto the half plane $\mathrm{Re}\,\xi \leqq \frac{4}{3}$ with a cut on the positive imaginary axis.

By analyzing in the similar way the domain $D_4 \cup L_3 \cup D_6 \cup l_4 \cup D_8 \cup L_0 \cup D_7$ is mapped onto the half plane with a cut on the negative imaginary axis (Fig. 3). According to Evgrafov-Fedoryuk [1] the domain which is mapped one-to-one onto the whole ξ-plane with cuts is called a canonical region for the differential equation (1.1). Therefore we could get two canonical regions(*)

$$\mathscr{D}_1 = D_5 \cup L_2 \cup D_3 \cup l_1 \cup D_1 \cup L_1 \cup D_2\,, \tag{3.1}$$
$$\mathscr{D}_2 = D_6 \cup L_3 \cup D_4 \cup l_2 \cup D_2 \cup L_1 \cup D_1$$

and two of half plane type which we call *semi*-canonical regions

(*) Precisely a canonical region is a subregion of $\mathscr{D}$, i.e., the canonical region does not contain neighborhood of boundaries of $\mathscr{D}$. But we can use a same symbol for the canonical region without confusion.

$$D_3 \cup L_2 \cup D_5 \cup l_3 \cup D_7 \cup L_0 \cup D_8\,, \tag{3.2}$$
$$D_4 \cup L_3 \cup D_6 \cup l_4 \cup D_8 \cup L_0 \cup D_7\,.$$

§ 4. Canonical regions for the case $p(x) = -(x-1)^2/x$.

In this case we consider the integral

$$\eta(1, x) = \int_1^x \left[-\frac{(x-1)^2}{x} \right]^{1/2} dx\,, \tag{4.1}$$

where one branch is chosen such that its imaginary part takes positive values for $x > 1$ and x moves on one sheet of complex planes with a branch cut along the negative real axis. Since the equation

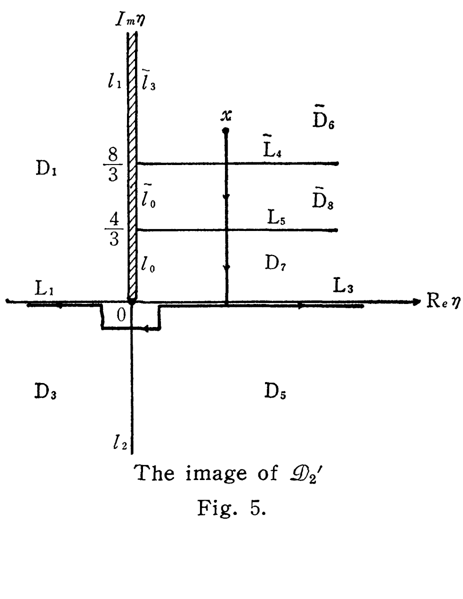
$$\eta(1, x) = -\mathrm{Im}\,\xi(1, x) + i\,\mathrm{Re}\,\xi\,(1, x)$$

holds the real and imaginary parts of the integral are respectively

$$\mathrm{Re}\,\eta(1, x) = -\frac{2}{3}(3a^2b - b^3 - 3b)\,, \qquad \mathrm{Im}\,\eta(1, x) = \frac{2}{3}(a^3 - 3ab^2 - 3a + 2)\,. \tag{4.2}$$

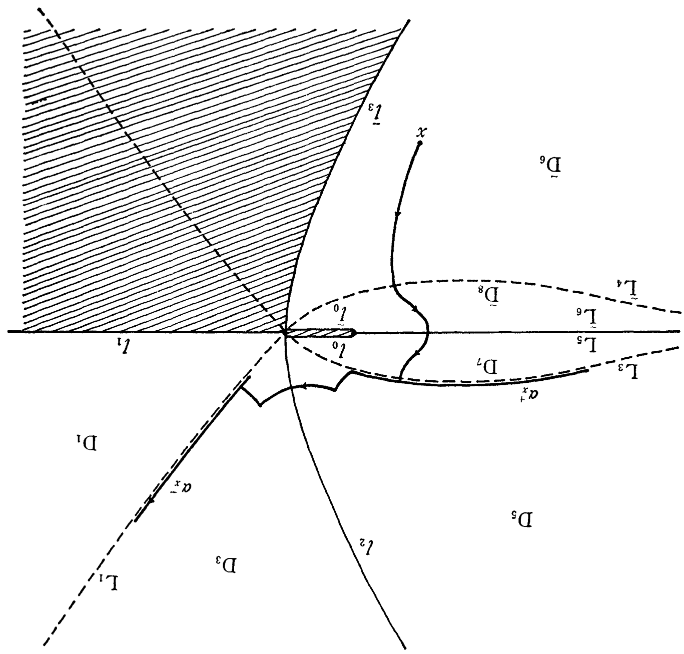
The level curves defined by $\mathrm{Re}\,\eta(1, x) = 0$ and $\mathrm{Im}\,\eta(1, x) = 0$ are the same one defined by $\mathrm{Im}\,\xi\,(1, x) = 0$ and $\mathrm{Re}\,\xi(1, x) = 0$ of the first case, respectively.

By the similar consideration to the former case we can get the figures of level curves (Fig. 2). On the line $L_1(L_2)$ $\mathrm{Re}\,\eta(1, x)$ takes negative (positive) values and $\mathrm{Im}\,\eta(1, x) = 0$, then the line $L_1(L_2)$ is mapped one-to-one onto the negative (positive) real axis of the η-plane. Since on the line $l_1(l_2$ and $l_3)$ $\mathrm{Re}\,\eta$ $(1, x) = 0$ and $\mathrm{Im}\,\eta(1, x)$ takes positive (negative) values, the image of $l_1(l_2$ and



The image of $\mathscr{D}_2'$

Fig. 5.



The canonical region $\mathcal{D}_2'$ and paths of integration

Fig. 6.

l_3) is a positive (negative) imaginary axis. The image of l_2 is assumed the left side of the negative imaginary axis and the image of l_3 is assumed the right side of it. Therefore the domain $D_3 \cup L_1 \cup D_1 \cup l_1 \cup D_2 \cup L_2 \cup D_4$ is mapped conformaly one-to-one onto the whole η-plane with a cut on the negative imaginary axis. Since $\operatorname{Im} \eta$ vanishes and $\operatorname{Re} \eta$ takes positive values on the line L_3, L_3 is mapped onto the positive real axis. Since on the segment l_0 $\operatorname{Re} \eta$ vanishes and $\operatorname{Im} \eta$ changes from 0 to 4/3, the segment l_0 is mapped onto the segment $(0, 4/3)$ of the positive imaginary axis as shown in Fig. 5.

Two lines L_5 and L_6 are the very same, but as the former case we assume that L_5 is the upper side and L_6 is the lower side of the negative real axis. On the line L_5 and L_6 $\operatorname{Im} \eta = 4/3$, and in $D_7(D_8)$ $\operatorname{Re} \eta$ takes positive (negative) values. Therefore the domain $D_1 \cup L_1 \cup D_3 \cup l_2 \cup D_5 \cup L_3 \cup D_7$ surrounded by l_1, l_0 and L_5 is mapped conformaly one-to-one onto the domain of the η-plane as shown in Fig. 5.

In the similar way we can show that the domain $D_8 \cup L_4 \cup D_6 \cup l_3 \cup D_4 \cup L_2 \cup D_2$ bounded by l_1, l_0 and L_6 is mapped as Fig. 5 but inverted. The domain $D_6 \cup L_4 \cup D_8 \cup l_0 \cup D_7 \cup L_3 \cup D_5$ is mapped onto a half plane $\operatorname{Im} \eta < 4/3$ with a cut $\operatorname{Re} \eta = 0$, $\operatorname{Im} \eta < 0$.

Thus we could get four canonical regions

$$\mathfrak{D}_1' = D_3 \cup L_1 \cup D_1 \cup l_1 \cup D_2 \cup L_2 \cup \Gamma_4, \tag{4.3}$$

$$\begin{cases} D_1 \cup L_1 \cup D_3 \cup l_2 \cup D_5 \cup L_3 \cup D_7, \\ D_8 \cup L_4 \cup D_6 \cup l_3 \cup D_4 \cup L_2 \cup D_2, \\ D_6 \cup L_4 \cup D_8 \cup l_0 \cup D_7 \cup L_3 \cup D_5. \end{cases} \tag{4.4}$$

The last three domains are not mapped onto the whole η-plane.

§ 5. Canonical regions for the case $p(x)=(x-1)^2/x$ (continued).

In the section 3 we got two "complete" canonical regions and two semi-canonical regions for the case $p(x)=(x-1)^2/x$, and in the section 4 we got one "complete" canonical regions and three semi canonical regions for the case $p(x)=-(x-1)^2/x$. In this and next sections we consider the given equation (1.1) on two sheets of complex planes, and we make semi-canonical regions (3.2) complete.

The origin is a singular branch point of the integrand. If the integral path C_1 is chosen a clockwise or counterclockwise unit circle around the origin, the integral takes value 8/3:

$$\int_{C_1} (x^{1/2} - x^{-1/2}) dx = \frac{8}{3}.$$

The Riemann surface of the integrand consists of two sheets of complex plane which are joined along the negative real axis.
If a point x in the second sheet is represented by $\tilde{x}$ the value of the integral from 1 to $\tilde{x}$ is given by

$$\begin{aligned} \tilde{\xi}(1, x) = \xi(1, \tilde{x}) &= \int_1^{\tilde{x}} (x^{1/2} - x^{-1/2}) dx \\ &= \left(\int_{C_1} + \int_{C_2}\right)(x^{1/2} - x^{-1/2}) dx \\ &= \frac{8}{3} - \left(\frac{2}{3}x^{3/2} - 2x^{1/2} + \frac{4}{3}\right), \end{aligned} \tag{5.1}$$

where C_2 is a path from $\tilde{1}$ to $\tilde{x}$ in the second sheet.
Thus we get

$$\tilde{\xi}(1, x) = \frac{8}{3} - \xi(1, x). \tag{5.2}$$

From the above equation (5.2) we can get level curves in the second sheet. The level curves represented by the equation $\mathrm{Re}\,\tilde{\xi} = 8/3$ have the same shapes as ones in the first sheet represented by the equation $\mathrm{Re}\,\xi = 0$. The level curves represented by the equation $\mathrm{Im}\,\tilde{\xi} = 0$ have the same shapes as ones in the first sheet represented by the equation $\mathrm{Im}\,\xi = 0$.

We represent these level curves in the second sheet by the same symbols in the first sheet with$\sim$. For example, on the curve l_1 in the first sheet $\mathrm{Re}\,\xi$ vanishes and $\mathrm{Im}\,\xi$ takes positive values, and on the curve $\tilde{l}_1$ in the second sheet $\mathrm{Re}\,\tilde{\xi}$ takes a value 8/3 only and $\mathrm{Im}\,\tilde{\xi}$ takes negative values. In the domain D_7 of the first sheet $\mathrm{Re}\,\xi$ takes values zero to 4/3 and $\mathrm{Im}\,\xi$ takes negative values. The domain $\tilde{D}_7$ of the second sheet is bounded by curves $\tilde{l}_3$, $\tilde{L}_0$ and $\tilde{l}_5$, and in $\tilde{D}_7$ $\mathrm{Re}\,\tilde{\xi}$ takes values 4/3 to 8/3 and $\mathrm{Im}\,\tilde{\xi}$ takes positive values, and so on. We consider the integral (5.1) as the mapping from the second x-plane into the ξ-plane. It is conformal. By analyzing similarly to the section 3, we get two semi-canonical regions:

$$\tilde{D}_8\cup\tilde{L}_0\cup\tilde{D}_7\cup\tilde{l}_3\cup\tilde{D}_5\cup\tilde{L}_2\cup\tilde{D}_3\,,\qquad \tilde{D}_7\cup\tilde{L}_0\cup\tilde{D}_8\cup\tilde{l}_4\cup\tilde{D}_6\cup\tilde{L}_3\cup\tilde{D}_4\,. \tag{5.3}$$

The former of (5.3) is a half plane $\mathrm{Re}\,\xi\geqq 4/3$ with a cut: $\mathrm{Re}\,\xi=8/3$, $\mathrm{Im}\,\xi<0$, and the latter is also the same half plane with a cut in the upper half plane.

By combining two kinds of semi-canonical regions (3.2) and (5.3), we get following four canonical regions:

$$\begin{aligned}
\mathscr{D}_3&=D_3\cup L_2\cup D_5\cup l_3\cup D_7\cup L_0\cup D_8\cup l_6\cup l_5\cup\tilde{D}_8\cup\tilde{L}_0\cup\tilde{D}_7\cup\tilde{l}_3\cup\tilde{D}_5\cup\tilde{L}_2\cup\tilde{D}_3\\
&\qquad\qquad (l_5=\tilde{l}_6,\ l_6=\tilde{l}_5)\,,\\
\mathscr{D}_4&=D_3\cup L_2\cup D_5\cup l_3\cup D_7\cup L_0\cup D_8\cup l_6\cup l_5\cup\tilde{D}_7\cup\tilde{L}_0\cup\tilde{D}_8\cup\tilde{l}_4\cup\tilde{D}_6\cup\tilde{L}_3\cup\tilde{D}_4\\
&\qquad\qquad (l_5=\tilde{l}_6,\ l_6=\tilde{l}_5)\,,\\
\mathscr{D}_5&=D_4\cup L_3\cup D_6\cup l_4\cup D_8\cup L_0\cup D_7\cup l_5\cup l_6\cup\tilde{D}_8\cup\tilde{L}_0\cup\tilde{D}_7\cup\tilde{l}_3\cup\tilde{D}_5\cup\tilde{L}_2\cup\tilde{D}_3\\
&\qquad\qquad (l_5=\tilde{l}_6,\ l_6=\tilde{l}_5)\,,\\
\mathscr{D}_6&=D_4\cup L_3\cup D_6\cup l_4\cup D_8\cup L_0\cup D_7\cup l_5\cup l_6\cup\tilde{D}_7\cup\tilde{L}_0\cup\tilde{D}_8\cup\tilde{l}_4\cup\tilde{D}_6\cup\tilde{L}_3\cup\tilde{D}_4\\
&\qquad\qquad (l_5=\tilde{l}_6,\ l_6=\tilde{l}_5)\,.
\end{aligned}\tag{5.4}$$

Each of above canonical regions has two cuts. Cuts of $\mathscr{D}_3$ are lines: $\mathrm{Re}\,\xi=0$, $\mathrm{Im}\,\xi>0$; $\mathrm{Re}\,\xi=8/3$, $\mathrm{Im}\,\xi<0$. Cuts of $\mathscr{D}_4$ are lines: $\mathrm{Re}\,\xi=0$, $\mathrm{Im}\,\xi>0$; $\mathrm{Re}\,\xi=8/3$, $\mathrm{Im}\,\xi>0$. Cuts of $\mathscr{D}_5$ are lines: $\mathrm{Re}\,\xi=0$, $\mathrm{Im}\,\xi<0$; $\mathrm{Re}\,\xi=8/3$, $\mathrm{Im}\,\xi<0$. Cuts of $\mathscr{D}_6$ are lines: $\mathrm{Re}\,\xi=0$, $\mathrm{Im}\,\xi<0$; $\mathrm{Re}\,\xi=8/3$, $\mathrm{Im}\,\xi>0$. Moreover following two regions:

$$\begin{aligned}
\mathscr{D}_7&=\tilde{D}_2\cup\tilde{L}_1\cup\tilde{D}_1\cup\tilde{l}_1\cup\tilde{D}_3\cup\tilde{L}_2\cup\tilde{D}_5\,,\\
\mathscr{D}_8&=\tilde{D}_1\cup\tilde{L}_1\cup\tilde{D}_2\cup\tilde{l}_2\cup\tilde{D}_4\cup\tilde{L}_3\cup\tilde{D}_6
\end{aligned}\tag{5.5}$$

are canonical. Thus we got canonical regions $\mathscr{D}_1,\cdots,\mathscr{D}_8$ for the case $p(x)=(x-1)^2/x$.

§ 6. Canonical regions for the case $p(x)=-(x-1)^2/x$ (continued).

To complete semi-canonical regions (4.4), we prepare the second complex plane and suppose that the point $\tilde{x}$ ranges in the second sheet. We consider the following equation derived from the integral (4.1)

$$\tilde{\eta}(1, x)=\eta(1, \tilde{x})=\mathrm{Im}\,\xi+i(8/3-\mathrm{Re}\,\xi)\,, \tag{6.1}$$
$$\mathrm{Re}\,\tilde{\eta}=\mathrm{Im}\,\xi\,, \qquad \mathrm{Im}\,\tilde{\eta}=8/3-\mathrm{Re}\,\xi\,.$$

From the last two equations we get level curves in the second sheet. The level curves $\mathrm{Re}\,\tilde{\eta}=0$ in the second sheet are of the same shapes as the ones in the first sheet defined by $\mathrm{Re}\,\eta=0$. The level curves $\mathrm{Im}\,\tilde{\eta}=8/3$ in the second sheet have same shapes as the curves in the first sheet defined by $\mathrm{Im}\,\eta=0$. We represent these level curves in the second sheet by the same symbols in the first sheet with $\sim$. For example, on $\tilde{l}_1$ $\mathrm{Re}\,\tilde{\eta}$ vanishes and $\mathrm{Im}\,\tilde{\eta}$ takes values less than 8/3. On $\tilde{L}_1$ $\mathrm{Re}\,\tilde{\eta}$ takes positive values and $\mathrm{Im}\,\tilde{\eta}$ takes a value 8/3. In the domain D_7 $\mathrm{Re}\,\eta$ takes positive values and $\mathrm{Im}\,\eta$ takes values zero to 4/3. In the domain $\tilde{D}_7$ $\mathrm{Re}\,\tilde{\eta}$ takes negative values and $\mathrm{Im}\,\tilde{\eta}$ takes values 4/3 to 8/3, and so on.

$\tilde{\eta}$ can be considered as a mapping defined on the second x-plane into the η-plane.

The domain $\tilde{D}_8\cup\tilde{L}_4\cup\tilde{D}_6$ is mapped by $\tilde{\eta}$ one-to-one onto a quater plane: $\mathrm{Re}\,\eta>0$, $\mathrm{Im}\,\eta>4/3$. Similarly, the domain $\tilde{D}_7\cup\tilde{L}_3\cup\tilde{D}_5$ is mapped one-to-one onto a quater plane $\mathrm{Re}\,\eta<0$, $\mathrm{Im}\,\eta>4/3$. The domain $\tilde{D}_5\cup\tilde{L}_3\cup\tilde{D}_7\cup\tilde{l}_0\cup\tilde{D}_8\cup\tilde{L}_4\cup\tilde{D}_6$ is mapped one-to-one onto the half plane $\mathrm{Im}\,\eta>4/3$ with a cut: $\mathrm{Re}\,\eta=0$, $\mathrm{Im}\,\eta>8/3$. These domains are also semi-canonical regions.

By combining them and (4.4) we get three canonical regions:

$$\mathcal{D}_2'=D_1\cup L_1\cup D_3\cup l_2\cup D_5\cup L_3\cup D_7\cup L_5\cup\tilde{D}_8\cup\tilde{L}_4\cup\tilde{D}_6\ (L_5=\tilde{L}_6)\,,$$
$$\mathcal{D}_3'=\tilde{D}_5\cup\tilde{L}_3\cup\tilde{D}_7\cup L_6\cup D_8\cup L_4\cup D_6\cup l_3\cup D_4\cup L_2\cup D_2\ (L_6=\tilde{L}_5)\,, \tag{6.2}$$
$$\mathcal{D}_4'=D_6\cup L_4\cup D_8\cup l_0\cup D_7\cup L_3\cup D_5\cup L_5\cup L_6\cup\tilde{D}_5\cup\tilde{L}_3\cup\tilde{D}_7\cup\tilde{l}_0\cup\tilde{D}_8\cup\tilde{L}_4\cup\tilde{D}_6\,.$$

Each of the canonical regions $\mathcal{D}_2'$ and $\mathcal{D}_3'$ has a cut on the positive imaginary axis. $\mathcal{D}_4'$ has two cuts on the imaginary axis: $\mathrm{Re}\,\eta=0$, $\mathrm{Im}\,\eta>8/3$; $\mathrm{Re}\,\eta=0$, $\mathrm{Im}\,\eta<0$. Moreover

$$\mathcal{D}_5=\tilde{D}_4\cup\tilde{L}_2\cup\tilde{D}_2\cup\tilde{l}_1\cup\tilde{D}_1\cup\tilde{L}_1\cup\tilde{D}_3 \tag{6.3}$$

is a canonical region with a cut $\mathrm{Re}\,\eta=0$, $\mathrm{Im}\,\eta>8/3$.

Thus we gained five canonical regions (4.3), (6.2) and (6.3) for the case $p(x)=-(x-1)^2/x$.

§ 7. Canonical paths.

To show asymptoticity of the WKB approximations (1.2) we consider some integral equation induced from the differential equation (1.1). Then paths of integration must have following properties. We denote the paths of integration by $\alpha_x^{\pm}$ starting from x.

1° Each of $\alpha_x^{\pm}$ corresponds to its image in the ξ-plane in one-to-one and continuous manner;

2° $\mathrm{Re}\,\xi(1, x)$ is non-decreasing along α_x^{+} and $\mathrm{Re}\,\xi(1, x)$ is non-increasing along $\alpha_x^{-(*)}$;

3° $\mathrm{Re}\,\xi(1, x) \to +\infty$ as $x \to \infty$ along α_x^{+}, and $\mathrm{Re}\,\xi(1, x) \to -\infty$ as $x \to \infty$ along α_x^{-}.

The paths of integration with above properties are called canonical paths. Once we get canonical regions the canonial paths are determined immediately. First we plot in the ξ-plane the image of the given point x, then we draw in the ξ-plane a direct line or a line consisting of segments or semi-circles avoiding the images of the turning and singular points (Figs. 3, 5). The canonical path is clearly obtained by taking the inverse image of the line in the ξ-plane (Figs. 4, 6).

For the two cases to be considered the quantities $p''/p^{3/2}, p'^2/p^{5/2}$ and $p'/p^{3/2}$ are bounded in the canonical regions. Therefore the approximation (1.2) are valid in each of canonical regions which are unbounded (Evgrafov-Fedoryuk [1], Nakano [3]).

§ 8. Canonical regions for the case $p(x)=(x-1)^2/x^3$.

The simplest case of the differential equation of the type (1.1) having a turning point at $x=1$ and two irregular singular points at $x=0, \infty$ is given by

$$\varepsilon^2 \frac{d^2y}{dx^2} - \frac{(x-1)^2}{x^3} y = 0 . \tag{8.1}$$

By analyzing in the similar way to the former cases, we can get canonical regions for the differential equation (8.1). Choosing one branch of $p(x)^{1/2}$ with positive value for $x>1$ and integrating it from 1 to x which moves on the one sheet of complex planes with a branch cut along the negative real axis, we get

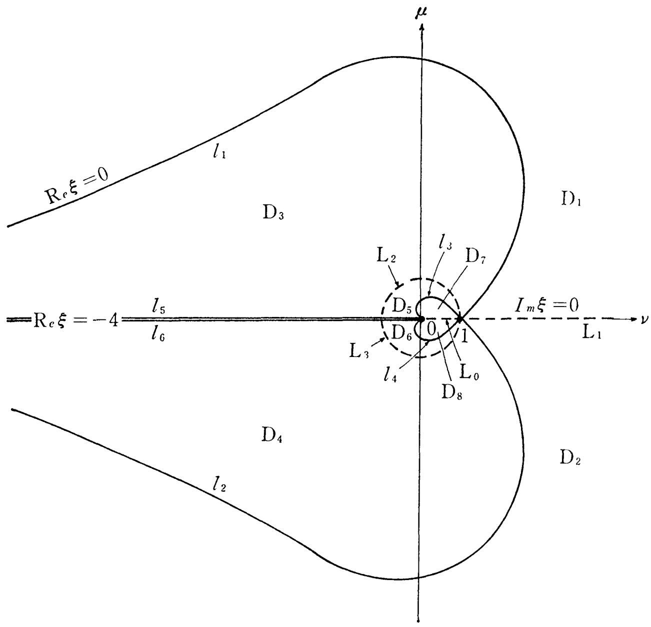
$$\xi(1, x) = \int_1^x \left[\frac{(x-1)^2}{x^3} \right]^{1/2} dx = 2(x^{1/2} + x^{-1/2} - 2). \tag{8.2}$$

In the polar coordinates $x = re^{i\theta}$ the real and imaginary parts of ξ are given by

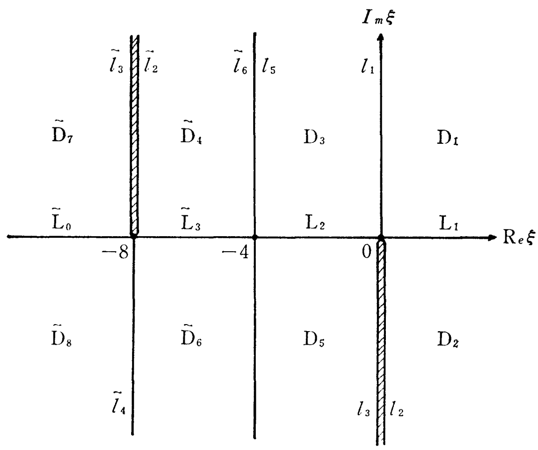
$$\mathrm{Re}\,\xi = \frac{2}{r}\left(r\cos\frac{\theta}{2} - 2\sqrt{r} + \cos\frac{\theta}{2} \right), \quad \mathrm{Im}\,\xi = 2(r^{1/2} - r^{-1/2})\sin\frac{\theta}{2} . \tag{8.3}$$

Thus we can draw curves in the plane defined by $\mathrm{Re}\,\xi=0$ and $\mathrm{Im}\,\xi=0$. $\mathrm{Re}\,\xi=$ represents two heart-shaped curves, $l_1, \cdots l_4$, and $\mathrm{Im}\,\xi=0$ represents a unit circle L_2, L_3 and the positive real axis L_0, L_1 (Fig. 7). $\mathrm{Re}\,\xi$ takes negative values in

(*) $\eta(1, x)$ is to be considered as $\xi(1, x)$.



Level curves for $(x-1)^2/x^3$
Fig. 7.



The image of $\mathscr{D}_1''\cup\tilde{\mathscr{D}}_4''$
Fig. 8.

the region between two heart-shaped curves, and Re ξ takes positive values in the other regions. Im ξ takes positive values in the upper half-plane except for the inside of the unit circle $r=1$, and in the unit disk in the lower-half plane $\operatorname{Im}\xi$ takes negative values in the other re gions. Two lines l_5 and l_6 are the same line, i.e., they are negative real axis. We suppose l_5 and l_6 the upper and the lower sides of the negative real axis respectively. On the both lines $\operatorname{Re}\xi$ takes a value -4.

Analyzing in detail we get following four semi-canonical regions:

$$
\begin{aligned}
&\mathscr{D}_1''=D_2\cup L_1\cup D_1\cup l_1\cup D_3\cup L_2\cup D_5\,,\\
&\mathscr{D}_2''=D_8\cup L_0\cup D_7\cup l_3\cup D_5\cup L_2\cup D_3\,,\\
&\mathscr{D}_3''=D_1\cup L_1\cup D_2\cup l_2\cup D_4\cup L_3\cup D_6\,,\\
&\mathscr{D}_4''=D_7\cup L_0\cup D_8\cup l_4\cup D_6\cup L_3\cup D_4\,.
\end{aligned}
\tag{8.4}
$$

To get canonical regions we must consider the Stokes curves in another complex plane. The point $e^{2\pi i}$ or $e^{-2\pi i}$ is a turning point of (8.1). The value of the integral between the turning point $x=1$ and the other is given by

$$\xi(1,\tilde{1})=-8\,.$$

If we denote points, curves and sets in the second leaf by same letters with $\sim$, we get an equation

$$\tilde{\xi}(1,x)=\xi(1,\tilde{1})+\xi(\tilde{1},\tilde{x})=-8-\xi(1,x)\,.$$

By this equation we can get informations in the second leaf from the first one. On the curves $\tilde{l}_1,\cdots,\tilde{l}_4$ $\operatorname{Re}\xi$ takes a value -8, and on the curve $\tilde{l}_5(=\tilde{l}_6)$ $\operatorname{Re}\xi=-4$. In the region between two heart-shaped curves $\operatorname{Re}\xi$ takes values -4 to -8, and in the other regions values of $\operatorname{Re}\xi$ are smaller than -8. $\operatorname{Im}\xi$ takes negative values in the upper half-plane except for the inside of the unit circle and in the unit disk in the lower-half plane.

Thus we get four semi-canonical regions:

$$
\begin{aligned}
&\tilde{\mathscr{D}}_1''=\tilde{D}_2\cup\tilde{L}_1\cup\tilde{D}_1\cup\tilde{l}_1\cup\tilde{D}_3\cup\tilde{L}_2\cup\tilde{D}_5\,,\\
&\tilde{\mathscr{D}}_2''=\tilde{D}_8\cup\tilde{L}_0\cup\tilde{D}_7\cup\tilde{l}_3\cup\tilde{D}_5\cup\tilde{L}_2\cup\tilde{D}_3\,,\\
&\mathscr{D}_3''=\tilde{D}_1\cup\tilde{L}_1\cup\tilde{D}_2\cup\tilde{l}_2\cup\tilde{D}_4\cup\tilde{L}_3\cup\tilde{D}_6\,,\\
&\mathscr{D}_4''=\tilde{D}_7\cup\tilde{L}_0\cup\tilde{D}_8\cup\tilde{l}_4\cup\tilde{D}_6\cup\tilde{L}_3\cup\tilde{D}_4\,.
\end{aligned}
\tag{8.5}
$$

Noticing boundaries of $\mathscr{D}_i''$ and $\tilde{\mathscr{D}}_i''$ and combining them appropriately we get eight canonical regions:

$$
\begin{aligned}
&\mathscr{D}_1''\cup\tilde{\mathscr{D}}_3'',\ \mathscr{D}_1''\cup\tilde{\mathscr{D}}_4''(l_5=\tilde{l}_6,\ l_7=\tilde{l}_8)\,,\\
&\mathscr{D}_2''\cup\tilde{\mathscr{D}}_3'',\ \mathscr{D}_2''\cup\tilde{\mathscr{D}}_4''(l_5=\tilde{l}_6,\ l_7=\tilde{l}_8)\,,
\end{aligned}
$$

$$\mathscr{D}_3'' \cup \tilde{\mathscr{D}}_1'',\ \mathscr{D}_3'' \cup \tilde{\mathscr{D}}_2''(l_6 = \tilde{l}_5,\ l_8 = \tilde{l}_7),$$
$$\mathscr{D}_4'' \cup \tilde{\mathscr{D}}_1'',\ \mathscr{D}_4'' \cup \tilde{\mathscr{D}}_2''(l_6 = \tilde{l}_5,\ l_8 = \tilde{l}_7). \tag{8.6}$$

Each of them has two cuts, for example, the negative imaginary axis and $\mathrm{Re}\,\xi = -8$, $\mathrm{Im}\,\xi > 0$ for the canonical region $\mathscr{D}_1 \cup \tilde{\mathscr{D}}_4$ (Fig. 8).

§ 9. Summary.

The principal parts of formal solutions of the differential equation (1.1) are called WKB (or LG) approximations (1.2) (Olver [5]). When ε tends to zero WKB approximations are asymptotic expansions (or approximations) of two independent solutions of (1.1). Asymptoticity is valid in subsets of complex planes which are called canonical regions. In the case $p(x) = (x-1)^2/x$ canonical regions are given in (3.1), (5.4) and (5.5). In the case $p(x) = -(x-1)^2/x$ they are given in (4.3), (6.2) and (6.3). In each of these cases the differential equation (1.1) has a turning point of order 2 at $x=1$, a regular singular point at $x=0$ and an irregular singular point at $x=\infty$. In the case $p(x) = (x-1)^2/x^3$ canonical regions are given in (8.6). This case is the simplest differential equation of type (1.1) having a turning point and two irregular singular points. To get "complete" canonical regions for general case is very complicated (Fedorjuk [2]), but in cases treated here all canonical regions are obtained and their union cover two sheets of complex planes. Appropricate pairs of them have parts overlapped. Once we get canonical regions, it is easy to determine canonical paths or paths of integral equations as given in § 7.

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