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HAYMAN DIRECTION OF MEROMORPHIC FUNCTIONS

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Abstract

Let f be meromorphic in the plane. Then f has a Hayman direction provided that

$$\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=\infty .$$

1. Introduction

We define a Hayman direction of a meromorphic function $f(z)$ to be a ray $\arg z=\theta$, $0\leqq\theta\leqq 2\pi$, such that for every positive integer l and positive $\varepsilon>0$,

$$\lim_{r\to\infty}[n(r,\ \theta-\varepsilon,\ \theta+\varepsilon,\ f=a)+n(r,\ \theta-\varepsilon,\ \theta+\varepsilon,\ f^{(l)}=b)]=\infty \tag{1}$$

holds for all $(a, b)\in C\times[C-0]$, where

$$n(r,\ \theta-\varepsilon,\ \theta+\varepsilon,\ g=\beta)$$

is the number of roots of $g-\beta=0$ in the region

$$[|z|<r]\cap[|\arg z-\theta|<\varepsilon].$$

Yang, Lo [1] proved that for given meromorphic function f there is a ray $\arg z=\theta$ which satisfies (1) provided that

$$\limsup_{r\to\infty}\frac{T(r,\ f)}{(\log r)^3}=\infty . \tag{2}$$

A problem posed in [3] asks whether (2) could be replaced by the usual existing condition of classical Julia directions

$$\limsup_{r\to\infty}\frac{T(r,\ f)}{(\log r)^2}=\infty . \tag{3}$$

In this paper we prove that there is a ray $\arg z=\theta$ satisfying (1) provided that (3) holds.

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2. Some lemmas

We use Ahlfors-Shimizu characteristic [2] for a given meromorphic function $f(z)$ as follows

$$T_0(r)=\int_0^r \frac{A(t)}{t}\,dt \tag{4}$$

where

$$A(r)=\frac{1}{\pi}\int_0^{2\pi} t\,dt\int_0^{2\pi}\frac{|f'|^2}{(1+|f|^2)^2}\,d\theta$$

and a known result about the relationship between $T_0(r)$ and $T(r, f)$ as follows.

LEMMA 1. *Suppose that $f(z)$ is meromorphic in the plane. Then*

$$|T(r, f)-T_0(r)-\log^+|f(0)||\leqq\frac{1}{2}\log 2$$

(for a proof, see [2], pp. 12-13).

LEMMA 2. *Given positive integer l and meromorphic function $f(z)$ in $|z|<1$. Suppose that in $|z|<1$, $f\neq 0$, $f^{(l)}\neq 1$. Then in $|z|<1/32$ either $|f|<1$ or $|f|>C_l$ uniformly, where C_l is a positive constant only depending on l.*
(for a proof, see [1]).

LEMMA 3. *Let $f(z)$ be meromorphic in the plane and α_ν, $\nu=1, 2, 3$, be three distinct finite complex numbers. Let F_0 be*

$$S^2-[\alpha_\nu,\ \nu=1, 2, 3],$$

such that

$$F_0\subset f(|z|<\infty).$$

Suppose that in $|z|\leqq R$, $f(z)$ has values in F_0. Then we have

$$A(r)<hL(r),\ (0<r<R)$$

where

$$L(r)=\int_0^{2\pi}\frac{|f'(re^{i\theta})|r}{1+|f(re^{i\theta})|^2}\,d\theta$$

and for $0<r<R$, we have

$$A(r)<\frac{2\pi^2h^2R}{R-r},$$

where h only depending on the geometric nature of conditions satisfied by $f(z)$.
(for a proof, see [2], pp. 137-144).

3. The main results

THEOREM 1. *Let $f(z)$ be meromorphic in the plane. Suppose that*

$$\limsup_{r\to\infty}\frac{T(r,\ f)}{(\log r)^2}=\infty\ . \tag{5}$$

Then there is a real $\theta\in[0,\ 2\pi]$, such that for every $\varepsilon>0$, and positive integer l,

$$\lim_{r\to\infty}[n(r,\ \theta-\varepsilon,\ \theta+\varepsilon,\ f=a)+n(r,\ \theta-\varepsilon,\ \theta+\varepsilon,\ f^{(l)}=b)]=\infty\ . \tag{6}$$

for all $(a,\ b)\in C\times[C-0]$.

THEOREM 2. *Let $f(z)$ be meromorphic in the plane. Suppose that*

$$\limsup_{r\to\infty}\frac{T(2r,\ f)}{T(r,\ f)}>1\ . \tag{7}$$

Then the conclusion of Theorem 1 is still true.

4. The proof of the theorems

Proof of Theorem 1. By hypothesis (5) and using Lemma 1, we can get a ray $\arg z=\theta$, such that for arbitrary positive δ

$$\limsup_{r\to\infty}\frac{T(r,\ \theta-\delta,\ \theta+\delta)}{(\log r)^2}=\infty \tag{8}$$

where

$$T(r,\ \theta-\delta,\ \theta+\delta)=\int_0^r\frac{A(t,\ \theta-\delta,\ \theta+\delta)}{t}dt$$

$$A(r,\ \theta-\delta,\ \theta+\delta)=\frac{1}{\pi}\iint_{\Delta_{r,\delta}}\frac{|f'|^2}{(1+|f|^2)^2}dxdy$$

is the Ahlfors charachteristic of $f(z)$ in the angle area

$$|\arg z-\theta|<\delta\ ,$$

where

$$\Delta_{r,\delta}=(z,\ |z|<r)\cap(z,\ |\arg z-\theta|<\delta)\ .$$

We prove the ray $\arg z=\theta$ is desired for Theorem 1. Otherwise, without loss of generality, we have complexes a, $b(b\neq 0)$ and some $\varepsilon>0$, and $l>0$, such that

$$n(r,\ \theta-\varepsilon,\ \theta+\varepsilon,\ f=a)+n(r,\ \theta-\varepsilon,\ \theta+\varepsilon,\ f^{(l)}=b)=0 \tag{9}$$

for any r, $0<r<\infty$.

Now given a sequence of positive real c_k which tends to infinitive, we find a sequence of circles D_k, $(k=1, 2, \cdots)$

$$D_k: |z-z_k|<\eta|z_k, \ z_k=|z_k|e^{i\theta}, \ \lim_{k\to\infty}|z_k|=\infty ,$$

with a sufficiently small $\eta>0$ such that

$$A(D_k)>c_k \log \frac{1}{1-\eta} . \tag{10}$$

$$(k=1, 2, \cdots)$$

We'll get D_k by induction.

After getting $D_1, D_2, \cdots, D_{k-1}$ if we cannot get D_k such that (10) holds, then for each $r>r_k^*=\text{Max}(|z_{k-1}|, k)$, we have

$$A(\Delta_0)\leqq c_k \log \frac{1}{1-\eta}=c_k \log \frac{r_0}{r_0-\eta r_0} , \tag{11}$$

where

$$r_0=r$$

$$\Delta_0: |z-r_0e^{i\theta}|<\eta r_0$$

and

$$A(\Delta_m)\leqq c_k \log \frac{1}{1-\eta}=c_k \log \frac{r_m}{r_m-\eta r_m} , \tag{12}$$

$$(m=1, 2, \cdots)$$

where

$$r_m=(1-\eta)r_{m-1}$$

$$\Delta_m: |z-r_me^{i\theta}|<\eta r_m . \tag{13}$$

Noting

$$r_0-r_1=\eta r_0$$

$$r_1-r_2=\eta(1-\eta)r_0$$

$$\cdots\cdots\cdots$$

$$r_{m-1}-r_m=\eta(1-\eta)^{m-1}r_0 \tag{14}$$

and

$$\eta[1+(1-\eta)+(1-\eta)^2+\cdots]r_0=r_0 , \tag{15}$$

we see there is a positive integer m such that

$$r_m\geqq r_k^* \tag{16}$$

and

$$r_{m+1}=r_m-\eta r_m\leqq r_k^*$$

i.e.

$$r_m \leqq \left(\frac{1}{1-\eta}\right) r_k^* . \tag{17}$$

On the other hand

$$E_m(r) = (z : r_m < |z| < r) \cap \left(z : |\arg z - \theta| < \frac{\eta}{2}\right) \subset \bigcup_{j=0}^{m} \Delta_j . \tag{18}$$

But by (11), (12), and (15), we have

$$\sum_{j=0}^{m} A(\Delta_j) \leqq c_k \sum_{j=0}^{m} \log \frac{r_j}{r_{j+1}} \leqq c_k \log \frac{r_0}{r_m} \leqq c_k \log \frac{r}{r_k^*} . \tag{19}$$

Then by (16), (17), (18) and (19), we have

$$\begin{aligned} A\left(r, \theta-\frac{\eta}{2}, \theta+\frac{\eta}{2}\right) &= A\left(r_m, \theta-\frac{\eta}{2}, \theta+\frac{\eta}{2}\right) + \frac{1}{\pi}\iint_{E_m(r)} \frac{|f'|^2}{(1+|f|^2)^2} dx dy \\ &\leqq A\left(\frac{1}{1-\eta} r_k^*, \theta-\frac{\eta}{2}, \theta+\frac{\eta}{2}\right) + \sum_{j=0}^{m} A(\Delta_j) \\ &\leqq A\left(\frac{1}{1-\eta} r_k^*, \theta-\frac{\eta}{2}, \theta+\frac{\eta}{2}\right) + c_k \log \frac{r}{r_k^*} . \end{aligned} \tag{20}$$

Since $r > r_k^*$ is arbitrary, we have

$$A\left(r, \theta-\frac{\eta}{2}, \theta+\frac{\eta}{2}\right) = O[(\log r)] . \tag{21}$$

Then

$$T\left(r, \theta-\frac{\eta}{2}, \theta+\frac{\eta}{2}\right) = \int_0^r \frac{A\left(t, \theta-\frac{\eta}{2}, \theta+\frac{\eta}{2}\right)}{t} dt = O(\log r)^2 \tag{22}$$

and this yields a contradiction to (8).

Take a sequence of circles $D_k : |z-z_k| < 64\eta|z_k|$, $(k=1, 2, \cdots)$. If $\eta < (1/32\pi)\varepsilon$, then $D_k \subset (z : |\arg z - \theta| < \varepsilon)$. By (9), functions

$$g_k(t) = \frac{f(z_k + 64\eta|z_k|t) - a}{b(64\eta|z_k|)^l}$$

are meromorphic in $|t| < 1$ and for each k, $g_k(t) \neq 0$, $g_k^{(l)}(t) \neq 1$. Then by Lemma 2 we have in $|t| \leqq 1/32$, either $|g_k(t)| \leqq 1$ or $|g_k(t)| > C_l$ uniformly.

There are two cases for each k as follows.

CASE 1. $|g_k(t)| > C_l$ holds uniformly in $|t| < 1/32$.
Denoting

$$f_k(t) = f(z_k + 64\eta|z_k|t),$$

this deduces

$$|f_k(t) - a| > C_l |b| (64\eta|z_k|)^l . \tag{23}$$

holds uniformly in $|t| < 1/32$.

CASE 2. $|g_k(t)|<1$ holds uniformly in $|t|<1/32$, and this deduces

$$|f_k(t)-a|<|b|(64\eta|z_k|)^l$$

for all t in $|t|<1/32$.

Now in case 1 we choose three distinct values α_1, α_2, α_3, such that

$$\alpha_1=a,\ |\alpha_\nu-a|=1,\quad \nu=2,3\,.$$

In case 2, we let

$$\alpha_1=a\ ,\ \alpha_2=e^{i\pi/2}\alpha_3\ ,$$

$$|\alpha_\nu|=2(|a|+|b|(64\eta|z_k|)^l)\,,\qquad(\nu=2,3)\,.$$

At the same time, without loss of generality, we assume that $[\alpha_\nu,\ \nu=1,2,3]$ includes all values (at most two) which are not taken by $f(z)$ in the plane. Then by the method of Theorem 5.3 in [2], in order to use our Lemma 3, we choose

$$F_0=S^2-[\alpha_\nu,\ \nu=1,2,3]$$

and clearly

$$F_0\subset f_k(|t|<\infty)\,,$$

since $f(z)$ is meromorphic in the plane and so is $f_k(t)$. Noting the geometric nature of conditions satisfied by $f(z)$ and α_ν on Riemann sphere, we have

$$f_k\Big(|t|\leqq\frac{1}{32}\Big)\subset F_0$$

for

$$f_k(t)\neq\alpha_\nu\,,\ \nu=1,2,3\,,$$

when $|t|\leqq1/32$. Then by Lemma 3 we have

$$A\Big(|t|\leqq\frac{1}{32}\Big)<hL\Big(|t|=\frac{1}{32}\Big)$$

$$A\Big(|t|\leqq\frac{1}{64}\Big)<\frac{2\pi^2h^2/32}{1/32-1/64}\leqq4\pi^2h^2\,,$$

where h is independent of k, since for both cases we can properly construct Jordan arcs β_ν ($\nu=1,2,3$) to join α_1, α_2, α_3 in turn just like in [2] such that the length of β_ν and the sphere area of F_0' and F_0'' are all greater than $C(a)$, where $C(a)$ is a positive constant only dependent of a, and

$$F_0'\cup F_0''=F_0$$

are two Jordan domains divided by β_ν, $\nu=1,2,3$.

Clearly

$$A\Big(|t|<\frac{1}{64}\Big)=A(D_k)\,.$$

So by (10), we have

$$c_k \log \frac{1}{1-\eta} < A(D_k) < 4\pi^2 h^2$$

i. e.

$$c_k < 4\pi^2 h^2 \left(\log \frac{1}{1-\eta}\right)^{-1}.$$

But this violates that c_k tends to infinitive. We are done.

Proof of Theorem 2. Since

$$T(r, f) = O(\log r)^2$$

means

$$T(2r, f) \sim T(r, f),$$

so by hypothesis of Theorem 2, we have

$$\limsup_{r\to\infty} \frac{T(r, f)}{(\log r)^2} = \infty.$$

Then we deduce Theorem 2 from Theorem 1 directly.

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