M. OZAWA
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COEFFICIENT ESTIMATES FOR THE CLASS Σ

Dedicated to Professor Y. Kusunoki on his sixtieth birthday

By Mitsuru Ozawa

1. Introduction.

Let Σ denote the class of functions $f(z)$ univalent in $|z|>1$, regular apart from a simple pole at the point at infinity and having the expansion

$$f(z)=z+\sum_{n=1}^{\infty}b_n z^{-n}$$

around there. Let us introduce quantities A_n, B_n by

$$A_n=\inf\{t:\ \Re(tb_1+b_n)\leqq t,\ {}^{\forall}f\in\Sigma\},$$

$$B_n=\inf\{t:\ \Re(tb_1-b_n)\leqq t,\ {}^{\forall}f\in\Sigma\},$$

respectively. It is evident that $A_{2n}=B_{2n}$. Kirwan made a conjecture that $B_n\leqq n$ seems to be true [3]. $B_2\leqq 2$ and $B_3\leqq 3$ were due to Garabedian and Schiffer [1] and Kirwan and Schober [2] proved $B_2=2$ and $B_3=3$.

In this paper we shall prove the following

THEOREM. $A_3\leqq 2$, $A_5\leqq(27+8\sqrt{3})/12$, $A_7\leqq 5.5$, $A_9<8$, $A_{11}<10$.

$A_n\leqq n-1$ for any odd $n\geqq 3$ seems to be true. Anyway it seems to be very difficult to decide A_n exactly as well as B_n. Our method of proof depends upon the Grunsky inequality. So to explain its related notions and relations is in order here.

Let $f(z)\in\Sigma$ and let $F_m(z)$ be the mth Faber polynomial of $f(z)$, which is defined by

$$F_m(f(z))=z^m+\sum_{n=1}^{\infty}a_{mn}z^{-n}.$$

Then Grunsky's inequality has the following form

$$\left|\sum_{m,n=1}^{N}na_{mn}x_mx_n\right|\leqq\sum_{n=1}^{N}n|x_n|^2$$

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for any N and for any complex vector $(x_1, \cdots, x_N)$. We have $na_{mn} = ma_{nm}$ and

$$
\begin{aligned}
&a_{1n} = b_n\,,\\
&a_{22} = 2b_3 + b_1^2,\\
&a_{24} = 2b_5 + 2b_1b_3 + b_2^2,\\
&a_{25} = 2b_6 + 2b_1b_4 + 2b_2b_3\,,\\
&a_{26} = 2b_7 + 2b_1b_5 + 2b_2b_4 + b_3^2,\\
&a_{33} = 3b_5 + 3b_1b_3 + 3b_2^2 + b_1^3,\\
&a_{35} = 3b_7 + 3b_1b_5 + 6b_2b_4 + 3b_3^2 + 3b_1^2b_3 + 3b_1b_2^2,\\
&a_{44} = 4b_7 + 4b_1b_5 + 8b_2b_4 + 6b_3^2 + 4b_1^2b_3 + 8b_1b_2^2 + b_1^4,\\
&a_{46} = 4b_9 + 4b_1b_7 + 8b_2b_6 + 12b_3b_5 + 4b_1^2b_5 + 6b_4^2 + 16b_1b_2b_4 + 8b_1b_3^2\\
&\qquad + 12b_2^2b^3 + 4b_1^3b_3 + 6b_1^2b_2^2,\\
&a_{55} = 5b_9 + 5b_1b_7 + 10b_2b_6 + 15b_3b_5 + 5b_1^2b_5 + 10b_4^2 + 20b_1b_2b_4 + 15b_1b_3^2\\
&\qquad + 20b_2^2b_3 + 5b_1^3b_3 + 15b_1^2b_2^2 + b_1^5,\\
&a_{66} = 6b_{11} + 6b_1b_9 + 12b_2b_8 + 18b_3b_7 + 24b_4b_6 + 6b_1^2b_7 + 24b_1b_2b_6 + 15b_5^2\\
&\qquad + 36b_1b_3b_5 + 24b_2^2b_5 + 6b_1^3b_5 + 24b_1b_4^2 + 72b_2b_3b_4 + 36_1^2b_2b_4 + 14b_3^3\\
&\qquad + 27b_1^2b_3^2 + 72b_1b_2^2b_3 + 6b_1^4b_3 + 9b_2^4 + 24b_1^3b_2^2 + b_1^6.
\end{aligned}
$$

In what follows we shall make us of the following notations:

$$
\begin{aligned}
b_1 &= p + ix',\\
b_2 &= y + iy',\\
b_3 &= \eta + i\eta',\\
b_4 &= \xi + i\xi',\\
b_5 &= \varphi + i\varphi',\\
b_6 &= \phi + i\phi',\\
b_7 &= \sigma + i\sigma',\\
b_8 &= \tau + i\tau',\\
b_9 &= \rho + \imath\rho'.
\end{aligned}
$$

2. Proof of Theorem.

(i) $A_3 \leqq 2$. By Grunsky's inequality

$$\left| b_3 + \frac{1}{2} b_1^2 \right| \leqq \frac{1}{2}.$$

Hence

$$\begin{aligned}\mathcal{R} b_3 &\leqq \frac{1}{2} - \frac{1}{2} \mathcal{R}(b_1^2) = \frac{1}{2}(1-p^2) + \frac{1}{2} x'^2 \\ &\leqq 1-p^2 \leqq 2(1-p)\end{aligned}$$

by the area theorem. Thus $\mathcal{R}\{2b_1+b_3\} \leqq 2$.

(ii) $A_5 \leqq (27+8\sqrt{3})/12$. We may assume that $\mathcal{R} b_1 = p \geqq 0$ by rotation. By Grunsky's inequality

$$\left| b_5 + b_1 b_3 + b_2^2 + \frac{1}{3} b_1^3 + \frac{2}{3} \alpha b_3 + \frac{\alpha^2}{9} b_1 \right| \leqq \frac{1}{3} + \frac{|\alpha|^2}{9}.$$

Hence by taking the real part

$$\mathcal{R} b_5 \leqq \frac{1}{3}(1-p^3) + \frac{\alpha^2}{9}(1-p) - \frac{2}{3} \alpha \eta - p\eta + x'\eta' + px'^2 + y'^2 - y^2.$$

Now we put $\alpha = -3p/2$. Then

$$\mathcal{R} b_5 \leqq \frac{1}{3}(1-p^3) + \frac{p^2}{4}(1-p) + px'^2 + x'\eta' + y'^2.$$

By the area theorem

$$\frac{3+2\sqrt{3}}{3} y'^2 \leqq \frac{3+2\sqrt{3}}{6}(1-p^2-x'^2-3\eta'^2).$$

Hence

$$\begin{aligned}\mathcal{R} b_5 &\leqq \frac{1}{3}(1-p^3) + \frac{p^2}{4}(1-p) + \frac{3+2\sqrt{3}}{6}(1-p^2) \\ &\quad - \left(\frac{3+2\sqrt{3}}{6} - p\right) x'^2 + x'\eta' - \frac{3+2\sqrt{3}}{2} \eta'^2 - \frac{-1+2\sqrt{3}}{3} y'^2 \\ &\leqq \frac{27+8\sqrt{3}}{12}(1-p).\end{aligned}$$

Thus $\mathcal{R}\{kb_1+b_5\} \leqq k$ with $k = (27+8\sqrt{3})/12$. Of course equality occurs only for $p=1$, that is, for $g(z) = z+1/z$.

(iii) $A_7 \leqq 5.5$.

LEMMA 1. *If* $p \leqq (nt^2-1)/(nt^2+1)$, *then* $\mathcal{R}\{tb_1+b_n\} \leqq t$.

Proof. By the area theorem $n(\mathcal{R} b_n)^2 \leqq 1-p^2$. Hence

$$\mathfrak{R}\{tb_1+b_n\}\leqq tp+\sqrt{(1-p^2)/n}\leqq t\,.$$

LEMMA 2. $-\eta\leqq 3(1-p)$.

Proof. Garabedian and Schiffer's inequality $|3b_1-b_3|\leqq 3$ implies the result immediately.

We may consider the case $0.99\leqq p\leqq 1$, since for $0\leqq p\leqq 0.99$ we have $\mathfrak{R}\{5.5b_1+b_7\}<5.5$ by Lemma 1. By Grunsky's inequality

$$\Big|b_7+b_1b_5+2b_2b_4+\frac{3}{2}b_3{}^2+b_1{}^2b_3+2b_1b_2{}^2+\frac{1}{4}b_1{}^4$$
$$+\Big(b_5+b_1b_3+\frac{1}{2}b_2{}^2\Big)\alpha+\Big(\frac{1}{4}b_3+\frac{1}{8}b_1{}^2\Big)\alpha^2\Big|\leqq\frac{1}{4}+\frac{1}{8}\alpha^2.$$

Taking the real part and putting $\alpha=-p$, we have

$$\mathfrak{R}b_7\leqq\frac{1}{4}(1-p^4)+\frac{1}{8}p^2(1-p^2)-\frac{1}{4}(p^2-4x'^2)\eta$$
$$-\frac{3}{2}py^2-2y\xi-\frac{3}{2}\eta^2+\frac{13}{8}p^2x'^2+px'\eta'+\frac{3}{2}\eta'^2+\frac{3}{2}py'^2$$
$$+2y'\xi'+4yx'y'+x'\varphi'.$$

Since $p\geqq 0.99$, $x'^2\leqq 1-p^2\leqq 0.0199$. Hence $p^2-4x'^2>0$. By Lemma 2

$$-\frac{1}{4}(p^2-4x'^2)\eta\leqq\frac{3}{4}(p^2-4x'^2)(1-p)\leqq\frac{3}{4}p^2(1-p)\,.$$

By the area theorem

$$\frac{14}{8}p^2x'^2\leqq\frac{14}{8}p^2(1-p^2-2y'^2-3\eta'^2-4\xi'^2-5\varphi'^2-2y^2-4\xi^2)\,.$$

Hence

$$\mathfrak{R}b_7\leqq\frac{1}{4}(1-p^4)+\frac{1}{8}(1-p^2)p^2+\frac{3}{4}p^2(1-p)+\frac{14}{8}p^2(1-p^2)$$
$$-\Big[\Big(\frac{3}{2}p+\frac{14}{4}p^2\Big)y^2+2y\xi+7p^2\xi^2+\frac{3}{2}\eta^2\Big]$$
$$-\Big[\frac{p^2}{8}x'^2-4yx'y'+\Big(\frac{7}{2}p^2-\frac{3}{2}p\Big)y'^2-2y'\xi'+7p^2\xi'^2$$
$$-px'\eta'+\Big(\frac{21}{4}p^2-\frac{3}{2}\Big)\eta'^2-x'\varphi'+\frac{35}{4}p^2\varphi'^2\Big].$$

The quadratic terms in two [ ] are positive definite for $0.99\leqq p\leqq 1$. Hence

$$\mathfrak{R}b_7\leqq 5.5(1-p)\,.$$

which gives the desired result.

(iv) $A_9<8$. By Grunsky's inequality

$$|5a_{55}+10a_{35}\alpha+10a_{15}\gamma+3a_{33}\alpha^2+a_{11}\gamma^2|\leqq 5+3\alpha^2+\gamma^2.$$

Taking the real part and putting $\alpha=-5p/6$ and $\gamma=-5p^2/8$, we have

$$\begin{aligned}\mathfrak{R}b_9\leqq &\frac{1}{5}(1-p^5)+\frac{p^2}{12}(1-p^3)+\frac{p^4}{64}(1-p)-\frac{1}{4}(p^3-8px'^2)\eta\\ &-\Big(\frac{9}{4}p^2-3x'^2\Big)y^2-4y^2\eta-2p\eta^2-2py\xi-2\xi^2-3\eta\varphi-2y\phi\\ &+\frac{9}{4}p^3x'^2+\Big(\frac{5}{4}p^2-x'^2\Big)x'\eta'+2p\eta'^2+px'\varphi'+3\eta'\varphi'+x'\sigma'\\ &+\Big(\frac{9}{4}p^2-3x'^2\Big)y'^2+2py'\xi'+2\xi'^2+2y'\phi'\\ &+x'^2\varphi+4x'y'\xi+4yx'\xi'+6\eta x'\eta'+8yy'\eta'+10pyx'y'.\end{aligned}$$

By Lemma 2

$$-\frac{1}{4}(p^3-8px'^2)\eta\leqq\frac{1}{4}(p^3-8px'^2)3(1-p)\leqq\frac{3}{4}p^3(1-p)\,.$$

By the area theorem

$$\begin{aligned}2.9x'^2\leqq 2.9(1&-p^2-2y^2-3\eta^2-4\xi^2-5\varphi^2-6\phi^2\\ &-2y'^2-3\eta'^2-4\xi'^2-5\varphi'^2-6\phi'^2-7\sigma'^2)\,.\end{aligned}$$

By the trivial inequalities

$$\begin{aligned}&x'^2\varphi\leqq 0.5(\varphi^2+x'^4)\,,\\ &4x'y'\leqq 2(0.1\xi^2+10x'^2y'^2)\,,\\ &4yx'\xi'\leqq 2(0.1y^2+10x'^2\xi'^2)\,,\\ &6\eta x'\eta'\leqq 3(0.1\eta^2+10x'^2\eta'^2)\,,\\ &10pyx'y'\leqq 5(0.1y^2+10x'^2y'^2)\end{aligned}$$

and

$$8yy'\eta'\leqq 4(0.1y^2+10y'^2\eta'^2)\leqq 0.4y^2+\frac{40}{3}(1-p^2-x'^2)y'^2.$$

What we want to prove is $\mathfrak{R}\{(7.8+1/64)b_1+b_9\}\leqq 7.8+1/64$. Hence by Lemma 1 it is sufficient to prove the result for $p\geqq 0.99$. Summing up the above facts, we have

$$\begin{aligned}\mathfrak{R}b_9\leqq &\frac{1}{5}(1-p^5)+\frac{p^2}{12}(1-p^3)+\frac{p^4}{64}(1-p)+\frac{3}{4}p^3(1-p)\\ &+2.9(1-p^2)-X-Y,\end{aligned}$$

where

$$X=(4.7+2.25p^2-3x'^2)y^2+4y^2\eta+(8.4+2p)\eta^2+2py\xi+13.4\xi^2$$
$$+3\eta\varphi+14\varphi^2+2y\phi+17.4\phi^2,$$
$$Y=(2.9-2.25p^3-0.5x'^2)x'^2-(1.25p^2-x'^2)x'\eta'+(6.7-30x'^2)\eta'^2$$
$$-px'\varphi'-3\eta'\varphi'+14.5\varphi'^2-x'\sigma'+20.3\sigma'^2$$
$$+\left(3.55-\frac{40}{3}(1-p^2)-\frac{161}{3}x'^2\right)y'^2-2py'\xi'+(9.6-20x'^2)\xi'^2$$
$$-2y'\phi'+17.4\phi'^2.$$

By making use of $x'^2\leqq 1-p^2\leqq 0.0199$ we can easily prove the positive definiteness of X and Y. Then

$$\Re b_9\leqq(1-p)\left(7.8+\frac{1}{64}\right).$$

Thus we have the desired result. Equality occurs only for $p=1$, that is, for $z+1/z$.

(v) $A_{11}<10$. By Grunsky's inequality

$$|6a_{66}+12a_{46}\alpha+12a_{26}\beta+4a_{44}\alpha^2+2a_{22}\beta^2|\leqq 6+4\alpha^2+2\beta^2.$$

Taking the real part and setting $\alpha=-3p/4$ and $\beta=3p^2/8$, we have

$$\Re b_{11}\leqq\frac{1}{6}(1-p^6)+\frac{p^2}{16}(1-p^4)+\frac{p^4}{128}(1-p^2)-\left(\frac{1}{4}+\frac{1}{64}\right)p^4\eta+\frac{13}{4}p^2x'^2\eta+x'^4\eta$$
$$-[(3p^3-10.5px'^2)y^2+2.25p^2y\xi+2.5p\xi^2+4\xi\phi-6x'^2y\xi+2py\phi+2y\tau$$
$$+(2.75p^2-4.5x'^2)\eta^2+3p\eta\varphi+2.5\varphi^2+3\eta\sigma]$$
$$+\frac{369}{128}p^4x'^2+(3p^3+13.5x'^2)y'^2+(2.25p^2-6x'^2)y'\xi'+2.5p\xi'^2$$
$$+(1.5p^3-3px'^2)x'\eta'+(2.75p^2-4.5x'^2)\eta'^2+(p^2-x'^2)x'\varphi'+3p\eta'\varphi'$$
$$+2.5\varphi'^2+2py'\phi'+px'\sigma'+4\xi'\phi'+3\eta'\sigma'+2y'\tau'+x'\rho'+\sigma x'^2+4\phi x'y'$$
$$+4yx'\phi'+6\varphi x'\eta'+8\xi x'\xi'+4\varphi y'^2+8yy'\varphi'+12y\eta'\xi'+12\xi y'\eta'$$
$$+8p\xi x'y'+8pyx'\xi'+18pyy'\eta'+19p^2yx'y'+12y^2x'\eta'-12y'^2x'\eta'$$
$$+24\eta yx'y'+9y^2y'^2+1.5y'^4-8yx'^3y'-4\varphi y^2-1.5y^4$$
$$+\eta\left(6x'\varphi'+12y'\xi'+7\eta'^2+14px'\eta'+9py'^2-12y\xi-\frac{7}{3}\eta^2-9py^2\right).$$

We may consider the case $0.998\leqq p\leqq 1$, since our desired result is $A_{11}<10$.

Firstly we consider the case $\eta\leqq 0$. Then

$$
\begin{aligned}
&3.25p^2x'^2\eta-\frac{7}{3}\eta^3-9p\eta y^2-12y\xi\eta+6x'\varphi'\eta+12y'\xi'\eta \\
&\quad+7\eta'^2\eta+14px'\eta'\eta+9py'^2\eta \\
&\leqq-\eta\ 5.5p[1-p^2-x'^2-2y'^2-3\eta'^2-4\xi'^2-5\varphi'^2-2y^2-4\xi^2] \\
&\quad+3.25p^2x'^2\eta+\left(16.5p-\frac{7}{3}\right)\eta^3-9py^2\eta-12y\xi\eta \\
&\quad+6\eta x'\varphi'+12\eta y'\xi'+7\eta\eta'^2+14p\eta x'\eta'+9p\eta y'^2 \\
&=-5.5p(1-p^2)\eta+\eta\left[(11p-9p)y^2-12y\xi+22p\xi^2+\left(16.5p-\frac{7}{3}\right)\eta^2\right] \\
&\quad+\eta[(3.25p^2+5.5p)x'^2+6x'\varphi'+27.5p\varphi'^2+14px'\eta' \\
&\qquad+(16.5p+7)\eta'^2+20py'^2+12y'\xi'+22p\xi'^2] \\
&\leqq-5.5p(1-p^2)\eta\,,
\end{aligned}
$$

since two terms in [ ] are positive definite for $0.998\leqq p\leqq 1$. By Lemma 2

$$
-\left\{\frac{17}{64}p^4+5.5p(1-p^2)\right\}\eta\leqq 3\left\{\frac{17}{64}p^4+5.5p(1-p^2)\right\}(1-p)\,.
$$

Further

$$
\begin{aligned}
&9y^2y'^2+1.5y'^4-12y'^2x'\eta' \\
&\leqq 4.5(1-p^2-x'^2-2y'^2-3\eta'^2)y'^2+1.5y'^4-12y'^2x'\eta' \\
&\leqq 4.5(1-p^2)y'^2.
\end{aligned}
$$

By the area theorem

$$
\begin{aligned}
3.8x'^2\leqq\,&3.8(1-p^2)-3.8(2y^2+3\eta^2+4\xi^2+5\varphi^2+6\phi^2+7\sigma^2+8\tau^2) \\
&-3.8(2y'^2+3\eta'^2+4\xi'^2+5\varphi'^2+6\phi'^2+7\sigma'^2+8\tau'^2+9\rho'^2)\,.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\Re b_{11}\leqq\,&\frac{1}{6}(1-p^2)+\frac{p^2}{16}(1-p^4)+\frac{p^4}{128}(1-p^2)+\frac{51}{64}p^4(1-p) \\
&+16.5p(1-p^2)(1-p)+3.8(1-p^2) \\
&-[(3p^3+7.6-10.5px'^2)y^2+(2.25p^2-6x'^2)y\xi+(15.2+2.5p)\xi^2 \\
&\quad+(2.25p^2+11.4-4.5x'^2)\eta^2+3p\eta\varphi+19\varphi^2+2py\phi+22.8\phi^2 \\
&\quad+4\xi\phi+3\eta\sigma+26.6\sigma^2+2y\tau+30.4\tau^2] \\
&-\left[\left(3.8-\frac{369}{128}p^4\right)x'^2+\{7.6-3p^3-4.5(1-p^2)-13.5px'^2\}y'^2\right. \\
&\quad-(2.25p^2-6x'^2)y'\xi'+(15.2-2.5p)\xi'^2-(1.5p^3-3px'^2)x'\eta' \\
&\quad+(11.4-2.75p^2+4.5x'^2)\eta'^2-(p^2-x'^2)x'\varphi'-3p\eta'\varphi'+19\varphi'^2
\end{aligned}
$$

$$-29y'\phi'-4\xi'\phi'+22.8\phi'^2-px'\sigma'-3\eta'\sigma'+26.6\sigma'^2-2y'\tau'$$
$$+30.4\tau'^2-x'\rho'+34.2\rho'^2\Big]$$
$$+\sigma x'^2+4\phi x'y'+4yx'\phi'+6\varphi x'\eta'+8\xi x'\xi'+4\varphi y'^2+8yy'\phi'$$
$$+12y\eta'\xi'+12\xi y'\eta'+8p\xi x'y'+8pyx'\xi'+18pyy'\eta'+19p^2yx'y'$$
$$+12y^2x'\eta'+24\eta yx'y'-8yx'^3y'-4\varphi y^2.$$

Now we make use of trivial inequalities with positive α_j:

$$\sigma x'^2\leqq 0.5\left(\alpha_1 x'^2\sigma^2+\frac{1}{\alpha_1}x'^2\right),$$
$$4\phi x'y'\leqq 2\left(\alpha_2 x'^2\phi^2+\frac{1}{\alpha_2}y'^2\right),$$
$$4yx'\phi'\leqq 2\left(\alpha_3 x'^2 y^2+\frac{1}{\alpha_3}\phi'^2\right),$$
$$6\varphi x'\eta'\leqq 3\left(\alpha_4 x'^2\varphi^2+\frac{1}{\alpha_4}\eta'^2\right),$$
$$8\xi x'\xi'\leqq 4\left(\alpha_5 x'^2\xi^2+\frac{1}{\alpha_5}\xi'^2\right),$$
$$4\varphi y'^2\leqq 2(\alpha_6 y'^2\varphi^2+\alpha_6^{-1}y'^2),$$
$$8yy'\varphi'\leqq 4(\alpha_7 y'^2y^2+\alpha_7^{-1}\varphi'^2),$$
$$12y\eta'\xi'\leqq 6(\alpha_8\eta'^2y^2+\alpha_8^{-1}\xi'^2),$$
$$12\xi y'\eta'\leqq 6(\alpha_9 y'^2\xi^2+\alpha_9^{-1}\eta'^2),$$
$$8p\xi x'y'\leqq 4p(\alpha_{10}x'^2\xi^2+\alpha_{10}^{-1}y'^2),$$
$$8pyx'\xi'\leqq 4p(\alpha_{11}x'^2y'^2+\alpha_{11}^{-1}\xi'^2),$$
$$18pyy'\eta'\leqq 9p(\alpha_{12}y'^2y^2+\alpha_{12}^{-1}\eta'^2),$$
$$19p^2yx'y'\leqq 9.5p^2(\alpha_{13}x'^2y^2+\alpha_{13}^{-1}y'^2),$$
$$12y^2x'\eta'\leqq 6(\alpha_{14}y^4+\alpha_{14}^{-1}x'^2\eta'^2),$$
$$24\eta yx'y'\leqq 12(\alpha_{15}y^2\eta^2+\alpha_{15}^{-1}x'^2y'^2),$$
$$-8yx'^3y'\leqq 4(\alpha_{16}x'^2y^2+\alpha_{16}^{-1}x'^4y'^2),$$
$$-4\varphi y^2\leqq 2(\alpha_{17}y^4+\alpha_{17}^{-1}\varphi^2).$$

The coefficient of y^2 is

$$3p^3+7.6-10.5px'^2-2\alpha_3x'^2-4\alpha_7y'^2-6\alpha_8\eta'^2-4p\alpha_{11}x'^2$$
$$-9p\alpha_{12}y'^2-9.5p^2\alpha_{13}x'^2-6\alpha_{14}y^2-12\alpha_{15}\eta^2-4\alpha_{16}x'^2-2\alpha_{17}y^2,$$

which is greater than

$$\begin{aligned}&3p^3+7.6-9.5\alpha_{13}(1-p^2)-(10.5p+2\alpha_3+4p\alpha_{11}+4\alpha_{16})x'^2\\&\quad+(19\alpha_{13}-4\alpha_7-9p\alpha_{12})y'^2+(28.5\alpha_{13}-6\alpha_8)\eta'^2+(19\alpha_{13}-6\alpha_{14}-2\alpha_{17})y^2\\&\quad+(28.5\alpha_{13}-12\alpha_{15})\eta^2.\end{aligned}$$

Now we put $\alpha_3=100$, $\alpha_7=150$, $\alpha_{11}=100$, $\alpha_{12}=140$, $\alpha_{13}=100$, $\alpha_{14}=230$, $\alpha_{15}=230$, $\alpha_{16}=100$ and $\alpha_{17}=250$. Then the above expression is greater than

$$3p^3+7.6-950(1-p^2)-(410.5p+600)x'^2\geqq 2.74$$

by $0.998\leqq p\leqq 1$ and $x'^2\leqq 1-p^2<0.004$. The coefficient of ξ^2 is

$$\begin{aligned}&15.2+2.5p-4\alpha_5x'^2-6\alpha_9y'^2-4\alpha_{10}x'^2\\&\geqq 15.2+2.5p-3\alpha_9(1-p^2)+(3\alpha_9-4\alpha_5-4\alpha_{10})x'^2\\&\geqq 15.2+2.5p-900(1-p^2)\,,\end{aligned}$$

if we put $\alpha_5=100$, $\alpha_9=300$ and $\alpha_{10}=100$. The last term is greater than 14.095 by $0.998\leqq p\leqq 1$. Similarly the coefficient of φ^2 is

$$\begin{aligned}&19-2\alpha_6y'^2-2\alpha_{17}^{-1}-3\alpha_4x'^2\\&\geqq 19-2/250-1200(1-p^2)\geqq 14.196\,,\end{aligned}$$

if we put $\alpha_4=400$, $\alpha_6=1200$ and $\alpha_{17}=250$. The coefficient of η^2 is

$$2.25p^2+11.4-4.5x'^2\geqq 13.623\,.$$

Now we put $\alpha_1=1000=\alpha_2$ and $\alpha_8=475$. Computations of coefficients of x'^2, y'^2 and η'^2 are now quite easy. Then we have

$$\mathfrak{R}b_{11}\leqq(1-p)P(p)-X-Y,$$

$$\begin{aligned}(1-p)P(p)=&\frac{1}{6}(1-p^6)+\frac{1}{16}p^2(1-p^4)+\frac{1}{128}p^4(1-p^2)\\&+\frac{51}{64}p^4(1-p)+16.5p(1-p^2)(1-p)+3.8(1-p^2)\,,\end{aligned}$$

$$\begin{aligned}X=&2.74y^2+(2.25p^2-6x'^2)y\xi+14.095\xi^2+2py\phi+4\xi\phi+22.78\phi^2\\&+2y\tau+30.4\tau^2+13.623\eta^2+3p\eta\varphi+14.196\varphi^2+3\eta\sigma+26.6\sigma^2,\end{aligned}$$

$$\begin{aligned}Y=&0.9166875x'^2-(1.5p^3-3px'^2)x'\eta'+8.5582\eta'^2-(p^2-x'^2)x'\varphi'\\&-3\eta'\varphi'+18.97\varphi'^2-px'\sigma'-3\eta'\sigma'+26.6\sigma'^2-x'\rho'+34.2\rho'^2\end{aligned}$$

$$+4.41645y'^2-(2.25p^2-6x'^2)y'\xi'+12.6\xi'^2-2py'\phi'-4\xi'\phi'$$
$$+22.78\phi'^2-2y'\tau'+30.4\tau'^2.$$

It is not so difficult to prove that X and Y are positive definite for $0.998 \leqq p \leqq 1$ and $x'^2 \leqq 0.004$. Thus $\mathcal{R}b_{11} \leqq (1-p)P(p)$. Now we set $p=1-x$. Then

$$10-P(p) \geqq 0.3375-22.5x+40x^2-10x^3-3x^4>0$$

for $0 \leqq x \leqq 0.002$. Hence $\mathcal{R}(b_{11}+10b_1)<10$.

Next we shall consider the case $\eta \geqq 0$. Then by the area theorem

$$\eta\left[\frac{13}{4}p^2x'^2+6x'\varphi'+12y'\xi'+7\eta'^2+14px'\eta'+9py'^2-12y\xi-\frac{7}{3}\eta^2-9py^2\right]$$
$$\leqq \eta\left[-\frac{27}{4}p^2x'^2+6x'\varphi'+12y'\xi'+7\eta'^2+14px'\eta'+9py'^2-12y\xi-\frac{7}{3}\eta^2-9py^2\right]$$
$$+10p^2\eta[1-p^2-2y'^2-3\eta'^2-5\xi'^2-2y^2-3\eta^2-4\xi^2]$$
$$=10p^2(1-p^2)\eta-\eta[(20p^2+9p)y^2+12y\xi+40p^2\xi^2+(30p^2+7/3)\eta^2]$$
$$-\eta\left[\frac{27}{4}p^2x'^2-14px'\eta'+(30p^2-7)\eta'^2-6x'\varphi'+50p^2\varphi'^2+(20p^2-9p)y'^2\right.$$
$$\left.-12y'\xi'+40p^2\xi'^2\right]$$
$$\leqq 10p^2(1-p^2)\eta\,.$$

Hence by

$$-\eta[(1/4+1/64)p^4-10p^2(1-p^2)-x'^4] \leqq 0$$

for $p^2 \geqq 0.996$ we can omit these terms. Therefore finally

$$\mathcal{R}b_{11} \leqq (1-p)Q(p)-X-Y,$$

where X and Y are the same as in the case $\eta \leqq 0$ and

$$(1-p)Q(p)=\frac{1}{6}(1-p^6)+\frac{1}{16}p^2(1-p^4)+\frac{p^4}{128}(1-p^2)+3.8(1-p^2)\,,$$

which is smaller than $(1-p)P(p)$. Hence we have the desired result.

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After the completion of this work the author has received two preprints [4] and [5]. In [4] they proved the existence of A_n and B_n with a crude estimate. In [5], $1<A_3\leqq 2$ was proved and a conjecture for the value of A_3 was stated.

DEPARTMENT OF MATHEMATICS
SCIENCE UNIVERSITY OF TOKYO
NODA, CHIBA, JAPAN