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KODAI MATH. J.
9 (1986), 77—83

SOME RESULTS IN GEOMETRY OF HYPERSURFACES

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0. Introduction.

In this paper we get several theorems about hypersurfaces in space forms.

In section 1, we show that if $x: M^n \to E^{n+1}$ is an isometric immersion of an n-dimensional complete non-compact Riemannian manifold whose sectional curvatures are greater than or equal to 0, then $x(M)$ is unbounded in E^{n+1}. We can prove this using Sacksteder theorem [12] which states that under the above condition $x(M)$ is the boundary of a convex body in E^{n+1}. But his proof is rather long and his theorem is more than what we need. do. Carmo and Lima [3] gave an independent proof of Sacksteder theorem, but it is also long. So we give a direct and easy proof using so-called Beltrami maps which are defined in do. Carmo and Warner [4].

In section 2, we show that if $x: M^n \to S^{n+1}(1)$ is an isometric immersion of an n-dimensional complete Riemannian manifold whose sectional curvatures are less than or equal to 1 and n is greater than 3, then $x(M)$ is totally geodesic. Ferus almost proved this result in [6], [7]. We consider higher codimensional cases.

All manifolds we consider in this paper are class C^∞, connected and have dimensions greater than or equal to 2. All immersions and vector fields are C^∞.

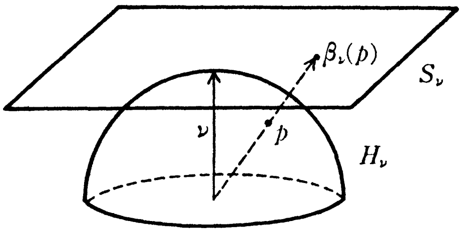
The author would like to express his hearty thanks to Professor S. Tanno for constant encouragement and advice.

1. Unboundedness of hypersurfaces.

The Beltrami maps are defined in M. do Carmo and F. Warner [2], and their properties are discussed fully.

Let $\nu \in S^{n+1}(1)$ ($\subset E^{n+2}$), and let H_ν denote the open hemisphere of $S^{n+1}(1)$ centered at ν. The Beltrami map β_ν is the diffeomorphism of H_ν onto the hyperplane $S_\nu \subset E^{n+2}$ tangent to $S^{n+1}(1)$ at ν obtained by central projection. We consider S_ν to be equipped with the canonical Riemannian structure induced from E^{n+2}. β_ν map great spheres of the sphere onto planes of S_ν, and vice versa. We call this Beltrami map as spherical Beltrami map.

Received March 26, 1985



The following proposition is in [4].

PROPOSITION 1. *Let $\nu \in S^{n+1}(1)$, let $X \subset H_\nu$ be a hypersurface, and let $\tilde{X}$ denote the hypersurface $\beta_\nu(X)$ in S_ν. Then $K_X \geqq 1$ everywhere if and only if $K_{\tilde{X}} \geqq 0$ everywhere.*

Now we get the following.

THEOREM 1. *Let M^n be a complete non-compact Riemannian manifold, and suppose that there is a compact subset C such that $K_M \geqq 0$ on $M \setminus C$. If $x: M^n \to E^{n+1}$ is an isometric immersion, then $x(M)$ is unbounded in E^{n+1}.*

Proof. Suppose $x(M)$ is bounded in E^{n+1}. We regard E^{n+1} as S_ν. We consider another Riemannian structure on M with respect to which

$$x: M^n \to S^{n+1}(1)$$

is an isometric immersion. We denote M with this Riemannian structure by $\tilde{M}$. It is easy to see that $\tilde{M}$ is complete. It follows from Proposition 1 that $K_{\tilde{M}} \geqq 1$ on $\tilde{M} \setminus C$. Using the same argument as in Bonnet theorem (cf. [2]), we conclude that $\tilde{M}$ is compact. This is a contradiction. (q. e. d.)

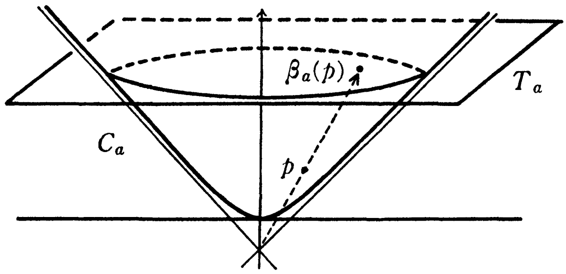
We define hyperbolic Beltrami map β_a. Put

$$H^{n+1}(-1) = \{(x^1, \cdots, x^{n+2}) \in \boldsymbol{R}^{n+2};\ (x^1)^2 + \cdots + (x^{n+1})^2 - (x^{n+2}+1)^2 = -1,\quad x^{n+2} \geqq 0\}$$

where $\boldsymbol{R}^{n+2}$ is endowed with indefinite metric $(dx^1)^2 + \cdots + (dx^{n+1})^2 - (dx^{n+2})^2$. For $a > 0$, we define the open cap C_a as

$$C_a = \{x \in H^{n+1}(-1);\ x^{n+2} < a\}$$

Let T_a be the $(n+1)$-dimensional plane which is perpendicular to the x^{n+2}-axis and contains $(0, \cdots, 0, a)$. The Beltrami map β_a is the diffeomorphism of C_a into the hyperplane T_a obtained by the projection from the center $(0, \cdots, 0, -1)$. For this Beltrami map, we have a proposition similar to Proposition 1.



THEOREM 2. *Let M^n be a complete non-compact Riemannian manifold, and suppose there is a compact subset C such that $K_M \geqq -1$ on $M \setminus C$. If $x: M^n \to H^{n+1}(-1)$ is an isometric immersion, then $x(M)$ is unbounded in $H^{n+1}(-1)$.*

Proof. Suppose $x(M)$ is bounded in $H^{n+1}(-1)$. We can assume $x(M)$ is contained in C_a. We define another Riemannian structure on M with respect to which

$$\beta_a \circ x : M^n \to T_a$$

is an isometric immersion. We denote M with this Riemannian structure by $\tilde{M}$. Then $\tilde{M}$ is complete and $K_{\tilde{M}} \geqq 0$ on $\tilde{M} \setminus C$, and $\tilde{M}$ is bounded in E^{n+1}. This contradicts theorem 1. (q. e. d.)

Next we turn to the negative curvature case.

Let $x: M^n \to N^m(c)$ be an isometric immersion of an n-dimensional Riemannian manifold M in an m-dimensional Riemannian manifold N with constant sectional curvature c. Let h denote the second fundamental form. For $x \in M$, define

$$T_0(x) = \{X \in T_x M;\ h(X, Y) = 0 \quad \text{for all } Y \in T_x M\}$$

$T_0(x)$ is called the space of relative nullity at x, and its dimension $\nu(x)$ is called the index of relative nullity at x. The minimal value ν_0 of ν on M is called the index of relative nullity of M. ν is upper-semicontinuous, and so the set G where $\nu = \nu_0$ holds is open. The following theorem is well-known.

THEOREM 3 ([5], [9]). *T_0 is integrable on G, and its integral manifolds are totally geodesic submanifolds of M. They are totally geodesically immersed in $N^m(c)$ by x. If M is complete, then the maximal integral manifolds of $T_0|_G$ are also complete.*

Now we show the following.

THEOREM 4. *Let M^n be a non-compact complete n-dimensional Riemannian manifold, and suppose there is a compact subset C such that $K_M \leqq 0$ on $M \setminus C$. If $x: M^n \to E^{n+1}$ is an isometric immersion and $n \geqq 3$, then $x(M)$ is unbounded in E^{n+1}.*

Proof. It follows from the curvature hypothesis that $\nu(x)\geqq n-2$ at every $x\in M\setminus C$. Put

$$G_0=\{x\in M\setminus C;\ \nu(x)=n-2\}$$

Suppose $\bar{G}_0$ (the closure of G_0) is not compact. Since G_0 is open, we can choose $p_k\in G_0$ for any integer $k>0$ such that $d_M(p_k, C)\geqq k$. It follows from theorem 3 that the $(n-2)$-dimensional totally geodesic manifold in M through p_k can be extended so far as it meets C. This totally geodesic submanifold is also totally geodesic in E^{n+1}, and so M is not bounded in E^{n+1}. If $\bar{G}_0$ is compact, put

$$G_1=\{x\in M\setminus(C\cup\bar{G}_0);\ \nu(x)=n-1\}$$

Suppose $\bar{G}_1$ is not compact. Since G_1 is open, the same argument as above holds good, and M is not bounded in E^{n+1}. If $\bar{G}_1$ is compact, then $M\setminus(C\cup\bar{G}_0\cup\bar{G}_1)$ is non-compact and $\nu\equiv n$ there. So we see that M is not bounded in E^{n+1}.

(q. e. d.)

We can show the following theorem in the same way.

THEOREM 5. *Let M^n be a non-compact complete n-dimensional Riemannian manifold, and suppose there is a compact subset C such that $K_M\leqq-1$ on $M\setminus C$. If $x: M^n\to H^{n+1}(-1)$ is an isometric immersion and $n\geqq 3$, then $x(M)$ is unbounded in $H^{n+1}(-1)$.*

2. Submanifold with $K_M\leqq 1$ in $S^{n+p}(1)$.

Consider the following question.

Let $x: M^n\to S^{n+p}(1)$ be an isometric immersion of a complete n-dimensional Riemannian manifold M with $K_M\leqq 1$ in $S^{n+p}(1)$. Is $x(M)$ totally geodesic?

Of course if $p\geqq n-1$ the flat torus gives negative answer to this question. In low codimension we can give a partial positive answer. First we consider the case $p=1$.

THEOREM 6. *If $n\geqq 4$ and $p=1$, then $x(M)$ is totally geodesic.*

Proof. Assume $n\geqq 5$. It is easily proved that the index of relative nullity $\nu_0\geqq n-2$. If $\nu_0\geqq n-1$, then $K_M\equiv 1$ and according to O'Neill and Stiel [10], we can conclude that $x(M)$ is totally geodesic. So we suppose $\nu_0=n-2$. Choose a point $x\in M$ which satisfies $\nu(x)=n-2$. The maximal integral manifold L_1 of T_0 through x is mapped to a $(n-2)$-dimensional great sphere in $S^{n+1}(1)$. Choose another point $y\in M$ which is not on L_1 and sufficiently near x. The maximal integral manifold L_2 of T_0 through y is also mapped to a $(n-2)$-great sphere in $S^{n+1}(1)$. We consider $S^{n+1}(1)$ as the unit hypersphere in E^{n+2}. Since L_1 and

L_2 do not intersect and L_1, L_2 are respectively on some $(n-1)$-planes through the origin in E^{n+2},

$$2(n-1) \leqq n+2$$

holds; that is $n \leqq 4$, this is a contradiction.

If $n=4$, we need the following theorem due to Ferus [7].

Let $\rho(t)$ denote the largest integer such that the fibration

$$V'_{t,\rho(t)} \to V'_{t,1}$$

of Stiefel manifolds has a global cross section (the points in $V'_{t,r}$ are the ordered r-tuples of linearly independent vectors in $\boldsymbol{R}^t$). For every integer n define ν_n to be the largest integer such that $\rho(n-\nu_n) \geqq \nu_n+1$.

THEOREM 7 ([7]). *Let M^n be an n-dimensional Riemannian manifold and T_0 a ν-dimensional, integrable distribution on M^n with the following properties;*

(1) *the maximal integral manifolds of T_0 are totally geodesic and complete.*

(2) *the sectional curvature of M has the same positive value k on all planes spanned by tangent vectors X, Y with $X \in T_0$ and $Y \in T_0^\perp$.*

then $\nu > \nu_n$ implies $\nu = n$.

We finish the proof of theorem 6. As $\nu_4=0$ [7] and $\nu \geqq 2$, the conclusion follows. (q. e. d.)

If $n=2$, 3, there are counter-examples.

$n=2$ $\qquad f: S^1(a) \times S^1(b) \to S^3(1) \qquad (a^2+b^2=1)$

$$n=3 \qquad \begin{cases} 2x_2^3+3(x_1^2+x_2^2)x_5-6(x_3^2+x_4^2)x_5 \\ \qquad +3\sqrt{3}(x_1^2-x_2^2)x_4+3\sqrt{3}\,x_1x_2x_3=2 \\ x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=1\,. \end{cases}$$

This is a homogeneous Riemannian manifold $SO(3)/\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ and its principal curvatures are equal to $\sqrt{3}$, 0, $-\sqrt{3}$ [13].

We consider higher codimentional cases.

LEMMA 1. *Let $x: M^n \to N^{n+p}(c)$ be an isometric immersion of an n-dimensional Riemannian manifold M^n with $K_M \leqq c$ in $(n+p)$-dimensional Riemannian manifold $N^{n+p}(c)$ with $K_N \equiv C$. If the normal connection is flat and $n > 2^p$, then the index of relative nullity ν_0 satisfies $\nu_0 \geqq n-2^p$.*

Proof. If $p=1$, lemma is clear. Suppose $p=2$. Since the normal connection is flat, there exist orthonormal normal vector fields ξ_1, ξ_2 such that A_α $(\alpha=1, 2)$ is simultaneously diagonalizable where we write $A_\alpha = A_{\xi_\alpha}$, the second fundamental forms associated with ξ_α. Let $\lambda_{\alpha,i}$ $(1 \leqq i \leqq n,\ 1 \leqq \alpha \leqq 2)$ be the eigenvalues of A_α corresponding to orthonormal eigenvectors E_i. Let p, q, r be the numbers of positive, zero and negative $\lambda_{1,i}$ $(1 \leqq i \leqq n)$. We may assume $p_1 \geqq r_1$ (by the change of the sign of ξ_1 if neccessary). We may assume $\lambda_{1,i} > 0$ $(1 \leqq i \leqq p_1)$, and $\lambda_{1,j} = 0$ $(p_1 + 1 \leqq j \leqq p_1 + q_1)$. We have

$$p_1 + q_1 \geqq n/2 > 2 \tag{1}$$

From Gauss equation and the curvature assumption, we have

$$\lambda_{1,i} \cdot \lambda_{1,j} + \lambda_{2,i} \cdot \lambda_{2,j} \leqq 0 \qquad (1 \leqq i < j \leqq p_1 + q_1) . \tag{2}$$

Since $\lambda_{1,i} \geqq 0$ $(1 \leqq i \leqq p_1 + q_1)$

$$\lambda_{2,i} \cdot \lambda_{2,j} \leqq -\lambda_{1,i} \cdot \lambda_{1,j} \leqq 0 \qquad (1 \leqq i < j \leqq p_1 + q_1) . \tag{3}$$

Then the same arguement of the $p=1$ case applies, we have $p_1 + q_1 - 2$ (>0) zeros in $\lambda_{2,j}$ $(1 \leqq j \leqq p_1 + q_1)$. If $p_1 \leqq 1$, then $q_1 = n - p_1 - r_1 \geqq n - 2$. So we may assume $p_1 > 1$. It follows from (2) that the zeros are in $\lambda_{2,j}$ $(p_1 + 1 \leqq j \leqq p_1 + q_1)$. So $q_1 \geqq p_1 + q_1 - 2$, that is $p_1 \leqq 2$. Since $p_1 \geqq r_1$, we have $r_1 \leqq 2$. Hence

$$q_1 = n - p_1 - r_1 \geqq n - 4 > 0 .$$

This proves $p=2$ case. General case can be proved in the same way. (q. e. d.)

THEOREM 8. *Let $x : M^n \to S^{n+p}(1)$ be an isometric immersion of an n-dimensional complete Riemannian manifold with $K_M \leqq 1$. If the normal connection is flat and $n \geqq 2^{p+1}$, then $x(M)$ is totally geodesic.*

Proof. According to Ferus [7],

$$\nu_n \leqq \frac{1}{2}(n-1) .$$

On the other hand, from lemma, we have

$$\nu_0 \geqq n - 2^p .$$

The hypothesis $n \geqq 2^{p+1}$ implies

$$\nu_0 \geqq n - 2^p > \frac{1}{2}(n-1) \geqq \nu_n .$$

Thus it follows from theorem 7 that $\nu_0 = n$, that is, $x(M)$ is totally geodesic.
(q. e. d.)

3. Remarks.

a) In theorem 1, 2 higher codimensional cases don't hold. It is easy to construct counter-examples.

b) Using lemma 1, we can slightly extend theorem 3, 4 to higher codimensional cases.

c) The case $n=2$ in theorem 4, that is, the existance in E^3 of a complete bounded surface of non-positive curvature, is completely open. A possible example is constructed by Rozendorn [11] which has a denumerable number of isolated singular points. Note that in this example $\inf K_M=-\infty$. This question is closely related to Jorge and Koutroufiotis [8].

d) When we almost finished this work, we found that Borisenko [1] had given positive answer to the question posed in section 2 under the condition that M is compact and $p<-1/2+\sqrt{1/4+n/2}$.

REFERENCES

[1] A. BORISENKO, Complete l-dimensional surfaces of non-positive extrinsic curvature in a Riemannian space, Math. USSR Sb. **33** (1977), 485-499.

[2] J. CHEEGER AND D. EBIN, Comparison theorems in Riemannian Geometry, North Holland (1975).

[3] M.P. do CARMO AND E. LIMA, Immersions of manifolds with non-negative sectional curvatures, Bol. Soc. Brasil Mat. **2** (1971), 9-22.

[4] M.P. do CARMO AND F.W. WARNAR, Rigidity and convexity of hypersurfaces in spheres, J. Diff. Geom. **4** (1970), 133-144.

[5] D. FERUS, On the completeness of nullity foliations, Michigan Math. J. **18** (1971), 61-64.

[6] D. FERUS, On the type number of hypersurfaces in spaces of constant curvature, Math. Ann. **187** (1970), 310-316.

[7] D. FERUS, Totally geodesic foliations, Math. Ann. **188** (1970), 313-316.

[8] L. JORGE AND D. KOUTROUFIOTIS, An estimate for the curvature of bounded submanifolds. Amer. J. Math. **103** (1981), 711-725.

[9] R. MALTZ, Isometric immersions into spaces of constant curvature, Illinois J. Math. **15** (1971), 490-502.

[10] B. O'NEILL AND E. STIEL, Isometric immersions of constant curvature manifolds, Michigan Math. J. **10** (1963), 335-339.

[11] E.R. ROZENDORN, Construction of bounded complete surface of nonpositive curvature, Usp. Math. Nauk. **16** (1961), 149-156. (in Russian)

[12] R. SACKSTEDER, On hypersurfaces with no negative sectional curvatures, Amer. J. Math. **82** (1960), 609-630.

[13] R. TAKAGI, Homogeneous hypersurfaces in a sphere with type number 2, Tōhoku Math. J. **23** (1971), 49-58.

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