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COADJOINT EQUIVARIANCY OF MOMENTUM MAPPING

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1. A coadjoint orbit of a Lie group G is a G-homogeneous symplectic manifold with the inclusion map as the coadjoint equivariant momentum mapping. If (M, Ω) is a G-homogeneous symplectic manifold with the coadjoint equivariant momentum mapping, then the coadjoint equivariant momentum mapping gives a symplectic covering mapping onto a coadjoint orbit. And we have a result of B. Kostant which classifies the (simply connected) homogeneous symplectic manifolds with coadjoint equivariant momentum mappings (cf. [1], [2], [5]). An action with a fixed point may be considered to be antipodal with a homogeneous action. As explained in [4], we have some theorems concerning with the existence of (coadjoint equivariant) momentum mappings. In particular, the following three guarantee the coadjoint equivariancy of a momentum mapping:

(1) $H^2(\mathfrak{g}, R)=0$, where $\mathfrak{g}$ is the Lie algebra of G (cf. [4], [5], [6]),

(2) the symplectic form is an exact form of a G-invariant 1-form (cf. [1]), and

(3) G is a semidirect product of G_1 by G_2, where G_1 and G_2 have coadjoint equivariant momentum mappings, $H^1(\mathfrak{g}_1, R)=0$ and G_1 is connected (cf. [3], [4]).

In this paper, we give a condition for the coadjoint equivariancy of momentum mappings. The result is

PROPOSITION. *Let (M, Ω) be a connected symplectic manifold, and let G be a symplectic action on (M, Ω) with a momentum mapping. If the action G has a fixed point, then G has a coadjoint equivariant momentum mapping.*

2. Let (M, Ω) be a connected symplectic manifold, that is, M is a connected smooth manifold with a non-degenerate closed 2-form Ω. Ω induces a bundle isomorphism $\Omega^\flat : TM \to T^*M$ between the tangent bundle and the cotangent bundle of M defined by

$$\Omega^\flat(v)=v \lrcorner \Omega .$$

Denote the inverse of $\Omega^\flat$ by $\Omega^\#$. $\Omega^\# : T^*M \to TM$ is also a bundle isomorphism. Let $C^\infty(M)$ (resp. $\mathrm{aut}(M, \Omega)$) be the set of all real valued smooth functions (resp. Hamiltonian vector fields i.e., vector field X satisfying $L_X\Omega=0$) on M.

For each $f \in C^\infty(M)$, define $\beta(f)$ by

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$$\beta(f)=\Omega^{\#}(df), \text{ or equivalently, } \Omega^{\flat}(\beta(f))=df.$$

Since we have

$$\begin{aligned} L_{\beta(f)}\Omega &= \beta(f) \lrcorner\, d\Omega + d(\beta(f) \lrcorner\, \Omega) \\ &= d(\Omega^{\flat}(\beta(f))) = d(df) = 0\,, \end{aligned}$$

$\beta(f) \in \operatorname{aut}(M, \Omega)$ for each $f \in C^{\infty}(M)$. For each f, h in $C^{\infty}(M)$, the Poisson bracket $\{f, h\}$ is defined by

$$\begin{aligned} \{f, h\} &= -\Omega(\beta(f), \beta(h)) \\ &= \beta(f) \lrcorner\, dh\,. \end{aligned}$$

$C^{\infty}(M)$ is a Lie algebra with the Poisson bracket $\{,\}$. For example, the Jacobi identity for $\{,\}$ comes from $d\Omega=0$. It is clear that $\operatorname{aut}(M, \Omega)$ is a Lie subalgebra of the Lie algebra of all smooth vector fields on M. Since we have

$$\begin{aligned} d\{f, h\} &= d(\beta(f) \lrcorner\, dh) \\ &= L_{\beta(f)} dh \\ &= L_{\beta(f)}(\beta(h) \lrcorner\, \Omega) \\ &= [\beta(f), \beta(h)] \lrcorner\, \Omega \\ &= \Omega^{\flat}[\beta(f), \beta(h)]\,, \end{aligned}$$

β satisfies $\beta\{f, h\}=[\beta(f), \beta(h)]$, that is, β is a Lie algebra homomorphism from $C^{\infty}(M)$ into $\operatorname{aut}(M, \Omega)$.

Since $L_X\Omega = dX \lrcorner\, \Omega = d(\Omega^{\flat}(X))$, for each Hamiltonian vector field X we have a closed 1-form $\Omega^{\flat}(X)$. Let $\gamma(X)$ be the de Rham cohomology class of $\Omega^{\flat}(X)$ for each X in $\operatorname{aut}(M, \Omega)$. We have

$$\begin{aligned} \Omega^{\flat}([X, Y]) &= [X, Y] \lrcorner\, \Omega \\ &= L_X(Y \lrcorner\, \Omega) \\ &= d(X \lrcorner\, Y \lrcorner\, \Omega) \qquad \text{for each } X, Y \text{ in } \operatorname{aut}(M, \Omega). \end{aligned}$$

This implies $\gamma[X, Y]=0$. Thus γ is a Lie algebra homomorphism from $\operatorname{aut}(M, \Omega)$ into $H^1(M, \boldsymbol{R})$ if we introduce the trivial Lie algebra structure in $H^1(M, \boldsymbol{R})$. We have the following short exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow R \hookrightarrow C^{\infty}(M) \xrightarrow{\beta} \operatorname{aut}(M, \Omega) \xrightarrow{\gamma} H^1(M, \boldsymbol{R}) \longrightarrow 0\,.$$

Let G be a Lie group. A smooth left action $\phi: G \times M \to M$ is called a symplectic action of a Lie group G on a symplectic manifold (M, Ω) if for each g in G the map $\phi_g: M \to M: m \mapsto \phi(g, m)$ is a symplectomorphism, that is, $\phi_g^*\Omega = \Omega$. A symplectic action (G, ϕ) on (M, Ω) induces a Lie algebra homomorphism

$$\rho : \mathfrak{g} \longrightarrow \mathrm{aut}(M, \Omega)$$

defined by

$$\rho(\xi)_m = (\psi_m)_{*e}(-\xi),$$

where $\mathfrak{g}$ is the Lie algebra of G consisting of all left-invariant vector fields on G, ψ_m is the orbit map of (G, ϕ) through m, that is, $\psi_m(g) = \phi(g, m)$ and e is the unit of G.

A momentum mapping for the symplectic action (G, ϕ) is a mapping $J: M \to \mathfrak{g}^*$ (=the dual space of $\mathfrak{g}$) such that

$$d\langle J, \xi\rangle = \rho(\xi) \lrcorner \Omega .$$

This condition is equivalent to

$$d\hat{J}(\xi) = \Omega^\flat(\rho(\xi))$$

or

$$\Omega^\sharp d\hat{J}(\xi) = \rho(\xi) \qquad \text{i.e.,} \qquad \beta(\hat{J}(\xi)) = \rho(\xi),$$

where $\hat{J}(\xi)$ in $C^\infty(M)$ is defined by

$$\hat{J}(\xi)(m) = \langle J(m), \xi\rangle .$$

A momentum mapping $J: M \to \mathfrak{g}^*$ for the symplectic action (G, ϕ) is coadjoint equivariant if and only if J satisfies

$$J(\phi_g(m)) = Ad(g^{-1})^* J(m)$$

for each g, m in G and M. This condition is equivalent to

$$\hat{J}\, Ad(g^{-1}) = \phi_g^* \hat{J} \qquad \text{for each } g \text{ in } G.$$

Differentiating the above, we have

$$\hat{J}[\xi, \eta] = \{\hat{J}(\xi), \hat{J}(\eta)\} \qquad \text{for each } \xi, \eta \text{ in } \mathfrak{g},$$

that is, $\hat{J}$ is a Lie algebra homomorphism of $\mathfrak{g}$ into $C^\infty(M)$. We know that if the Lie group G is connected and if $\hat{J}$ is a Lie algebra homomorphism, then J is coadjoint equivariant.

3. At first, we prove

LEMMA (cf. [1]). *For a symplectic action (G, ϕ) on a symplectic manifold (M, Ω)* we have

(1) $\phi_g^* \Omega^\flat \phi_{g*} = \Omega^\flat$
(2) $\phi_{g*} \Omega^\sharp \phi_g^* = \Omega^\sharp$
(3) $\phi_{g*} \beta(f) = \beta(\phi_{g^{-1}}^* f)$
(4) $\phi_g^* \{f, h\} = \{\phi_g^* f, \phi_g^* h\}$
(5) $\phi_{g*} \rho(\xi) = \rho(Ad(g)\xi)$

for each g in G, f, h in $C^\infty(M)$ and ξ in $\mathfrak{g}$. These mean that the diagrams below are commutative.

$$\begin{tikzcd} T^*M \arrow[r, "\Omega^\#"] & TM \arrow[d, "\phi_{g*}"] \\ T^*M \arrow[u, "\phi_g^*"] \arrow[r, "\Omega^\#"] & TM \end{tikzcd} \qquad \begin{tikzcd} C^\infty(M) \arrow[r, "\beta"] & \mathrm{aut}(M, \Omega) \arrow[d, "\phi_{g*}"] & \mathfrak{g} \arrow[l, "\rho"'] \arrow[d, "\mathrm{Ad}(g)"] \\ C^\infty(M) \arrow[u, "\phi_g^*"] \arrow[r, "\beta"] & \mathrm{aut}(M, \Omega) & \mathfrak{g} \arrow[l, "\rho"] \end{tikzcd}$$

Proof of Lemma. For each g in G, $\phi_g^*\Omega=\Omega$ implies

$$\Omega(\phi_{g*}v,\ \phi_{g*}w)=\Omega(v,\ w) \qquad \text{for any } v,\ w \text{ in } T_mM.$$

This means

$$(\phi_g^*\Omega^\flat(\phi_{g*}v))(w)=(\Omega^\flat(v))(w)$$

and proves (1). (2) is only another expression of (1). (2) implies (3), since

$$\begin{aligned}\phi_{g*}\beta(f)&=\phi_{g*}\Omega^\#(df)=\Omega^\#(\phi_{g^{-1}}^*df)=\Omega^\#(d\phi_{g^{-1}}^*f)\\ &=\beta(\phi_{g^{-1}}^*f)\,.\end{aligned}$$

The proof of (4) is

$$\begin{aligned}\phi_g^*\{f,\ h\}&=\phi_g^*(\beta(f)\lrcorner\, dh)=(\phi_{g^{-1}*}\beta(f))\lrcorner\,\phi_g^*dh\\ &=\beta(\phi_g^*f)\lrcorner\, d(\phi_g^*h)=\{\phi_g^*f,\ \phi_g^*h\}\end{aligned}$$

using (3). (5) comes from

$$\begin{aligned}\phi_{g*}(\rho(\xi)_m)&=\phi_{g*}(\psi_m)_{*e}(-\xi)=(\phi_g\psi_m)_{*e}(-\xi)\\ &=(\psi_{\phi(g,m)}I_g)_{*e}(-\xi)=(\psi_{\phi(g,m)})_{*e}(-Ad(g)\xi)\\ &=\rho(Ad(g)\xi)_{\phi(g,m)}\,,\end{aligned}$$

where I_g is the inner automorphism of G induced by g and $(I_g)_{*e}=Ad(g)$.

Now we prove our Proposition. Let J be a given momentum mapping for the symplectic action $(G,\ \phi)$, that is, $\hat{J}$ is linear and satisfies $\beta\hat{J}=\rho$. Using Lemma, we have

$$\begin{aligned}\beta(\phi_g^*\hat{J}(\xi))&=\phi_{g^{-1}}^*\beta\hat{J}(\xi)=\phi_{g^{-1}}^*\rho(\xi)=\rho(Ad(g^{-1})\xi)\\ &=\beta\hat{J}(Ad(g^{-1})\xi) \qquad \text{for each } g \text{ in } G.\end{aligned}$$

Therefore

$$\phi_g^*\hat{J}(\xi)-\hat{J}(Ad(g^{-1})\xi)$$

is a constant function on M, and so

$$\phi_g^* \hat{J}(\xi) - \hat{J}(Ad(g^{-1})\xi) = (\phi_g^* \hat{J}(\xi))(m_0) - \hat{J}(Ad(g^{-1})\xi)(m_0)$$

for any given point m_0 in M. Let m_0 be a fixed point for the symplectic action (G, ϕ), that is, $\phi_g(m_0) = m_0$ for each g in G. Then we have

$$\phi_g^* \hat{J}(\xi) - \hat{J}(Ad(g^{-1})\xi) = \hat{J}(\xi)(m_0) - \hat{J}(Ad(g^{-1})\xi)(m_0)$$

for each g in G. Define $\lambda : \mathfrak{g} \to C^\infty(M)$ by

$$\lambda(\xi) = \hat{J}(\xi) - \hat{J}(\xi)(m_0).$$

Then λ is linear and satisfies $\beta\lambda = \rho$. Moreover we have

$$\begin{aligned} \phi_g^* \lambda(\xi) &= \phi_g^* \hat{J}(\xi) - \hat{J}(\xi)(m_0) \\ &= \hat{J}(Ad(g^{-1})\xi) - \hat{J}(Ad(g^{-1})\xi)(m_0) \\ &= \lambda(Ad(g^{-1})\xi). \end{aligned}$$

Thus λ defines a coadjoint equivariant momentum mapping for the symplectic action (G, ϕ). This completes the proof of our Proposition.

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