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SOME FUNCTIONS ON THE SET OF TRIANGLES OR QUADRANGLES

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Dedicated to Professor Shigeru Ishihara on his 60 th Birthday

1. Introduction.

Let $M=(A_1A_2\cdots A_k)$ be a polygon on a Euclidean 2-space E^2 with the canonical coordinate system (x, y), and let $P=(x, y)$ be an arbitrary point of E^2. We define a function $f(x, y)=f_M(P)$ corresponding to M by

$$f_M(P)=\sum_{i=1}^{k}|\Delta PA_iA_{i+1}|^2,$$

where $A_{k+1}=A_1$ and $|\Delta PAB|$ denotes the area of the triangle determined by three points P, A and B. Now we define $F(M)$ by

$$F(M)=\int e^{-4r^2f(x,y)}dE^2,$$

where r denotes a non-zero real number. For a triangle M we get the following equality:

$$F(M)=\frac{\pi}{2\sqrt{3}\,r^2S(M)}e^{-4r^2S(M)^2/3},$$

where $S(M)$ denotes the area of M (cf. Theorem 2.1). The main result of this paper is the following (cf. Theorem 3.1.).

THEOREM A. *Let M be a quadrangle on E^2. Then*

$$F(M)\leqq\frac{\pi}{2r^2S(M)}e^{-r^2S(M)^2}$$

holds, where the equality holds if and only if M is a parallelogram. Therefore, for any parallelogram M of the fixed area $S=S(M)$, $F(M)$ is independent of the shape of M.

Let $Q=(x, y, z)$ be a point of a Euclidean 3-space E^3 and define a function $g(x, y, z)=g_M(P)$ corresponding to a polygon M by

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$$g_M(P) = \sum_{i=1}^{k} |\Delta QA_iA_{i+1}|^2 .$$

Then we get a function $G(M)$ similarly (cf. § 4).

The author is grateful to Professor Yoichiro Takahashi for introducing us the function $G(M)$ for quadrangles M.

2. The axis of a triangle.

Let $M=\Delta ABC$ be a triangle on a Euclidean 2-space E^2. We can assume that M is placed so that the origin is the center of gravity of M and $A=(0, b_1)$, $B=(-a, b_2)$ and $C=(a, b_3)$ with $a>0$. Then

$$b_1+b_2+b_3=0 .$$

The equations of lines are given by

$$AB:\ (b_1-b_2)x-ay+ab_1=0 ,$$

$$BC:\ (b_2-b_3)x+2ay-a(b_2+b_3)=0 ,$$

$$CA:\ (b_1-b_3)x+ay-ab_1=0 .$$

Then, for a point $P=(x, y)$ of E^2 we obtain

$$\begin{aligned} 4f_M(P) &= 4(|\Delta PAB|^2+|\Delta PBC|^2+|\Delta PCA|^2) \\ &= [(b_1-b_2)x-ay+ab_1]^2+[(b_2-b_3)x+2ay+ab_1]^2 \\ &\quad +[(b_1-b_3)x+ay-ab_1]^2 \\ &= 6[(b_1^2+b_1b_2+b_2^2)x^2+a(b_1+2b_2)xy+a^2y^2]+3a^2b_1^2 . \end{aligned}$$

If $b_1+2b_2\neq 0$, we define an orthonormal base (e_1, e_2) by $e_1=(e_{11}, e_{12})$ and $e_2=(e_{21}, e_{22})$, where

$$e_{12}/e_{11}=(a^2-b_1^2-b_1b_2-b_2^2+\theta)/a(b_1+2b_2) ,$$

$$e_{22}/e_{21}=(a^2-b_1^2-b_1b_2-b_2^2-\theta)/a(b_1+2b_2) ,$$

$$\theta=[(a^2+b_1^2+b_1b_2+b_2^2)^2-3a^2b_1^2]^{1/2} .$$

Let (x^*, y^*) be a new coordinate system with respect to (e_1, e_2). Then we obtain

$$4f_M(P)=3(\lambda_1 x^{*2}+\lambda_2 y^{*2})+3a^2b_1^2 , \tag{2.1}$$

where

$$\lambda_1,\ \lambda_2=a^2+b_1^2+b_1b_2+b_2^2\pm\theta .$$

If $b_1+2b_2=0$, then $\lambda_1=3b_2^2$ and $\lambda_2=a^2$ in (2.1) with respect to (x, y). In any case, we obtain $\lambda_1\lambda_2=3a^2b_1^2$. The area $S(M)$ of M is given by $S(M)=3ab_1/2$

and hence $3a^2b_1^2=4S(M)^2/3$. Therefore

$$\int e^{-4r^2 f(x,y)}dE^2=\int e^{-3r^2\lambda_1 x^2}dx\int e^{-3r^2\lambda_2 y^2}dy\, e^{-3r^2a^2b_1^2}$$

$$=\frac{\pi}{3r^2(\lambda_1\lambda_2)^{1/2}}e^{-4r^2S(M)^2/3}.$$

Thus, we get

THEOREM 2.1. *For a triangle M on E^2,*

$$F(M)=\frac{\pi}{2\sqrt{3}\,r^2S(M)}e^{-4r^2S(M)^2/3}$$

holds, i.e., $F(M)$ depends only on the area of M.

We may call e_1-direction (e_2-direction, resp.) the axis of the triangle ΔABC. If ΔABC is an equilateral triangle, then $b_1=-2b_2=-2b_3$, $a^2=3b_2^2$ and $\theta=0$ hold, and hence e_1 is not definite. In this case any direction through the center of gravity of ΔABC is an axis.

A geometric meaning of the axes of a triangle is (2.1). It is an open question if there are other geometric meanings of axes of a triangle.

3. $F(M)$ for a quadrangle.

Let $M=(ABCD)$ be a quadrangle on E^2. We can assume that M is placed so that

$$A=(-a,0),\qquad B=(e,-d),$$

$$C=(b,0),\qquad D=(0,c),$$

where $b>0$, $a+b>0$ and $c>0$.

The equations of lines are given by

$$AB:\ dx+(a+e)y+ad=0,$$

$$BC:\ dx-(b-e)y-bd=0,$$

$$CD:\ cx+by-bc=0,$$

$$DA:\ cx-ay+ac=0.$$

Then, for a point $P=(x,y)$ of E^2 we obtain

$$4f_M(P)=[dx+(a+e)y+ad]^2+[dx-(b-e)y-bd]^2$$

$$+[cx+by-bc]^2+[cx-ay+ac]^2$$

$$=2(c^2+d^2)x^2+2[a^2+b^2+e^2+(a-b)e]y^2$$
$$+2[-(a-b)(c-d)+2de]xy+2(a-b)(c^2+d^2)x$$
$$+2[-(a^2+b^2)(c-d)+(a-b)de]y+(a^2+b^2)(c^2+d^2)\,.$$

Putting $x=\bar{x}-\alpha$, $y=\bar{y}-\beta$, we get

$$4f_M(P)=2(c^2+d^2)\bar{x}^2+2[a^2+b^2+e^2+(a-b)e]\bar{y}^2 \tag{3.1}$$
$$+2[-(a-b)(c-d)+2de]\overline{xy}+C(M)\,,$$

where

$$\alpha=(a-b)(a^2+b^2)(c+d)^2/K+2(a-b)c^2e^2/K$$
$$+[(a-b)^2(2c^2+d^2+cd)+2(a^2+b^2)(c-d)d]e/K\,,$$
$$\beta=-(a+b)^2(c^2+d^2)(c-d)/K\,,$$
$$K=3(a^2+b^2)(c^2+d^2)+2[ab(c-d)^2+(a^2+b^2)cd]$$
$$+4(a-b)c(c+d)e+4c^2e^2$$
$$=2(a+b)^2(c^2+d^2)+[2ce+(a-b)(c+d)]^2>0\,,$$
$$C(M)=-\alpha(a-b)(c^2+d^2)-\beta[-(a^2+b^2)(c-d)+(a-b)de]$$
$$+(a^2+b^2)(c^2+d^2)$$
$$=S(M)^2+T(M)\,,$$

where $S(M)=(a+b)(c+d)/2$ is the area of M and

$$T(M)=(a+b)^2(c-d)^2c^2e^2/K+(a-b)(a+b)^2c(c-d)(c^2-d^2)e/K \tag{3.2}$$
$$+(1/4)(a-b)^2(a+b)^2(c-d)^2(c+d)^2/K$$
$$=(a+b)(c-d)^2[ce+(1/2)(a-b)(c+d)]^2/K\geqq 0\,.$$

$T(M)=0$ holds if and only if $c=d$ or $2ce=-(a-b)(c+d)$. The first three terms of the right hand side of (3.1) is written as

$$\lambda_1 x^{*2}+\lambda_2 y^{*2}$$

where λ_1 and λ_2 are given by

$$\lambda_1,\ \lambda_2=a^2+b^2+c^2+d^2+e^2+(a-b)e\pm L\,,$$
$$L^2=[a^2+b^2+c^2+d^2+e^2+(a-b)e]^2$$
$$-[3(a^2+b^2)(c^2+d^2)+2ab(c^2+d^2)+2(a-b)^2cd$$
$$+4(a-b)(c+d)ce+4c^2e^2]\,.$$

Consequently,

$$\lambda_1\lambda_2=3(a^2+b^2)(c^2+d^2)+2ab(c^2+d^2)+2(a-b)^2cd$$
$$+4(a-b)(c+d)ce+4c^2e^2$$
$$=4S(M)^2+R(M),$$

where

(3.3) $$R(M)=[2ce+(a-b)(c+d)]^2+(a+b)^2(c-d)^2\geqq 0.$$

$R(M)=0$ holds if and only if $c=d$ and $c=b-a$, i.e., M is a parallelogram.

Summarizing the above we obtain the following.

THEOREM 3.1. *Let M be a quadrangle on E^2. Let $S(M)$ denote the area of M and define $T(M)$ and $R(M)$ by* (3.2) *and* (3.3), *respectively. Then*

$$\int e^{-4r^2f(x,y)}dE^2=\frac{\pi}{r^2[4S(M)^2+R(M)]^{1/2}}e^{-r^2(S(M)^2+T(M))}$$
$$\leqq\frac{\pi}{2r^2S(M)}e^{-r^2S(M)^2},$$

where $T(M)=R(M)=0$ holds if and only if M is a parallelogram.

COROLLARY 3.2. *Among parallelograms on E^2, $F(M)$ does not depend on the shape of M, but only on the area of M.*

4. $G(M)$ for M on E^2 in E^3.

Let $M=(ABCD)$ be a quadrangle on E^2 and let $Q=(x, y, z)$ be a point of a Euclidean 3-space E^3. By $P=(x, y)$ we denote the image of Q by the natural projection: $E^3\to E^2$. Then

$$4|\Delta QAB|^2=|AB|^2z^2+4|\Delta PAB|^2$$

holds, where $|AB|$ denotes the length of the segment AB. We define functions $g(x, y, z)=g_M(Q)$ and $G(M)$ by

$$g_M(Q)=|\Delta QAB|^2+|\Delta QBC|^2+|\Delta QCD|^2+|\Delta QDA|^2,$$

$$G(M)=\int e^{-4r^2g(x,y,z)}dE^3.$$

Then

$$G(M)=\left(\int e^{-r^2V(M)z^2}dz\right)F(M)$$
$$=\sqrt{\pi}\,r^{-1}V(M)^{-1/2}F(M),$$

where we have put

$$V(M)=|AB|^2+|BC|^2+|CD|^2+|DA|^2$$

$$=2[a^2+b^2+c^2+d^2+e^2+(a-b)e]$$

$$=4S(M)+U(M)\,,$$

$$2U(M)=2(a-d)^2+2(b-d)^2+(a+b-2c)^2+(a-b+2e)^2\geqq 0\,,$$

where $a, b, c, d, e, S(M)$ and $F(M)$ are ones used in § 3. $U(M)=0$ holds if and only if M is a square. Applying Theorem 3.1, we obtain

THEOREM 4.1. *Let M be a quadrangle on E^2. Then $G(M)$ satisfies the following.*

$$G(M)=\frac{\pi^{3/2}}{r^3[(4S(M)+U(M))(4S(M)^2+R(M)]^{1/2}}e^{-r^2(S(M)^2+T(M))}$$

$$\leqq\frac{\pi^{3/2}}{4r^3S(M)^{3/2}}e^{-r^2S(M)^2}\,,$$

where $R(M)$, $S(M)$ and $T(M)$ are defined in § 3, and $R(M)=T(M)=U(M)=0$ holds if and only if M is a square.

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