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ON SOLUTIONS OF A HOMOGENEOUS LINEAR MATRIX EQUATION WITH VARIABLE COMPONENTS

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1. We denote the totality of real numbers by $\mathbf{R}$ and the totality of complex numbers by $\mathbf{C}$.

Let I be a closed interval $[\alpha, \beta]=\{t \mid \alpha \leqq t \leqq \beta, t \in \mathbf{R}\}$. We denote by $C^{\mu}(I, \mathbf{R})$ the totality of real-valued functions defined and of class C^{μ} on I ($\mu=0, 1, \cdots, \infty$), and hereafter we fix some μ.

A complex-valued function $f(t)$ defined on I is called a function of class C^{μ} on I if $\operatorname{Re} f(t) \in C^{\mu}(I, \mathbf{R})$ and $\operatorname{Im} f(t) \in C^{\mu}(I, \mathbf{R})$. We denote by $C^{\mu}(I, \mathbf{C})$ the totality of complex-valued functions defined and of class C^{μ} on I.

A d-dimensional row vector $\boldsymbol{x}$ with components $x_{\rho} \in \mathbf{C}$ ($\rho=1, 2, \cdots, d$) will be denoted by

$$\boldsymbol{x}=(x_1, x_2, \cdots, x_d)$$

and a d'-dimensional column vector $\boldsymbol{y}$ with components $y_{\sigma} \in \mathbf{C}$ ($\sigma=1, 2, \cdots, d'$) by

$$\boldsymbol{y}=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{d'} \end{pmatrix}=\operatorname{col}(y_1, y_2, \cdots, y_{d'}) .$$

Now, let $B(t)$ be a square matrix of degree n:

$$B(t)=\begin{pmatrix} b_{11}(t) & b_{12}(t) \cdots b_{1n}(t) \\ b_{21}(t) & b_{22}(t) \cdots b_{2n}(t) \\ \vdots & \vdots \quad\quad \vdots \\ b_{n1}(t) & b_{n2}(t) \cdots b_{nn}(t) \end{pmatrix},$$

where $b_{jk}(t) \in C^{\mu}(I, \mathbf{C})$ ($j, k=1, 2, \cdots, n$), and let us assume, throughout this paper, that for a positive integer $s: 1 \leqq s \leqq n-1$, a condition

$$\text{(1)} \qquad \operatorname{rank} B(t)=n-s \ (=r)$$

is satisfied on the interval I, and further let us consider a homogeneous linear matrix equation

$$\text{(2)} \qquad B(t)P(t)=O ,$$

where $P(t)$ is an $n \times s$ matrix:

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$$P(t)=\begin{pmatrix} p_{11}(t) & p_{12}(t) \cdots p_{1s}(t) \\ p_{21}(t) & p_{22}(t) \cdots p_{2s}(t) \\ \vdots & \vdots \quad\quad \vdots \\ p_{n1}(t) & p_{n2}(t) \cdots p_{ns}(t) \end{pmatrix}.$$

The purpose of this paper is to establish the existence of solutions $P(t)$ of the equation (2) on I, such that every component $p_{jk}(t)$ of $P(t)$ belongs to $C^{\mu}(I, \mathbf{C})$ and

$$\operatorname{rank} P(t)=s$$

on I.

Y. Sibuya treated such a problem in a paper [1], provided that $B(t)$ was periodic on $-\infty<t<+\infty$. But as a certain part of the proof was omitted in his paper, we will, in this paper, give a detailed proof of this part to clarify the matter.

2. Let I_1 and I_2 be two open intervals (α_1, β_1) and (α_2, β_2) contained in the interval I, such that $\alpha_1<\alpha_2<\beta_1<\beta_2$, and but let us consider $I_1=[\alpha_1, \beta_1)$ if $\alpha_1=\alpha$ and $I_2=(\alpha_2, \beta_2]$ if $\beta_2=\beta$.

Put $r=n-s$ and let $B\begin{pmatrix} j_1\ j_2 \cdots j_r \\ k_1\ k_2 \cdots k_r \end{pmatrix}$ denote a minor of degree r of $B(t)$ such that

$$B\begin{pmatrix} j_1\ j_2 \cdots j_r \\ k_1\ k_2 \cdots k_r \end{pmatrix}=\begin{vmatrix} b_{j_1k_1}(t) & b_{j_1k_2}(t) \cdots b_{j_1k_r}(t) \\ b_{j_2k_1}(t) & b_{j_2k_2}(t) \cdots b_{j_2k_r}(t) \\ \vdots & \vdots \quad\quad \vdots \\ b_{j_rk_1}(t) & b_{j_rk_2}(t) \cdots b_{j_rk_r}(t) \end{vmatrix},$$

$$\begin{pmatrix} 1\leqq j_1< j_2< \cdots < j_r\leqq n \\ 1\leqq k_1<k_2< \cdots <k_r\leqq n \end{pmatrix}.$$

Let us now assume that the condition (1) is satisfied on $I_1\cup I_2$ and further that a condition

$$B\begin{pmatrix} j_1\ j_2 \cdots j_r \\ k_1\ k_2 \cdots k_r \end{pmatrix}\neq 0 \tag{3}$$

is satisfied on I_1 and a condition

$$B\begin{pmatrix} l_1\ l_2 \cdots l_r \\ m_1\ m_2 \cdots m_r \end{pmatrix}\neq 0 \tag{4}$$

is satisfied on I_2.

Under these assumptions, it is clear that there exist two $n\times s$ matrices $P(t)$ and $Q(t)$ having the following properties:

(I) $P(t)$ and $Q(t)$ satisfy matrix equations

$$B(t)P(t)=O \qquad \text{on} \quad I_1 \tag{5}$$

and

(6) $$B(t)Q(t)=O \quad \text{on} \quad I_2$$

respectively, and every component $p_{jk}(t)$ of $P(t)$ belongs to $C^{\mu}(I_1, \mathbf{C})$ and every component $q_{jk}(t)$ of $Q(t)$ belongs to $C^{\mu}(I_2, \mathbf{C})$;

(II) rank $P(t)=s$ on I_1 and rank $Q(t)=s$ on I_2.

However, we here wish to look over the structure of these matrices $P(t)$ and $Q(t)$.

Let us define s-tuple $(j'_{r+1}, j'_{r+2}, \cdots, j'_n)$ for $1 \leqq j_1 < j_2 < \cdots < j_r \leqq n$ in such a manner that $1 \leqq j'_{r+1} < j'_{r+2} < \cdots < j'_n \leqq n$ and $\{j_1, \cdots, j_r, j'_{r+1}, \cdots, j'_n\} = \{1, 2, \cdots, n\}$. That is, $j_1 < j_2 < \cdots < j_r$ and $j'_{r+1} < j'_{r+2} < \cdots < j'_n$ form a complete system of indices $\{1, 2, \cdots, n\}$. s-tuples $(k'_{r+1}, k'_{r+2}, \cdots, k'_n)$, $(l'_{r+1}, l'_{r+2}, \cdots, l'_n)$ and $(m'_{r+1}, m'_{r+2}, \cdots, m'_n)$ are defined for $1 \leqq k_1 < k_2 < \cdots < k_r \leqq n$, $1 \leqq l_1 < l_2 < \cdots < l_r \leqq n$ and $1 \leqq m_1 < m_2 < \cdots < m_r \leqq n$ in the same manner.

By virtue of Cramer's rule, we see the following fact.

We can take arbitrarily all components $p_{k'_\sigma g}(t)$ of s row vectors

$$\hat{\boldsymbol{p}}_{k'_\sigma}(t)=(p_{k'_\sigma 1}(t), p_{k'_\sigma 2}(t), \cdots, p_{k'_\sigma s}(t)) \qquad (\sigma=r+1, r+2, \cdots, n),$$

or s column vectors

$$\tilde{\boldsymbol{p}}_g(t)=\mathrm{col}\,(p_{k'_{r+1} g}(t), p_{k'_{r+2} g}(t), \cdots, p_{k'_n g}(t)) \qquad (g=1, 2, \cdots, s)$$

under restrictions that $p_{k'_\sigma g}(t) \in C^{\mu}(I_1, \mathbf{C})$ and

(7) $$\det(\tilde{\boldsymbol{p}}_1(t), \tilde{\boldsymbol{p}}_2(t), \cdots, \tilde{\boldsymbol{p}}_s(t))=\det\begin{pmatrix} \hat{\boldsymbol{p}}_{k'_{r+1}}(t) \\ \hat{\boldsymbol{p}}_{k'_{r+2}}(t) \\ \vdots \\ \hat{\boldsymbol{p}}_{k'_n}(t) \end{pmatrix} \neq 0 \qquad \text{on} \quad I_1 .$$

We can further express other r row vectors

$$\hat{\boldsymbol{p}}_{k_\rho}(t)=(p_{k_\rho 1}(t), p_{k_\rho 2}(t), \cdots, p_{k_\rho s}(t)) \qquad (\rho=1, 2, \cdots, r)$$

by linear combinations of $\hat{\boldsymbol{p}}_{k'_\sigma}(t)$ $(\sigma=r+1, r+2, \cdots, n)$:

$$\hat{\boldsymbol{p}}_{k_\rho}(t)=\sum_{\sigma=r+1}^{n} \xi_{\rho\sigma}(t)\hat{\boldsymbol{p}}_{k'_\sigma}(t) \qquad (\rho=1, 2, \cdots, r),$$

whose coefficients $\xi_{\rho\sigma}(t)$ belong to $C^{\mu}(I_1, \mathbf{C})$.

More exactly,

$$\xi_{\rho\sigma}(t)=-\frac{B_{\rho\sigma}\begin{pmatrix} j_1\, j_2 \cdots j_r \\ k_1\, k_2 \cdots k_r \end{pmatrix}}{B\begin{pmatrix} j_1\, j_2 \cdots j_r \\ k_1\, k_2 \cdots k_r \end{pmatrix}} \quad \begin{pmatrix} \rho=1, 2, \cdots, r; \\ \sigma=r+1, r+2, \cdots, n \end{pmatrix},$$

where

$$B_{\rho\sigma}\begin{pmatrix} j_1 & j_2 & \cdots & j_r \\ k_1 & k_2 & \cdots & k_r \end{pmatrix} = \begin{array}{c} \\ \\ \end{array} \begin{array}{c} \rho\text{-th column} \\ \begin{vmatrix} b_{j_1k_1}(t) & b_{j_1k_2}(t) \cdots & b_{j_1k'_\sigma}(t) \cdots & b_{j_1k_r}(t) \\ b_{j_2k_1}(t) & b_{j_2k_2}(t) \cdots & b_{j_2k'_\sigma}(t) \cdots & b_{j_2k_r}(t) \\ \vdots & \vdots & \vdots & \vdots \\ b_{j_rk_1}(t) & b_{j_rk_2}(t) \cdots & b_{j_rk'_\sigma}(t) \cdots & b_{j_rk_r}(t) \end{vmatrix} \end{array}.$$

Of course, we see rank $P(t)=s$ on I_1.

The matrix $Q(t)$:

$$Q(t)=\begin{pmatrix} q_{11}(t) & q_{12}(t) \cdots & q_{1s}(t) \\ q_{21}(t) & q_{22}(t) \cdots & q_{2s}(t) \\ \vdots & \vdots & \vdots \\ q_{n1}(t) & q_{n2}(t) \cdots & q_{ns}(t) \end{pmatrix}$$

also has a structure similar to $P(t)$.

We can choose arbitrarily all components $q_{m'_\sigma g}(t)$ of s row vectors

$$\hat{\boldsymbol{q}}_{m'_\sigma}(t)=(q_{m'_\sigma 1}(t),\ q_{m'_\sigma 2}(t),\ \cdots,\ q_{m'_\sigma s}(t)) \qquad (\sigma=r+1,\ r+2,\ \cdots,\ n)$$

or s column vectors

$$\tilde{\boldsymbol{q}}_g(t)=\mathrm{col}\,(q_{m'_{r+1}g}(t),\ q_{m'_{r+2}g}(t),\ \cdots,\ q_{m'_n g}(t)) \qquad (g=1,\ 2,\ \cdots,\ s)$$

under restrictions that $q_{m'_\sigma g}(t)\in C^\mu(I_2,\mathbf{C})$ and

(8) $$\det(\tilde{\boldsymbol{q}}_1(t),\ \tilde{\boldsymbol{q}}_2(t),\ \cdots,\ \tilde{\boldsymbol{q}}_s(t))=\det\begin{pmatrix} \hat{\boldsymbol{q}}_{m'_{r+1}}(t) \\ \hat{\boldsymbol{q}}_{m'_{r+2}}(t) \\ \vdots \\ \hat{\boldsymbol{q}}_{m'_n}(t) \end{pmatrix}\neq 0 \quad \text{on} \quad I_2.$$

Other r row vectors

$$\hat{\boldsymbol{q}}_{m_\rho}(t)=(q_{m_\rho 1}(t),\ q_{m_\rho 2}(t),\ \cdots,\ q_{m_\rho s}(t)) \qquad (\rho=1,\ 2,\ \cdots,\ r)$$

can be expressed by linear combinations of $\hat{\boldsymbol{q}}_{m'_\sigma}(t)$ $(\sigma=r+1,\ r+2,\ \cdots,\ n)$:

$$\hat{\boldsymbol{q}}_{m_\rho}(t)=\sum_{\sigma=r+1}^{n}\eta_{\rho\sigma}(t)\hat{\boldsymbol{q}}_{m'_\sigma}(t) \qquad (\rho=1,\ 2,\ \cdots,\ r),$$

whose coefficients $\eta_{\rho\sigma}(t)$ belong to $C^\mu(I_2,\mathbf{C})$.

Of course, we have rank $Q(t)=s$ on I_2.

3. We use the same notations as in Nos. 1-2, and will prove the following:

LEMMA 1. *Assume that the condition* (1) *is satisfied on* $I_1\cup I_2$ *and the conditions* (3) *and* (4) *are satisfied on* I_1 *and on* I_2 *respectively. Then we have*

$$B\begin{pmatrix} l_1 & l_2 \cdots & l_r \\ k_1 & k_2 \cdots & k_r \end{pmatrix} \neq 0; \tag{9}$$

$$B\begin{pmatrix} j_1 & j_2 \cdots & j_r \\ m_1 & m_2 \cdots & m_r \end{pmatrix} \neq 0 \tag{10}$$

on $I_1 \cap I_2$.

Proof. The lemma shall be principally proved for (9). If we put

$$\hat{\boldsymbol{b}}_{l_\rho}(t) = (b_{l_\rho m_1}(t),\ b_{l_\rho m_2}(t),\ \cdots,\ b_{l_\rho m_r}(t)) \qquad (\rho = 1, 2, \cdots, r),$$

then, by virtue of the conditions (1) and (4), other vectors

$$\hat{\boldsymbol{b}}_{l'_\sigma}(t) = (b_{l'_\sigma m_1}(t),\ b_{l'_\sigma m_2}(t),\ \cdots,\ b_{l'_\sigma m_r}(t)) \qquad (\sigma = r+1, r+2, \cdots, n)$$

can be expressed by linear combinations of $\hat{\boldsymbol{b}}_{l_\rho}(t)$ $(\rho = 1, 2, \cdots, r)$ with coefficients $\varphi_{\rho\sigma}(t)$ belonging to $C^\mu(I_2, \mathbf{C})$. That is,

$$\hat{\boldsymbol{b}}_{l'_\sigma}(t) = \sum_{\rho=1}^{r} \varphi_{\rho\sigma}(t) \hat{\boldsymbol{b}}_{l_\rho}(t) \qquad (\sigma = r+1, r+2, \cdots, n),$$

where

$$\varphi_{\rho\sigma}(t) = \frac{B^{\rho\sigma}\begin{pmatrix} l_1 & l_2 \cdots & l_r \\ m_1 & m_2 \cdots & m_r \end{pmatrix}}{B\begin{pmatrix} l_1 & l_2 \cdots & l_r \\ m_1 & m_2 \cdots & m_r \end{pmatrix}} \qquad \begin{pmatrix} \rho = 1, 2, \cdots, r; \\ \sigma = r+1, r+2, \cdots, n \end{pmatrix}$$

and

$$B^{\rho\sigma}\begin{pmatrix} l_1 & l_2 \cdots & l_r \\ m_1 & m_2 \cdots & m_r \end{pmatrix} = \begin{array}{c} \begin{vmatrix} b_{l_1 m_1}(t) & b_{l_1 m_2}(t) \cdots & b_{l_1 m_r}(t) \\ b_{l_2 m_1}(t) & b_{l_2 m_2}(t) \cdots & b_{l_2 m_r}(t) \\ \vdots & \vdots & \vdots \\ b_{l'_\sigma m_1}(t) & b_{l'_\sigma m_2}(t) \cdots & b_{l'_\sigma m_r}(t) \\ \vdots & \vdots & \vdots \\ b_{l_r m_1}(t) & b_{l_r m_2}(t) \cdots & b_{l_r m_r}(t) \end{vmatrix} \end{array} \begin{array}{l} \\ \\ \\ \rho\text{-th row}. \\ \\ \\ \end{array}$$

In this case, any component $b_{l'_\sigma m'_\upsilon}(t)$ $(\sigma, \upsilon = r+1, r+2, \cdots, n)$ which does not lie on the m_τ-th column $(\tau = 1, 2, \cdots, r)$ of $B(t)$ also can be expressed by a linear combination with the common coefficients $\varphi_{\rho\sigma}(t)$ as follows:

$$b_{l'_\sigma m'_\upsilon}(t) = \sum_{\rho=1}^{r} \varphi_{\rho\sigma}(t) b_{l_\rho m'_\upsilon}(t) \qquad (\sigma, \upsilon = r+1, r+2, \cdots, n),$$

otherwise it contradicts the fact that the condition (1) holds on I_2.

Next if we put

$$\tilde{\boldsymbol{b}}_{k_\tau}(t) = \mathrm{col}\,(b_{j_1 k_\tau}(t),\ b_{j_2 k_\tau}(t),\ \cdots,\ b_{j_r k_\tau}(t)) \qquad (\tau = 1, 2, \cdots, r),$$

then, by virtue of the conditions (1) and (3), other vectors

$$\tilde{\boldsymbol{b}}_{k'_\upsilon}(t)=\mathrm{col}\,(b_{j_1 k'_\upsilon}(t),\ b_{j_2 k'_\upsilon}(t),\ \cdots,\ b_{j_r k'_\upsilon}(t)) \qquad (\upsilon=r+1,\ r+2,\ \cdots,\ n)$$

can be expressed by linear combinations of $\tilde{\boldsymbol{b}}_{k_\tau}(t)$ $(\tau=1, 2, \cdots, r)$ with coefficients belonging to $C^\mu(I_1, \mathbf{C})$. That is,

$$\tilde{\boldsymbol{b}}_{k'_\upsilon}(t)=\sum_{\tau=1}^{r}\phi_{\tau\upsilon}(t)\tilde{\boldsymbol{b}}_{k_\tau}(t) \qquad (\upsilon=r+1,\ r+2,\ \cdots,\ n)\,,$$

where

$$\phi_{\tau\upsilon}(t)=\frac{B_{\tau\upsilon}\begin{pmatrix} j_1\ j_2 \cdots\ j_r \\ k_1\ k_2 \cdots\ k_r\end{pmatrix}}{B\begin{pmatrix} j_1\ j_2 \cdots\ j_r \\ k_1\ k_2 \cdots\ k_r\end{pmatrix}} \qquad \begin{pmatrix}\tau=1,\ 2,\ \cdots,\ r\,; \\ \upsilon=r+1,\ r+2,\ \cdots,\ n\end{pmatrix}$$

and

$$B_{\tau\upsilon}\begin{pmatrix} j_1\ j_2 \cdots\ j_r \\ k_1\ k_2 \cdots\ k_r\end{pmatrix}=\begin{array}{c} \rho\text{-th column} \\ \begin{vmatrix} b_{j_1k_1}(t) & b_{j_1k_2}(t) \cdots b_{j_1k'_\upsilon}(t) \cdots b_{j_1k_r}(t) \\ b_{j_2k_1}(t) & b_{j_2k_2}(t) \cdots b_{j_2k'_\upsilon}(t) \cdots b_{j_2k_r}(t) \\ \vdots & \vdots \qquad\quad \vdots \qquad\quad \vdots \\ b_{j_rk_1}(t) & b_{j_rk_2}(t) \cdots b_{j_rk'_\upsilon}(t) \cdots b_{j_rk_r}(t)\end{vmatrix}\end{array}.$$

In this case, any component $b_{j'_\sigma k'_\upsilon}(t)$ $(\sigma,\ \upsilon=r+1,\ r+2,\ \cdots,\ n)$ which does not lie on the j_ρ-th row $(\rho=1, 2, \cdots, r)$ of $B(t)$ also can be expressed by a linear combination with the common coefficients $\phi_{\tau\upsilon}(t)$ as follows:

$$b_{j'_\sigma k'_\upsilon}(t)=\sum_{\tau=1}^{r}\phi_{\tau\upsilon}(t)b_{j'_\sigma k_\tau}(t) \qquad (\sigma,\ \upsilon=r+1,\ r+2,\ \cdots,\ n)\,,$$

otherwise it contradicts the fact that the condition (1) holds on I_1.

Under these circumstances, we arrive at the following conclusion.

Multiplying an adequate function which belongs to $C^\mu(I_2, \mathbf{C})$, to each l_ρ-th row vector $(\rho=1, 2, \cdots, r)$ of $B(t)$ and adding these row vectors to each l'_σ-th row vector $(\sigma=r+1, r+2, \cdots, n)$ of $B(t)$, we can make each l'_σ-th row vector be the zero vector. And further, multiplying an adequate function which belongs to $C^\mu(I_1, \mathbf{C})$, to each k_τ-th column vector $(\tau=1, 2, \cdots, r)$ of $B(t)$ and adding these column vectors to each k'_υ-th column vector $(\upsilon=r+1, r+2, \cdots, n)$ of $B(t)$, we can make each k'_υ-th column vector be the zero vector. After all, there remain only the components $b_{l_\rho k_\tau}(t)$ $(\rho,\ \tau=1, 2, \cdots, r)$ in $B(t)$, and hence the condition (1) implies

$$B\begin{pmatrix} l_1\ l_2 \cdots\ l_r \\ k_1\ k_2 \cdots\ k_r\end{pmatrix}\neq 0 \qquad \text{on} \quad I_1\cap I_2\,.$$

We also can prove, on the same lines, that

$$B\begin{pmatrix} j_1 & j_2 \cdots & j_r \\ m_1 & m_2 \cdots & m_r \end{pmatrix} \neq 0 \quad \text{on} \quad I_1 \frown I_2 .$$

4. We use the same notations as in Nos. 1–2, and for the solutions $P(t)$ and $Q(t)$ obtained in No. 2, we will prove the following:

LEMMA 2. *Suppose that the condition* (1) *holds on* $I_1 \cup I_2$ *and the condition* (3) *holds on* I_1 *and the condition* (4) *holds on* I_2. *Let* $P(t)$ *and* $Q(t)$ *be the solutions of the equations* (5) *and* (6) *respectively, obtained in No.* 2. *Then there exists a square matrix* $C(t)$ *of degree* s *such that*

(I) *Every component* $c_{jk}(t)$ *of* $C(t)$ *belongs to* $C^{\mu}(I_1 \frown I_2, \mathbf{C})$;

(II) $\operatorname{rank} C(t) = s$ *on* $I_1 \frown I_2$;

(III) $P(t) = Q(t)C(t)$ *on* $I_1 \frown I_2$.

Proof. By rearranging the row vectors of $P(t)$ and $Q(t)$, we put

$$\hat{P}(t) = \begin{pmatrix} \hat{\boldsymbol{p}}_{k_1}(t) \\ \vdots \\ \hat{\boldsymbol{p}}_{k_r}(t) \\ \hat{\boldsymbol{p}}_{k'_{r+1}}(t) \\ \vdots \\ \hat{\boldsymbol{p}}_{k'_n}(t) \end{pmatrix} \quad \text{and} \quad \hat{Q}(t) = \begin{pmatrix} \hat{\boldsymbol{q}}_{k_1}(t) \\ \vdots \\ \hat{\boldsymbol{q}}_{k_r}(t) \\ \hat{\boldsymbol{q}}_{k'_{r+1}}(t) \\ \vdots \\ \hat{\boldsymbol{q}}_{k'_n}(t) \end{pmatrix}.$$

Then, by virtue of the condition (9) proved in Lemma 1, we can express the vectors $\hat{\boldsymbol{p}}_{k_1}(t), \hat{\boldsymbol{p}}_{k_2}(t), \cdots, \hat{\boldsymbol{p}}_{k_r}(t)$ by linear combinations of $\hat{\boldsymbol{p}}_{k'_{r+1}}(t), \hat{\boldsymbol{p}}_{k'_{r+2}}(t), \cdots, \hat{\boldsymbol{p}}_{k'_n}(t)$ and the vectors $\hat{\boldsymbol{q}}_{k_1}(t), \hat{\boldsymbol{q}}_{k_2}(t), \cdots, \hat{\boldsymbol{q}}_{k_r}(t)$ by linear combinations of $\hat{\boldsymbol{q}}_{k'_{r+1}}(t), \hat{\boldsymbol{q}}_{k'_{r+2}}(t), \cdots, \hat{\boldsymbol{q}}_{k'_n}(t)$ with the same coefficients belonging to $C^{\mu}(I_1 \frown I_2, \mathbf{C})$. That is,

$$(11) \qquad \begin{cases} \hat{\boldsymbol{p}}_{k_\rho}(t) = \sum_{\sigma=r+1}^{n} \zeta_{\rho\sigma}(t) \hat{\boldsymbol{p}}_{k'_\sigma}(t) & (\rho = 1, 2, \cdots, r); \\ \hat{\boldsymbol{q}}_{k_\rho}(t) = \sum_{\sigma=r+1}^{n} \zeta_{\rho\sigma}(t) \hat{\boldsymbol{q}}_{k'_\sigma}(t) & (\rho = 1, 2, \cdots, r), \end{cases}$$

where $\zeta_{\rho\sigma}(t) \in C^{\mu}(I_1 \frown I_2, \mathbf{C})$ and more exactly

$$\zeta_{\rho\sigma}(t) = -\frac{B_{\rho\sigma}\begin{pmatrix} l_1 & l_2 \cdots & l_r \\ k_1 & k_2 \cdots & k_r \end{pmatrix}}{B\begin{pmatrix} l_1 & l_2 \cdots & l_r \\ k_1 & k_2 \cdots & k_r \end{pmatrix}} \qquad \begin{pmatrix} \rho = 1, 2, \cdots, r; \\ \sigma = r+1, r+2, \cdots, n \end{pmatrix}$$

and

$$B_{\rho\sigma}\begin{pmatrix} l_1 & l_2 & \cdots & l_r \\ k_1 & k_2 & \cdots & k_r \end{pmatrix} = \begin{array}{c} \begin{array}{ccccc} & & \rho\text{-th column} & & \end{array} \\ \begin{vmatrix} b_{l_1k_1}(t) & b_{l_1k_2}(t) & \cdots b_{l_1k'_\sigma}(t) & \cdots b_{l_1k_r}(t) \\ b_{l_2k_1}(t) & b_{l_2k_2}(t) & \cdots b_{l_2k'_\sigma}(t) & \cdots b_{l_2k_r}(t) \\ \vdots & \vdots & \vdots & \vdots \\ b_{l_rk_1}(t) & b_{l_rk_2}(t) & \cdots b_{l_rk'_\sigma}(t) & \cdots b_{l_rk_r}(t) \end{vmatrix} \end{array}.$$

Now, we shall verify

$$\det\begin{pmatrix} \hat{\boldsymbol{q}}_{k'_{r+1}}(t) \\ \hat{\boldsymbol{q}}_{k'_{r+2}}(t) \\ \cdot \\ \hat{\boldsymbol{q}}_{k'_n}(t) \end{pmatrix} \neq 0 \qquad \text{on} \quad I_1 \cap I_2 . \tag{12}$$

If the above determinant vanishes at some $t_0 \in I_1 \cap I_2$, then there exists a set of s complex numbers: $(c_1, c_2, \cdots, c_s) \neq (0, 0, \cdots, 0)$ such that

$$c_1 \hat{\boldsymbol{q}}_{k'_{r+1}}(t_0) + c_2 \hat{\boldsymbol{q}}_{k'_{r+2}}(t_0) + \cdots + c_s \hat{\boldsymbol{q}}_{k'_n}(t_0) = \boldsymbol{o} .$$

This fact and the fact that $\hat{\boldsymbol{q}}_{k_\rho}(t)$ $(\rho = 1, 2, \cdots, r)$ can be expressed by linear combinations of $\hat{\boldsymbol{q}}_{k'_\sigma}(t)$ $(\sigma = r+1, r+2, \cdots, n)$, imply

$$\operatorname{rank}\begin{pmatrix} \hat{\boldsymbol{q}}_{k_1}(t_0) \\ \vdots \\ \hat{\boldsymbol{q}}_{k_r}(t_0) \\ \hat{\boldsymbol{q}}_{k'_{r+1}}(t_0) \\ \vdots \\ \hat{\boldsymbol{q}}_{k'_n}(t_0) \end{pmatrix} < s$$

which contradicts that $\operatorname{rank} Q(t) = s$ on I_2.

Thus there exists a square matrix $C(t)$ of degree s such that

$$\begin{pmatrix} \hat{\boldsymbol{p}}_{k'_{r+1}}(t) \\ \hat{\boldsymbol{p}}_{k'_{r+2}}(t) \\ \vdots \\ \hat{\boldsymbol{p}}_{k'_n}(t) \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{q}}_{k'_{r+1}}(t) \\ \hat{\boldsymbol{q}}_{k'_{r+2}}(t) \\ \vdots \\ \hat{\boldsymbol{q}}_{k'_n}(t) \end{pmatrix} C(t) \qquad \text{on} \quad I_1 \cap I_2$$

and every component of $C(t)$ belongs to $C^\mu(I_1 \cap I_2, \mathbf{C})$. It follows furthermore from (7) that $\det C(t) \neq 0$ on $I_1 \cap I_2$, and hence $\operatorname{rank} C(t) = s$ on $I_1 \cap I_2$.

By rearranging the row vectors of $\hat{P}(t)$ and $\hat{Q}(t)$, and by observing the relations (11), we have

$$P(t) = Q(t)C(t) \qquad \text{on} \quad I_1 \cap I_2 .$$

Thus, the lemma has been completely proved.

5. Now we will prove the following:

THEOREM. *Let I be a closed interval $[\alpha, \beta]$ of a real variable t and let $B(t)=(b_{jk}(t))$ be a square matrix of degree n whose components $b_{jk}(t)$ belong to $C^{\mu}(I, \mathbf{C})$ and further assume that the condition* (1) *is satisfied on I. Then there exists an $n\times s$ matrix $P(t)=(p_{jk}(t))$ such that $P(t)$ satisfies the equation* (2) *on I and all components $p_{jk}(t)$ of $P(t)$ belong to $C^{\mu}(I, \mathbf{C})$ and* rank $P(t)=s$ *on I.*

Proof. Let us put $r=n-s$. Then, for any point $t_0\in[\alpha, \beta]$, there exists, by assumption, a nonzero minor of degree r of $B(t_0)$. Furthermore, there exists, by virtue of the continuity of functions, a neighborhood $U(t_0)$ of t_0 such that the above-mentioned minor does not vanish on $I\cap U(t_0)$.

Since we can find such neighborhoods $U(t)$ for all points $t\in I$ and since we see $I\subset\bigcup_{t\in I}U(t)$, this open covering $\{U(t)\}_{t\in I}$ has, by the Heine-Borel theorem, a finite subcovering $\{U(t_\kappa)\}_{\kappa=1}^{\kappa_0}$. By use of this subcovering, we can form, without loss of generality, a set $\{I_\kappa\}_{\kappa=1}^{\kappa_0}$ of intervals possessing the following properties:

(i) $I=\sum_{\kappa=1}^{\kappa_0} I_\kappa$;

(ii) $I_1=[\alpha_1, \beta_1)$, $I_{\kappa_0}=(\alpha_{\kappa_0}, \beta_{\kappa_0}]$, $\alpha_1=\alpha$, $\beta_{\kappa_0}=\beta$,

$I_\kappa=(\alpha_\kappa, \beta_\kappa)$ $(\kappa=2, 3, \cdots, \kappa_0-1)$;

(iii) $I_\kappa\cap I_{\kappa+1}\neq\emptyset$ $(\kappa=1, 2, \cdots, \kappa_0-1)$,

$I_\kappa\cap I_{\kappa'}=\emptyset$ $(\kappa+1<\kappa', \kappa=1, 2, \cdots, \kappa_0-2)$,

that is, $\alpha_1<\alpha_2<\beta_1<\cdots<\alpha_\kappa<\beta_{\kappa-1}<\alpha_{\kappa+1}<\beta_\kappa$

$<\cdots<\beta_{\kappa_0-2}<\alpha_{\kappa_0}<\beta_{\kappa_0-1}<\beta_{\kappa_0}$ $(\kappa=2, 3, \cdots, \kappa_0-1)$;

(iv) For each I_κ, there exists a minor of degree r of $B(t)$ which does not vanish on I_κ.

We consider first the intervals I_1 and I_2, and we choose two minors $B\begin{pmatrix} j_1\ j_2 \cdots j_r \\ k_1\ k_2 \cdots k_r \end{pmatrix}$ and $B\begin{pmatrix} l_1\ l_2 \cdots l_r \\ m_1\ m_2 \cdots m_r \end{pmatrix}$ of degree r of $B(t)$ such that

$$B\begin{pmatrix} j_1\ j_2 \cdots j_r \\ k_1\ k_2 \cdots k_r \end{pmatrix}\neq 0 \text{ on } I_1 \text{ and } B\begin{pmatrix} l_1\ l_2 \cdots l_r \\ m_1\ m_2 \cdots m_r \end{pmatrix}\neq 0 \text{ on } I_2.$$

As seen in No. 2, there exist, in this case, the matrices $P(t)$ and $Q(t)$ satisfying the equations (5) and (6) respectively, such that all components $p_{jk}(t)$ of $P(t)$ belong to $C^{\mu}(I_1, \mathbf{C})$ and all components $q_{jk}(t)$ of $Q(t)$ belong to $C^{\mu}(I_2, \mathbf{C})$ and further

$$\operatorname{rank} P(t)=s \quad \text{on} \quad I_1 \qquad \text{and} \quad \operatorname{rank} Q(t)=s \quad \text{on} \quad I_2 .$$

Now it follows from Lemma 2 that there exists a square matrix $C(t)$ of degree s possessing the properties (I), (II), (III) stated in Lemma 2.

We next use a method adopted in the paper of Y. Sibuya [1]. Since the matrix $C(t)$ is non-singular on $I_1\cap I_2$, if we choose an arbitrary point $t_1\in I_1\cap I_2$, and if we choose a sufficiently small positive number ε, then any square matrix C of degree s satisfying $\|C-C(t_1)\|<\varepsilon$, is non-singular, where $\|\cdot\|$ denotes the Euclidean norm of a matrix.

There exists, by virtue of the continuity of functions, a positive number δ such that $\|C(t)-C(t_1)\|<\varepsilon$ whenever $|t-t_1|<\delta$ and $t\in I_1\cap I_2$.

Let t_1' be a point belonging to $I_1\cap I_2$ such that $0<t_1'-t_1<\delta$ and let γ be a small positive number fulfilling the inequality $t_1+\gamma<t_1'-\gamma$. Further let $\chi(t)$ be a real-valued function defined and of class C^∞ on $-\infty<t<+\infty$, such that $0\leqq\chi(t)\leqq 1$ for all t, $\chi(t)=1$ for $t\leqq t_1+\gamma$ and $\chi(t)=0$ for $t\geqq t_1'-\gamma$.

And we make a square matrix $\tilde{C}(t)$ of degree s in the following manner:

$$\tilde{C}(t)=\begin{cases} C(t) & \text{for} \quad \alpha_2<t\leqq t_1\,, \\ \chi(t)(C(t)-C(t_1'))+C(t_1') & \text{for} \quad t_1\leqq t\leqq t_1'\,, \\ C(t_1') & \text{for} \quad t_1'\leqq t<+\infty\,. \end{cases}$$

Since

$$\begin{aligned} &\|\tilde{C}(t)-C(t_1)\| \\ &\qquad =\|\chi(t)(C(t)-C(t_1))+(1-\chi(t))(C(t_1')-C(t_1))\| \\ &\qquad \leqq\chi(t)\|C(t)-C(t_1')\|+(1-\chi(t))\|C(t_1')-C(t_1)\| \\ &\qquad <\chi(t)\varepsilon+(1-\chi(t))\varepsilon=\varepsilon \end{aligned}$$

for $t_1\leqq t\leqq t_1'$, we see that $\tilde{C}(t)$ is non-singular on $\alpha_2<t<+\infty$.

Further we can easily verify that all components of $\tilde{C}(t)$ are of class C^μ on $\alpha_2<t<+\infty$.

We now put $P^{(1)}(t)=P(t)$ on I_1 and we define $P^{(2)}(t)$ on $I_1\cup I_2$ as follows:

$$P^{(2)}(t)=\begin{cases} P^{(1)}(t) & \text{for} \quad \alpha_1\leqq t\leqq\alpha_2\,, \\ Q(t)\tilde{C}(t) & \text{for} \quad \alpha_2<t<\beta_2\,. \end{cases}$$

Then $P^{(2)}(t)$ is a solution of the equation (2) on $I_1\cup I_2$ and all components of $P^{(2)}(t)$ belong to $C^\mu(I_1\cup I_2, \mathbf{C})$ and further

$$\operatorname{rank} P^{(2)}(t)=s \qquad \text{on} \quad I_1\cup I_2\,.$$

By repeating the above-mentioned process for the intervals I_κ ($\kappa=1, 2, \cdots, \kappa_0$) successively, we can construct the desired solution $P(t)$ of the equation (2) on I.

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