D. MATSUSHITA
KODAI MATH. J.
25 (2002), 54–60

SIMULTANEOUS MINIMAL MODELS OF HOMOGENEOUS TORIC DEFORMATIONS

Daisuke Matsushita

Abstract

Every flat family of Du Val singularities admits a simultaneous minimal resolution after a finite base change. We investigate a flat family of isolated Gorenstein toric singularities and prove that there exists a simultaneous partial resolution.

1. Introduction

For a flat family of surfaces $f : X \to S$, a birational morphism $\tau : \tilde{X} \to X$ is said to be *simultaneous minimal resolution* if τ satisfies the following two conditions:

(1) $f \circ \tau$ is a flat morphism.
(2) $\tilde{X}_s := (f \circ \tau)^{-1}(s)$ $(s \in S)$ is the minimal resolution of X_s.

Let $f : X \to S$ be a flat morphism whose central fibre $f^{-1}(0)$ has only Du Val singularities. Brieskorn [3, 4] and Tyurina [11] proved that there exists an open set $0 \in U$, $(U \subset S)$ and a finite surjective morphism $U' \to U$ such that a flat morphism $f' : X \times_U U' \to U'$ admits a simultaneous minimal resolution. We consider an analogous problem for a flat family of isolated Gorenstein toric singularities. According to the Minimal Model Theory, it is natural to consider an existence of "simultaneous terminalization" for a flat family of higher dimensional singularities.

Definition 1. Let $f : X \to S$ be a flat morphism. It is said that f admits a simultaneous terminalization if there exists a birational morphism $\tau : \tilde{X} \to X$ which satisfies the following conditions:

(1) $f \circ \tau$ is a flat morphism.
(2) $\tilde{X}_s := (f \circ \tau)^{-1}(s)$ $(s \in S)$ has only terminal singularities.
(3) $K_{\tilde{X}_s}$ is τ-nef.

By [2, Theorem 8.1], an n-dimensional isolated toric singularity is rigid if $n \geq 4$ or it is not Gorenstein. Hence we investigate a flat family of 3-dimensional isolated Gorenstein toric singularities. Our result is the following:

1991 *Mathematics Subject Classification*: Primary 14E30; Secondary 14B07, 14M25.
Received April 16, 2001; revised July 9, 2001.

THEOREM 1.1. *Let $f : X \to S$ be a flat morphism such that the central fibre $f^{-1}(0)$ has only 3-dimensional isolated Gorenstein toric singularities and the base space S is reduced. Then there exist an open neighbourhood $0 \in U$, $(U \subset S)$ and a birational morphism $\tau : \tilde{X} \to X \times_S U$ which satisfy the following conditions*:

(1) *$f \circ \tau : \tilde{X} \to U$ is a flat morphism.*

(2) *The fibre $\tilde{X}_s$ has only hypersurface singularities in cyclic quotient space. Moreover those singularities are defined by*

$$\{xy - zw = 0\} \subset \boldsymbol{C}^4/G, \quad G \cong \boldsymbol{Z}/n\boldsymbol{Z},$$

where the action of G is given by

$$(x, y, z, w) \to (\zeta x, \zeta^{-1} y, \zeta^a z, \zeta^{-a} w),$$

(ζ is an n-th root of unity).

(3) *$K_{\tilde{X}_s}$ is τ-nef.*

Remark 1. The singularity of Theorem 1.1 (3) is not terminal singularity if G is not trivial. We construct an example of a flat family of isolated Gorenstein singularity which admits no simultaneous terminalization even if after finite base change. Please see Remark 4 in section 3.

This note is organised as follows: We recall the definition of homogeneous toric deformation according to K. Altmann in section 2. Theorem 1 is proved in section 3.

2. Homogeneous toric deformation

The following definition of homogeneous toric deformation is introduced by K. Altmann in [1, Definition 3.1].

DEFINITION 2. A flat morphism $f : X \to \boldsymbol{C}^m$ is called a *homogeneous toric deformation* if the following conditions are satisfied:

(1) $X := \operatorname{Spec} \boldsymbol{C}[\sigma^\vee \cap M]$ is an affine toric variety.

(2) f is defined by m equations $x^{r_i} - x^{r_0} = t_i$ $(1 \leq i \leq m)$, where $r_i \in \sigma^\vee \cap M$ and $t_1, \dots, t_m$ are coordinates of $\boldsymbol{C}^m$.

(3) Let $L := \bigoplus_{i=1}^{i=m} \boldsymbol{Z}(r_i - r_0)$ be the sublattice of M. The central fibre $Y := f^{-1}(0, \dots, 0)$ is isomorphic to $\operatorname{Spec} \boldsymbol{C}[\bar{\sigma}^\vee \cap \overline{M}]$ where $\bar{\sigma} = \sigma \cap L^\perp$ and $\overline{M} := M/L$.

(4) $i : Y \to X$ sends the closed orbit in Y isomorphically onto the closed orbit in X.

In this note, we consider a homogeneous toric deformation with some additional conditions:

DEFINITION 3. We call homogeneous toric deformation $f : X \to \boldsymbol{C}^m$ a *Gorenstein homogeneous toric deformation* if it satisfies the following two conditions:

(1) Y has only Gorenstein singularities.
(2) Kodaira-Spencer map $\boldsymbol{C}^m \to T^1_Y$ is nontrivial.

Remark 2. We list some examples of Gorenstein homogeneous toric deformation.

(1) The simplest example is $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ defined by $x - y = t$.
(2) Let $g : \mathscr{X} \to S$ be a versal deformation space of Du Val singularity of type A_n. The space $\mathscr{X}$ is defined by the equation
$$\mathscr{X} = (xy + z^{n+1} + t_1 z^{n-1} + \cdots + t_{n-1} z + t_n = 0)$$
in $\boldsymbol{C}^{n+3}$ and g is the projection. Let α_i $(0 \le i \le n)$ be the $i+1$-th elementary symmetric polynomials of $(n+1)$-variables and H a hyperplane in $\boldsymbol{C}^{n+1}$ defined by $\sum_{i=0}^n s_i = 0$, where $s_0, \dots, s_n$ are coordinates of $\boldsymbol{C}^{n+1}$. We take a base change by $\alpha : H \to \boldsymbol{C}^n$ $(\alpha^* t_i = \alpha_i(s_0, \dots, s_n))$.
$$\begin{array}{ccc} \mathscr{X} \times_{\boldsymbol{C}^n} H & \longrightarrow & \mathscr{X} \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ H & \longrightarrow & \boldsymbol{C}^n. \end{array}$$
Then $\mathscr{X} \times_{\boldsymbol{C}^n} H$ can be described
$$\left\{ xy + \prod_{i=0}^n (z + s_i), \sum_{i=0}^n s_i = 0 \right\} \subset \boldsymbol{C}^{n+4}.$$
Using new coordinates $z_i := z + s_i$, $\mathscr{X} \times_{\boldsymbol{C}^n} H$ is written as
$$\mathscr{X} \times_{\boldsymbol{C}^n} H = \left(xy + \prod_{i=0}^n z_i = 0 \right) \subset \boldsymbol{C}^{n+3}$$
and $f = (z_1 - z_0, \dots, z_n - z_0)$. Thus $f : \mathscr{X} \times_{\boldsymbol{C}^n} H \to H$ is a Gorenstein homogeneous toric deformation.
(3) Let $g : \mathscr{X} \to \mathscr{M}$ be a versal deformation space of an n-dimensional $(n \ge 3)$ isolated Gorenstein toric singularity. We denote by $\mathscr{S}$ an irreducible component of $\mathscr{M}$ and by $\mathscr{S}_{\mathrm{red}}$ its reduced structure. By [2, Theorem 8.1], the base change $f : \mathscr{X}_{\mathrm{red}} := \mathscr{X} \times_{\mathscr{S}} \mathscr{S}_{\mathrm{red}} \to \mathscr{S}_{\mathrm{red}}$ is a Gorenstein homogeneous toric deformation.

3. Simultaneous minimal model of Gorenstein homogeneous toric deformation

Theorem 1.1 is obtained as a corollary of the following theorem.

Theorem 3.1. *Let* $f : X := \operatorname{Spec} \boldsymbol{C}[\sigma^\vee \cap M] \to \boldsymbol{C}^m$ *be a Gorenstein homogeneous toric deformation and* $\tau : \tilde{X} \to X$ *a toric minimal model of* X. *Assume that* $\dim X = n + m$. *Then*

(1) *$f \circ \tau : \tilde{X} \to \boldsymbol{C}^m$ is a flat morphism,*
(2) *$K_{\tilde{X}_t}$ is τ-nef,*
(3) *$\tilde{X}_t$ has only hypersurface singularities in a quotient space. Moreover these singularities are defined by*

$$(F_i - F_0 = 0) \subset \boldsymbol{C}^{n+m}/G, \quad (1 \le i \le m)$$

where
(a) *G is an abelian group which acts on $\boldsymbol{C}^{n+m}$ diagonally,*
(b) *$\boldsymbol{C}^{n+m}/G$ has only Gorenstein terminal singularities,*
(c) *Each F_i is written as*

$$F_i = \prod_{j=p_i+1}^{p_{i+1}} x_j \quad (0 \le i \le m)$$

$$0 = p_0 < p_1 < p_2 < \cdots < p_m < p_{m+1} = n + m$$

where x_j is the j-th coordinate of $\boldsymbol{C}^{n+m}$. Moreover F_i are invariant monomials under the action of G.

Remark 3. If $\dim X = 2 + m$ (i.e. Every fibre of f is 2-dimensional), then F_i, $(1 \le i \le m)$ is written as

$$F_i = x_{i+1} \ \ (0 \le i \le m-1), \quad F_m = x_m x_{m+1}$$

by changing indices if necessary. Because F_i are invariant monomials under the action of G, the action of each element of $g \in G$ is nontrivial only on coordinates x_m and x_{m+1}. Since $\boldsymbol{C}^{2+m}/G$ has only Gorenstein terminal singularities, the action of G must be trivial. Thus each fibre of $f \circ \tau$ is smooth and τ gives a simultaneous resolution of f.

Proof of Theorem 1.1. Since S is reduced, there exists an open set $0 \in U$, $(U \subset S)$ which satisfies the following commutative diagram:

$$\begin{tikzcd}
X \times_S U \cong \eta^* \mathscr{X}_{\mathrm{red}} \arrow[r] \arrow[d] & \mathscr{X}_{\mathrm{red}} \arrow[d] \\
U \arrow[r, "\eta"'] & \mathscr{S}_{\mathrm{red}}
\end{tikzcd}$$

where η is an open immersion and $\mathscr{X}_{\mathrm{red}} \to \mathscr{S}_{\mathrm{red}}$ is the restriction of a versal deformation space to some irreducible component with its reduced structure. For Theorem 1.1, it is enough to prove that there exists a birational morphism $\tau : \widetilde{\mathscr{X}_{\mathrm{red}}} \to \mathscr{X}_{\mathrm{red}}$ which satisfies the assertions of Theorem 1.1. By [2, Theorem 8.1], we describe $\mathscr{X}_{\mathrm{red}} \to \mathscr{S}_{\mathrm{red}}$ as a Gorenstein homogeneous toric deformation. Then by Theorem 3.1, there exists a birational morphism $\widetilde{\mathscr{X}_{\mathrm{red}}} \to \mathscr{X}_{\mathrm{red}}$ which satisfies assertions (1) and (2) of Theorem 1.1. We check the assertion (3). Because $\dim \widetilde{\mathscr{X}_{\mathrm{red}}} = \dim \mathscr{S}_{\mathrm{red}} + 3$, F_i is written as

$$F_i = x_i \ \ (0 \le i \le m-2), \quad F_{m-1} = x_{m-1} x_m, \quad F_m = x_{m+1} x_{m+2}$$

or

$$F_i = x_i \ (0 \le i \le m-1), \quad F_m = x_m x_{m+1} x_{m+2}.$$

Each F_i are invariant monomials under the action of G. Hence, in the latter case, singularities of a fibre is isomorphic to $\boldsymbol{C}^3/G$. There exists no 3-dimensional Gorenstein quotient terminal singularities. Thus G is trivial. Therefore the central fibre has only the following singularities:

$$\{x_{m-1}x_m - x_{m+1}x_{m+2} = 0\} \subset \boldsymbol{C}^4/G.$$

Again there exists no 3-dimensional Gorenstein quotient terminal singularities. Hence $\boldsymbol{C}^4/G$ has only isolated singularities. The proof of Theorem 1.1 is completed by the classification of 4-dimensional isolated Gorenstein toric singularities [8, Theorem 2.4]. □

Proof of Theorem 3.1. By [1, Theorem 3.5, Remark 3.6], the construction of σ is as follows:

(1) σ is defined by $\sigma = \boldsymbol{R}_{\ge 0}P$, where P is an $(n+m-1)$-dimensional polygon given by

$$P := \mathrm{Conv}\left(\bigcup_{i=0}^{m} R_i \times e_i\right).$$

Note that R_i $(0 \le i \le m)$ are integral polytopes in $\boldsymbol{R}^{n-1}$ and

$$R_i \times e_i := \{(x_1, \dots, x_{n-1}, 0, \dots, 1, \dots, 0) \in \boldsymbol{R}^{n+m} \mid (x_1, \dots, x_{n-1}) \in R_i\}.$$

(2) f is defined by $(x^{r_i} - x^{r_0})$ $(1 \le i \le m)$, where $r_i : N_{\boldsymbol{R}} = \boldsymbol{R}^{n+m} \to \boldsymbol{R}$ is the $(n+i)$-th projection.

Thus, all primitive one dimensional generators of σ are contained in the hyperplane in $N_{\boldsymbol{R}}$ defined by $r_0 + \cdots + r_m = 1$. By [9, Theorem 0.2], there exists a toric minimal model $\tilde{X}$. Let $\sigma = \bigcup \sigma_\lambda$ be the corresponding cone decomposition. By [9, Definition 1.11], these cones satisfy the following three conditions:

(1) σ_λ is a simplex.

(2) One dimensional primitive generators $k_1, \dots, k_{n+m}$ of σ_λ are contained in the hypersurface defined by $r_0 + \cdots + r_m = 1$.

(3) The polytope

$$\Delta_\lambda := \sum_{j=0}^{n+m} \alpha_j k_j, \quad \sum_{j=0}^{n+m} \alpha_j \le 1, \ \ \alpha_j \ge 0$$

contains no lattice points except its vertices.

Let $X_\lambda := \mathrm{Spec}\, \boldsymbol{C}[\sigma_\lambda^\vee \cap M]$ and let $k_j^\vee$ $(1 \le j \le n+m)$ be the dual vectors of k_j. By (1), X_λ can be written as follows:

$$X_\lambda \cong \boldsymbol{C}^{n+m}/G$$

where $G := N/\bigoplus_{j=1}^{n+m} \boldsymbol{Z}k_j$ and the action of G is diagonal. Because each k_j are contained in the hypersurface defined by $r_0 + \cdots + r_m = 1$ and $\langle r_i, k_j \rangle \ge 0$ $(r_i \in \sigma^\vee)$,

$$\begin{cases} \langle r_i, k_j \rangle = 1 & \text{for } p_i < j \le p_{i+1} \\ \langle r_i, k_j \rangle = 0 & \text{other } j \end{cases}$$

where $0 = p_0 < p_1 < p_2 < \cdots < p_m < p_{m+1} = n+m$. Thus x^{r_i} is written as

$$x^{r_i} = \prod_{j=p_i+1}^{p_{i+1}} x_j$$

where $x_j = x^{k_j^\vee}$ is the j-th coordinate of $\boldsymbol{C}^{n+m}$. The monomials x^{r_i} are invariant under the action of G, because $r_i \in \sigma_\lambda^\vee \cap M$. Thus if we set $F_i = x^{r_i}$, the proof of Theorem 3.1 is completed. □

Remark 4. There exists an example of a flat family of isolated Gorenstein toric singularity which has no simultaneous terminalization even after finite base change.

LEMMA 3.2. *Let Y be a hypersurface singularity in a cyclic quotient space defined by*

$$\{x_1x_2 - x_3x_4 = 0\} \subset \boldsymbol{C}^4/G, \quad G \cong \boldsymbol{Z}/l\boldsymbol{Z}.$$

where the action of G given by

$$(x_1, \dots, x_4) \to (\zeta^{a_1}x_1, \dots, \zeta^{a_4}x_{n+1}), \quad (0 < a_i < l)$$

$$a_1 + a_2 \equiv a_3 + a_4 \equiv 0 \pmod{l}.$$

Note that ζ is a primitive l-root of unity and a_i's are coprime. Let X be the subvariety $\boldsymbol{C}^4/G \times \boldsymbol{C}$ defined by

$$x_1x_2 - x_3x_4 = t$$

and $f : X \to \boldsymbol{C}$ the projection. Then Y has only isolated Gorenstein toric singularities and f admits no simultaneous terminalization even after any finite base change.

Proof. It is easy to see that Y has only isolated toric singularities. Since the residue form

$$\mathrm{Res}\, \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{x_1x_2 - x_3x_4}$$

is G-equivariant, Y has only isolated Gorenstein toric singularities. Because

$$\sum_{i=1}^{4} a_i \ge 2l,$$

$\boldsymbol{C}^4/G$ has only Gorenstein terminal singularities. We derive a contradiction assuming that there exists a simultaneous terminalization after some finite base change. Let Z be the subvariety $\boldsymbol{C}^4/G \times \boldsymbol{C}$ defined by

$$x_1x_2 - x_3x_4 = t^m.$$

From the assumption, there exists a simultaneous terminalization $\tau : \mathscr{X} \to Z$. Let Z' be the subvariety in $\boldsymbol{C}^5$ defined by

$$x_1x_2 - x_3x_4 = t^m.$$

Then there exists a finite morphism $Z' \to Z$. Since Z' has only hypersurface singularities whose singular locus has codimension four, it is $\boldsymbol{Q}$-factorial by [7, XI.3.13]. By [6, Lemma 5.16], Z is again $\boldsymbol{Q}$-factorial. Because a general fibre of $f : X \to \boldsymbol{C}$ is smooth, the codimension of exceptional set of τ is greater than two. That contradicts to [5, VI 1.5 Theorem].

References

[1] K. Altmann, Minkowski sums and homogeneous deformations of toric varieties, Tôhoku Math. J. (2), **47** (1995), 151–184.

[2] K. Altmann, The versal deformation of an isolated toric Gorenstein singularity, Invent. Math., **128** (1997), 443–479.

[3] E. Brieskorn, Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen, Math. Ann., **166** (1966), 76–102.

[4] E. Brieskorn, Die Auflösung der rationalen Singularitäten holomorpher Abbildungen, Math. Ann., **178** (1968), 255–270.

[5] J. Kollár, Rational Curves on Algebraic Varieties, Ergeb. Math. Grenzgeb. (3) 32, Springer-Verlag, Berlin, 1996.

[6] J. Kollár and S. Mori, Birational Geometry of Algebraic Varieties, Cambridge Tracts in Math. 134, Cambridge University Press, Cambridge, 1998.

[7] A. Grothendieck, Cohomologie Locale des Faisceaux Cohérents et Théorèmes de Lefschetz Locaux et Globaux (SGA2), North-Holland, Amsterdam, 1968.

[8] D. R. Morrison and G. Stevens, Terminal quotient singularities in dimensions three and four, Proc. Amer. Math. Soc., **90** (1984), 15–20.

[9] M. Reid, Decomposition of toric morphisms, Arithmetic and Geometry Volume II (M. Artin and J. Tate eds.), Progress in Math. 36, Birkhäuser, Boston, 1983, 395–418.

[10] M. Reid, Young person's guide to canonical singularities, Algebraic Geometry Bowdoin 1985 (S. J. Bloch ed.), Proc. Sympos. Pure Math. 46-1, AMS, Providence, 1987, 345–414.

[11] G. N. Tyurina, Resolution of singularities of plane deformations of double rational points, Functional Anal. Appl., **4** (1970), 68–73.

Division of Mathematics
Graduate School of Science
Hokkaido University
Sapporo, 060-0810 Japan
e-mail: matusita@math.sci.hokudai.ac.jp