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CHARACTERS OF AUTOMORPHISM GROUPS ASSOCIATED WITH KÄHLER CLASSES AND FUNCTIONALS WITH COCYCLE CONDITIONS

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1. Introduction

Let M be a connected compact Kähler manifold. An obvious necessary condition for M to admit a Kähler-Einstein metric is that the first Chern class $c_1(M)$ is either negative, zero or positive, where a real 2-dimensional de Rham cohomology class is said to be negative (resp. positive) if it is represented by a negative (resp. positive) definite $(1,1)$-form. Conversely, if $c_1(M)$ is negative or zero then M admits a Kähler-Einstein metric by the solution to the Calabi conjectures (Aubin [1], Yau [21]).

In the remaining case where $c_1(M)$ is positive, in which case M is often called a Fano manifold, there are further necessary conditions. First of all the Lie algebra $\mathfrak{h}(M)$ of all holomorphic vector fields on a Kähler-Einstein Fano manifold M is reductive (Matsushima [14]). Secondly a Lie algebra character $f : \mathfrak{h}(M) \to \boldsymbol{C}$ introduced in [10] must vanish on a Kähler-Einstein Fano manifold. It was also proven by Bando-Mabuchi [5] that if M admits a Kähler-Einstein metric then certain functional, called K-energy, of M is bounded from below. This analytic necessary condition played a theoretically important role in the later studies. In fact Ding and Tian [9] extended the results of [10] and [5] to obtain a necessary condition applicable to manifolds which do not carry any non-zero holomorphic vector fields. Tian [20] further extended these ideas to define certain notions of stability, called K-stability and CM-stability, and presented an example of a Fano manifold with no non-zero holomorphic vector fields and no Kähler-Einstein metrics. On the other hand there are known sufficient conditions for the existence of positive Kähler-Einstein metrics by Aubin [2], Ding [8], Siu [17], Tian [18] and Nadel [15].

Now one would hope to have a necessary and sufficient condition for the existence of positive Kähler-Einstein metrics. To state such a condition, Tian [20] introduced a notion of properness for the K-energy and a functional introduced by Ding [8]. Combining [20] and [4] one can show, at least when $\mathfrak{h}(M) = 0$, that a Fano manifold admits a Kähler-Einstein metric if and only if

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Ding's functional (or K-energy) is proper, and thus the properness of Ding's functional and the properness of K-energy are equivalent. It can be checked directly that if the sufficient condition obtained in, for example, [18] is satisfied then the both functionals are proper, and that if either of the functionals is proper then the necessary conditions obtained in [10], [9], [20] are satisfied.

The purpose of this paper is to show that there is a family of functionals satisfying cocyle conditions and that both of the K-energy and Ding's functional can be derived from the family. It is constructed as follows. Given a Kähler manifold M with a Kähler class Ω, there exists a Lie algebra character $f_\Omega : \mathfrak{h}(M) \to \boldsymbol{C}$ with the property that if M admits a Kähler metric of constant scalar curvature in Ω then $f_\Omega = 0$ (cf. [11], [7], [3]). Of course when M is a Fano manifold and $\Omega = c_1(M)$, then f_Ω coincides with the above f. We try to lift f_Ω to a character of the group of automorphisms which preserve Ω. This can be performed successfully when Ω is a Hodge class and the group action lifts to a holomorphic line bundle L whose Chern class coincides with Ω, see Nakagawa [16]. We can give an explicit formula of the character in terms of the Chern-Simons invariants of certain virtual bundles. A merit of this formula is that it is written using Hermitian metrics of L and the anti-canonical bundle K_M^{-1} and that we may choose these two metrics independently. Note that if in certain situations the character can be written in terms of a Kähler form ω and the pull-back form $\omega' = \sigma^*\omega$ by an automorphism σ, then we obtain a functional written in terms of ω and ω' satisfying cocycle conditions.

Now return to the situation where $\Omega = c_1(M)$ and $L = K_M^{-1}$. In section 4 we will see that if we choose a Kähler form $\omega \in \Omega$ and then choose a fiber metric of L so that its Chern form is equal to ω, then the formula of the group character yields the K-energy. On the other hand if we choose metrics for L and K_M^{-1} equal, the formula yields Ding's functional.

2. Review of characteristic classes of foliations

A transeversely holomorphic foliation $\mathscr{F}$ of complex codimension m on a smooth manifold W of real dimension $2m+n$ is given by a system of local charts $\{z^1,\dots,z^m,x^1,\dots,x^n\}$ where $\{x^1,\dots,x^n\}$ is real coordinates along the leaves and $\{z^1,\dots,z^m\}$ are complex coordinates in the normal directions, such that for any neighboring local charts $\{w^1,\dots,w^m,y^1,\dots,y^n\}$, the w^i's are holomorphic functions of z^i's. Then there is a subbundle $T^*_{1,0}$ of $TW^* \otimes \boldsymbol{C}$ spanned by $\{dz^1,\dots,dz^m\}$ in local charts. Note that the definition of $T^*_{1,0}$ is independent of the choice of local charts. A section of $T^*_{1,0}$ will be called a differential form of type $(1,0)$. Let S be the subbundle of $TW \otimes \boldsymbol{C}$ annihilated by $T^*_{1,0}$. The quotient bundle $\nu(\mathscr{F}) = (TW \otimes \boldsymbol{C})/S$ is called the normal bundle of $\mathscr{F}$.

Let $E \to W$ be a complex vector bundle of rank r over W. A *basic connection* of E is a linear connection whose connection form is of type $(1,0)$. Not every vector bundle admits a basic connection. But, for example, an argument using partition of unity shows that the normal bundle $\nu(\mathscr{F})$ carries basic connections ([6]). It is obvious from the dimension reasons that, for a multi-index α with $|\alpha| > m$, the Chern form $c^{\alpha}(E, \nabla)$ vanishes identically if ∇ is a basic connection. Thus we can define characteristic classes of foliations, which we review next.

The differential graded algebra WU_m is defined as

$$WU_m = \bigwedge(u_1, \dots, u_m) \otimes \{\boldsymbol{C}[c_1, \dots, c_m]/\deg > 2m\} \otimes \{\boldsymbol{C}[\bar{c}_1, \dots, \bar{c}_m]/\deg > 2m\}$$

where $\bigwedge(u_1, \dots, u_m)$ is the exterior algebra in $u_1, \dots, u_m$ with $\deg u_i = i$, $\boldsymbol{C}[c_1, \dots, c_m]$ is a polynomial algebra in $c_1, \dots, c_m$ with $\deg c_i = 2i$ and similarly for $\boldsymbol{C}[\bar{c}_1, \dots, \bar{c}_m]$, and where the differential d is defined by $du_i = c_i - \bar{c}_i$ and $dc_i = d\bar{c}_i = 0$.

Suppose that E carries basic connections. Let $\Omega^* W$ be the de Rham complex of W. We define a differential graded algebra map $\lambda_W : WU_m \to \Omega^* W$ as follows.

Choose an Hermitian metric of E and take an arbitrary metric connection ∇^0, and a basic connection ∇^1 of E. Let $p : W \times I \to W$ be the projection, where I denotes the unit interval. Then $\nabla^{0,1} = s\nabla^1 + (1-s)\nabla^0$ is a connection of p^*E. Denote by $c_i(\nabla^0), c_i(\nabla^1)$ and $c_i(\nabla^{0,1})$ respectively the i-th Chern forms with respect to ∇^0, ∇^1 and $\nabla^{0,1}$. Set $h_i = p_* c_i(\nabla^{0,1})$. Then we have

$$dh_i = c_i(\nabla^1) - c_i(\nabla^0),$$

and

$$dh_i - d\bar{h}_i = c_i(\nabla^1) - \overline{c_i(\nabla^1)}$$

since ∇^0 is a metric connection and its Chern forms are real forms. From this it follows that the map λ_W defined by

$$\lambda_W(u_i) = h_i - \bar{h}_i, \quad \lambda_W(c_i) = c_i(\nabla^1), \quad \lambda_W(\bar{c}_i) = \overline{c_i(\nabla^1)}$$

is a DGA-map. It is well-known that the induced homomorphism $\lambda_W^* : H^*(WU_m) \to H^*_{\mathrm{DR}}(W; \boldsymbol{C})$ is independent of the choice of the Hermitian connection ∇^0 and the basic connection ∇^1. We note that $\sum_{k=0}^m c_1^k u_1 \bar{c}_1^{m-k}$ is closed in WU_m, and thus we have

$$\lambda_W^* \left(\sum_{k=0}^m c_1^k u_1 \bar{c}_1^{m-k} \right) \in H^{2m+1}(W; \boldsymbol{C}).$$

3. The case of suspension foliations

Let M be a compact Kähler manifold and σ an automophism of M. Suppose that σ generates an infinite cyclic group $G \cong \boldsymbol{Z}$. Let $E \to M$ be a holomorphic vector bundle. We assume that the action of G lifts to E. We set $E_\sigma := (\boldsymbol{R} \times E)/G$ and $M_\sigma := (\boldsymbol{R} \times M)/G$, where G acts on $\boldsymbol{R} \times E$ by

$$\sigma^n(v, t) = (t - n, \sigma^n(v))$$

and on $\boldsymbol{R} \times M$ similarly. There is a natural transversely holomorphic foliation on $\boldsymbol{R} \times M$ with leaf dimension 1, and it descends to M_σ.

LEMMA 3.1. *The complex vector bundle $E_\sigma \to M_\sigma$ carries basic connections.*

Proof. E_σ is obtained as follows. Consider $I \times E$ and identify $\{1\} \times E$ and $\{0\} \times E$ by the relation $(1, v) \sim (o, \sigma(v))$. M_σ is also obtained similarly. The leaves of the foliation are of the form $\{(t, p) | t \in I\}$. Remark that the vector bundle E_σ is flat along the leaf direction, i.e. the transition functions do not involve the leaf coordinate.

Let $\phi(t)$ be a smooth function on I such that $\phi(t) \equiv 0$ near $t = 0$ and $\phi(t) \equiv 1$ near $t = 1$. Choose any Hermitian metric h of the line bundle $E \to M$. We define an Hermitian metric of $E_\sigma \to M_\sigma$ by

$$\tilde{h}_t = (1 - \phi(t))h + \phi(t)\sigma^* h.$$

Then by the above remark $\tilde{h}_t^{-1}\partial\tilde{h}_t$ defines a basic connection, where ∂ denotes the $(1,0)$-part of the exterior differentiation, namely

$$\partial = \sum_{i=1}^{m} dz^i \frac{\partial}{\partial z^i}$$

it terms of normal holomorphic coordinates $z^1, \dots, z^m$. $\square$

So we can define $\hat{f}_E : G \to \boldsymbol{R}$ by

$$\hat{f}_E(\sigma) = i\lambda_W^* \left(\sum_{k=0}^{m} c_1^k u_1 \bar{c}_1^{m-k} \right) [W]. \tag{1}$$

Recall that, given an Hermitian metric h on a holomorphic vector bundle E, the Ricci form ρ_h is given by

$$\rho_h := \frac{i}{2\pi} \bar{\partial}\partial \log \det h.$$

It represents the first Chern class $c_1(E)$, and its coefficients

$$R_{i\bar{j}} := -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det h$$

are called the Ricci curvature of h. When an automorphism σ of M lifts to an action on E, we have

$$\sigma^* \rho_h = \rho_{\sigma^* h}.$$

THEOREM 3.2. *Let h be an Hermitian metric of the holomorphic vector bundle $E \to M$. Then $\hat{f}_E(\sigma)$ can be given by*

$$\hat{f}_E(\sigma) = \frac{1}{2\pi}\int_M \log \frac{\det \sigma^* h}{\det h} \sum_{k=0}^{m} \sigma^* \rho_h^k \wedge \rho_h^{m-k}. \tag{2}$$

Moreover $\hat{f}_E : G \to \boldsymbol{R}$ *is a group character.*

Remark 3.3. From the independence of the choice of Hermitian connections ∇^0 and basic connections ∇^1, the above expression of f_E is also independent of the choice of Hermitian metrics h.

Proof. Let $\tilde{h}_t$ be as in the proof of Lemma 3.1, and put

$$h_t = \left(\frac{(\det h)^{1-\phi(t)}(\det \sigma^* h)^{\phi(t)}}{\det \tilde{h}_t}\right)^{1/r} \tilde{h}_t,$$

where r is the rank of E. Then the Ricci form ρ_{h_t} is written as

$$\rho_{h_t} = (1-\phi(t))\rho_h + \phi(t)\rho_{\sigma^* h}. \tag{3}$$

It is obvious that

$$\theta^1 = h_t^{-1}\partial h_t$$

also defines a basic connection, and it is easy to check that

$$\theta^0 = h_t^{-1}\partial h_t + \frac{1}{2} h_t^{-1}\partial_t h_t$$

defines an Hermitian connection, where

$$\partial_t = dt\frac{\partial}{\partial t}.$$

From the definition of u_1 we easily get

$$\lambda_W^* u_1 = \frac{i}{2\pi}\partial_t \log \det h_t.$$

We also have

$$\lambda_W^* c_1 = \frac{i}{2\pi}(\partial_t + \bar{\partial})\partial \log \det h_t.$$

It follows that, for all k, we have

$$\lambda_W^*(c_1^k u_1 \bar{c}_1^{m-k}) = \frac{i}{2\pi}\partial_t \log \det h_t \wedge \left(\frac{i}{2\pi}\bar{\partial}\partial \log \det h_t\right)^m. \tag{4}$$

From (3) and (4) we have

$$
\begin{aligned}
i\lambda_W^* c_1^k u_1 \bar{c}_1^{m-k}[W] &= (m+1)\frac{1}{2\pi}\int_{M\times I} \phi'(t) \log \frac{\det \sigma^* h}{\det h}\, dt \wedge (\rho_h + \phi(t)(\rho_{\sigma^* h} - \rho_h))^m \\
&= (m+1)\frac{1}{2\pi}\int_M \log \frac{\det \sigma^* h}{\det h} \sum_{k=0}^m \binom{m}{k} \frac{1}{k+1} (\rho_h)^{m-k} \wedge (\rho_{\sigma^* h} - \rho_h)^k \\
&= \frac{1}{2\pi}\int_M \log \frac{\det \sigma^* h}{\det h} \sum_{k=0}^m \rho_h^k \wedge \rho_{\sigma^* h}^{m-k}.
\end{aligned}
$$

This completes the proof of the first half.

From (2) and the fact that the right hand side of (2) is independent of the choice of the metric h it is easy to check

$$\hat{f}_E(\sigma\tau) = \hat{f}_E(\sigma) + \hat{f}_E(\tau).$$

This completes the proof of the second half. An alternate proof can also be given as follows. There is a fibration $p : W = M_G = EG \times_G M \to BG = S^1$. The integration over the fibers gives

$$p_*(i\lambda_W^* c_1^k u_1 \bar{c}_1^{m-k}) \in H^1(BG; \boldsymbol{R}) \cong \mathrm{Hom}(G, \boldsymbol{R}).$$

The interpretation of the last isomorphism shows that $\hat{f}_E$ is a homomorphism. □

4. A Lie algebra character and its lifting to a group character

In this section we first review the Lie algebra character obtained as an obstruction to the existence of Kähler metric of constant scalar curvature ([11], [7], [3]), and give an explicit formula of its lifting to a group character.

Let Ω be a fixed Kähler class on an m-dimensional connected compact Kähler manifold M, and $\omega = i/(2\pi)\sum g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$ a Kähler form which represents Ω. The Ricci form of ω will be denoted by

$$\rho_\omega = \frac{i}{2\pi}\sum R_{i\bar{j}}\, dz^i \wedge d\bar{z}^j = \frac{i}{2\pi}\bar{\partial}\partial \log \det g.$$

We put

$$V := \Omega^m[M],$$

$$\mu = \frac{1}{V}\int_M \frac{s_\omega}{m}\omega^m,$$

where $s_\omega = g^{i\bar{j}}R_{i\bar{j}}$ denotes the scalar curvature of ω. Then there is a smooth function F_ω uniquely determined up to constant such that

$$s_\omega - \mu m = \Delta F_\omega.$$

Define $f : \mathfrak{h}(M) \to \boldsymbol{C}$ by

$$f(X) = \frac{1}{2\pi} \int_M X F_\omega \omega^m.$$

It is known that f is independent of the choice of $\omega \in \Omega$. This implies that f is invariant under automorphisms of M and is a Lie algebra character. Obviously if M admits a Kähler metric of constant scalar curvature then we have $F_\omega = 0$ and f vanishes.

Let g be a Hodge metric, Ω its Kähler class, and L a holomorphic line bundle with $c_1(L) = \Omega$. We assume that the action of an automorphism σ lifts to L. Consider the character defined by

$$\begin{aligned}\hat{f} &= \frac{1}{2^{m+1}(m+1)!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \hat{f}_{K_M^{-1} \otimes L^{m-2j}} \\ &\quad - \frac{1}{2^{m+1}(m+1)!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \hat{f}_{K_M \otimes L^{m-2j}} \\ &\quad - \frac{\mu m}{2^{m+1}(m+1)!(m+1)} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \hat{f}_{L^{m+1-2j}}.\end{aligned}$$

The following theorem has been proved by the second author [16].

Theorem 4.1. *Let $X \in \mathfrak{h}(M)$. Then we have*

$$\Re f(X) = \left.\frac{d}{dt}\right|_{t=0} \hat{f}(\exp(t \Re X)).$$

where $\Re f(X)$ denotes the real part of $f(X)$.

The proof of this theorem will follow from later computations. Combining Remark 3.3 and Theorem 4.1 we obtain the following.

Corollary 4.2. *The Lie algebra character $\Re f$ can be lifted to a group character $\hat{f}$ which can be written explicitly using Hermitian metrics of L and K_M. Moreover $\hat{f}$ is independent of the choice of these metrics in each of the terms of $\hat{f}_{K_M^{-1} \otimes L^{m-2j}}$, $\hat{f}_{K_M \otimes L^{m-2j}}$ and $\hat{f}_{L^{m+1-2j}}$.*

In the rest of this paper we show that a suitable choice of Hermitian metrics of L and K_M^{-1} yields the K-energy, and when $L = K_M^{-1}$ a different choice yields Ding's functional.

Let ω be a Hodge metric of M and h an Hermitian metric of L. As before ρ_ω and ρ_h respectively denotes the Ricci forms of ω and h. By the assumption we have $[\rho_h] = c_1(L) = [\omega]$. From Theorem 3.2 we have

$$(5) \qquad 2\pi\hat{f}(\sigma) = \frac{1}{2^{m+1}(m+1)!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\int_M\left(\log\frac{\sigma^*\omega^m}{\omega^m}+(m-2j)\log\frac{\sigma^*h}{h}\right)$$

$$\sum_{k=0}^{m}\sigma^*(\rho_\omega+(m-2j)\rho_h)^k \wedge (\rho_\omega+(m-2j)\rho_h)^{m-k}$$

$$-\frac{1}{2^{m+1}(m+1)!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\int_M\left(-\log\frac{\sigma^*\omega^m}{\omega^m}+(m-2j)\log\frac{\sigma^*h}{h}\right)$$

$$\sum_{k=0}^{m}\sigma^*(-\rho_\omega+(m-2j)\rho_h)^k \wedge (-\rho_\omega+(m-2j)\rho_h)^{m-k}$$

$$-\frac{\mu m}{2^{m+1}(m+1)!(m+1)}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}(m+1-2j)^{m+1}$$

$$\times\int_M\log\frac{\sigma^*h}{h}\sum_{k=0}^{m}\sigma^*\rho_h^k \wedge \rho_h^{m-k}.$$

Using

$$\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}(\ell-2j)^k = 0 \quad \text{for } k \neq \ell,$$

$$\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}(\ell-2j)^\ell = 2^\ell \ell!$$

c.f. [19], and putting

$$\gamma = \frac{\sigma^*\omega^m}{\omega^m}, \quad \tilde{\gamma} = \frac{\sigma^*h}{h},$$

we obtain

$$(6) \qquad 2\pi(m+1)\hat{f}(\sigma) = \frac{1}{2}\int_M \log\gamma\sum_{k=0}^{m}\sigma^*\rho_h^k \wedge \rho_h^{m-k}$$

$$+\log\tilde{\gamma}\sum_{k=0}^{m}k\sigma^*\rho_\omega \wedge \sigma^*\rho_h^{k-1} \wedge \rho_h^{m-k}$$

$$+\log\tilde{\gamma}\sum_{k=0}^{m}\sigma^*\rho_h^k \wedge (m-k)\rho_\omega \wedge \rho_h^{m-k-1}$$

$$-\frac{1}{2}\int_M(-\log\gamma)\sum_{k=0}^{m}\sigma^*\rho_h^k \wedge \rho_h^{m-k}$$

$$
\begin{aligned}
&+\log\tilde{\gamma}\sum_{k=0}^{m}k(-\sigma^*\rho_\omega)\wedge\sigma^*\rho_h^{k-1}\wedge\rho_h^{m-k}\\
&+\log\tilde{\gamma}\sum_{k=0}^{m}\sigma^*\rho_h^{k}\wedge(m-k)(-\rho_\omega)\wedge\rho_h^{m-k-1}\\
&-\mu m\int_M\log\tilde{\gamma}\sum_{k=0}^{m}\sigma^*\rho_h^{k}\wedge\rho_h^{m-k}\\
=&\int_M\log\gamma\sum_{k=0}^{m}\sigma^*\rho_h^{k}\wedge\rho_h^{m-k}+\log\tilde{\gamma}\sum_{k=0}^{m}k\sigma^*\rho_\omega\wedge\sigma^*\rho_h^{k-1}\wedge\rho_h^{m-k}\\
&+\log\tilde{\gamma}\sum_{k=0}^{m}\sigma^*\rho_h^{k}\wedge(m-k)\rho_\omega\wedge\rho_h^{m-k-1}\\
&-\mu m\int_M\log\tilde{\gamma}\sum_{k=0}^{m}\sigma^*\rho_h^{k}\wedge\rho_h^{m-k}.
\end{aligned}
$$

Suppose now that $\rho_h=\omega$. We put

$$\varphi:=-\log\tilde{\gamma}=-\log\frac{\sigma^*h}{h}.$$

Then we have

$$\frac{i}{2\pi}\partial\bar{\partial}\varphi=-\frac{i}{2\pi}\partial\bar{\partial}\log\frac{\sigma^*h}{h}=\sigma^*\rho_h-\rho_h=\sigma^*\omega-\omega.$$

So if we put $\omega_\varphi:=\sigma^*\omega$, then

$$\omega_\varphi=\omega+\frac{i}{2\pi}\partial\bar{\partial}\varphi\quad\textit{and}\quad\sigma^*\rho_\omega=\rho_{\omega_\varphi}.$$

Define a functional $\mathscr{M}(\omega,\omega_\varphi)$ by

$$\mathscr{M}(\omega,\omega_\varphi)=\frac{2\pi}{V}\hat{f}(\sigma).$$

THEOREM 4.3. *$\mathscr{M}(\omega,\omega_\varphi)$ coincides with the K-energy.*

Proof. From (6) we have

$$
\begin{aligned}
(m+1)V\mathscr{M}(\omega,\omega_\varphi)=&\int_M\log\frac{\omega_\varphi^m}{\omega^m}\sum_{k=0}^{m}\omega_\varphi^k\wedge\omega^{m-k}-\int_M\varphi\sum_{k=0}^{m}k\rho_{\omega_\varphi}\wedge\omega_\varphi^{k-1}\wedge\omega^{m-k}\\
&-\int_M\varphi\sum_{k=1}^{m+1}\omega_\varphi^{k-1}\wedge(m+1-k)\rho_\omega\wedge\omega^{m-k}
\end{aligned}
$$

$$+ \mu m \int_M \varphi \sum_{k=0}^{m} \omega_\varphi^k \wedge \omega^{m-k}$$

$$= \int_M \log \frac{\omega_\varphi^m}{\omega^m} \sum_{k=0}^{m} \omega_\varphi^k \wedge \omega^{m-k}$$

$$- (m+1) \int_M \varphi \sum_{k=1}^{m} \omega_\varphi^{k-1} \wedge \rho_\omega \wedge \omega^{m-k}$$

$$- \int_M \varphi \sum_{k=0}^{m} k(\rho_{\omega_\varphi} - \rho_\omega) \wedge \omega_\varphi^{k-1} \wedge \omega^{m-k}$$

$$+ \mu m \int_M \varphi \sum_{k=0}^{m} \omega_\varphi^k \wedge \omega^{m-k}.$$

The third term of the right hand side is equal to

$$\int_M \log \frac{\omega_\varphi^m}{\omega^m} \sum_{k=0}^{m} k(\omega_\varphi - \omega) \wedge \omega_\varphi^{k-1} \wedge \omega^{m-k}$$

$$= \int_M \log \frac{\omega_\varphi^m}{\omega^m} (\omega_\varphi \wedge \omega^{m-1} - \omega^m + 2(\omega_\varphi^2 \wedge \omega^{m-2} - \omega_\varphi \wedge \omega^{m-1})$$

$$+ \cdots + m(\omega_\varphi^m - \omega_\varphi^{m-1} \wedge \omega))$$

$$= - \int_M \log \frac{\omega_\varphi^m}{\omega^m} (\omega^m + \omega_\varphi \wedge \omega^{m-1} + \cdots + \omega_\varphi^m) + (m+1) \int_M \log \frac{\omega_\varphi^m}{\omega^m} \omega_\varphi^m.$$

Thus we obtain

$$V\mathscr{M}(\omega, \omega_\varphi) = - \int_M \varphi \sum_{k=0}^{m-1} \omega_\varphi^k \wedge \rho_\omega \wedge \omega^{m-k-1} \tag{7}$$

$$+ \int_M \log \frac{\omega_\varphi^m}{\omega^m} \omega_\varphi^m + \frac{\mu m}{m+1} \int_M \varphi \sum_{k=0}^{m} \omega_\varphi^k \wedge \omega^{m-k}.$$

This last expression is equal to the K-energy (c.f. [4], §5). □

Proof of Theorem 4.1. Let $X \in \mathfrak{h}(M)$, and define φ_t by

$$(\exp(t\Re X))^* \omega - \omega = \frac{i}{2\pi} \partial \bar{\partial} \varphi,$$

where $\Re X$ denotes the real part of X. Then

$$
\begin{aligned}
2\pi \frac{d}{dt}\bigg|_{t=0} \hat{f}(\exp(t\Re X)) &= \frac{d}{dt}\bigg|_{t=0} V\mathscr{M}(\omega, \omega_\varphi) \\
&= -\int_M \dot{\varphi}|_{t=0}(s_\omega - \mu m)\omega^m \\
&= -\int_M \Delta(\dot{\varphi}|_{t=0})F_\omega \omega^m.
\end{aligned}
$$

Since X is holomorphic, $\bar{\partial} i(X)\omega = 0$. Hence the harmonic integration theory shows

$$i(X)\omega = i\bar{\partial}u + i\alpha$$

where $\alpha = \alpha_{\bar{j}}\, d\bar{z}^j$ is a harmonic $(0,1)$-form. Therefore we have

$$L_{1/2(X+\bar{X})}\omega = i\partial\bar{\partial}(\Re u).$$

It follows that

$$\dot{\varphi}|_{t=0} = \Re u$$

modulo constant. Using $\bar{\partial}^* \alpha = 0$, we have

$$
\begin{aligned}
-\int_M \Delta(\dot{\varphi}|_{t=0})F_\omega \omega^m &= -\int_M \Delta(\Re(u))F_\omega \omega^m \\
&= -\int_M \Re(\mathrm{div}(X))F_\omega \omega^m \\
&= 2\pi \Re(f(X)).
\end{aligned}
$$

This completes the proof of Theorem 4.1. □

Suppose that $c_1(M) > 0$ and that $\rho_\omega = \omega + (i/2\pi)\partial\bar{\partial}F_\omega$. Then we have

$$
\begin{aligned}
\mathscr{M}(\omega, \omega_\varphi) = &-\frac{1}{V}\int_M \varphi \sum_{k=0}^{m-1} \omega_\varphi^k \wedge \left(\omega + \frac{i}{2\pi}\partial\bar{\partial}F_\omega\right) \wedge \omega^{m-k-1} + \frac{1}{V}\int_M \log \frac{\omega_\varphi^m}{\omega^m}\omega_\varphi^m \\
&+ \frac{m}{(m+1)V}\int_M \varphi \sum_{k=0}^{m} \omega_\varphi^k \wedge \omega^{m-k} \\
= &-\frac{1}{(m+1)V}\int_M \varphi \sum_{k=0}^{m} \omega_\varphi^k \wedge \omega^{m-k} + \frac{1}{V}\int_M \log \frac{\omega_\varphi^m}{\omega^m}\omega_\varphi^m \\
&- \frac{1}{V}\int_M F_\omega(\omega_\varphi^m - \omega^m) + \frac{1}{V}\int_M \varphi\omega_\varphi^m.
\end{aligned}
$$

This last expression is the K-energy for Fano manifolds used in [20].

Suppose again that $c_1(M) > 0$ and that $L = K_M^{-1}$. We choose $h = \omega^m$. Then from (6) we have

$$2\pi\hat{f}(\sigma) = \int_M \log \frac{\omega_\varphi^m}{\omega^m} \sum_{k=0}^m \rho_{\omega_\varphi}^k \wedge \rho_\omega^{m-k},$$

where we have put $\omega_\varphi := \sigma^*\omega$. We assume $\eta := \rho_\omega > 0$, $\eta' := \rho_{\omega_\varphi} > 0$, and put $\eta' = \eta + (i/2\pi)\partial\bar{\partial}\psi$. Then of course $\psi = -\log \omega_\varphi^m/\omega^m$, and thus

$$2\pi\hat{f}(\sigma) = -\int_M \psi \sum_{k=0}^m \eta'^k \wedge \eta^{m-k}. \tag{8}$$

Here ψ is normalized by

$$\int_M e^{-\psi}\omega^m = V.$$

We wish to rewrite the right hand side of (8) in the form invariant under the change of ψ into $\psi + constant$. If we define F_η by

$$\rho_\eta = \eta + \frac{i}{2\pi}\partial\bar{\partial}F_\eta, \quad \int_M e^{F_\eta}\eta^m = V,$$

then

$$F_\eta = \log \frac{\omega^m}{\eta^m}$$

and

$$\frac{1}{V}\int_M e^{-\psi + F_\eta}\eta^m = 1.$$

Hence

$$\frac{2\pi}{V}\hat{f}(\sigma) = -\frac{1}{(m+1)V}\int_M \psi \sum_{k=0}^m \eta'^k \wedge \eta^{m-k} - \log\left(\frac{1}{V}\int_M e^{-\psi+F_\eta}\eta^m\right).$$

We define

$$\mathscr{F}(\omega, \omega_\varphi) = -\frac{1}{(m+1)V}\int_M \varphi \sum_{k=0}^m \omega_\varphi^k \wedge \omega^{m-k} - \log\left(\frac{1}{V}\int_M e^{-\varphi+F_\omega}\omega^m\right),$$

where F_ω is normalized by $1/V \int_M e^{F_\omega}\omega^m = 1$.

Theorem 4.4. *$\mathscr{F}(\omega, \omega_\varphi)$ coincides with Ding's functional.*

Proof. Ding's functional ([8]), which we denote by $\mathscr{D}(\omega, \omega_\varphi)$, is defined by

$$\mathscr{D}(\omega, \omega_\varphi) = J_\omega(\varphi) - \frac{1}{V}\int_M \varphi\omega^m - \log\left(\frac{1}{V}\int_M e^{F_\omega - \varphi}\omega^m\right),$$

where $J_\omega(\varphi)$ is defined by

$$I_\omega(\varphi) = \frac{1}{V}\int_M \varphi(\omega^m - \omega_\varphi^m),$$
$$J_\omega(\varphi) = \int_0^1 \frac{I_\omega(s\varphi)}{s}\, ds.$$

But one computes

$$\begin{aligned} J_\omega(\varphi) &= \frac{1}{V}\int_0^1\int_M \varphi(\omega^m - \omega_{s\varphi}^m)\, ds \\ &= \frac{1}{V}\int_M \varphi\omega^m - \frac{1}{V}\int_0^1\int_M \varphi\left(\omega + \frac{i}{2\pi}\partial\bar{\partial}s\varphi\right)^m ds \\ &= \frac{1}{V}\int_M \varphi\omega^m - \frac{1}{(m+1)V}\int_M \varphi\sum_{k=0}^m \omega_\varphi^{m-k}\wedge\omega^k. \end{aligned}$$

This completes the proof. □

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