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Mosaic and trace formulae of log-hyponormal operators

Dedicated to Professor Michiaki Watanabe in celebration of his having been honoured as an Emeritus Professor of Niigata University

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Abstract. The purpose of this paper is to introduce mosaics of log-hyponormal operators and give a Helton-Howe type trace formula.

1. Introduction.

J. D. Pincus and D. Xia, in [**14**], studied mosaics and principal functions of semi-hyponormal operators and gave the trace formula. In [**19**], Xia announced trace fomulae for semi-hyponormal operators. In [**4**], we gave trace formulae of p-hyponormal operators for $0 < p \leq 1/2$. In particular we proved a Helton-Howe type trace formula (cf. [**13**], p. 240, Theorem 2.4). In this paper, we introduce mosaics and principal functions of log-hyponormal operators and prove a Helton-Howe type trace formula of it.

Let $\mathscr{H}$ be a complex separable Hilbert space and $B(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. An operator $T \in B(\mathscr{H})$ is said to be p-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$. If $p = 1$, T is called hyponormal and if $p = 1/2$, T is called semi-hyponormal. The set of all semi-hyponormal operators in $B(\mathscr{H})$ is denoted by SH. Let SHU denote the set of all operators in SH with equal defect and nullity (cf. [**19**], p. 4). Hence we may assume that the operator U in the polar decomposition $T = U|T|$ is unitary if $T \in \mathrm{SHU}$. An operator $T \in B(\mathscr{H})$ is said to be log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$. Since the function $\log(\cdot)$ is operator monotone, an operator T is log-hyponormal if T is an invertible p-hyponormal operator. In [**15**] K. Tanahashi gave a counter example of log-hyponormal operator which is not p-hyponormal. When $\log|T| \geq 0$, he also proved that $T' = U\log|T|$ is semi-hyponormal if $T = U|T|$ is log-hyponormal. If $T = U|T|$ is log-hyponormal, then we can choose a number $c > 0$ such that $\log((1/c)|T|) \geq 0$. Indeed, it is $c = \inf\{r : r \in \sigma(|T|)\}$. Hence

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we have $U\log((1/c)|T|) \in \mathrm{SUH}$. We often use this property and the following result.

Theorem A (Tanahashi [**16**], Lemma 6). *Let $T = U|T|$ be log-hyponormal with $\log|T| \geq 0$ and $T' = U\log|T|$. Then*

$$\sigma(T) = \{e^r \cdot e^{i\theta} : re^{i\theta} \in \sigma(T')\}.$$

Let $\boldsymbol{T} = \{e^{i\theta} \mid 0 \leq \theta < 2\pi\}$, Σ be the set of all Borel sets in $\boldsymbol{T}$, m be a measure on the measurable space $(\boldsymbol{T}, \Sigma)$ such that $dm(\theta) = (1/2\pi)\,d\theta$ and $\mathscr{D}$ be a separable Hilbert space. The Hilbert space of all vector-valued, strongly-measurable and square-integrable functions with values in $\mathscr{D}$ and with inner product

$$(f, g) = \int_{\boldsymbol{T}} (f(e^{i\theta}), g(e^{i\theta}))_{\mathscr{D}}\, dm(\theta)$$

is denoted by $L^2(\mathscr{D})$; Hardy space is denoted by $H^2(\mathscr{D})$, and the projection from $L^2(\mathscr{D})$ to $H^2(\mathscr{D})$, by $\mathscr{P}$. If $f \in L^2(\mathscr{D})$, then

$$(\mathscr{P}(f))(e^{i\theta}) = \lim_{r \to 1-0} \frac{1}{2\pi i} \int_{|z|=1} f(z)(z - re^{i\theta})^{-1}\, dz.$$

Let ν be a singular measure on $(\boldsymbol{T}, \Sigma)$, $F \in \Sigma$ be a set such that $\nu(\boldsymbol{T} \backslash F) = 0$ and $m(F) = 0$. Put $\mu = m + \nu$. Let $R(\cdot)$ be a standard operator-valued strongly-measurable function defined on $\Omega = (\boldsymbol{T}, \Sigma, \mu)$ with values being the projection in $\mathscr{D}$, $L^2(\Omega, \mathscr{D})$ be a Hilbert space of all $\mathscr{D}$-valued strongly measurable and square-integrable functions on Ω with inner product $(f, g) = \int_{\boldsymbol{T}} (f(e^{i\theta}), g(e^{i\theta}))_{\mathscr{D}}\, d\mu$, and

$$\tilde{H} = \{f : f \in L^2(\Omega, \mathscr{D}), R(e^{i\theta}) f(e^{i\theta}) = f(e^{i\theta}), e^{i\theta} \in \boldsymbol{T}\}.$$

Then $\tilde{H}$ is a subspace of $L^2(\Omega, \mathscr{D})$. The space $L^2(\mathscr{D})$ is identified with a subspace of $L^2(\Omega, \mathscr{D})$. Hence $\mathscr{P}$ extends to $L^2(\Omega, \mathscr{D})$ such that

$$\mathscr{P} f = 0 \quad \text{for } f \in L^2(\Omega, \mathscr{D}) \ominus L^2(\mathscr{D}).$$

We define an operator $\mathscr{P}_0$ from $L^2(\Omega, \mathscr{D})$ to $\mathscr{D}$ as follows:

$$\mathscr{P}_0(f) = \int f(e^{i\theta})\, dm(\theta).$$

Then $\mathscr{P}_0$ is the projection from $L^2(\Omega, \mathscr{D})$ to $\mathscr{D}$ (cf. [**19**], p. 50). Let $\alpha(\cdot)$ and $\beta(\cdot)$ be operator valued, uniformly bounded, and strongly measurable functions on Ω such that $\alpha(e^{i\theta})$ and $\beta(e^{i\theta})$ are linear operators in $\mathscr{D}$, satisfying

$$R(e^{i\theta})\alpha(e^{i\theta}) = \alpha(e^{i\theta})R(e^{i\theta}) = \alpha(e^{i\theta}), \quad R(e^{i\theta})\beta(e^{i\theta}) = \beta(e^{i\theta})R(e^{i\theta}) = \beta(e^{i\theta})$$

and $\beta(e^{i\theta}) \geq 0$.

Furthermore, suppose that $\alpha(e^{i\theta}) = 0$ if $e^{i\theta} \in F$. And we denote $(\alpha f)(e^{i\theta}) = \alpha(e^{i\theta})f(e^{i\theta})$. An operator $\tilde{U}$ in $\tilde{\mathscr{H}}$ is defined by

$$(\tilde{U}f)(e^{i\theta}) = e^{i\theta}f(e^{i\theta}).$$

Since $\beta(e^{i\theta}) \geq 0$ and $\mathscr{P}$ is a projection on $L^2(\mathscr{D})$, we have

$$(\alpha(e^{i\theta})^*(\mathscr{P}(\alpha f))(e^{i\theta}) + \beta(e^{i\theta})f(e^{i\theta}), f(e^{i\theta}))_{\mathscr{D}} \geq 0.$$

See details [**19**]. And the following results hold.

Theorem B (Xia [**17**], Theorem 6). *With the above notations, let $\tilde{T}$ be an operator in $\tilde{\mathscr{H}}$ defined by*

$$(\tilde{T}f)(e^{i\theta}) = e^{i\theta}(Af)(e^{i\theta}),$$

where $(Af)(e^{i\theta}) = \alpha(e^{i\theta})^*(\mathscr{P}(\alpha f))(e^{i\theta}) + \beta(e^{i\theta})f(e^{i\theta})$. Then $\tilde{T}$ is semi-hyponormal and the corresponding polar differential operator $|\tilde{T}| - \tilde{U}|\tilde{T}|\tilde{U}^*$ is

$$((|\tilde{T}| - \tilde{U}|\tilde{T}|\tilde{U}^*)f)(e^{i\theta}) = \alpha(e^{i\theta})^*\mathscr{P}_0(\alpha f).$$

Theorem C (Xia [**17**], Theorem 7). *Let $T = U|T|$ be a semi-hyponormal operator in $\mathscr{H}$ such that U is unitary. Then there exist a function space $\tilde{\mathscr{H}}$, and operators $\tilde{T}$ and $\tilde{U}$ in $\tilde{\mathscr{H}}$ which have the forms in Theorem B such that*

$$WTW^{-1} = \tilde{T} \quad \text{and} \quad WUW^{-1} = \tilde{U},$$

where W is a unitary operator from $\mathscr{H}$ to $\tilde{\mathscr{H}}$. Moreover $\alpha(\cdot) \geq 0$.

$\tilde{T}$ is said to be the *singular integral model* of T.

2. Mosaic of log-hyponormal operators.

By the singular integral model of a semi-hyponormal operator $T = U|T|$, it holds the following

Theorem D (Xia [**19**], Theorem V.2.5). *With the above notations, let $T = U|T|$ be in* SHU *and $\alpha(\cdot)$, $\beta(\cdot)$ be of Theorems B and C of the singular integral model of T. Then the following statements hold.*

(1) *There exists a unique $B(\mathscr{D})$-valued measurable function of two variables,* $\mathrm{B}(e^{i\theta}, r)$ $(e^{i\theta} \in \boldsymbol{T}, r \in [0, \infty))$, *satisfying*

$$0 \leq \mathrm{B}(e^{i\theta}, r) \leq I$$

such that

$$I + \alpha(e^{i\theta})(\beta(e^{i\theta}) - \ell)^{-1}\alpha(e^{i\theta}) = \exp \int_0^\infty \frac{\mathrm{B}(e^{i\theta}, r)}{r - \ell} dr.$$

(2) *For any bounded Baire function* ψ *on* $\sigma(|T|)$, *the function* $\mathrm{B}(e^{i\theta}, r)$ *has*

$$\int \psi(r)\mathrm{B}(e^{i\theta}, r)\, dr = \alpha(e^{i\theta}) \int_0^1 \psi(\beta(e^{i\theta}) + k \cdot \alpha(e^{i\theta})^2)\, dk\alpha(e^{i\theta}).$$

Especially, it holds

$$\int \frac{\mathrm{B}(e^{i\theta}, r)}{r - \ell} dr = \alpha(e^{i\theta}) \int_0^1 (\beta(e^{i\theta}) + k \cdot \alpha(e^{i\theta})^2 - \ell)^{-1}\, dk\alpha(e^{i\theta}).$$

Remark 1. The function $\mathrm{B}(e^{i\theta}, r)$ is defined on $[0, \infty]$. But, following Theorems V 2.4 and 2.5 of [**19**], we may assume that $\mathrm{B}(e^{i\theta}, t) = 0$ for $t < \inf\{r : r \in \sigma(|T|)\}$.

Definition 1. For $T \in \mathrm{SHU}$, the function $\mathrm{B}(\cdot, \cdot)$ in Theorem D is said to be the *mosaic* of T. We denote the mosaic of T by $\mathrm{B}_T(\cdot, \cdot)$.

Definition 2. Let $T = U|T|$ be a log-hyponormal operator and $T' = U \log|T|$. Let $c = \inf\{r : r \in \sigma(|T|)\} > 0$. Since $U(\log|T| - \log c) = U \log((1/c)|T|) \in \mathrm{SHU}$, there exists the mosaic $\mathrm{B}_{U \log((1/c)|T|)}(\cdot, \cdot)$ of $U \log((1/c)|T|)$ and by Remark 1 we define

$$\boldsymbol{B}_{T'}(e^{i\theta}, r) := \mathrm{B}_{U \log((1/c)|T|)}(e^{i\theta}, r - \log c)$$

and

$$\mathscr{B}_T(e^{i\theta}, r) := \begin{cases} \boldsymbol{B}_{T'}(e^{i\theta}, \log r) & \text{if } r \geq c \\ 0 & \text{if } r < c. \end{cases}$$

For a log-hyponormal operator T, we call $\mathscr{B}_T(\cdot, \cdot)$ and $\boldsymbol{B}_{T'}(\cdot, \cdot)$ the *mosaics* of T and T', respectively.

Let t be $t \geq 0$. For an operator $T = U|T| \in \mathrm{SHU}$, since $U(|T| + t) \in \mathrm{SHU}$, by Theorem D (1) it holds

$$\exp \int_0^\infty \frac{\mathrm{B}_{U(|T|+t)}(e^{i\theta}, r)}{r - \ell} dr = I + \alpha(e^{i\theta})(\beta(e^{i\theta}) + t - \ell)^{-1}\alpha(e^{i\theta})$$

$$= \exp \int_0^\infty \frac{\mathrm{B}_T(e^{i\theta}, r)}{r - (\ell - t)} dr = \exp \int_t^\infty \frac{\mathrm{B}_T(e^{i\theta}, r - t)}{r - \ell} dr.$$

Hence, by the uniqueness of the mosaic in Theorem D (1) and Remark 1 we have

$$\mathrm{B}_{U(|T|+t)}(e^{i\theta}, r) = \mathrm{B}_{U|T|}(e^{i\theta}, r - t). \qquad (*)$$

Theorem 1. *Let* $T = U|T|$ *be a* log-*hyponormal. For* $0 < k \le c = \inf\{r : r \in \sigma(|T|)\}$, *it holds that*

$$\mathrm{B}_{U\log((1/c)|T|)}(e^{i\theta}, r - \log c) = \mathrm{B}_{U\log((1/k)|T|)}(e^{i\theta}, r - \log k).$$

Proof. Since $U(\log((1/k)|T|)) = U(\log|T| - \log k)$ is semi-hyponormal, we have

$$\begin{aligned}\mathrm{B}_{U(\log(1/k)|T|)}(e^{i\theta}, r - \log k) &= \mathrm{B}_{U(\log((1/c)|T|)+\log(c/k))}(e^{i\theta}, r - \log k)\\ &= \mathrm{B}_{U(\log((1/c)|T|))}\left(e^{i\theta}, r - \log k - \log\frac{c}{k}\right) \quad \left(\text{by } (*) \text{ and } \log\frac{c}{k} > 0\right)\\ &= \mathrm{B}_{U(\log((1/c)|T|))}(e^{i\theta}, r - \log c).\end{aligned}$$

Hence the proof is complete. □

By Theorem 1, the mosaic $\mathscr{B}_T(e^{i\theta}, r)$ of a log-hyponormal operator T is independent from the choice of $\mathrm{B}_{U\log((1/k)|T|)}(e^{i\theta}, r - \log k)$ $(0 < k \le c)$. Therefore, if a log-hyponormal operator $T = U|T|$ satisfies $\log|T| \ge 0$, then we may take $c = 1$. From now on, let $c = \inf\{r : r \in \sigma(|T|)\}$.

Remark 2. For a log-hyponormal operator $T = U|T|$ with $\log|T| \ge 0$, by $(*)$

(1) if $r \ge c$, $\mathscr{B}_T(e^{i\theta}, r) = \boldsymbol{B}_{T'}(e^{i\theta}, \log r) = \mathrm{B}_{U(\log|T|-\log c)}(e^{i\theta}, \log r - \log c)$ $= \mathrm{B}_{U\log|T|}(e^{i\theta}, \log r)$,

(2) if $r < c$, $\mathscr{B}_T(e^{i\theta}, r) = 0 = \mathrm{B}_{U\log|T|}(e^{i\theta}, \log r)$ (because by Remark 1 and $\log r < \inf\{\rho : \rho \in \sigma(\log|T|)\}$).

Hence in this case two mosaics of $T' = U\log|T|$ in Definitions 1 and 2 are the same.

Definition 3.

(1) If $T \in \mathrm{SHU}$, then the *determining set* $\mathrm{D}(T)$ of T is defined by

$$\mathrm{D}(T) = \boldsymbol{C} - \bigcup\{\mathrm{G} : \mathrm{G} \text{ is open in } \boldsymbol{C} \text{ and } \mathrm{B}_T(e^{i\theta}, r) = 0 \text{ for a.e. } re^{i\theta} \in \mathrm{G}\}.$$

(2) If T is a log-hyponormal operator, then the *determining set* $\boldsymbol{D}(T)$ of T is defined by

$$\boldsymbol{D}(T) = \boldsymbol{C} - \bigcup\{\mathrm{G} : \mathrm{G} \text{ is open in } \boldsymbol{C} \text{ and } \mathscr{B}_T(e^{i\theta}, r) = 0 \text{ for a.e. } re^{i\theta} \in \mathrm{G}\}.$$

For a log-hyponormal operator $T = U|T|$, since $S = U\log((1/c)|T|) \in \mathrm{SHU}$, we have

$$\mathrm{D}(S) = \{(\log(r/c)) \cdot e^{i\theta} : re^{i\theta} \in \boldsymbol{D}(T)\}. \qquad (**)$$

An operator T is called completely nonnormal if it has no nontrivial reducing subspace on which it is normal. We show the following

Theorem 2. *Let* $T = U|T|$ *be a* log-*hyponormal operator. Then*

$$\boldsymbol{D}(T) \subseteq \sigma(T).$$

Moreover, if T *is completely nonnormal, then* $\boldsymbol{D}(T) = \sigma(T)$.

Proof. Let $c = \inf\{r : r \in \sigma(|T|)\}$. (1) Let r be $0 \le r < c$. Then it is well known $re^{i\theta} \notin \sigma(T)$. By the definition we have $\mathscr{B}_T(e^{i\theta}, r) = 0$. Hence, we have $re^{i\theta} \notin \boldsymbol{D}(T) \cup \sigma(T)$. (2) Let r be $r \ge c$ and $T' = U \log|T|$. Since

$$\mathscr{B}_T(e^{i\theta}, r) = \boldsymbol{B}_{T'}(e^{i\theta}, \log r) = \mathrm{B}_{U\log((1/c)|T|)}\left(e^{i\theta}, \log\frac{r}{c}\right),$$

by $(**)$ we have

$$\mathrm{D}\left(U\log\left(\frac{1}{c}|T|\right)\right) = \left\{\left(\log\frac{r}{c}\right) \cdot e^{i\theta} : re^{i\theta} \in \boldsymbol{D}(T)\right\}.$$

Since $U\log((1/c)|T|) \in \mathrm{SHU}$, by Theorem V.3.2 of [**19**] we have

$$\mathrm{D}\left(U\log\left(\frac{1}{c}|T|\right)\right) \subseteq \sigma\left(U\log\left(\frac{1}{c}|T|\right)\right).$$

By Theorem A,

$$\sigma\left(U\left(\frac{1}{c}|T|\right)\right) = \sigma\left(U\exp\left(\log\left(\frac{1}{c}|T|\right)\right)\right) = \left\{e^r e^{i\theta} : re^{i\theta} \in \sigma\left(U\log\left(\frac{1}{c}|T|\right)\right)\right\}.$$

Hence if $re^{i\theta} \in \boldsymbol{D}(T)$, then

$$\frac{r}{c} \cdot e^{i\theta} = e^{\log(r/c)} e^{i\theta} \in \sigma\left(U\exp\left(\log\left(\frac{1}{c}|T|\right)\right)\right) = \frac{1}{c} \cdot \sigma(U|T|),$$

so that

$$\boldsymbol{D}(T) \subseteq \sigma(T).$$

If T is completely nonnormal, then by Theorem 3 of [**7**] it holds that $U\log((1/c)|T|)$ is completely nonnormal. Since $U\log((1/c)|T|)$ is semi-hyponormal, it holds that $\mathrm{D}(U\log((1/c)|T|)) = \sigma(U\log((1/c)|T|))$ by Theorem V.3.2 of [**19**]. By the above it holds that

$$re^{i\theta} \in \boldsymbol{D}(T) \Leftrightarrow \left(\log\frac{r}{c}\right) \cdot e^{i\theta} \in \mathrm{D}\left(U\log\left(\frac{1}{c}|T|\right)\right)$$

and

$$re^{i\theta} \in \sigma(T) \Leftrightarrow \left(\log\frac{r}{c}\right) \cdot e^{i\theta} \in \sigma\left(U \log\left(\frac{1}{c}|T|\right)\right).$$

Hence we have $\boldsymbol{D}(T) = \sigma(T)$. So the proof is complete. □

THEOREM 3. *Let $T = U|T|$ be a* log-*hyponormal operator. Then*

$$\|\log|T| - \log|T^*|\| \leq \frac{1}{2\pi}\iint_{\boldsymbol{D}(T)} r^{-1}\, dr d\theta.$$

PROOF. Let $c = \inf\{r : r \in \sigma(|T|)\}$. Since $U\log((1/c)|T|)$ is semi-hyponormal, by Theorem V.3.5 of [**19**] it holds that

$$\left\|\log\left(\frac{1}{c}|T|\right) - \log\left(\frac{1}{c}|T^*|\right)\right\| \leq \frac{1}{2\pi}\iint_{\mathrm{D}(U\log((1/c)|T|))} d\rho d\theta.$$

Since

$$\mathrm{D}\left(U\log\left(\frac{1}{c}|T|\right)\right) = \left\{e^{i\theta} \cdot \left(\log\frac{r}{c}\right) : re^{i\theta} \in \boldsymbol{D}(T)\right\}$$

and $\|\log((1/c)|T|) - \log((1/c)|T^*|)\| = \|\log|T| - \log|T^*|\|$, by the transformation $\rho = \log(r/c)$, we have

$$\|\log|T| - \log|T^*|\| \leq \frac{1}{2\pi}\iint_{\boldsymbol{D}(T)} r^{-1}\, dr d\theta.$$

So the proof is complete. □

Hence we have the following corollary.

COROLLARY 4. *Let T be a* log-*hyponormal operator with $m_2(\boldsymbol{D}(T)) = 0$. Then T is normal, where $m_2(\cdot)$ is the planar Lebesgue measure.*

3. Trace formulae of log-hyponormal operators.

For the trace formula of a log-hyponormal operator T, we define the principal function of T.

DEFINITION 4. Let $\mathrm{Tr}_{\mathscr{D}}(\cdot)$ be the trace on $\mathscr{D}$.

(1) For $T \in \mathrm{SHU}$, the *principal function* $g_T(e^{i\theta}, r)$ of T is defined by

$$g_T(e^{i\theta}, r) = \mathrm{Tr}_{\mathscr{D}}(\mathrm{B}_T(e^{i\theta}, r)).$$

(2) For a log-hyponormal operator $T = U|T|$, put $T' = U\log|T|$. The *principal functions* $g_T(e^{i\theta}, r)$ and $g_{T'}(e^{i\theta}, r)$ of T and T' are defined by

$$g_T(e^{i\theta}, r) = \mathrm{Tr}_{\mathscr{D}}(\mathscr{B}_T(e^{i\theta}, r)) \quad \text{and} \quad g_{T'}(e^{i\theta}, r) = \mathrm{Tr}_{\mathscr{D}}(\boldsymbol{B}_{T'}(e^{i\theta}, r))$$

where $\mathscr{B}_T(\cdot,\cdot)$ and $\boldsymbol{B}_{T'}(\cdot,\cdot)$ are the mosaics of T and T', respectively. Subscripts will usually be suppressed when clear from the context.

Remark 3. For a log-hyponormal operator $T = U|T|$, let $c = \inf\{r : r \in \sigma(|T|)\}$, $T' = U \log|T|$ and $S = U \log((1/c)|T|)$. Let $g_T(e^{i\theta}, r)$, $g_{T'}(e^{i\theta}, r)$ and $g_S(e^{i\theta}, r)$ be the principal functions of T, T' and S, respectively. Then by Definition 3 we have

$$g_T(e^{i\theta}, r) = g_{T'}(e^{i\theta}, \log r) = g_S(e^{i\theta}, \log r - \log c).$$

Theorem 5. *Let $T = U|T|$ and $S = V|S|$ be* log-*hyponormal operators. If T and S are unitarily equivalent, then*

$$g_T(e^{i\theta}, r) = g_S(e^{i\theta}, r).$$

Proof. Let k be $0 < k \le \inf\{r : r \in \sigma(|T|) \cup \sigma(|S|)\}$. By Theorem 1, we may consider the principal functions corresponding to the operators $T' = U \log((1/k)|T|)$ and $S' = V \log((1/k)|S|)$. Since theorem holds for semi-hyponormal operators by Theorem VII.2.4 of [**19**], we may only prove that T' and S' are unitarily equivalent. We assume that $W^*TW = S$ for a unitary operator W. Since $W^*|T|W = |S|$, we have $W^*(\log((1/k)|T|))W = \log((1/k)|S|)$ and

$$W^*UW|S| = W^*UWW^*|T|W = W^*TW = S = V|S|.$$

Hence $W^*UWx = Vx$ for $x \in \mathrm{ran}(|S|)$. Since $|S|$ is invertible, we have $W^*UW = V$. Therefore, we have

$$\begin{aligned} W^*T'W &= W^*U\left(\log\left(\frac{1}{k}|T|\right)\right)W = W^*UWW^*\left(\log\left(\frac{1}{k}|T|\right)\right)W \\ &= W^*UW\left(\log\left(\frac{1}{k}|S|\right)\right) = V\left(\log\left(\frac{1}{k}|S|\right)\right) = S'. \end{aligned}$$

So the proof is complete. □

Hence, the principal function $g_T(\cdot,\cdot)$ of T is independent of the concrete model of T.

Here we denote the trace class of operators by $\mathscr{C}_1$. For operators A and B, the commutator $AB - BA$ is denoted by $[A, B]$. By $\mathscr{A}_2$, we denote the linear space of all Laurent polynomials $p(x, y)$ of two variables such that $p(x, y) = \sum_{j=0}^{N} \sum_{k=-N}^{N} a_{jk} x^j y^k$, where N is an arbitrary positive integer. For an operator X and an invertible operator Y, we define $p(X, Y)$ by

$$p(X, Y) = \sum_{j,k} a_{jk} X^j Y^k.$$

For $p(x, y), q(x, y) \in \mathscr{A}_2$, we denote the Jacobian $\frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x}$ by $J(p, q)$ and $(J(p,q))(r, e^{i\theta}) = \left(\frac{\partial p}{\partial x}\right)(r, e^{i\theta}) \cdot \left(\frac{\partial q}{\partial y}\right)(r, e^{i\theta}) - \left(\frac{\partial p}{\partial y}\right)(r, e^{i\theta}) \cdot \left(\frac{\partial q}{\partial x}\right)(r, e^{i\theta})$.

Then in [**4**] we proved the following

Theorem E (Chō and Huruya [**4**], Theorem 9). *Let $T = U|T| \in \mathrm{SHU}$ and $g_T(\cdot,\cdot)$ be the principal function of T and $[|T|, U] \in \mathscr{C}_1$. Then, for $p, q \in \mathscr{A}_2$,*

$$\mathrm{Tr}([p(|T|, U), q(|T|, U)]) = \iint (J(p,q))(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r)\, dr dm(\theta).$$

We show two trace foumulae associated with a log-hyponormal operator. First one is

Theorem 6. *Let $T = U|T|$ be a* log-*hyponormal operator such that $[\log|T|, U] \in \mathscr{C}_1$. Let $T' = U \log|T|$ and $g_{T'}$ be the principal function of T'. Then, for $p, q \in \mathscr{A}_2$,*

$$\mathrm{Tr}([p(\log|T|, U), q(\log|T|, U)]) = \int_{\log c}^{\infty} \left(\int_{\boldsymbol{T}} J(p,q)(r, e^{i\theta}) e^{i\theta} g_{T'}(e^{i\theta}, r)\, dm(\theta) \right) dr,$$

where $c = \inf\{r : r \in \sigma(|T|)\}$.

Proof. Put $S = U \log((1/c)|T|)$. Then $S \in \mathrm{SHU}$ and $[|S|, U] = [\log|T|, U] \in \mathscr{C}_1$. Put $\tilde{p}(x, y) = p(x + \log c, y)$ and $\tilde{q}(x, y) = q(x + \log c, y)$. Then it holds

$$\mathrm{Tr}([p(\log|T|, U), q(\log|T|, U)]) = \mathrm{Tr}([\tilde{p}(|S|, U), \tilde{q}(|S|, U)]).$$

By Theorem E, we have

$$\begin{aligned}
\mathrm{Tr}([\tilde{p}(|S|, U), \tilde{q}(|S|, U)]) &= \iint_{\mathrm{D}(S)} J(\tilde{p}, \tilde{q})(r, e^{i\theta}) e^{i\theta} g_S(e^{i\theta}, r)\, dr dm(\theta) \\
&= \int_0^{\infty} \left(\int_{\boldsymbol{T}} J(\tilde{p}, \tilde{q})(r, e^{i\theta}) e^{i\theta} g_S(e^{i\theta}, r)\, dm(\theta) \right) dr \\
&= \int_0^{\infty} \left(\int_{\boldsymbol{T}} J(p, q)(r + \log c, e^{i\theta}) e^{i\theta} g_S(e^{i\theta}, r)\, dm(\theta) \right) dr. \quad (\dagger)
\end{aligned}$$

By the transformation $t = r + \log c$, from Remark 3 we have

$$(\dagger) = \int_{\log c}^{\infty} \left(\int_{\boldsymbol{T}} J(p,q)(t, e^{i\theta}) e^{i\theta} g_S(e^{i\theta}, t - \log c)\, dm(\theta) \right) dt$$
$$= \int_{\log c}^{\infty} \left(\int_{\boldsymbol{T}} J(p,q)(t, e^{i\theta}) e^{i\theta} g_{T'}(e^{i\theta}, t)\, dm(\theta) \right) dr.$$

So the proof is complete. □

For the second one, we prepare the following

Theorem 7. *Let* $T = U|T| \in \mathrm{SHU}$ *and* $g_T(\cdot)$ *be the principal function of* T. *Let* $[|T|, U] \in \mathscr{C}_1$. *Then, for* $p, q \in \mathscr{A}_2$,

$$\mathrm{Tr}([p(\exp(|T|), U), q(\exp(|T|), U)]) = \iint J(p,q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, \log r)\, dr dm(\theta).$$

Proof. For $m = 1, 2, \ldots$ and $n = \pm 1, \pm 2, \ldots$, by Theorem 8 of [**4**] we have

$$\text{(1)} \qquad \mathrm{Tr}([|T|^m, U^n]) = \iint mne^{in\theta} r^{m-1} g_T(e^{i\theta}, r)\, dr dm(\theta)$$
$$= \iint ne^{in\theta} \frac{d}{dr}(r^m) g_T(e^{i\theta}, r)\, dr dm(\theta)$$

and by the proof of Theorem 9 of [**4**]

$$\text{(2)} \qquad \mathrm{Tr}(|T|^m - U|T|^m U^{-1}) = \iint mr^{m-1} g_T(e^{i\theta}, r)\, dr dm(\theta)$$
$$= \iint \frac{d}{dr}(r^m) g_T(e^{i\theta}, r)\, dr dm(\theta).$$

For an operator S, we denote the trace norm of S by $\|S\|_1$. Since

$$[|T|^m, U^n] = |T|^{m-1}[|T|, U^n] + |T|^{m-2}[|T|, U^n]|T| + \cdots + [|T|, U^n]|T|^{m-1}$$

and

$$|T|^m - U|T|^m U^{-1} = [|T|^m, U]U^{-1},$$

we have

$$\|[|T|^m, U^n]\|_1 \le m \||T|\|^{m-1} \|[|T|, U^n]\|_1$$

and

$$\||T|^m - U|T|^m U^{-1}\|_1 \le m\|T\|^{m-1} \|[|T|, U]\|_1.$$

Since $\mathscr{C}_1$ is complete, in $\mathscr{C}_1$ we have

$$\lim_{\ell\to\infty}\left[\left(\sum_{h=0}^{\ell}\frac{1}{h!}|T|^h\right)^m, U^n\right] = [(\exp(|T|))^m, U^n]$$

and

$$\lim_{\ell\to\infty}\left\{\left(\sum_{h=0}^{\ell}\frac{1}{h!}|T|^h\right)^m - U\left(\sum_{h=0}^{\ell}\frac{1}{h!}|T|^h\right)^m U^{-1}\right\} = (\exp(|T|))^m - U(\exp(|T|))^m U^{-1}.$$

Since $|\mathrm{Tr}(D)| \le \|D\|_1$ for $D \in \mathscr{C}_1$, by (1) we obtain

$$\mathrm{Tr}([(\exp(|T|))^m, U^n]) = \lim_{\ell\to\infty}\iint ne^{in\theta}\cdot\frac{d}{dr}\left(\sum_{h=0}^{\ell}\frac{1}{h!}r^h\right)^m g_T(e^{i\theta}, r)\,drdm(\theta)$$

$$= \iint ne^{in\theta}\cdot me^{mr} g_T(e^{i\theta}, r)\,drdm(\theta)$$

and similarly by (2)

$$\mathrm{Tr}([(\exp(|T|))^m - U(\exp(|T|))^m U^{-1}]) = \iint me^{mr} g_T(e^{i\theta}, r)\,drdm(\theta).$$

Putting $e^r = s$, we have

$$\mathrm{Tr}([(\exp(|T|))^m, U^n]) = \iint ne^{in\theta}\cdot ms^{m-1} g_T(e^{i\theta}, \log s)\,dsdm(\theta) \tag{3}$$

and

$$\mathrm{Tr}((\exp(|T|))^m - U(\exp|T|)^m U^{-1}) = \iint ms^{m-1} g_T(e^{i\theta}, \log s)\,dsdm(\theta). \tag{4}$$

Define a bilinear form $(\cdot,\cdot)$ on $\mathscr{A}_2$ by

$$(p,q) = \mathrm{Tr}([p(\exp(|T|), U), q(\exp(|T|), U)])$$

for $p, q \in \mathscr{A}_2$. Let $p_2(x,y) = y$. Then by the proof of Theorem 9 of [**4**] we can define a linear functional ℓ on $\mathscr{A}_2$ by, for $q \in \mathscr{A}_2$,

$$\ell\left(\frac{\partial q}{\partial y}\right) = (p_2, q).$$

Then by the similar way of the proof of Theorem E we have

$$(p,q) = -\ell(J(p,q)). \tag{5}$$

Next we define a linear functional ℓ_0 on $\mathscr{A}_2$ by, for $p \in \mathscr{A}_2$,

$$\ell_0(p) = \iint p(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, \log r)\, dr dm(\theta).$$

Since (3) and (4) hold, by the similar way of the proof of Theroem E we have

$$\ell_0 = -\ell. \tag{6}$$

Therefore, by (5) and (6) we have

$$\begin{aligned}\mathrm{Tr}([p(\exp(|T|), U), q(\exp(|T|), U)]) &= (p, q) = \ell_0(J(p, q)) \\ &= \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, \log r)\, dr dm(\theta)\end{aligned}$$

for $p, q \in \mathscr{A}_2$. So the proof is complete. □

Hence, we have the following

Theorem 8. *Let $T = U|T|$ be a* log-*hyponormal operator with* $\log|T| \geq 0$ *and $g_T(\cdot, \cdot)$ be the principal function of T. Assume that* $[\log|T|, U] \in \mathscr{C}_1$. *Then, for any $p, q \in \mathscr{A}_2$, it holds that*

$$\mathrm{Tr}([p(|T|, U), q(|T|, U)]) = \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r)\, dr dm(\theta).$$

Proof. Let $T' = U \log|T|$ and $g_{T'}$ be the principal function of T'. Since $T' \in \mathrm{SHU}$, by Theorem 7 we have

$$\mathrm{Tr}([p(\exp(|T'|), U), q(\exp(|T'|), U)]) = \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_{T'}(e^{i\theta}, \log r)\, dr dm(\theta).$$

Since $g_{T'}(e^{i\theta}, \log r) = g_T(e^{i\theta}, r)$ and $\exp(|T'|) = |T|$, we have

$$\mathrm{Tr}([p(|T|, U), q(|T|, U)]) = \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r)\, dr dm(\theta).$$

So the proof is complete. □

Finally, we show the following main result.

Theorem 9. *Let $T = U|T|$ be a* log-*hyponormal operator and $g_T(\cdot, \cdot)$ be the principal function of T. Assume that* $[\log|T|, U] \in \mathscr{C}_1$. *Then, for any $p, q \in \mathscr{A}_2$, it holds that*

$$\mathrm{Tr}([p(|T|, U), q(|T|, U)]) = \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r)\, dr dm(\theta).$$

Proof. For $c = \inf\{r : r \in \sigma(|T|)\}$, let $R = U((1/c)|T|)$. Put $\tilde{p}(r, z) = p(c \cdot r, z)$ and $\tilde{q}(r, z) = q(c \cdot r, z)$. Then we have

$$\mathrm{Tr}([p(|T|,U),q(|T|,U)]) = \mathrm{Tr}([\tilde{p}(|R|,U),\tilde{q}(|R|,U)]).$$

Since R is log-hyponormal with $\log|R| = \log((1/c)|T|) \geq 0$, by Theorem 8 we have

$$\begin{aligned}\mathrm{Tr}([\tilde{p}(|R|,U),\tilde{q}(|R|,U)]) &= \iint J(\tilde{p},\tilde{q})(t,e^{i\theta})e^{i\theta}g_R(e^{i\theta},t)\,dtdm(\theta)\\ &= \iint c\cdot J(p,q)(c\cdot t,e^{i\theta})e^{i\theta}g_R(e^{i\theta},t)\,dtdm(\theta). \quad (\dagger\dagger)\end{aligned}$$

By the transformation $r = c\cdot t$, we have

$$\begin{aligned}(\dagger\dagger) &= \iint J(p,q)(r,e^{i\theta})e^{i\theta}g_R\left(e^{i\theta},\frac{r}{c}\right)drdm(\theta)\\ &= \iint J(p,q)(r,e^{i\theta})e^{i\theta}g_T(e^{i\theta},r)\,drdm(\theta),\end{aligned}$$

because $g_T(e^{i\theta},r) = \mathrm{Tr}(\mathrm{B}_{U\log((1/c)|T|)}(e^{i\theta},\log(r/c))) = g_R(e^{i\theta},(r/c))$ by Definition 2 and Remark 3. So the proof is complete. $\square$

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