J. Math. Soc. Japan
Vol. 57, No. 3, 2005

Ends of leaves of Lie foliations

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(Received Sept. 21, 2004)

Abstract. Let G be a simply connected Lie group and consider a Lie G foliation $\mathscr{F}$ on a closed manifold M whose leaves are all dense in M. Then the space of ends $\mathscr{E}(F)$ of a leaf F of $\mathscr{F}$ is shown to be either a singleton, a two points set, or a Cantor set. Further if G is solvable, or if G has no cocompact discrete normal subgroup and $\mathscr{F}$ admits a transverse Riemannian foliation of the complementary dimension, then $\mathscr{E}(F)$ consists of one or two points. On the contrary there exists a Lie $\widetilde{SL}(2,\boldsymbol{R})$ foliation on a closed 5-manifold whose leaf is diffeomorphic to a 2-sphere minus a Cantor set.

1. Introduction.

Let G be a connected Lie group. A Lie G foliation $\mathscr{F}$ on a closed manifold M is a foliation locally modelled on the geometry (G,G). That is, $\mathscr{F}$ is a foliation defined by distinguished charts taking values in G, with transition functions the restrictions of the left translations by elements of G. See Section 2 for the precise definition.

Lie G foliations form a very special class of foliations and satisfy various strong properties. For example, any left invariant Riemannian metric of G gives rise to a metric on the normal bundle of $\mathscr{F}$, invariant by the holonomy pseudogroup; that is, $\mathscr{F}$ is a Riemannian foliation. Each leaf of $\mathscr{F}$ has trivial holonomy and they are mutually Lipshitz diffeomorphic. Conversely by the work of P. Molino [**14**], the study of Riemannian foliations reduces to that of Lie foliations. See [**15**] for detailed accounts.

Classical examples of Lie G foliations are:

EXAMPLE 1.1. a) Let H be a Lie group admitting a (e.g. surjective) homomorphism $f : H \to G$, and let Γ be a uniform lattice of H. Then the right action of $N = \mathrm{Ker}(f)$ on the quotient space $\Gamma \setminus H$ gives rise to a Lie G foliation, with leaves diffeomorphic to $(\Gamma \cap N) \setminus N$.

b) Assume G is compact, let B be a closed manifold, and let $\varphi : \pi_1(B) \to G$ be a group homomorphism. The suspension gives a fibration $G \to M \to B$ together with a Lie G foliations transverse to the fibers; the leaf is the $\mathrm{Ker}(\varphi)$-normal covering of B.

Only very few essentially different examples are known, and in these examples G is solvable ([**13**]).

This shortage of examples makes the study of Lie foliations difficult. Little is known

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2000 *Mathematics Subject Classification.* Primary 57R30; Secondary 57D30, 37C85.
Key Words and Phrases. foliations, Lie G foliations, Riemannian foliations, leaf, space of ends, developing map, holonomy homomorphism.
[1] Partially supported by Grand-in-Aid for Scientific Research (No. (A)13304005), Japan Society for the Promotion of Science.
[2] Partially supported by Japan Society for the Promotion of Science, (No. L03514).

about the topology of their leaves. Our purpose here is to investigate the end set of leaves F of Lie foliations. It is known, easy to show, that the closure of F is a submanifold of M and $\mathscr{F}$ is a minimal Lie foliation there. Thus it is no loss of generality to assume that the leaves of $\mathscr{F}$ are dense in M.

Throughout this paper we assume that *the manifold M is closed and the leaves of the foliation $\mathscr{F}$ are dense in M*. Denote by $\mathscr{E}(F)$ the space of the ends of F. (See [**4**], [**11**], [**5**].) The following is a consequence of results on general foliations and laminations ([**8**], [**1**]).

Proposition 1.2. *The space of the ends $\mathscr{E}(F)$ either is a singleton, consists of two points, or is a Cantor set.*

A proof of this proposition is included in Section 4, since the proof is very short in our frame, and is the first step for showing the following Theorem 1.3.

Consider the case where the Lie group G is compact, or more generally G admits a cocompact discrete normal subgroup Γ. If the quotient compact group G/Γ admits a free subgroup on two generators $\boldsymbol{Z} * \boldsymbol{Z}$, it is easy to construct a Lie G foliation whose leaves have a Cantor set of ends: Let B be a surface of genus ≥ 2, and $\psi : \pi_1(B) \to G/\Gamma$ a homomorphism whose image is isomorphic to $\boldsymbol{Z}*\boldsymbol{Z}$. In the suspended foliation (Example 1.1 b)) $\boldsymbol{Z}*\boldsymbol{Z}$ acts on a leaf F freely, properly discontinuously and cocompactly. Therefore $\mathscr{E}(F)$ is a Cantor set.

Let us state the main results of the present paper.

Theorem 1.3. *If the Lie group G is solvable, then $\mathscr{E}(F)$ is a finite set.*

Theorem 1.3 is already known by H. Winkelnkemper [**20**] and E. Ghys [**7**] when G is abelian. Also the argument of [**7**] can be extended to the case of nilpotent Lie groups.

Theorem 1.4. *Suppose the Lie group G is simply connected. Then $\mathscr{E}(F)$ consists of two points if and only if G is isomorphic to $\boldsymbol{R}^n$ and the holonomy group is of rank $n+1$. In this case the classifiant* ([**9**]) *is a linear flow on T^{n+1}.*

Theorem 1.4 generalizes the result of P. Caron-Y. Carrière [**2**] for Lie flows, which was used in the classification of Riemannian flows on 3-dimensional manifolds ([**3**]). Also in [**6**], study was done about Lie flows on 4-dimensional manifolds. Our proof of Theorem 1.4 is an elaboration of the method of [**2**].

On the other hand there are also Lie G foliations with a Cantor set of ends for such Lie groups G which do not admit any cocompact discrete normal subgroup.

Theorem 1.5. *There exists a minimal Lie $PSL(2, \boldsymbol{R})$ foliation on a compact 5-dimensional manifold whose leaf is a 2-sphere minus a Cantor set.*

The method deserves to be published, because hopefully it has a generalization. We first produce a "homogeneous Lie G foliation", with the Lie group H in Example 1.1 a), the product of G and a p-adic Lie group. Then the p-adic leaves are changed to real manifolds. We will also construct a minimal transversely homogeneous foliation, modeled on the Poincaré disk modulo its isometries, thus transversely holomorphic, whose leaves are the 2-sphere minus a Cantor set.

In the proof of Proposition 1.2, we define a "quasi-action" (Section 3) of the holonomy group on the leaf. When it is an isometric action, we can say more. Namely we have:

THEOREM 1.6. *Suppose that the Lie group G does not admit a cocompact discrete normal subgroup and that $\mathscr{F}$ has a transverse Riemannian foliation of the complementary dimension. Then $\mathscr{E}(F)$ is a finite set.*

In Theorem 1.6, the assumption on the Lie group G is necessary, as has been shown in Example 1.1.a). Notice that the Lie G foliation there has a transverse Riemannian foliation, the bundle foliation.

For the proof of Theorems 1.4 and 1.5, we need the following standard fact. Since there is no reference, its proof is included in Appendix A.

PROPOSITION 1.7. *The end set $\mathscr{E}(F)$ is a transverse invariant. That is, $\mathscr{E}(F)$ does not depend on the manifold M or on the foliation $\mathscr{F}$, but only on the holonomy group Γ as a subgroup of G.*

In view of these results, even more than for general foliations and laminations, the ends of leaves of Lie foliations present strong analogies with the ends of finitely generated groups. However our situation is subtler, because the holonomy group does not act on the leaf honestly, there is only a quasi-action, and because one lacks in a natural simplicial object with the same ends as the leaf. In this spirit, the main question left open by the present paper is that of a structure theorem for the holonomy group of Lie foliations with infinitely many ends, analogous to Stallings'. See Section 6 for the discussion.

This paper is prepared while the third author is visiting the University of Tokyo. He expresses his graditude to the institute and especially to Takashi Tsuboi for their hospitality.

2. Preliminaries.

First of all we summarize the conventions we use about the fundamental group $\pi_1(X)$ of a space X and the group of the deck transformations on the universal covering space $\widetilde{X}$.

CONVENTION 2.1.

(a): *X and $\widetilde{X}$ are pointed spaces. The covering map sends the base point to the base point.*

(b): *For paths $\sigma, \sigma' : [0,1] \to X$ such that $\sigma(1) = \sigma'(0)$, their composite is denoted by $\sigma \cdot \sigma'$. The group operation of the fundamental group $\pi_1(X)$ is also defined by this rule.*

(c): *Consequently, by the usual definition, $\pi_1(X)$ acts on $\widetilde{X}$ from the left.*

(d): *Once the lift $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ of a map $f : X \to Y$ is chosen, the homomorphism $\widetilde{f}_* : \pi_1(X) \to \pi_1(Y)$ is defined.*

Let $\mathscr{F}$ be a foliation on a closed manifold M.

DEFINITION 2.2. The foliation $\mathscr{F}$ is called a *Lie G foliation* if there exists a family $\{(U_i, p_i), g_{ij}\}$ which satisfies the following.

(a): $\{U_i\}$ is an open covering of M, $p_i : U_i \to G$ is a submersion, and g_{ij} is an element of G.
(b): p_i maps each leaf of $\mathscr{F}|_{U_i}$ to a point.
(c): For $x \in U_i \cap U_j$, we have $p_i(x) = g_{ij} p_j(x)$.

Fix, once and for all, a submersion $p_i : U_i \to G$. On U_j which intersects U_i, consider the map $g_{ij} \cdot p_j$. Then the domain of definition of p_i is extended to $U_i \cup U_j$. Continuing this way, we obtain a globally defined submersion. The domain of definition is the universal covering space. This submersion, denoted by $\widetilde{D} : \widetilde{M} \to G$, is called a *developing map*. The lift $\widetilde{\mathscr{F}}$ of $\mathscr{F}$ is the pull back by $\widetilde{D}$ of the point foliation.

Associated with this, we can also define in an analogous way a homomorphism $\widetilde{\phi} : \pi_1(M) \to G$, called the *holonomy homomorphism*, which satisfy the following equivariance relation:

$$\widetilde{D}(\alpha x) = \widetilde{\phi}(\alpha)\widetilde{D}(x), \quad \forall \alpha \in \pi_1(M), \quad \forall x \in \widetilde{M}. \tag{1}$$

The subgroup $\Gamma = \mathrm{Image}(\widetilde{\phi})$ is called the *holonomy group* of the Lie foliation $\mathscr{F}$. It is an important invariant of the Lie G foliation.

Conversely given a submersion $\widetilde{D} : \widetilde{M} \to G$ and a homomorphism $\widetilde{\phi} : \pi_1(M) \to G$ which satisfy (1), one obtains a Lie G foliation on M.

Let $\theta : G \to H$ be a Lie group homomorphism which is a covering map onto a Lie group H. Given a Lie G foliation $\mathscr{F}$ defined by

$$(\widetilde{\phi}, \widetilde{D}) : (\pi_1(M), \widetilde{M}) \to (G, G),$$

one can give $\mathscr{F}$ a structure of the Lie H foliation by

$$(\theta \circ \widetilde{\phi}, \theta \circ \widetilde{D}) : (\pi_1(M), \widetilde{M}) \to (H, H).$$

On the other hand if

$$(\widetilde{\phi}_1, \widetilde{D}_1) : (\pi_1(M), \widetilde{M}) \to (H, H)$$

defines a Lie H foliation, then there exists a lift $\widetilde{D} : \widetilde{M} \to G$ of $\widetilde{D}_1$. For a fixed $\alpha \in \pi_1(M)$ and for any $x \in \widetilde{M}$, we have

$$\theta \circ \widetilde{D}(\alpha x) = \widetilde{\phi}_1(\alpha) \cdot \theta \circ \widetilde{D}(x).$$

Choose $\beta \in G$ such that $\theta(\beta) = \widetilde{\phi}_1(\alpha)$. Then we have $\theta(\widetilde{D}(\alpha x)) = \theta(\beta \cdot \widetilde{D}(x))$. Changing β suitably within its class mod $\mathrm{Kernel}(\theta)$, one gets $\widetilde{D}(\alpha x) = \beta \cdot \widetilde{D}(x)$. Notice that β is independent of x since $\mathrm{Kernel}(\theta)$ is discrete. This way we obtain a map $\widetilde{\phi} : \pi_1(M) \to G$, which is shown to be a homomorphism since $\widetilde{\phi}$ and $\widetilde{D}$ satisfy (1). Thus a Lie H foliation can be viewed as a Lie G foliation. Therefore in the rest of this paper, we assume that *the Lie group G is simply connected*.

The following is well known and easy to show using the left invariant Riemannian metric on G.

PROPOSITION 2.3. *The developing map $\widetilde{D} : \widetilde{M} \to G$ is a locally trivial bundle map whose fibers are the leaves of $\widetilde{\mathscr{F}}$.*

Now consider the action of the normal subgroup $\mathrm{Kernel}(\widetilde{\phi})$ of $\pi_1(M)$ and the quotient space $\overline{M} = \mathrm{Kernel}(\widetilde{\phi}) \backslash \widetilde{M}$. The foliation $\widetilde{\mathscr{F}}$ projects down to a foliation $\overline{\mathscr{F}}$ on $\overline{M}$, and the developing map $\widetilde{D}$ yields a map $D : \overline{M} \to G$, which is also called the *developing map*. The group $\pi_1(M)/\mathrm{Kernel}(\widetilde{\phi})$ acts on $\overline{M}$ and the quotient of this action is M. $\overline{M}$ is called the *holonomy covering space* associated with $\mathscr{F}$. The covering map is denoted by $p : \overline{M} \to M$.

By certain abuse, we identify $\pi_1(M)/\mathrm{Kernel}(\widetilde{\phi})$ with the holonomy group Γ. Thus on one hand Γ acts on $\overline{M}$ as the deck transformations, and on the other hand $\Gamma \subset G$ acts on G as the left translations.

PROPOSITION 2.4.

(a): *$D : \overline{M} \to G$ is a locally trivial bundle map, with fibers the leaves of $\overline{\mathscr{F}}$. D is Γ-equivariant; That is,*

$$D(\gamma x) = \gamma D(x), \quad \forall \gamma \in \Gamma, x \in \overline{M}. \tag{2}$$

(b): *Each leaf of $\overline{\mathscr{F}}$ is mapped homeomorphically onto a leaf of $\mathscr{F}$ by the covering map p.*

(c): *The holonomy group Γ is dense in G.*

PROOF. By Proposition 2.3, $\mathrm{Kernel}(\widetilde{\phi})$ acts on $\widetilde{M}$ in such a way that it leaves all the leaves of $\widetilde{\mathscr{F}}$ invariant. Therefore the leaves of $\overline{\mathscr{F}}$ are the quotient of the leaves of $\widetilde{\mathscr{F}}$ by the action of $\mathrm{Kernel}(\widetilde{\phi})$. It is now clear that D is a bundle map with fibers the leaves of $\overline{\mathscr{F}}$. The property (2) follows immediately from (1), showing (a).

On the other hand, as for the action of Γ on $\overline{M}$, we have that no leaf of $\overline{\mathscr{F}}$ is left invariant by a nontrivial element of Γ. Clearly this shows (b). The condition (c) is what we assumed in Section 1. □

Denote by e the identity of G and let $F = D^{-1}(e)$, a leaf of $\overline{\mathscr{F}}$. Let us recall the definition of the set of ends $\mathscr{E}(F)$ of F. Our definition, fit for a manifold F, is a bit different from the usual one ([**4**], [**11**]). But it is easy to show that the both definitions are in fact equivalent.

Let $\mathscr{K}(F)$ be the set of those K which satisfy the following conditions.

(a): K is a codimension zero connected compact submanifold of F.
(b): No component of the complement of K is precompact.
(c): For distinct components A_0, A_1 of ∂K, there does not exist a path σ in F such that $\sigma(0) \in A_0$, $\sigma(1) \in A_1$ and that $\sigma(]0,1[) \cap K = \varnothing$.

Let $\mathscr{P}(F)$ be the set of closures P of all components of the complements of all $K \in \mathscr{K}(F)$. An *escaping sequence* is a decreasing sequence $\{P_n\}_{n \in \boldsymbol{N}}$ of elements $P_n \in \mathscr{P}(F)$

such that $\bigcap_n P_n = \varnothing$. Two escaping sequences $\{P_n\}$ and $\{P'_m\}$ are said to be *equivalent* if for each n, there exists m such that $P'_m \subset P_n$. It is easy to show that this is in fact an equivalence relation. An equivalence class $\epsilon = [\{P_n\}]$ is called an *end* of F. Denote by $\mathscr{E}(F)$ the set of all the ends of F.

The proof of the following lemma is left to the reader.

Lemma 2.5. *Card* $\mathscr{E}(F) \geq n$ *if and only if there exists* $K \in \mathscr{K}(F)$ *which has at least* n *boundary components.*

For $\epsilon = [\{P_n\}] \in \mathscr{E}(F)$ and $P \in \mathscr{P}(F)$, write $\epsilon \lhd P$ if $P_n \subset P$ for some $n \in \boldsymbol{N}$. Denote by $\mathscr{E}(P)$ the set of the ends ϵ such that $\epsilon \lhd P$.

It is well known that the set $\mathscr{E}(P)$ is nonempty for any $P \in \mathscr{P}(F)$, and thus a topology of $\mathscr{E}(F)$ is defined with basis $\mathscr{E}(P)$ for $P \in \mathscr{P}(F)$. It is known, easy to show, that $\mathscr{E}(F)$ is a totally disconnected, compact and Hausdorff space.

For a boundary component A of $K \in \mathscr{K}(F)$, let us denote by $\underline{A}$ the closure of the component of the complement of K such that $K \cap \underline{A} = A$. As a matter of fact, if A and A' are distinct boundary components of K, then we have $\underline{A} \cap \underline{A'} = \varnothing$.

Definition 2.6. Let K_0 and K_1 be mutually disjoint elements of $\mathscr{K}(F)$ and let A_0 be a boundary component of K_0. We say that A_0 is *the exit to* K_1 if $K_1 \subset \underline{A_0}$.

The proof of the following two lemmas are omitted.

Lemma 2.7. *Let* A_i *be a boundary component of some* $K_i \in \mathscr{K}(F)$ $(i \in \boldsymbol{Z}/2\boldsymbol{Z})$ *such that* $K_0 \cap K_1 = \varnothing$.

(a): A_1 *is the exit to* K_0 *and* A_0 *is not the exit to* K_1 *if and only if* $\underline{A_0} \subset \underline{A_1}$.
(b): A_i *is the exit to* K_{i+1} $(\forall i)$ *if and only if* $\underline{A_0} \cup \underline{A_1} = F$.
(c): A_i *is not the exit to* K_{i+1} $(\forall i)$, *if and only if* $\underline{A_0} \cap \underline{A_1} = \varnothing$.

Lemma 2.8. *Let* K_i *and* A_i *be as in the previous lemma. Assume that* K_i *has at least three boundary components. Then* A_1 *is the exit to* K_0 *and* A_0 *is not the exit for* K_1 *if and only if* $\mathscr{E}(\underline{A_0}) \subset \mathscr{E}(\underline{A_1})$.

3. Action of Γ on $\mathscr{E}(F)$.

Let us define a Riemannian metric dm of M which is convenient for the study of $\mathscr{F}$. Let TM be the tangent bundle of M and let $T\mathscr{F}$ (resp. $N\mathscr{F}$) be the tangent (resp. normal) bundle of $\mathscr{F}$, where $N\mathscr{F}$ is to be a subbundle of TM. Likewise define subbundles $T\overline{\mathscr{F}} = p^*T\mathscr{F}$ and $N\overline{\mathscr{F}} = p^*N\mathscr{F}$ of $T\overline{M}$.

Fix once and for all a left invariant Riemannian metric dm_0 on G. The pull-back $D^*(dm_0)$ is a metric on $N\overline{\mathscr{F}}$, invariant by the deck transformations, and therefore induces a metric of $N\mathscr{F}$, which extends to a Riemannian metric dm on TM, called a *bundle-like* metric for $\mathscr{F}$. The lift of dm to the holonomy covering $\overline{M}$ is denoted by $\overline{dm}$. The distances in M and $\overline{M}$ obtained from dm and $\overline{dm}$ are both denoted by the same letter d.

For $g \in G$, denote by F_g the inverse image $D^{-1}(g)$, also with the same convention $F = F_e$ as before. The leaf F_g is equipped with the distance function d_{F_g} obtained from the Riemannian metric on F_g which is the restriction of $\overline{dm}$.

We shall define something like an action (but much more rough) of the holonomy group Γ on F. Let us define

$$\mathscr{H}(F) = \{f \in \mathrm{Homeo}(F) \mid f \text{ and } f^{-1} \text{ are Lipschitz}\},$$
$$\mathscr{B}(F) = \left\{ f \in \mathscr{H}(F) \mid \sup_{x \in F} d_F(f(x), x) < \infty \right\}.$$

The proof of the following lemma is left to the reader.

LEMMA 3.1. *$\mathscr{B}(F)$ is a normal subgoup of the group $\mathscr{H}(F)$.*

DEFINITION 3.2. A homomorphism $\psi : \Gamma \to \mathscr{H}(F)/\mathscr{B}(F)$ is called a *quasi-action* of Γ on F.

Our purpose is to construct a natural quasi-action of Γ on F.

Let $\Omega(G)$ be the set of the piecewise smooth paths $\sigma : [0,1] \to G$ such that $\sigma(0) = e$. Given $\sigma \in \Omega(G)$, let us define an $\mathscr{F}$-preserving diffeomorphism f_σ of M. In the first place, for any point $x \in \overline{M}$, define the curve σ_x in $\overline{M}$ by;

$$\begin{array}{lll} \text{(a)} & D(\sigma_x(t)) = D(x)\sigma(t) & \forall t \in [0,1], \\ \text{(b)} & d_\pm \sigma_x(t) \in N\overline{\mathscr{F}} & \forall t \in [0,1], \\ \text{(c)} & \sigma_x(0) = x, & \end{array}$$

where $d_\pm$ denotes the right or left derivative. The path σ_x is well-defined on the whole of $[0,1]$ and is unique.

For an element $g \in G$ and a path $\tau \in \Omega(G)$, define the path $g\tau$ by

$$(g\tau)(t) = g(\tau(t)), \quad \forall t \in [0,1].$$

The same notation is used for an element of Γ and a path in $\overline{M}$. It is a routine work to establish the following lemma.

LEMMA 3.3.

(a): *For any $\gamma \in \Gamma$, $x \in \overline{M}$ and $\sigma \in \Omega(G)$, we have $\sigma_{\gamma x} = \gamma \sigma_x$.*
(b): *For any $x \in \overline{M}$ and $\sigma, \tau \in \Omega(G)$, we have $(\sigma \cdot \sigma(1)\tau)_x = \sigma_x \cdot \tau_{\sigma_x(1)}$.*

Define a map $\overline{f}_\sigma : \overline{M} \to \overline{M}$ by $\overline{f}_\sigma(x) = \sigma_x(1)$, $\forall x \in \overline{M}$. Then we have;

LEMMA 3.4.

(a): *$\overline{f}_\sigma$ is an $\overline{\mathscr{F}}$-preserving diffeomorphism.*
(b): *$\overline{f}_\sigma$ is Γ-equivariant.*
(c): *We have $\overline{f}_{\sigma \cdot \sigma(1)\tau} = \overline{f}_\tau \circ \overline{f}_\sigma$.*

PROOF. The last two statements are immediate consequences of Lemma 3.3. That $\overline{f}_\sigma$ has a left inverse follows from (c) by taking τ to be the path defined by

$$\tau(t) = \sigma(1)^{-1}\sigma(1-t).$$

By an analogous argument, one can show that $\overline{f}_\sigma$ has a right inverse. It is clear by the definition that $\overline{f}_\sigma$ preserves $\overline{\mathscr{F}}$. □

Corollary 3.5. *The homomorphism $\overline{f}_\sigma$ induces an $\mathscr{F}$-preserving diffeomorphism f_σ of M.*

Lemma 3.6. *$\overline{f}_\sigma|_F : F \to F_{\sigma(1)}$ is a Lipschitz diffeomorphism.*

Proof. Identify the leaves F and $F_{\sigma(1)}$ with their image by $p : \overline{M} \to M$. Instead of considering $\overline{f}_\sigma$, we need only show that $f_\sigma|_F$ is Lipschitz. But since M is compact, we clearly have $\|Df_\sigma\|_{T\mathscr{F}} < \infty$, showing Lemma 3.6. □

Next we shall show that if $\sigma(1) = e$, then $\overline{f}_\sigma|_F \in \mathscr{B}(F)$. But in later section, we need a bit more.

Lemma 3.7. *For any $r > 0$, there exists $R > 0$ such that for any $\sigma \in \Omega(G)$ with $\sigma(1) = e$ and $\mathrm{length}(\sigma) < r$ and for any $x \in \overline{M}$, we have $d_{L_x}(x, \overline{f}_\sigma(x)) < R$, where L_x is the $\overline{\mathscr{F}}$-leaf through x. In particular, we have $\overline{f}_\sigma|_F \in \mathscr{B}(F)$.*

Proof. Since we consider a bundle-like metric, we have $d(x, \overline{f}_\sigma(x)) \le \mathrm{length}(\sigma_x) = \mathrm{length}(\sigma)$. Therefore Lemma 3.7 follows from the following sublemma. □

Sublemma 3.8. *For any $r > 0$, there exists $R > 0$ such that for any $x, y \in \overline{M}$ such that $d(x,y) < r$ and that x and y lie on the same leaf L of $\overline{\mathscr{F}}$, we have $d_L(x,y) < R$.*

Proof. Assume for contradiction that for some $r > 0$, there exist sequences of points x_n and y_n of $\overline{M}$ such that $d(x_n, y_n) < r$, x_n and y_n lie on the same leaf L_n of $\overline{\mathscr{F}}$ and that $d_{L_n}(x_n, y_n) \to \infty$ $(n \to \infty)$. Replacing x_n and y_n with their image by the action of a certain element of Γ if necessary, one may assume that all the points x_n lie in some fixed fundamental domain. Hence we may assume that $x_n \to x_0$ and that $y_n \to y_0$. But then since $\overline{\mathscr{F}}$ is a bundle foliation, we have that x_0 and y_0 lie on the same leaf L_0. Choose an arc τ in L_0 joining x_0 and y_0. Then for any large n, there is an arc in L_n joining x_n and y_n of length smaller than $\mathrm{length}(\tau) + 1$. A contradiction. □

Definition 3.9. Let us define the quasi-action $\psi : \Gamma \to \mathscr{H}(F)/\mathscr{B}(F)$ by

$$\psi(\gamma) = \gamma \circ \overline{f}_{\sigma_\gamma}, \quad \forall \gamma \in \Gamma,$$

where $\sigma_\gamma \in \Omega(G)$ is an arbitrary path such that $\sigma_\gamma(1) = \gamma^{-1}$.

Proposition 3.10. *ψ is a well defined homomorphism.*

Proof. It follows at once from Lemmas 3.6 and 3.7 that $\psi(\gamma)$ is well defined and independent of the choice of σ_γ. To show that ψ is a homomorphism, we have

$$\begin{aligned}\psi(\gamma'\gamma) &= \gamma' \circ \overline{f}_{\sigma_{\gamma'}} \circ \gamma \circ \overline{f}_{\sigma_\gamma} \\ &= \gamma' \circ \gamma \circ \overline{f}_{\sigma_{\gamma'}} \circ \overline{f}_{\sigma_\gamma} \\ &= (\gamma'\gamma) \circ \overline{f}_\tau,\end{aligned}$$

where $\tau = \sigma_\gamma \cdot \sigma_\gamma(1)\sigma_{\gamma'}$. Now we have

$$\tau(1) = \sigma_\gamma(1)\sigma_{\gamma'}(1) = \gamma^{-1}\gamma'^{-1} = (\gamma'\gamma)^{-1}.$$

The proof is now complete. □

There is a natural homomorphism $\alpha : \mathscr{H}(F)/\mathscr{B}(F) \to \mathrm{Homeo}(\mathscr{E}(F))$. Composing with ψ, we obtain

$$\alpha \circ \psi : \Gamma \longrightarrow \mathrm{Homeo}(\mathscr{E}(F)),$$

an action on $\mathscr{E}(F)$.

4. Proof of Proposition 1.1 and Theorem 1.2.

Let $\mathfrak{g}$ be the Lie algebra of G. The left invariant Riemannian metric dm_0 of G gives rise to an inner product of $\mathfrak{g}$. Choose $r > 0$ small enough so that the exponential map exp is injective on the ball $\mathfrak{g}_r = \{X \in \mathfrak{g} \mid |X| \le r\}$.

Let U_0 be an open symmetric neighbourhood of e in G such that $U_0 \subset \exp(\mathfrak{g}_r)$. (Symmetric means that $U_0^{-1} = U_0$.) For $g = \exp(X) \in U_0$, $X \in \mathfrak{g}_r$, define a path $\tau_g \in \Omega(G)$ by $\tau_g(t) = \exp(tX)$. Define a diffeomorphism $\chi : F \times U_0 \to D^{-1}(U_0)$ by $\chi(x,g) = f_{\tau_g}(x)$. (See Section 3, for the definition of f_{τ_g}.) Identify $D^{-1}(U_0)$ with $F \times U_0$ by χ and denote a point in $D^{-1}(U_0)$ by the coordinates $(x,g) \in F \times U_0$.

For arbitrary $K \in \mathscr{K}(F)$, choose a symmetric open neighbourhood $U \subset U_0$ further smaller so that the covering map $p : \overline{M} \to M$ is injective on $K \times U$. For $\gamma \in \Gamma \cap U$, $\psi(\gamma) \in \mathscr{H}(F)/\mathscr{B}(F)$ has a representative $\overline{\psi}(\gamma) \in \mathscr{H}(F)$, defined by

$$\overline{\psi}(\gamma) = \gamma \circ f_{\tau_{\gamma^{-1}}}.$$

Using the above coordinates, one can describe

$$\overline{\psi}(\gamma)(x) = \gamma(x, \gamma^{-1}),$$

where F is identified with $F \times \{e\}$.

For simplicity, denote by $\gamma * K$ the subset $\gamma(K \times \{\gamma^{-1}\})$ of F. For a boundary component A of K, we also use the notation $\gamma * A$. By the definition of the action of Γ on $\mathscr{E}(F)$, we have

$$\mathscr{E}(\underline{\gamma * A}) = \gamma\mathscr{E}(\underline{A}). \tag{3}$$

Since p is injective on $K \times U$, the $\gamma * K$'s are mutually disjoint for $\gamma \in \Gamma \cap U$.

Lemma 4.1. *Let $\delta, \gamma, \gamma\delta \in \Gamma \cap U$ and let A be a boundary component of K. Suppose K has at least three boundary components. Then A is the exit to $\delta * K$ if and only if $\gamma * A$ is the exit to $\gamma\delta * K$.*

Proof. Suppose A is the exit to $\delta * K$. Let $\delta * B$ be a boundary component of $\delta * K$, not the exit to K. Then by Lemma 2.8, we have $\delta\mathscr{E}(\underline{B}) = \mathscr{E}(\underline{\delta * B}) \subset \mathscr{E}(\underline{A})$. Therefore

$$\mathscr{E}(\underline{\gamma\delta * B}) = \gamma\delta\mathscr{E}(\underline{B}) \subset \gamma\mathscr{E}(\underline{A}) = \mathscr{E}(\underline{\gamma * A}),$$

showing that $\gamma * A$ is the exit to $\gamma\delta * K$.

The converse can be shown likewise. $\square$

Lemma 4.2. *For any neighbourhood $V \subset U$ of e and for any $P \in \mathscr{P}(F)$, there exists $\gamma \in \Gamma \cap V$ such that $\gamma * K$ is contained in P.*

Proof. For an escaping sequence $\{P_i\}$ such that $P_1 = P$, we have that $\bigcap_i \mathrm{Cl}p(P_i)$ is a nonempty $\mathscr{F}$-saturated closed subset of M. Since all leaves of $\mathscr{F}$ are dense in M, we get $\bigcap_i \mathrm{Cl}p(P_i) = M$. Especially we have $\mathrm{Cl}\, p(P) = M$. That is, the orbit by Γ of P is dense in $\overline{M}$, and in particular in $K \times V$. Since ∂P is compact, we have $\gamma(\partial P) \cap (K \times V) = \varnothing$ for but finite number of $\gamma \in \Gamma$. Therefore there exists $\gamma \in \Gamma \cap V$ such that $K \times \{\gamma^{-1}\} \subset \gamma^{-1}(P)$. Applying γ, we get that $\gamma * K \subset P$. $\square$

Proof of Proposition 1.2. Suppose that $\mathscr{E}(F)$ has at least three ends. Then there is an element $K \in \mathscr{K}(F)$ such that K has at least three boundary components. For an arbitrary element P of $\mathscr{P}(F)$, we have that $\gamma * K$ is contained in P ($\exists \gamma \in \Gamma$). By Lemma 2.7, for suitable two boundary components A and B of K, we have $\mathscr{E}(\underline{\gamma * A}) \cup \mathscr{E}(\underline{\gamma * B}) \subset \mathscr{E}(P)$. That is, any open set of $\mathscr{E}(F)$ contains at least two points. Therefore $\mathscr{E}(F)$ is perfect and hence a Cantor set. $\square$

The rest of this section is devoted to the proof of Theorem 1.3. Let us assume that the Lie group G is solvable. Define $G_0 = G$ and $G_i = [G_{i-1}, G_{i-1}]$ for $i \geq 1$ and assume that $G_n = \{e\}$. By Proposition 1.2, assume for contradiction that $\mathscr{E}(F)$ is a Cantor set. Then there exists $K \in \mathscr{K}(F)$ which has at least $2^n + 1$ boundary components.

In what follows, all neighbourhoods of e in G are to be symmetric. As before a neighbourhood $U \subset U_0$ of e is chosen in such a way that p is injective on $K \times U$.

Lemma 4.3. *For any neighbourhood $V \subset U$ of e, there exists a boundary component A of K and an element $\gamma \in \Gamma \cap V$ such that A is not the exit to $\gamma * K$ and that $\gamma * A$ is the exit to K.*

Proof. Choose a neighbourhood W of e such that $W^2 \subset V$. Let A and B be distinct boundary components of K. By Lemma 4.2, there exists $\alpha \in \Gamma \cap W$ such that $\alpha * K \subset \underline{B}$. If $\alpha * A$ is the exit to K, then put $\gamma = \alpha$.

If not, choose $\beta \in \Gamma \cap W$ such that $\beta * K \subset \underline{\alpha * A}$. If $\beta * A$ is the exit to $\alpha * K$, then it is also the exit to K. So we may put $\gamma = \beta$.

In the remaining case, $\beta * A$ is not the exit to $\alpha * K$ and $\alpha * A$ is the exit to $\beta * K$. Put $\gamma = \beta^{-1}\alpha \in \Gamma \cap V$. By Lemma 4.1, we are done. $\square$

LEMMA 4.4. *For any neighbourhood $V \subset U$ of e, there exist three distinct boundary components A, B and C of K and elements γ and δ of $\Gamma \cap V$ such that $\gamma * K \subset \underline{B}$, $\delta * K \subset \underline{C}$ and that $\gamma * A$ and $\delta * A$ are the exit to K.*

PROOF. Choose a neighbourhood W of e such that $W^2 \subset V$. By Lemma 4.3, we already know the existence of two distinct boundary components A' and B' of K and an element $\gamma' \in \Gamma \cap W$ such that $\gamma' * K \subset \underline{B'}$ and that $\gamma' * A'$ is the exit to K. Let C' be any other boundary component of K. Take $\alpha \in \Gamma \cap W$ such that $\alpha * K \subset \underline{C'}$.

Suppose $\alpha * B'$ is not the exit to K. Then let $A = A'$, $B = B'$, $C = C'$, $\gamma = \gamma'$ and $\delta = \alpha\gamma'$. We clearly have that $\alpha\gamma' * A$ is the exit to $\alpha * K$, hence of K and that $\alpha\gamma' * K \subset \underline{\alpha * B} \subset \underline{C}$.

On the other hand, suppose that $\alpha * B'$ is the exit to K. Then we have by Lemma 4.1

$$\underline{B'} \subset \underline{{\gamma'}^{-1}B'}, \quad \underline{B'} \subset \underline{\alpha * B'}, \quad {\gamma'}^{-1} * K \subset \underline{A'}, \quad \alpha * K \subset \underline{C'}.$$

Therefore put $A = B'$, $B = A'$, $C = C'$, $\gamma = {\gamma'}^{-1}$ and $\delta = \alpha$. $\square$

LEMMA 4.5. *For any neighbourhood $V \subset U$ of e, and integers $1 \leq j \leq 2^n$, there exist elements $\gamma_j \in \Gamma \cap V$ and distinct boundary components B_j and A of K such that $\gamma_j * K \subset \underline{B_j}$ and $\gamma_j * A$ is the exit to K.*

PROOF. Choose a neighbourhood W of e such that $W^2 \subset V$. Apply Lemma 4.4 for W. We get components A, B and C and elements γ and δ as in Lemma 4.4. Let B_j ($1 \leq j \leq 2^n$) be arbitrary boundary components distinct from A. Choose $\alpha_j \in \Gamma \cap W$ so that $\alpha_j * K \subset \underline{B_j}$. Then one of the components $\alpha_j * B$ and $\alpha_j * C$ is not the exit to K. If $\alpha_j * B$ (resp. $\alpha_j * C$) is not the exit to K, put $\gamma_j = \alpha_j\gamma$ (resp. $\gamma_j = \alpha_j\delta$). It is easy to show the required properties. $\square$

The following lemma, a variant of Klein's criterion, will play an essential role in what follows.

LEMMA 4.6. *Let $\mathscr{E}_0^-$, $\mathscr{E}_0^+$, $\mathscr{E}_1^-$ and $\mathscr{E}_1^+$ be mutually disjoint nonempty subset of $\mathscr{E}(F)$. Suppose $\alpha_j(\mathscr{E}(F) - \mathscr{E}_j^-) = \mathscr{E}_j^+$ for some $\alpha_j \in \Gamma$ ($j = 0, 1$). Then α_0 and α_1 generate a free subgroup of Γ.*

PROOF. For each integer $i \geq 1$, let j_i be either 0 or 1 and ϵ_i either 1 or -1.

CLAIM. *For any nontrivial reduced word $w = \alpha_{j_n}^{\epsilon_n} \cdots \alpha_{j_2}^{\epsilon_2}\alpha_{j_1}^{\epsilon_1}$, we have*

$$\alpha_{j_n}^{\epsilon_n} \cdots \alpha_{j_2}^{\epsilon_2}\alpha_{j_1}^{\epsilon_1}\left(\mathscr{E}(F) - \mathscr{E}_{j_1}^{-\epsilon_1}\right) \subset \mathscr{E}_{j_n}^{\epsilon_n}.$$

Clearly this claim shows Lemma 4.6, since any nontrivial reduced word of α_0 and α_1 represents a nontrivial element of Γ.

The proof of Claim is by an induction. For $n = 1$, this is clear by the assumption. Assume

$$\alpha_{j_{n-1}}^{\epsilon_{n-1}} \cdots \alpha_{j_2}^{\epsilon_2} \alpha_{j_1}^{\epsilon_1} \big(\mathscr{E}(F) - \mathscr{E}_{j_1}^{-\epsilon_1}\big) \subset \mathscr{E}_{j_{n-1}}^{\epsilon_{n-1}}.$$

Since the word w is reduced, we have $(j_n, -\epsilon_n) \neq (j_{n-1}, \epsilon_{n-1})$. That is, $\mathscr{E}_{j_{n-1}}^{\epsilon_{n-1}} \subset \mathscr{E}(F) - \mathscr{E}_{j_n}^{-\epsilon_n}$. Therefore we get

$$\alpha_{j_n}^{\epsilon_n} \alpha_{j_{n-1}}^{\epsilon_{n-1}} \cdots \alpha_{j_1}^{\epsilon_1} \big(\mathscr{E}(F) - \mathscr{E}_{j_1}^{-\epsilon_1}\big) \subset \alpha_{j_n}^{\epsilon_n} \big(\mathscr{E}_{j_{n-1}}^{\epsilon_{n-1}}\big) \subset \alpha_{j_n}^{\epsilon_n} \big(\mathscr{E}(F) - \mathscr{E}_{j_n}^{-\epsilon_n}\big) \subset \mathscr{E}_{j_n}^{\epsilon_n}.$$

This completes the proof of the claim. □

Lemma 4.7. *Let i be an integer such that $0 \leq i \leq n$. For any neighbourhood $V \subset U$ of e and integers $1 \leq j \leq 2^{n-i}$, there exist elements $\gamma_j \in \Gamma \cap V \cap G_i$ and distinct boundary components B_j and A of K such that $\gamma_j * K \subset \underline{B_j}$ and $\gamma_j * A$ is the exit to K.*

Proof. The case where $i = 0$ has been dealt with in Lemma 4.5. We will show the other cases by an induction. Choose a neighbourhood W of e such that $W^4 \subset V$. By the induction assumption, for $1 \leq j \leq 2^{n-i+1}$, there exist $\delta_j \in \Gamma \cap W \cap G_{i-1}$ and distinct boundary components A and C_j such that $\delta_j * K \subset \underline{C_j}$ and $\delta_j * A$ is the exit to K.

Notice first of all that $\delta_2\delta_1 \neq \delta_1\delta_2$, since $\delta_2\delta_1 * K \subset \underline{C_2}$ and $\delta_1\delta_2 * K \subset \underline{C_1}$.

Consider the two sets $\underline{\delta_2^{-1}\delta_1^{-1} * A}$ and $\underline{\delta_1^{-1}\delta_2^{-1} * A}$. By Lemma 2.7, we have the following three possibilities.

Case 1: $\underline{\delta_2^{-1}\delta_1^{-1} * A} \cup \underline{\delta_1^{-1}\delta_2^{-1} * A} = F$.
Case 2: $\underline{\delta_2^{-1}\delta_1^{-1} * A} \cap \underline{\delta_1^{-1}\delta_2^{-1} * A} = \varnothing$.
Case 3: $\underline{\delta_2^{-1}\delta_1^{-1} * A} \subset \underline{\delta_1^{-1}\delta_2^{-1} * A}$ or $\underline{\delta_1^{-1}\delta_2^{-1} * A} \subset \underline{\delta_2^{-1}\delta_1^{-1} * A}$.

Now Case 1 is impossible since we have

$$\underline{\delta_2^{-1}\delta_1^{-1} * A} \subset \underline{\delta_2^{-1} * A} \subset \underline{A}, \quad \underline{\delta_1^{-1}\delta_2^{-1} * A} \subset \underline{\delta_1^{-1} * A} \subset \underline{A}$$

and $\underline{A}$ is a proper subset of F.

Consider Case 2. Let $\mathscr{E}_0^- = \mathscr{E}(\underline{\delta_2^{-1}\delta_1^{-1} * A})$, $\alpha_0 = \delta_2\delta_1\delta_2$ and $\mathscr{E}_0^+ = \mathscr{E}(F) - \mathscr{E}(\underline{\delta_2 * A})$. Also let $\mathscr{E}_1^- = \mathscr{E}(\underline{\delta_1^{-1}\delta_2^{-1} * A})$, $\alpha_1 = \delta_1\delta_2\delta_1$ and $\mathscr{E}_1^+ = \mathscr{E}(F) - \mathscr{E}(\underline{\delta_1 * A})$. Now they satisfy the condition of Lemma 4.6. Thus Γ and hence G has a free group on two generators as a subgroup. But then G cannot be solvable. Therefore Case 2 is also impossible.

Now we need only consider Case 3. Assume, to fix the idea, that $\underline{\delta_2^{-1}\delta_1^{-1} * A} \subset \underline{\delta_1^{-1}\delta_2^{-1} * A}$ holds.

Since C_1 is the exit to $\delta_1 * K$, we have $\delta_2^{-1}\delta_1^{-1} * C_1$ is the exit to $\delta_2^{-1} * K$, hence of K. From this it follows that $\delta_2^{-1}\delta_1^{-1} * C_1$ is the exit to $\delta_1^{-1}\delta_2^{-1} * K$. Let $\gamma_1 = \delta_1\delta_2\delta_1^{-1}\delta_2^{-1}$ and let $B_1 = C_1$. Then we have $\gamma_1 \in \Gamma \cap V \cap G_i$, $\gamma_1 * K \subset \underline{B_1}$ and $\gamma_1 * A$ is the exit to K.

The other elements γ_j, $j \geq 2$ can be constructed likewise from δ_{2j-1} and δ_{2j}. □

Proof of Theorem 1.3. Lemma 4.7 assures the existence of γ_1 for $i = n$. But this is absurd since $G_n = \{e\}$. □

5. Case of two ends.

This section is devoted to the proof of Theorem 1.4. First we assume $\mathscr{E}(F) = \{\epsilon^+, \epsilon^-\}$, where $F = D^{-1}(e)$ as before, and will show that G is abelian.

If Γ acts on $\mathscr{E}(F)$ nontrivially, then replace Γ by a subgroup of index two which acts on $\mathscr{E}(F)$ trivially. The manifold M is replaced by a double covering. The holonomy group is still dense in G. Therefore we assume in this section that Γ acts on $\mathscr{E}(F)$ trivially.

Fix once and for all an element $K \in \mathscr{K}(F)$ with two boundary components. Let $P^{\pm}$ be the closure of the component of $F - K$ such that $\mathscr{E}(P^{\pm}) = \{\epsilon^{\pm}\}$.

Choose a neighbourhood U of e in G with the following properties. As before all neighbourhoods are to be symmetric.

(a): U is small enough so that the product structure $D^{-1}(U) = F \times U$ is defined as in Section 4.
(b): For any $g \in G$, $gU \cap U$ is either empty or connected.
(c): For distinct α, $\beta \in \Gamma$, we have $\alpha(K \times U) \cap \beta(K \times U) = \varnothing$.

The proof of the existence of such U is left to the reader.

Our method is to define a total order in $\Gamma \cap W$ for some small neighbourhood W of e and to show that it is invariant by the left and right translations. First of all, we need the following lemma.

Lemma 5.1. *For α, $\beta \in \Gamma \cap U$, the following conditions are equivalent.*

(a): $\alpha(K \times U) \cap \beta(P^- \times U) \neq \varnothing$.
(b): $\alpha(K \times U) \cap \beta(P^+ \times U) = \varnothing$.
(c): $\alpha(P^+ \times U) \cap \beta(K \times U) \neq \varnothing$.
(d): $\alpha(P^- \times U) \cap \beta(K \times U) = \varnothing$.
(e): $\alpha(P^- \times \{u\}) \subset \beta(P^- \times U)$, $\forall u \in U \cap \alpha^{-1}\beta U$.
(f): $\alpha(P^- \times \{u\}) \subset \beta(P^- \times U)$, $\exists u \in U \cap \alpha^{-1}\beta U$.

Proof. Notice that, since U is symmetric, we have $e \in \alpha U \cap \beta U$. That is, $\alpha(F \times U) \cap \beta(F \times U) \neq \varnothing$. The rest of the proof is omitted. □

Definition 5.2. For α, $\beta \in \Gamma \cap U$, define $\alpha \prec \beta$ if the conditions of Lemma 5.1 hold.

Let V be a neighbourhood of e in G such that $V^2 \subset U$.

Lemma 5.3. *$(\Gamma \cap V, \prec)$ is a totally ordered set, isomorphic to $(\boldsymbol{Z}, \leq)$. For α, β, $\gamma \in \Gamma \cap V$, if $\alpha \prec \beta$, then $\gamma\alpha \prec \gamma\beta$.*

Proof. Suppose $\alpha \prec \beta \prec \gamma$ for α, β, $\gamma \in \Gamma \cap V$. Then by (e) of Lemma 5.1, we have:

$$\alpha(P^- \times \{e\}) \subset \beta(P^- \times \{\beta^{-1}\alpha\}),$$
$$\beta(P^- \times \{\beta^{-1}\alpha\}) \subset \gamma(P^- \times U).$$

Then by (f) of Lemma 5.1, we have $\alpha \prec \gamma$.

Since $\alpha(F \times U) \cap \beta(F \times U) \neq \varnothing$ for any α, $\beta \in \Gamma \cap V$, $(\Gamma \cap V, \prec)$ is a totally ordered set. For any compact subset $A \subset \overline{M}$, $\gamma(K \times U) \cap A \neq \varnothing$ for but finite $\gamma \in \Gamma \cap V$. This shows that $(\Gamma \cap V, \prec)$ is isomorphic to $(\boldsymbol{Z}, \leq)$. The last statement can be shown by using e.g., (b) of Lemma 5.1. □

The rest of this section is mainly devoted to the proof of the invariance of the order $\prec$ by the right translation of an element γ of Γ which is very near to e. For this, we need show that the quasi-action by such γ on F is like a translation. This is a phenomenon essentially not difficult to imagine. However to show it, we need some preparations.

Let $\alpha_0 = e$ and place all the elements of $\Gamma \cap V$ in order as follows;

$$\cdots \alpha_{n-1} \prec \alpha_n \prec \alpha_{n+1} \cdots.$$

Let $K_n = \alpha_n(K \times V) \cap F$ and $P_n^{\pm} = \alpha_n(P^{\pm} \times V) \cap F$.

Lemma 5.4. *There exists $a > 0$ such that for any $n \in \boldsymbol{Z}$,*

$$\mathrm{dist}_F(K_n, K_{n+1}) < a.$$

Proof. Consider the subset $p(K \times V) \subset M$. For $x \in M$, denote by L_x the $\mathscr{F}$-leaf through x. The function $x \mapsto \mathrm{dist}_{L_x}(x, p(K \times V))$ is an upper semi-continuous function defined on a compact manifold M and therefore has an upper bound. This shows Lemma 5.4. □

Lemma 5.5. *For any $b > 0$, there exists a neighbourhood W of e in G such that for any subset $Q \subset F$ with $\mathrm{diam}_F(Q) < b$, p is injective on $Q \times W$.*

Proof. Suppose the contrary. Then for some $b > 0$ and for any $n \in \boldsymbol{N}$, there exist a neighbourhood W_n of e in G and a subset $Q_n \subset F$, such that $W_n \to \{e\}$ $(n \to \infty)$, $\mathrm{diam}_F(Q_n) < b$ and p is not injective on $Q_n \times W_n$. For a suitable element $\gamma_n \in \Gamma$, one may assume that $\gamma_n(Q_n \times W_n)$ meets the same fundamental domain. Also notice that the diameter of $\gamma_n(Q_n \times W_n)$ is bounded. Thus all $\gamma_n(Q_n \times W_n)$ are contained in some compact subset of $\overline{M}$. One may assume that $\mathrm{Cl}(\gamma_n(Q_n \times W_n))$ converge to some compact Q_0 in the Hausdorff topology. Then $D(\gamma_n(Q_n \times W_n)) = \gamma_n(W_n)$ converge to a point in G. That is, Q_0 is contained in a leaf of $\overline{\mathscr{F}}$ and therefore admits an open neighbourhood N on which p is injective. For any sufficiently large n, $\gamma_n(Q_n \times W_n)$ is contained in N. This implies that p is injective on $Q_n \times W_n$. A contradiction. □

Now let us recall that in Section 4, for any $g \in U$, we have chosen a path τ_g and using it, defined a product structure of $D^{-1}(U) = F \times U$. We also defined a representative $\overline{\psi}(\gamma) \in \mathscr{H}(F)$ of $\psi(\gamma) \in \mathscr{H}(F)/\mathscr{B}(F)$ for any $\gamma \in \Gamma \cap U$. Recall that $K_n = \alpha_n(K \times V) \cap F = \overline{\psi}(\alpha_n)(K)$.

LEMMA 5.6. *There exists a constant $c > 0$ such that for any α, $\beta \in \Gamma \cap V$, we have*

$$c(\alpha, \beta) = \sup_{x \in F} d_F(\overline{\psi}(\alpha)\overline{\psi}(\beta)(x), \overline{\psi}(\alpha\beta)(x)) < c.$$

PROOF. We have

$$\begin{aligned}\overline{\psi}(\alpha)\overline{\psi}(\beta)(x) &= \alpha \overline{f}_{\tau_{\alpha^{-1}}} \beta \overline{f}_{\tau_{\beta^{-1}}}(x) \\ &= \alpha\beta \overline{f}_{\tau_{\alpha^{-1}}} \overline{f}_{\tau_{\beta^{-1}}}(x) \\ &= \alpha\beta \overline{f}_{\zeta}(x),\end{aligned}$$

where $\zeta = \tau_{\beta^{-1}} \cdot \beta^{-1}\tau_{\alpha^{-1}} \in \Omega(G)$. On the other hand, we have

$$\overline{\psi}(\alpha\beta) = \alpha\beta \overline{f}_{\xi},$$

where $\xi = \tau_{\beta^{-1}\alpha^{-1}}$.

Since $\alpha\beta$ keeps invariant the distance along the leaves of $\overline{\mathscr{F}}$, we have

$$\begin{aligned}c(\alpha, \beta) &= \sup\{d_F(\overline{f}_{\zeta}(x), \overline{f}_{\xi}(x)) \mid x \in F\} \\ &= \sup\{d_L(\overline{f}_{\xi}^{-1}\overline{f}_{\zeta}(y), y) \mid y \in L = F_{\alpha^{-1}\beta^{-1}}\} \\ &= \sup\{d_L(\overline{f}_{\eta}(y), y) \mid y \in L = F_{\alpha^{-1}\beta^{-1}}\}\end{aligned}$$

for some path η, for which we have

$$\text{length}(\eta) = \text{length}(\tau_{\alpha^{-1}}) + \text{length}(\tau_{\beta^{-1}}) + \text{length}(\tau_{\beta^{-1}\alpha^{-1}}).$$

Thus length(η) has an upper bound independent of the choice of α and β in $\Gamma \cap V$. Lemma 5.6 follows from Lemma 3.7 of Section 3. $\square$

Let $Q_n = \mathrm{Cl}(F - (P_{n-1}^- \cup P_{n+1}^+))$. Clearly $\mathrm{diam}_F(K_n)$ is bounded. Together with Lemma 5.4, this implies that $\mathrm{diam}_F(Q_n)$ is bounded. Now let c be the constant in Lemma 5.6 and let R_n be the c-neighbourhood of Q_n. We have $\mathrm{diam}_F(R_n) < b$ for some $b > 0$. Choose a neighbourhood $W \subset V$ of e in G for this b as in Lemma 5.5. Now we have;

LEMMA 5.7. *For any $\gamma \in \Gamma \cap W - \{e\}$, we have either $\gamma\alpha_n \succ \alpha_n$ $(\forall n)$ or $\gamma\alpha_n \prec \alpha_n$ $(\forall n)$.*

PROOF. First notice that $\gamma\alpha_n$ may not be in $\Gamma \cap V$, i.e. not one of the α_m's. Assume the contrary. Then since $\gamma\alpha_n \prec \alpha_n$ is equivalent to $\alpha_n \prec \gamma^{-1}\alpha_n$, replacing γ with γ^{-1} if neccesary, one may assume that for some n, $\gamma\alpha_n \succ \alpha_n$ and $\gamma\alpha_{n+1} \prec \alpha_{n+1}$. Now we have

$$\overline{\psi}(\gamma)K_n = \gamma(K_n \times \{\gamma^{-1}\}) \subset \gamma(R_n \times W).$$

Now p is injective on $R_n \times W$. Therefore we have $\overline{\psi}(\gamma)(K_n) \cap R_n = \varnothing$. In other words, $\overline{\psi}(\gamma)(K_n)$ is at least c-apart from Q_n.

On the other hand, $\overline{\psi}(\gamma\alpha_n)(K)$ is contained in the c-neighbourhood of $\overline{\psi}(\gamma)\overline{\psi}(\alpha_n)(K) = \overline{\psi}(\gamma)(K_n)$. Therefore $\overline{\psi}(\gamma\alpha_n)(K) \cap Q_n = \varnothing$. Therefore either $\overline{\psi}(\gamma\alpha_n)(K)$ is contained in P^-_{n-1} or P^+_{n+1}, that is, either $\gamma\alpha_n \prec \alpha_{n-1}$ or $\gamma\alpha_n \succ \alpha_{n+1}$. But since we assume $\gamma\alpha_n \succ \alpha_n$, we have $\gamma\alpha_n \succ \alpha_{n+1}$.

By the same argument one can deduce from the assumption $\gamma\alpha_{n+1} \prec \alpha_{n+1}$, that $\gamma\alpha_{n+1} \prec \alpha_n$. But then $\gamma\alpha_{n+1} \prec \alpha_n \prec \alpha_{n+1} \prec \gamma\alpha_n$. This contradicts the left invariance of $\prec$ (Lemma 5.3). □

Lemma 5.7 asserts in particular that for $\alpha \in V \cap \Gamma$ and for $\gamma \in W \cap \Gamma$, we have $\gamma\alpha \prec \alpha \Longleftrightarrow \gamma \prec e$. Let W_0 be an *open* neighbourhood of e in G such that $W_0^4 \subset W$. It is easy to show the following.

COROLLARY 5.8. *For any α, β, $\gamma \in \Gamma \cap W_0$, we have*

$$\alpha \prec \beta \Longleftrightarrow \gamma\alpha \prec \gamma\beta \Longleftrightarrow \alpha\gamma \prec \beta\gamma.$$

We also have

$$\alpha \prec \beta \Longleftrightarrow \beta^{-1} \prec \alpha^{-1}.$$

PROOF OF THEOREM 1.4. Assume F has two ends. First of all let us show that G is abelian. Since Γ is dense in G, it suffices to show that Γ is abelian. Denote the center of Γ by $C(\Gamma)$. Suppose in way of contradiction that Γ is not abelian. Then since $\Gamma \cap W_0$ generates Γ, we have $B = (\Gamma - C(\Gamma)) \cap W_0 \neq \varnothing$. Clearly we have $B^{-1} = B$. As we have already shown in Lemma 5.3, the ordered set $(\Gamma \cap W_0, \prec)$ is isomorphic to $(\boldsymbol{Z}, \leq)$. Thus there exists a minimum element ω in $B \cap \{\alpha > e\}$. Choose a small neighbourhood W_1 of e in G such that $W_1\omega W_1 \subset W_0$. ($W_0$ was chosen to be open.) Then for any $\gamma \in \Gamma \cap W_1$, we have by Corollary 5.8 that $\gamma\omega\gamma^{-1} \in B \cap \{\alpha > e\}$. Therefore by the minimality of ω, it follows that $\gamma\omega\gamma^{-1} \succ \omega$.

Replacing γ by γ^{-1}, we get by the same argument that $\gamma^{-1}\omega\gamma \succ \omega$. Taking the conjugate, we have $\gamma\omega\gamma^{-1} \prec \omega$. Therefore $\gamma\omega\gamma^{-1} = \omega$. Since $\Gamma \cap W_1$ generates Γ, we have $\omega \in C(\Gamma)$. This contradiction shows that G is abelian.

Let us show the rest of Theorem 1.4. Since G is simply connected, we have $G = \boldsymbol{R}^n$. Since Γ is dense in $\boldsymbol{R}^n$, we have that $m = \mathrm{rank}(\Gamma) \geq n+1$. Suppose for contradiction that $m > n+1$. Then one can construct a linear Lie $\boldsymbol{R}^n$ foliation on T^m, with leaves homeomorphic to $\boldsymbol{R}^{m-n}$, having the holonomy group Γ. By Proposition 1.7, we get that $\mathscr{E}(F)$ is a singleton for the foliation $\mathscr{F}$ in question.

On the contrary if $\mathrm{rank}(\Gamma) = n+1$ for a dense subgroup Γ of $\boldsymbol{R}^n$, then one can construct a linear flow on T^{n+1} with the holonomy group Γ. Again by Proposition 1.7, any foliation $\mathscr{F}$ whose holonomy group is Γ has leaves with two ends.

That the linear flow is the classifiant follows immediately from the criterion of Haefliger ([**9**]). □

6. Leaves with infinitely many ends.

In this section we shall show Theorem 1.5. We begin with a discrete, cocompact, irreducible subgroup Γ in $PSL(2, \boldsymbol{R}) \times PSL(2, \boldsymbol{Q}_2)$, where $\boldsymbol{Q}_2$ is the locally compact, totally disconnected field of diadic numbers.

The construction of such a lattice Γ by arithmetic means is classical. Here is a guideline for the geometer reader. For details and proofs, see for example [**19**] and in particular theorem 1.1 of chapter IV.

Fix nonzero a, $b \in \boldsymbol{Z}$, and, for every commutative ring R, consider the *quaternionic algebra* over R:

$$H(R) = R[i, j]/i^2 = a,\ j^2 = b,\ ji = -ij\,,$$

which is free of dimension 4 over R, with basis $(1,\, i,\, j,\, ij)$, and has norm:

$$N(x + yi + zj + tij) = (x + yi + zj + tij)(x - yi - zj - tij) = x^2 - ay^2 - bz^2 + abt^2.$$

The elements of norm 1 form a multiplicative group $H(R)^1$.

A first fact is that if the quadratic form N is isotropic over a field k, then $H(k)$ is isomorphic to the algebra of 2×2 matrices with entries in k. This isomorphism carries $H(k)^1$ onto $SL(2, k)$.

On another side, consider the ring $\boldsymbol{Z}[1/2]$, and the *order* $H(\boldsymbol{Z}[1/2])$. Then the diagonal image of $H(\boldsymbol{Z}[1/2])^1$ into $H(\boldsymbol{R})^1 \times H(\boldsymbol{Q}_2)^1$ is a discrete subgroup of finite covolume. It is *strongly irreducible* in the sense that its intersection with each slice, if nonempty, is a singleton. Finally $H(\boldsymbol{Z}[1/2])^1$ is cocompact if N is not isotropic over $\boldsymbol{Q}$.

So to get our cocompact irreducible lattice, we just have to choose a and b such that N is isotropic over $\boldsymbol{R}$ and over $\boldsymbol{Q}_2$ but not over $\boldsymbol{Q}$. With the help of elementary arithmetics (for example [**17**]) the reader will easily verify that for $a = b = 3$ the quadratic form N is isotropic over $\boldsymbol{Q}_2$ and of course over $\boldsymbol{R}$, but not over $\boldsymbol{Q}_5$ and thus nor over $\boldsymbol{Q}$. Projecting the image of $H(\boldsymbol{Z}[1/2])^1$ to $PSL(2, \boldsymbol{R}) \times PSL(2, \boldsymbol{Q}_2)$, we get a required lattice Γ.

Since the lattice Γ is strongly irreducible, its projections to factors are injective, with dense images $\Gamma(\boldsymbol{R})$, $\Gamma(\boldsymbol{Q}_2)$. We may moreover assume Γ is without torsion, changing if necessary Γ to a finite index subgroup. Here we have used the following lemma: if a topological group G is infinite and simple, then every finite index subgroup Δ' of every dense subgroup Δ is also dense. (On the contrary, notice that $\boldsymbol{Z}$ is dense in the 2-adic Lie group $\boldsymbol{Z}_2$, but $2\boldsymbol{Z}$ is not.) Proof: one may assume Δ' normal in Δ, thus its closure $\overline{\Delta}'$ is normal in G. Since G is infinite, $\overline{\Delta}'$ is not trivial. Since G is simple, $\overline{\Delta}' = G$.

Recall, see for example [**18**], that there is a natural action of $PSL(2, \boldsymbol{Q}_2)$ on the homogenous trivalent simplicial tree T. This action is continuous, proper, without edge inversion, transitive on the edges, and with two orbits on the vertices. Since $\Gamma(\boldsymbol{Q}_2)$ is dense in $PSL(2, \boldsymbol{Q}_2)$, it is also transitive on the edges and has two orbits on the vertices. In other words, fixing any edge $e = (v, w)$, the group Γ splits as an amalgamated product $\Gamma = \Gamma_v *_{\Gamma_e} \Gamma_w$, where Γ_i is the stabilizer of $i = e,\ v,\ w$. Conversely given an amalgamated product $\Pi = A *_C B$, its tree is defined as follows. The set of vertices are $\Pi/A \cup \Pi/B$, and two vertices πA and $\pi' B$ are joined by an edge if and only if $\pi A \cap \pi' B$ is nonempty. One can define a left action of Π on the tree. Then the tree of the amalgamated product $\Gamma_v *_{\Gamma_e} \Gamma_w$ is equivariantly homeomorphic to T. One can construct also a tree T' using left cosets. The group Γ acts on T' from the right.

On the other hand, since Γ is discrete and cocompact in $PSL(2, \boldsymbol{R}) \times PSL(2, \boldsymbol{Q}_2)$, obviously $\Gamma_v(\boldsymbol{R})$, $\Gamma_w(\boldsymbol{R})$ and $\Gamma_e(\boldsymbol{R})$ are three commensurable cocompact discrete subgroups in $PSL(2, \boldsymbol{R})$. Let $\boldsymbol{D}$ denote the Poincaré disk and, for $i = v, w, e$, let $\Sigma_i = \Gamma_i(\boldsymbol{R}) \setminus \boldsymbol{D}$, a closed smooth surface since $\Gamma(\boldsymbol{R})$ is without torsion. Let

$$c_v : \Sigma_e \to \Sigma_v, \quad c_w : \Sigma_e \to \Sigma_w$$

be the corresponding three-fold coverings. Then for each $i = v, w$, there is a map $\zeta_i : \Sigma_e \to \boldsymbol{D} \subset \boldsymbol{S}^2$ such that the map $g_i = (c_i, \zeta_i) : \Sigma_e \to \Sigma_i \times \boldsymbol{S}^2$ is an embedding. This is a consequence of the following lemma.

LEMMA 6.1. *For every* 3*-fold covering* $c : \bar{\Sigma} \to \Sigma$ *of a closed surface, there exists a smooth map* $z : \bar{\Sigma} \to \boldsymbol{D}$ *such that the product map* (c, z) *is an embedding of* $\bar{\Sigma}$ *into* $\Sigma \times \boldsymbol{D}$.

This topological lifting lemma lies in turn on the following algebraic lifting lemma, that will be proved in Appendix B. Consider $B_3 \to S_3$, the natural projection from the braid group on three strings onto the symmetric group on three letters.

LEMMA 6.2. *Every homomorphism from the fundamental group of a closed surface into* S_3*, lifts to* B_3.

PROOF OF LEMMA 6.1. We have two normal S_3 coverings. First, $\Sigma' \to \Sigma$, the S_3-principal bundle associated to $\bar{\Sigma}$. In other words, $\bar{\Sigma}$ is isomorphic to Σ'/S_2 where

$$S_2 = \{\mathrm{id}, (23)\} \subset S_3.$$

Second, $E \to E/S_3$, where E is the set of triples (z_1, z_2, z_3) of three distinct points in $\boldsymbol{D}$. The fundamental group of E/S_3 is B_3.

The normal covering Σ' induces a homomorphism $\pi_1(\Sigma) \to S_3$, which lifts to B_3 after Lemma 6.2. Since Σ is a surface, this implies the existence of an S_3-equivariant smooth map $Z : \Sigma' \to E$. Write for every $x' \in \Sigma'$:

$$Z(x') = (Z_1(x'), Z_2(x'), Z_3(x')) \in E.$$

Since the map Z is S_3-equivariant, the map Z_1 is S_2-invariant, and thus quotients to a map $z : \bar{\Sigma} \to \boldsymbol{D}$. Consider, above every $x \in \Sigma$, the fibres $\bar{\Sigma}_x$ and Σ'_x. Choose $x' \in \Sigma'_x$. Since the map z is a quotient of Z_1 and since Z is S_3-equivariant,

$$z(\bar{\Sigma}_x) = Z(\Sigma'_x) = \{Z_1(x'), Z_2(x'), Z_3(x')\}$$

are three *distinct* points in $\boldsymbol{D}$. Thus (c, z) is one-to-one, thus an embedding. $\square$

For $i = v, w$ the product manifold $\hat{M}_i = \Sigma_i \times \boldsymbol{S}^2$ is equipped with the trivial foliation $\mathscr{F}_i$ with spherical leaves. It is a transversely homogeneous $(PSL(2, \boldsymbol{R}), \boldsymbol{D})$ foliation with holonomy group $\Gamma_i(\boldsymbol{R})$. Since ζ_i is homotopic to the constant map, the normal bundle of the image of g_i is trivial. Let $N(g_i)$ be a tubular neighbourhood of the image. Then the foliation $\mathscr{F}_i$ restricted to $N(g_i)$ is a trivial foliation by discs. Thus there is an identification of $\partial N(g_i)$ with $\Sigma_e \times \boldsymbol{S}^1$. Define $M_i = \hat{M}_i - \mathrm{Int}(N(g_i))$ and $M_e = \Sigma_e \times \boldsymbol{S}^1 \times [0,1]$, the latter being equipped with the trivial foliation $\mathscr{F}_e$ by annular leaves. By identifying the two boundary components of M_e with the boundaries of M_v and M_w, we obtain a foliated manifold $(M, \mathscr{F})$. The foliation $\mathscr{F}$ is a transversely homogeneous $(PSL(2, \boldsymbol{R}), \boldsymbol{D})$ foliation. Its holonomy group is $\Gamma(\boldsymbol{R})$.

Indeed, obviously Σ_e is a total transversal and the holonomy pseudogroup $\mathscr{H}_e$ of $\mathscr{F}$ on Σ_e is generated by those local diffeomorphisms of Σ_e that project to the identity either on Σ_v or on Σ_w. In other words, consider the pseudogroup $\mathscr{H}$ of local diffeomorphisms of $\boldsymbol{D}$, pullback of $\mathscr{H}_e$. On one hand, $\mathscr{H}$ is Haefliger equivalent to $\mathscr{H}_e$. On the other hand, it is generated by those local diffeomorphisms of $\boldsymbol{D}$ which project to the identity either on Σ_v or on Σ_w. Thus the holonomy group of $\mathscr{F}$ is generated by $\Gamma_v(\boldsymbol{R}) \cup \Gamma_w(\boldsymbol{R})$, hence coincides with $\Gamma(\boldsymbol{R})$.

Any transversely homogeneous $(G, G/K)$ foliation $\mathscr{F}$, where G is a semi-simple Lie group and K the maximal compact subgroup, can be changed into a Lie G foliation with the same generic leaf and the same holonomy group as $\mathscr{F}$. Namely let L, $\overline{M}$, Γ denote the leaf, the holonomy covering and the holonomy group of $\mathscr{F}$, and $D : \overline{M} \to G/K$ the developing map. Define $\overline{M}'$ as the fiber product of $\overline{M}$ and G over G/K:

$$\overline{M}' = \{(x, g) \in \overline{M} \times G \mid D(x) = gK\}.$$

Obviously the diagonal action of Γ on $\overline{M}'$ is properly discontinuous, free and cocompact, and $D' : (x, g) \mapsto g$ is a Γ-equivariant fibration of $\overline{M}'$ onto G, with the same fiber L. Thus we get a Lie foliation with the desired property.

The fact that the leaves have infinitely many ends now follows from Remark 8.11; or alternatively by Remark 6.3 they have more than one end and by Theorem 1.4 they cannot have two ends. More directly, let F be a leaf of $\mathscr{F}$ which passes through $\Gamma_e(\boldsymbol{R}) \times \boldsymbol{S}^1 \times I \subset M_e$, where $\Gamma_e(\boldsymbol{R})$ is a point of $\Sigma_e \cong \Gamma_e(\boldsymbol{R}) \setminus \boldsymbol{D}$. A connected component of $F \cap M_i$ $(i = v, w)$ is a thrice punctured sphere and a connected component of $F \cap M_e$ is an annulus. Shrinking thrice punctured spheres to points and annuli to edges, one gets a trivalent graph. It is easy to show that this graph coincides with the tree T' of the amalgamated product $\Gamma_v *_{\Gamma_e} \Gamma_w$ constructed from the left cosets.

Stallings' theorem for Lie foliations? The proof of Theorem 1.5, together with Proposition 1.2 and Theorem 1.4, suggests an analogue, for Lie foliations, of Stallings' theorem for groups. Recall that a group Γ is said to *split* over a subgroup Λ if there are either two subgroups A, B *strictly containing* Λ such that Γ is the amalgamated product:

$$\Gamma = A *_\Lambda B$$

or a subgroup A *containing* Λ and an embedding $\phi : \Lambda \to A$ such that Γ is the HNN-extension:

$$\Gamma = A *_{\Lambda,\phi} .$$

Question. *If the leaves of a Lie foliation have more than one end, does the holonomy group necessarily split over a subgroup which is discrete and cocompact in the Lie group?*

Remark 6.3. The converse of the above question holds true.

Proof. Assume that the holonomy group Γ of some Lie G foliation $\mathscr{F}$ splits over a subgroup Λ which is discrete and cocompact in G. First of all consider the simplest case where the indices of A and B over Λ are finite. Then as is mentioned in the proof above of Theorem 1.5 (in the case of an amalgamated product), the group Γ acts on a locally finite tree T, cocompactly, so that Λ is the stabilizer of some edge. Since Λ is discrete and cocompact in G, it follows that the diagonal action of Γ on $T \times G$ is properly discontinuous and cocompact. After Remark 8.11 and Proposition 8.10, the leaf of the foliation has the same space of ends as the tree T, thus more than one.

Unfortunately this argument does not work for the case where either A or B has infinite index over Λ, since then the tree T is not locally finite. But still the group Γ acts on T without edge inversion, transitively on the edges, such that Λ is the stabilizer of some edge e_0. Also the group Γ has a finite generating set $\mathscr{T}$ such that τe_0 is adjacent to e_0 for any $\tau \in \mathscr{T}$. Denote by $d_{\mathscr{T}}$ the left invariant word distance on Γ with respect to $\mathscr{T}$.

Since Λ is discrete and cocompact in G, there is a relatively compact open neighbourhood U of e in G such that $U \cdot \Lambda = G$. We shall use materials of Appendix A. Let $\mathscr{U}$ be a U-generating set of Γ (Definition 8.4). Clearly $\mathscr{U}$ generates Γ, and the left invariant word metric $d_{\mathscr{U}}$ is equivalent to $d_{\mathscr{T}}$. That is, there is a constant $K \geq 1$ such that for any α and β in Γ, we have

$$K^{-1} d_{\mathscr{U}}(\alpha, \beta) \leq d_{\mathscr{T}}(\alpha, \beta) \leq K d_{\mathscr{U}}(\alpha, \beta).$$

Let $\{e_n\}_{n \in \boldsymbol{Z}}$ be the ordered sequence of edges which lie on some geodesic line passing through the edge e_0. For any e_n, there is an element u_n of Γ which sends e_0 to e_n. Multiplying if necessary by some element of Λ from the right, one may assume that $u_n \in U$. Consider, as in Appendix A, the graph $X(\Gamma, U, \mathscr{U})$. Let c_n be a minimal geodesic arc in $X(\Gamma, U, \mathscr{U})$ which links e to u_n. Then we have $d_{\mathscr{U}}(e, u_n) \to \infty$ as $n \to \pm\infty$. Indeed, if we endow the set E of edges of the tree T with the standard metric, then the map $\Gamma \ni \gamma \mapsto \gamma e_0 \in E$ is 1-Lipschitz for the metric $d_{\mathscr{T}}$, and thus K-Lipschitz for $d_{\mathscr{U}}$.

Now $d_{\mathscr{U}}(e, u_n) \to \infty$ implies that $\mathrm{length}(c_n) \to \infty$ as $n \to \pm\infty$, and thus there are admissible subsequences (Appendix A) $\{u_{n(i)}\}_{i \geq 0}$ of $\{u_n\}_{n \geq 0}$, and $\{u_{m(i)}\}_{i \geq 0}$ of $\{u_m\}_{m \leq 0}$. To verify that they are not equivalent (Appendix A), consider *any* arc in the graph $X(\Gamma, U, \mathcal{U})$ joining $u_{n(i)}$ to $u_{m(i)}$, say

$$(\alpha_1, \alpha_2, \cdots, \alpha_p), \quad (\alpha_{k+1}\alpha_k^{-1} \in \mathscr{U},\ \alpha_1 = u_{n(i)},\ \alpha_p = u_{m(i)}).$$

We are going to show that there is a fixed finite set in Γ which intersects this arc $(\alpha_1, \cdots, \alpha_p)$ for any i.

For each $1 \leq k \leq p-1$, choose a minimizing geodesic a_k from α_k to α_{k+1} in the metric space $(\Gamma, d_{\mathscr{T}})$. Let $(\beta_1, \beta_2, \cdots, \beta_q)$ be the sequence of vertices of the composite arc $a_1 a_2 \cdots a_{p-1}$. In the tree T, the edges $\beta_l e_0$ and $\beta_{l+1} e_0$ are adjacent for any $1 \leq l \leq q-1$.

Since e_0 lies between $e_{n(i)} = \beta_1 e_0$ and $e_{m(i)} = \beta_q e_0$, there is some l such that $\beta_l e_0 = e_0$, equivalently $\beta_l \in \Lambda$. Let a_k be the arc that contains β_l. Then $d_{\mathscr{T}}(\beta_l, \alpha_k) \leq K$, in other words $\alpha_k \in \beta_l B$, where B is the $d_{\mathscr{T}}$-ball in Γ centered at e and of radius K. Finally the arc $(\alpha_1, \cdots, \alpha_p)$ has to intersect a fixed set $\Lambda B \cap U$, which is finite since Λ is discrete, B finite, and U relatively compact. This shows that the two admissible sequences $\{u_{n(i)}\}$ and $\{u_{m(i)}\}$ are not equivalent, and that the graph $X(\Gamma, U, \mathscr{U})$ has at least two ends. The proof is now complete by Proposition 8.10. $\square$

Remark 6.4. The answer to the question above is positive when the leaves have two ends.

Proof. This follows from Theorem 1.4, since $\boldsymbol{Z}^{n+1}$ splits as a HNN-extension of $\boldsymbol{Z}^n$ over $(\boldsymbol{Z}^n, \mathrm{id})$. $\square$

Remark 6.5. The answer is also positive when the Lie group G is compact, or more generally when the holonomy group of the foliation contains a subgroup which is discrete, cocompact and normal in G.

Proof. In that case, the positive answer follows from Stallings' theorem. Indeed, assume that $\mathscr{F}$ is a Lie G foliation whose holonomy group Γ contains a subgroup Δ which is discrete, cocompact and normal in G. Consider $\mathscr{F}$ as a Lie G' foliation, where $G' = G/\Delta$. Following Section 2, and remembering that Δ is contained in Γ, one sees that its holonomy group is $\Gamma' = \Gamma/\Delta$ and that its holonomy covering $\bar{M}'$ (whose deck transformation group is Γ') is equipped with a developing map $D' : \bar{M}' \to G'$ whose fibre is connected and projects one-to-one onto the leaf F on $\mathscr{F}$. Endow as usual M with a bundle-like metric (See Section 3) and lift it to $\bar{M}'$. Then D' is a Riemannian fibration whose basis is compact, thus its total space $\bar{M}'$ is quasi-isometric to the fibre, namely the leaf F; thus the holonomy group Γ' is also quasi-isometric to F; thus Γ' has more than one end. After Stallings' theorem, Γ' splits over a finite group Λ'. In other words Γ splits over Λ, the preimage of Λ' in G, which contains Δ as a subgroup of finite index, and thus is discrete and cocompact in G. $\square$

Remark 6.6. The answer is also positive when $\mathscr{F}$ admits a transverse Riemannian foliation of the complementary dimension.

Proof. In the two-ends case, Remark 6.4 concludes. In the Cantor-of-ends case, Theorem 1.6 gives a cocompact lattice Δ, which is contained in Γ (see the proof of the theorem). Thus the statement follows from the previous remark. $\square$

7. Proof of Theorem 1.6.

For a while let F be a complete connected Riemannian manifold with more than two ends, and let $\mathrm{Isom}(F; \mathscr{E}(F)) \subset \mathrm{Isom}(F)$ be the subgroup of isometries which act trivially on $\mathscr{E}(F)$.

Lemma 7.1. *The subgroup* $\mathrm{Isom}(F;\mathscr{E}(F))$ *is normal, compact and open in* $\mathrm{Isom}(F)$*, and the action of* $\mathrm{Isom}(F)/\mathrm{Isom}(F;\mathscr{E}(F))$ *on the quotient space* $\mathrm{Isom}(F;\mathscr{E}(F))\setminus F$ *is properly discontinuous.*

Proof. Obviously $\mathrm{Isom}(F;\mathscr{E}(F))$ is normal and closed in $\mathrm{Isom}(F)$. The group $\mathrm{Isom}(F)$ is a Lie group possibly with infinite components by the compact–open topology ([**16**], [**12**] Chapter II). Denote by $B(F)$ the orthonormal frame bundle of F, and choose a base frame $u \in B(F)$. Then the map $\iota : \mathrm{Isom}(F) \to B(F)$ defined by $\iota(f) = df(u)$ is an embedding onto a closed submanifold ([**12**], p. 41).

Let $K \in \mathscr{K}(F)$ be a submanifold with more than two boundary components. Then for a base point $x_0 \in K$ and for any $f \in \mathrm{Isom}(F;\mathscr{E}(F))$ we have $d(x_0, f(x_0)) \leq 3\,\mathrm{diam}(K)$. For otherwise f would displace K outside itself, which implies that the action of f on $\mathscr{E}(F)$ is nontrivial. Now the subgroup $\mathrm{Isom}(F;\mathscr{E}(F))$ is shown to be compact.

Since $\mathrm{Isom}(F;\mathscr{E}(F))$ contains the identity component of $\mathrm{Isom}(F)$, $\mathrm{Isom}(F;\mathscr{E}(F))$ is open. Since $\mathrm{Isom}(F;\mathscr{E}(F))$ is open and since $\mathrm{Isom}(F)$ acts properly on F, it follows that $\mathrm{Isom}(F)/\mathrm{Isom}(F;\mathscr{E}(F))$ acts properly discontinuously on $\mathrm{Isom}(F;\mathscr{E}(F))\setminus F$. □

Proof of Theorem 1.6. We assume that the Lie G foliation $\mathscr{F}$ admits a transverse Riemannian foliation $\mathscr{G}$ of the complementary dimension. In this situation, the quasi-action of Γ on F of Section 3 becomes a true action by isometries.

Choose a Riemann metric of M which is bundle-like both for $\mathscr{F}$ and $\mathscr{G}$. Let $\overline{\mathscr{G}}$ be the lift of $\mathscr{G}$ to $\overline{M}$. The developing map $D : \overline{M} \to G$ restricted to a leaf of $\overline{\mathscr{G}}$, being a local isometry between complete Riemannian manifolds, is a covering map. Since G is assumed to be simply connected (Section 2), it is a homeomorphism. Thus $\overline{\mathscr{F}}$ and $\overline{\mathscr{G}}$ gives a product structure $\overline{M} = F \times G$.

Now we get a projection $q : \overline{M} = F \times G \to F$. The action of Γ on $\overline{M}$ projects down to an isometric action on F. This yields a homomorphism $\overline{\psi} : \Gamma \to \mathrm{Isom}(F)$. One knows immediately that $\overline{\psi}$ represents the quasi-action $\psi : \Gamma \to H(F)/\mathscr{B}(F)$ in Section 3.

Assume F has more than two ends and consider the diagonal action of Γ on $F \times G = \overline{M}$. Let $\Delta = \overline{\psi}^{-1}(\mathrm{Isom}(F;\mathscr{E}(F)))$. Since $\mathrm{Isom}(F;\mathscr{E}(F))$ is compact and since Γ acts properly discontinuously on $\overline{M}$, it follows that Δ is discrete in G. Since $\mathrm{Isom}(F;\mathscr{E}(F))$ is normal in $\mathrm{Isom}(F)$, Δ is normal in Γ. Since Γ is dense in G, its subgroup Δ is normal in G. Finally, consider the diagonal action of Γ/Δ on $(\Delta \setminus G) \times (\mathrm{Isom}(F;\mathscr{E}(F)) \setminus F)$. It is cocompact since Γ acts cocompactly on $\overline{M}$. Since by Lemma 7.1 it is properly discontinuous on the factor $\mathrm{Isom}(F;\mathscr{E}(F))\setminus F$, the other factor, $\Delta\setminus G$ must be compact. This completes the proof of Theorem 1.6. □

8. Appendix A.

Think of a Cayley graph $C(\Gamma)$ of Γ embedded in $\overline{M}$. Thus $C(\Gamma)$ is mapped by D to G. Recall that the developing map $D : \overline{M} \to G$ is a bundle map with fiber F. This implies intuitively that for a precompact connected neighbourhood U of e in G and for an appropriate choice of the generators of Γ, the inverse image $D^{-1}(U) \cap C(\Gamma)$ is connected. This is the property called compact generation in [**10**]. We will observe further that the set of the ends of $D^{-1}(U) \cap C(\Gamma)$ coincides with $\mathscr{E}(F)$.

Let $\varLambda$ be a dense subgroup of G, and U and V precompact neighbourhoods of e in G.

DEFINITION 8.1. A finite subset $\mathscr{S}$ of $\varLambda$ is called a (U,V)-*generating set* for $\varLambda$, if for any λ, $\lambda' \in \varLambda \cap U$, there exists a sequence $s_1, s_2, \cdots, s_m$ of elements of $\mathscr{S}$ such that $\lambda' = \lambda s_1 s_2 \cdots s_m$ and that $\lambda s_1 \cdots s_r \in V$ $(1 \leq r \leq m-1)$.

We shall show that the existence of such generating set does not depend on the choice of U and V. Let U' and V' be other precompact neighbourhoods.

LEMMA 8.2. *Let $\mathscr{S}$ be a (U,V)-generating set. Then we have the following.*
(1) *There exists a (U,V')-generating set.*
(2) *There exists a (U',V)-generating set.*

PROOF. (1): Since V is precompact, there exists a finite subset P of $\varLambda$ which contains the unit e and satisfies $V \subset \bigcup\{V'p^{-1} \mid p \in P\}$. Now let

$$\mathscr{T} = \{p^{-1}sp' \mid s \in \mathscr{S}, p, p' \in P\}.$$

By the assumption, we have for any λ, $\lambda' \in \varLambda \cap U$, there exists a sequence $s_1, s_2, \cdots, s_m$ of elements of $\mathscr{S}$ such that $\lambda' = \lambda s_1 s_2 \cdots s_m$ and that $\lambda s_1 \cdots s_r \in V$ $(1 \leq r \leq m-1)$. Then for any r, there exists $p_r \in P$ such that $\lambda s_1 \cdots s_r p_r \in V'$. Thus we have

$$\lambda' = \lambda s_1 p_1 \cdot p_1^{-1} s_2 p_2 \cdot \cdots \cdot p_{m-2}^{-1} s_{m-1} p_{m-1} \cdot p_{m-1}^{-1} s_m.$$

This shows that $\mathscr{T}$ is a (U,V')-generating set.

(2): There exists a finite symmetric subset Q of $\varLambda$ such that

$$U' \subset \bigcup\{(U \cap V)q \mid q \in Q\}.$$

Let $\mathscr{T} = \mathscr{S} \cup Q$. Given any λ, $\lambda' \in U'$, there exist q, $q' \in Q$ such that λq, $\lambda' q'^{-1} \in U \cap V$. Now it is clear that $\mathscr{T}$ is a (U',V)-generating set. □

DEFINITION 8.3. The subgroup $\varLambda$ is called *compactly generated* if there exists a (U,V) generating set for some (and thus for any) precompact neighbourhoods U, V of e in G.

In what follows, we shall define the set of the vertical ends of a compactly generated group. For this purpose, instead of defining an end using escaping sequences of connected open subsets, an equivalent definition based upon point sequences is preferable.

Let X be a locally finite simplicial complex. A sequence of subsets of X is called *divergent* if for any compact subset K of X, only finitely many of them meet K. A point sequence $\{x_n\}$ in X is called *admissible* if there exists a divergent sequence of arcs c_n joining x_n to x_{n+1}. Two admissible sequences $\{x_n\}$ and $\{y_n\}$ are called *equivalent* if there exists a divergent sequence of arcs c'_n joining x_n and y_n. Then an equivalence class corresponds to an end. See [**11**] for more details.

Let $\Lambda \subset G$ be a compactly generated subgroup, U a precompact open neighbourhood of e in G.

Definition 8.4. A finite subset $\mathscr{S} \subset \Lambda$ is called a *U-generating set of Λ* if for any $u,\ u' \in \mathrm{Cl}(U)$ in the same class of G/Λ, there are elements $s_1, s_2, \ldots, s_m \in \mathscr{S}$ such that $us_1 \cdots s_m = u'$ and $us_1 \cdots s_r \in U$ $(1 \le r \le m-1)$.

Proposition 8.5. *A compactly generated subgroup Λ admits a U-generating set $\mathscr{S}$ for any open precompact neighbourhood U.*

Proof. Let V be a neighbourhood of e such that $V^{-1} = V$ and $V^2 \subset U$, and let W be an open precompact set in G such that $\mathrm{Cl}(U) \subset W$. Let $\mathscr{S}$ be a (W, V)-generating set. Given two elements $u, u' = u\lambda \in \mathrm{Cl}(U)$, where $\lambda \in \Lambda$, there is an element λ_u in $V \cdot u \cap \Lambda \cap W$ such that $\lambda_{u'} = \lambda_u \lambda \in W$. Now for some $s_1, \cdots s_m \in \mathscr{S}$, $\lambda_{u'} = \lambda_u s_1 \cdots s_m$ and $\lambda_u s_1 \cdots s_r \in V$ $(1 \le r \le m-1)$. Also there is $v \in V$ such that $u = v\lambda_u$ and thus $u' = v\lambda_{u'}$. Finally since $V^2 \subset U$, we have $v\lambda_u s_1 \cdots s_r \in U$. □

Henceforth, we only consider a *symmetric* U-generating set.

For a compactly generated subgroup Λ, a precompact open neighbourhood U and a U-generating set of $\mathscr{S}$, define a graph $X = X(\Lambda, U, \mathscr{S})$ as follows. A vertex of X is an element of $\Lambda \cap U$ and an edge of X is a pair (λ, λ') of vertices such that $\lambda^{-1}\lambda' \in \mathscr{S}$.

Clearly $X(\Lambda, U, \mathscr{S})$ is a locally finite connected graph. We shall compare the set of the ends $\mathscr{E}(X(\Lambda, U, \mathscr{S}))$ for different choices of U and $\mathscr{S}$. Let us call a sequence $\lambda_1, \lambda_2, \cdots, \lambda_m$ of vertices of $X(\Lambda, U, \mathscr{S})$ an *edge path,* if for each $1 \le r \le m-1$, there exists an edge joining λ_r and λ_{r+1}. Their *length* and *endpoints* are defined in an obvious manner. We also consider infinite edge paths.

Lemma 8.6. *Let $\mathscr{T}$ be another U-generating set. Then there exists $N > 0$ such that for any edge $(\lambda, \lambda t)$ of $X(\Lambda, U, \mathscr{T})$, there exists an edge path in $X(\Lambda, U, \mathscr{S})$ of length $\le N$ joining the two endpoints of $(\lambda, \lambda t)$.*

Proof. Fix $t \in \mathscr{T}$. Choose any point $\mu \in \mathrm{Cl}(U) \cap \mathrm{Cl}(U)t^{-1}$. Since $\mathscr{S}$ is a U-generating set, there exists a sequence $s_1, s_2, \cdots, s_m$ of elements of $\mathscr{S}$ such that $\mu t = \mu s_1 s_2 \cdots s_m$ and that $\mu s_1 \cdots s_r \in U$ $(1 \le r \le m-1)$. Since U is open, there exists a neighbourhood N_μ of μ such that for any $\mu' \in N_\mu$ we have $\mu' s_1 \cdots s_r \in U$ $(1 \le r \le m-1)$.

Since $\mathrm{Cl}(U) \cap \mathrm{Cl}(U)t^{-1}$ is compact, it is covered by finitely many N_μ. This shows that the two endpoints of an edge $(\lambda, \lambda t)$ are joined by an edge path in $X(\Lambda, U, \mathscr{S})$ of bounded length. Since there are only finitely many t, the proof is complete. □

Corollary 8.7. *Suppose $\mathscr{S} \subset \mathscr{T}$. Then the inclusion map*

$$X(\Lambda, U, \mathscr{S}) \to X(\Lambda, U, \mathscr{T})$$

induces a homeomorphism between the sets of ends.

Lemma 8.8. *Let $U \subset V$ be an open neighbourhood of e in G and let $\mathscr{S}$ be a U-generating set and a V-generating set such that $e \in \mathscr{S}$ and that $V \subset \bigcup\{Us \mid s \in \mathscr{S}\}$. Then the inclusion map $\iota : X(\Lambda, U, \mathscr{S}) \to X(\Lambda, V, \mathscr{S})$ induces a homeomorphism between the sets of the ends.*

Proof. An end of $X(\Lambda, V, \mathscr{S})$ can be represented by an admissible edge path $\{\lambda_n\}_{n\geq 1}$, where $\lambda_n \in \Lambda \cap V$. By the choice of $\mathscr{S}$, each λ_n is joined by an edge in $X(\Lambda, V, \mathscr{S})$ to a vertex λ'_n of $X(\Lambda, U, \mathscr{S})$. Consider the U-generating set $\mathscr{S}^3$. ($\mathscr{S} \subset \mathscr{S}^3$ since $e \in \mathscr{S}$.) Then the sequence $\{\lambda'_n\}$ is an admissible edge path of $X(\Lambda, U, \mathscr{S}^3)$. Therefore by Lemma 8.6, $\{\lambda'_n\}$ is an admissible sequence in $X(\Lambda, U, \mathscr{S})$. This shows that the inclusion ι induces a surjection of the sets of ends. The injectivity is proved by an analogous argument. $\square$

Thus we have shown that the set of ends $\mathscr{E}(X(\Lambda, U, \mathscr{S}))$ is independent of the choice of a precompact open neighbourhood U and a U-generating set $\mathscr{S}$.

Definition 8.9. For a compactly generated subgroup Λ of G, the set of ends $\mathscr{E}(X(\Lambda, U, \mathscr{S}))$ is called the *set of vertical ends* of Λ and is denoted by $\mathscr{E}_V(\Lambda)$.

Proposition 8.10. *The holonomy group Γ of a Lie G foliation $\mathscr{F}$ is compactly generated and $\mathscr{E}(F)$ is homeomorphic to $\mathscr{E}_V(\Gamma)$.*

Proof. Choose a precompact symmetric connected open neighbourhood U of e in G. Since $D^{-1}(U)$ is mapped by p onto M, one can find a precompact open connected subset $A \subset D^{-1}(U)$ such that $D(A) = U$ and $p(A) = M$. Then $\{\gamma A \mid \gamma \in \Gamma\}$ is an open covering of the holonomy covering space $\overline{M}$. Let

$$\mathscr{S} = \{s \in \Gamma \mid s\mathrm{Cl}(A) \cap \mathrm{Cl}(A) \neq \varnothing\}.$$

Let us show that $\mathscr{S}$ is a U-generating set. For any $v \in G$ and $\gamma \in \Gamma$, we have

$$v\gamma \in U \Leftrightarrow \gamma^{-1}v^{-1} \in U \Leftrightarrow v^{-1} \in \gamma U \Leftrightarrow F_{v^{-1}} \cap \gamma A \neq \varnothing,$$

where $F_{v^{-1}} = D^{-1}(v^{-1})$. The same thing is true for $\mathrm{Cl}(U)$ and $\mathrm{Cl}(A)$.

For any elements $v, v\gamma \in \mathrm{Cl}(U)$, $F_{v^{-1}} \cap A \neq \varnothing$ and $F_{v^{-1}} \cap \gamma A \neq \varnothing$. Since $\{\gamma A \mid \gamma \in \Gamma\}$ is an open covering of $F_{v^{-1}}$, there exist $s_1, \cdots, s_m \in \mathscr{S}$ such that $\gamma = s_1 s_2 \cdots s_m$ and $F_{v^{-1}} \cap s_1 \cdots s_r A \neq \varnothing$ $(1 \leq r \leq m-1)$. This shows that $\mathscr{S}$ is a U-generating set.

To prove the latter part, let

$$A^* = \bigcup \{\gamma \mathrm{Cl}(A) \mid \gamma \in \Gamma \cap U\}.$$

Clearly there are proper inclusions $F \subset A^* \subset D^{-1}(\mathrm{Cl}(U)^2)$. The composite of the induced maps $\mathscr{E}(F) \to \mathscr{E}(A^*) \to \mathscr{E}(D^{-1}(\mathrm{Cl}(U)^2))$ is the identity.

Since $\mathrm{Cl}(A)$ is compact, any point of $\mathrm{Cl}(A)$ is joined to some point in $\mathrm{Cl}(A) \cap F$ by a path in $\mathrm{Cl}(A)$ of bounded length. The same is true for A^*. This shows that the map $\mathscr{E}(F) \to \mathscr{E}(A^*)$ is surjective. Therefore we have that $\mathscr{E}(F) \approx \mathscr{E}(A^*)$.

Now choosing a path σ_s in A^* joining the base point $a_0 \in \mathrm{Cl}(A)$ to sa_0 for $s \in \mathscr{S}$, one obtains a proper map of $X(\Gamma, U, \mathscr{S})$ into A^*. It is a routine work to show that this map induces a homeomorphism between the sets of ends. $\square$

Remark 8.11. Suppose there is a locally finite simplicial complex T on which a dense subgroup Γ of G acts in such a way that the diagonal action on $T \times G$ of Γ is properly discontinuous and cocompact. Then the same argument shows that $\mathscr{E}_V(\Gamma) = \mathscr{E}(T)$.

9. Appendix B.

Here we give a proof of Lemma 6.2. For a, $b \in S_3$ define $\epsilon(a,b) \in \{-1,0,1\}$ by $[a,b] = aba^{-1}b^{-1} = (231)^{\epsilon(a,b)}$.

Claim 1. *Given lifts λ, $\mu \in B_3$ of the permutations* (23), (12) *respectively, every pair a, $b \in S_3$ admits lifts α, $\beta \in B_3$ such that $[\alpha,\beta] = [\lambda,\mu]^{\epsilon(a,b)}$.*

Indeed, the case $\epsilon(a,b) = 0$ is immediate since a and b then lie in a same cyclic subgroup. So let $\epsilon(a,b) = \pm 1$, and in particular a or b is odd. Changing if necessary a to ab or b to ba, one can assume that both are odd. The case $\epsilon(a,b) = -1$ reduces to the case $+1$ by permuting a and b. There remain three possibilities, namely three pairs of distinct transpositions, on which the cyclic group generated by (231) acts transitively. Thus $a = (231)^{\epsilon}(23)(231)^{-\epsilon}$ and $b = (231)^{\epsilon}(12)(231)^{-\epsilon}$ for some ϵ; and we have the solution $\alpha = [\lambda,\mu]^{\epsilon}\lambda[\lambda,\mu]^{-\epsilon}$ and $\beta = [\lambda,\mu]^{\epsilon}\mu[\lambda,\mu]^{-\epsilon}$.

Claim 2. *The transposition* (23) *admits three lifts λ_1, λ_2, λ_3, and* (12) *admits three lifts μ_1, μ_2, μ_3, such that $\gamma_1 = [\lambda_1,\mu_1]$, $\gamma_2 = [\lambda_2,\mu_2]$ and $\gamma_3 = [\lambda_3,\mu_3]$ verify $\gamma_1\gamma_2\gamma_3 = e$.*

Indeed, consider σ_1 and σ_2 the standard lifts of (12) and (23) (the standard generator of B_3), recall that $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, and verify that $\lambda_1 = \sigma_2^{-1}$ and $\lambda_2 = \lambda_3 = \sigma_2$ and $\mu_1 = \mu_2 = \sigma_1^{-1}$ and $\mu_3 = \sigma_1$ work.

End of the proof of lemma 6.2. Given a_i, $b_i \in S_3$ $(i = 1,\dots,g)$ such that $[a_1,b_1]\dots[a_g,b_g] = e$ we seek for lifts α_i, $\beta_i \in B_3$ $(i = 1,\dots,g)$ such that $[\alpha_1,\beta_1]\dots[\alpha_g,\beta_g] = e$. Use an induction on g.

If $\epsilon(a_i,b_i) = 0$ for some i, then after claim 1 the permutations a_i, b_i lift to α_i, β_i such that $[\alpha_i,\beta_i] = e$ and we are reduced to the $g-1$ case. Likewise, if $\epsilon(a_{i+1},b_{i+1}) = -\epsilon(a_i,b_i)$ for some i, then after claim 1 the permutations a_i, b_i, a_{i+1}, b_{i+1} lift to α_i, β_i, α_{i+1}, β_{i+1} such that $[\alpha_i,\beta_i] = \gamma_1^{\pm 1} = [\alpha_{i+1},\beta_{i+1}]^{-1}$ and we are reduced to the $g-2$ case. In the remaining cases $\epsilon(a_i,b_i) = \epsilon$ does not depend on i; in particular g is a multiple of 3. In the case $\epsilon = +1$ (resp. -1), after claim 1 the permutations a_i, b_i $(i = 1,2,3)$ lift to α_i, β_i such that $[\alpha_i,\beta_i] = \gamma_i$ (resp. γ_{4-i}^{-1}) thus $[\alpha_1,\beta_1][\alpha_2,\beta_2][\alpha_3,\beta_3] = e$ and we are reduced to the $g-3$ case. □

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