J. Math. Soc. Japan
Vol. 57, No. 4, 2005

An analogue of Connes-Haagerup approach for classification of subfactors of type III_1

By Toshihiko MASUDA

(Received Nov. 14, 2003)
(Revised Nov. 4, 2004)

Abstract. Popa proved that strongly amenable subfactors of type III_1 with the same type II and type III principal graphs are completely classified by their standard invariants. In this paper, we present a different proof of this classification theorem based on Connes and Haagerup's arguments on the uniqueness of the injective factor of type III_1.

1. Introduction.

One of main problems in subfactor theory, initiated by V. F. R. Jones ([**18**]), is classification of subfactors, and significant contribution to this problem has been made by S. Popa from an analytic viewpoint. The main theorem in [**32**] says that strongly amenable subfactors of type II_1 possess the generating property, showing especially that such subfactors are completely classified by their standard invariants. Furthermore, in [**31**], [**33**], he considered notions of approximate innerness and central freeness for subfactors to obtain classification from a different viewpoint and indeed showed that strongly amenable subfactors with these two properties can be classified by the same invariant (see [**31**] for the type II_1 case and [**33**] for the type III case).

The most important application of the main result in [**33**] is classification of strongly amenable subfactors of type III_1. In fact, Popa proved the approximate innerness and central freeness for inclusions of approximately finite dimensional (AFD) type III_1 subfactors with the identical type II and type III principal graphs. We remark that the assumption on the graphs here is automatic for subfactors of finite depth as was shown in [**16**] for example.

The main purpose of this paper is to present an alternative proof for the abovementioned classification result for subfactors of type III_1 by a different approach (although Popa's classification of strongly amenable subfactors of type II_1 in [**32**] also plays a crucial role in our arguments). Our approach is based on [**6**] and [**11**] instead, where the uniqueness of an injective factor of type III_1 is shown. More precisely, in [**6**] A. Connes showed that an injective factor of type III_1 with the trivial bicentralizer is necessarily isomorphic to the Araki-Woods factor of type III_1. Then, in [**11**] U. Haagerup proved the triviality of the bicentralizer for every injective factor of type III_1 (and hence the desired uniqueness). Roughly speaking, this property means the existence of states with large centralizers. Also see [**41**], or [**37**, Chapter XVIII] for their theory.

2000 *Mathematics Subject Classification.* Primary 46L37; Secondary 46L40.
Key Words and Phrases. subfactors of type III_1, standard invariants, symmetric enveloping algebras, relative bicentralizer, modular automorphisms.

To be able to employ the arguments in [**6**], [**11**] in the subfactor setting, we have to begin by formulating a "relative bicentralizer" and its triviality. In [**33**] as a crucial step Popa proved the existence of states with large centralizer for a certain class of subfactors of type III_1. His method is similar to that in [**11**], and this result corresponds to the triviality of the relative bicentralizer in the current approach.

In [**6**], after reducing the uniqueness problem to the approximate innerness of modular automorphisms, Connes established the latter (under the triviality assumption of the bicentralizer). In our classification problem (for subfactors of type III_1) it is also possible to reduce the problem to the approximate innerness (in the subfactor sense) of modular automorphisms (see the last part of §3).

The key fact for the proof for the above-mentioned approximate innerness in [**6**] is the equivalence between the semi-discreteness introduced by Effros and Lance ([**8**]) and the injectivity ([**5**], see also [**38**]). Though a notion of semi-discreteness is missing in the subfactor setting, the Effros-Lance type characterization of amenability for subfactors of type II_1 was worked out by Popa. His characterization is in terms of symmetric enveloping algebra ([**35**]) and was used to study various aspects of amenability and rigidity results for subfactors. What we need here is a similar characterization in the type III_1 subfactor setting, and the so-called Longo-Rehren construction ([**26**]) as well as symmetric enveloping algebras will be used. These will be used to show the approximate innerness of modular automorphisms in the subfactor setting.

The paper is organized as follows: In §2 basic facts on classification results on subfactors and their automorphisms are collected. In §3 our classification problem (for subfactors of type III_1) is reduced to that for torus actions on subfactors of type III_λ $(0 < \lambda < 1)$. More precisely, we observe: what is really needed is the approximate innerness of modular automorphisms. In §4 we consider a relative version of the bicentralizer together with its fundamental properties. In §5 construction of symmetric enveloping algebras for type III_1 subfactors is discussed, where the relationship to the Longo-Rehren construction has to be clarified. In §6 a (type III_1) analogue of the Effros-Lance type characterization for strongly amenable subfactors is obtained in terms of symmetric enveloping algebras. In §7 the desired approximate innerness of modular automorphisms is established based on this characterization and hence the main result (Theorem 2.2) is proved.

Acknowledgements. The author would like to thank Professor Izumi, Professor Kawahigashi, Professor Kosaki and Professor Ueda for helpful suggestions on this work. He is also grateful to the editor and the referee for various comments on this paper. The author is supported by Grant-Aid for Scientific Research, Japan Society for the Promotion of Science.

2. Preliminaries and notations.

Let $\mathscr{R}_0$, $\mathscr{R}_\lambda$, $\mathscr{R}_\infty$ be the injective factors of type II_1, type III_λ $(0 < \lambda < 1)$ and type III_1 respectively. (However we remark that we never use the uniqueness of an injective factor of type III_1 in our argument.) Our standard reference for general theory of von Neumann algebras is [**37**], for subfactor theory, [**9**] and [**22**], and for sector theory, [**15**] and [**25**].

2.1. Main theorem.

First we recall Popa's classification results on strongly amenable subfactors in [**32**], [**31**] and [**33**].

Let $\mathscr{N} \subset \mathscr{M}$ be an inclusion of factors with $[\mathscr{M} : \mathscr{N}] < \infty$, $\mathscr{E}_0$ the minimal conditional expectation for $\mathscr{N} \subset \mathscr{M}$, and $\mathscr{N} \subset \mathscr{M}(=: \mathscr{M}_0) \subset \mathscr{M}_1 \subset \mathscr{M}_2 \subset \cdots$ the Jones tower for $\mathscr{N} \subset \mathscr{M}$ with the k-th Jones projection $e_k \in \mathscr{M}_k$. (Throughout this paper, we only deal with minimal conditional expectations in the sense of [**13**], and inclusions of type II factors are always extremal in the sense of [**32**] by either assumptions or constructions.) By definition, the standard invariant for $\mathscr{N} \subset \mathscr{M}$ is the lattice of its relative commutants $\{\mathscr{M}_i' \cap \mathscr{M}_j\}_{i \leq j}$.

Take a tunnel $\cdots \subset \mathscr{N}_k \subset \cdots \subset \mathscr{N}_1 := \mathscr{N} \subset \mathscr{M}$ for $\mathscr{N} \subset \mathscr{M}$. We define $\mathscr{N}^{\mathrm{st}} \subset \mathscr{M}^{\mathrm{st}} := \bigvee_k (\mathscr{N}_k' \cap \mathscr{N}) \subset \bigvee_k (\mathscr{N}_k' \cap \mathscr{M})$, and call it the model inclusion.

Popa introduced several properties related to amenability of subfactors. In this paper, we say the standard invariant of $\mathscr{N} \subset \mathscr{M}$ is amenable if its principal graph satisfies the Følner type condition in [**31**, Definition 3.1], ergodic if $\mathscr{N}^{\mathrm{st}}$ and $\mathscr{M}^{\mathrm{st}}$ are factors, and strongly amenable if it is amenable and ergodic.

Let $\mathscr{N} \subset \mathscr{M}$ be an inclusion of AFD factors of type II$_1$ with a strongly amenable standard invariant. In [**32**], Popa showed that $\mathscr{N} \subset \mathscr{M}$ has the generating property, i.e, there exists a choice of a tunnel $\cdots \subset \mathscr{N}_k \subset \cdots \subset \mathscr{N}_1 := \mathscr{N} \subset \mathscr{M}$ such that $\mathscr{N}^{\mathrm{st}} \subset \mathscr{M}^{\mathrm{st}} = \mathscr{N} \subset \mathscr{M}$. Especially, this means that strongly amenable inclusions of AFD factors of type II$_1$ can be classified by their standard invariants.

In [**31**] and [**33**], Popa gave another classification theorem as follows.

THEOREM 2.1 ([**31**, Theorem 4.1], [**33**, Theorem 5.1]). *Let $\mathscr{N} \subset \mathscr{M}$ be an approximately inner, centrally free, strongly amenable subfactor with $\mathscr{N} \cong \mathscr{N} \otimes \mathscr{R}_0$. Then $\mathscr{N} \subset \mathscr{M}$ is isomorphic to $(\mathscr{N}^{\mathrm{st}} \subset \mathscr{M}^{\mathrm{st}}) \otimes \mathscr{N}$.*

When $\mathscr{N}$ and $\mathscr{M}$ are AFD factors of type II$_1$, the above theorem gives an alternative proof of the main theorem in [**32**]. (In fact, it is easily shown that $\mathscr{N}^{\mathrm{st}} \subset \mathscr{M}^{\mathrm{st}}$ has the generating property (see [**32**, Remark 1.4.4]), and $\mathscr{N}^{\mathrm{st}} \subset \mathscr{M}^{\mathrm{st}} \cong \mathscr{N}^{\mathrm{st}} \otimes \mathscr{R}_0 \subset \mathscr{M}^{\mathrm{st}} \otimes \mathscr{R}_0$ by the relative McDuff type theorem [**2**].) In the rest of this paper, if we say a strongly amenable subfactor, we always assume involved factors are injective (hence AFD).

If $\mathscr{N} \subset \mathscr{M}$ is an inclusion of AFD type III$_1$ factors with identical type II and type III principal graphs, then $\mathscr{N} \subset \mathscr{M}$ is shown to be approximately inner and centrally free in [**33**]. These facts imply

THEOREM 2.2. *Let $\mathscr{N} \subset \mathscr{M}$ be a strongly amenable subfactor of type III$_1$ with the identical type II and type III principal graphs. Then $\mathscr{N} \subset \mathscr{M}$ is isomorphic to $\mathscr{N}^{\mathrm{st}} \otimes \mathscr{R}_\infty \subset \mathscr{M}^{\mathrm{st}} \otimes \mathscr{R}_\infty$.*

As explained in Introduction, the aim of this paper is to show Theorem 2.2 based on the arguments in [**6**] and [**11**] instead.

2.2. Automorphism groups of subfactors.

Let $\mathrm{Aut}(\mathscr{M}, \mathscr{N})$ be the set of automorphisms of $\mathscr{M}$ preserving the inclusion globally. Every $\alpha \in \mathrm{Aut}(\mathscr{M}, \mathscr{N})$ can be extended to $\mathrm{Aut}(\mathscr{M}_k)$ canonically by setting $\alpha(e_k) = e_k$. The inner automorphism group of $\mathscr{N} \subset \mathscr{M}$ is defined by $\mathrm{Int}(\mathscr{M}, \mathscr{N}) := \{\mathrm{Ad}\, u | u \in U(\mathscr{N})\}$. It is easy to see that the extension of $\mathrm{Ad}\, u \in \mathrm{Int}(\mathscr{M}, \mathscr{N})$ to $\mathscr{M}_k$ is also given by the same unitary $\mathrm{Ad}\, u$. We denote by $\widetilde{\mathscr{N}} \subset \widetilde{\mathscr{M}} := \mathscr{N} \rtimes_{\sigma^\varphi} \boldsymbol{R} \subset \mathscr{M} \rtimes_{\sigma^{\varphi\circ\mathscr{E}_0}} \boldsymbol{R}$ the common continuous crossed product of a subfactor $\mathscr{N} \subset \mathscr{M}$ of type III. For $\alpha \in \mathrm{Aut}(\mathscr{M})$, we denote by $\tilde{\alpha}$ the canonical extension of α in the sense of [**12**].

Definition 2.3 ([**24**, Section 5]). Define $\Phi(\alpha) := \{\alpha|_{\mathscr{M}'\cap\mathscr{M}_k}\}_{k=0}^\infty$. We call $\Phi(\alpha)$ the Loi invariant for α.

Definition 2.4 ([**3**, Definition 1], [**30**, Definition 1.5.1]). For $\alpha \in \mathrm{Aut}(\mathscr{M}, \mathscr{N})$, we say α is strongly outer if α satisfies the following property; if $a \in \bigcup_k \mathscr{M}_k$ satisfies $\alpha(x)a = ax$ for every $x \in \mathscr{M}$, then we have $a = 0$.

The strong outerness of automorphisms can be characterized by the language of sector theory as follows.

Theorem 2.5 ([**3**, Theorem 2], [**20**, Theorem 3]). *Let $\mathscr{N} \subset \mathscr{M}$ be an inclusion of factors of type III, and γ Longo's canonical endomorphism. Then α is non-strongly outer if and only if α appears as an irreducible component of γ^k for some $k \geq 1$.*

Definition 2.6 ([**39**, Definition 3.1]). Let $\mathscr{N} \subset \mathscr{M}$ be a subfactor of type III. For $\alpha \in \mathrm{Aut}(\mathscr{M}, \mathscr{N})$, we say α is strongly free if α satisfies the following property; if $a \in \bigcup_k \widetilde{\mathscr{M}_k}$ satisfies $\tilde{\alpha}(x)a = ax$ for every $x \in \widetilde{\mathscr{M}}$, then we have $a = 0$.

We denote by $\mathrm{Cnt_o}(\mathscr{M}, \mathscr{N})$ (resp. $\mathrm{Cnt_f}(\mathscr{M}, \mathscr{N})$) be the set of all non-strongly outer (resp. non-strongly free) automorphisms. Both sets are normal subgroups in $\mathrm{Aut}(\mathscr{M}, \mathscr{N})$.

When $\mathscr{N} = \mathscr{M}$, we have $\mathrm{Cnt_o}(\mathscr{M}, \mathscr{M}) = \mathrm{Int}(\mathscr{M})$ and $\mathrm{Cnt_f}(\mathscr{M}, \mathscr{M}) = \mathrm{Cnt_r}(\mathscr{M})$. It is well-known that every element in $\mathrm{Cnt_r}(\mathscr{M})$ is expressed as the composition of an inner automorphism and an (extended) modular automorphism. A subfactor analogue of this fact has been obtained by Kosaki as the following theorem.

Theorem 2.7 ([**21**, Theorem 19]). Let $\mathscr{N} \subset \mathscr{M}$ be a subfactor of type III_λ, $\lambda \neq 0$, and α a non-strongly free automorphism for $\mathscr{N} \subset \mathscr{M}$. Then $\alpha = \beta \circ \sigma_t^\varphi$ for some non-strongly outer automorphism β.

We recall several important definitions. The approximately inner automorphism group $\overline{\mathrm{Int}}(\mathscr{M}, \mathscr{N})$ is the closure of $\mathrm{Int}(\mathscr{M}, \mathscr{N})$ in $\mathrm{Aut}(\mathscr{M})$ equipped with the usual u-topology. Define $C(\mathscr{M}, \mathscr{N}) := \{\{x_n\} \in \ell^2(\boldsymbol{N}, \mathscr{N}) |\lim_{n\to\infty} \|[\psi, x_n]\| = 0$ for every $\psi \in \mathscr{M}_*\}$. By definition, an automorphism $\alpha \in \mathrm{Aut}(\mathscr{M}, \mathscr{N})$ is centrally trivial if and only if $\{\alpha(x_n) - x_n\}$ converges to 0 σ-strongly* for any $\{x_n\} \in C(\mathscr{M}, \mathscr{N})$. We denote by $\mathrm{Cnt}(\mathscr{M}, \mathscr{N})$ the set of all centrally trivial automorphisms. Note that for a II_1 subfactor $\mathscr{N} \subset \mathscr{M}$, $\{x_n\}$ is in $C(\mathscr{M}, \mathscr{N})$ if and only if $\{\|[x_n, a]\|_2\}$ converges to 0 for every $a \in \mathscr{M}$.

2.3. Automorphism groups of strongly amenable subfactors of type II.

In the study of automorphism groups of operator algebras, the most important classes of automorphisms are approximately inner ones and centrally trivial ones. For strongly amenable subfactors of type II, these two classes are characterized in the terms of the Loi invariant and non-strong outerness.

Theorem 2.8 ([**24**, Theorem 5.4]). *Let $\mathscr{N} \subset \mathscr{M}$ be a strongly amenable subfactor of type II_1. Then $\mathrm{Ker}\, \Phi = \overline{\mathrm{Int}}(\mathscr{M}, \mathscr{N})$.*

Corollary 2.9 ([**40**, Theorem 4.6]). *Let $\mathscr{N} \subset \mathscr{M}$ be a strongly amenable subfactor of type II_∞. Then* $\mathrm{Ker}\,\Phi \cap \mathrm{Ker}\,\mathrm{mod} = \overline{\mathrm{Int}}(\mathscr{M}, \mathscr{N})$.

Theorem 2.10 ([**30**, Theorem 1.6]). *Let $\mathscr{N} \subset \mathscr{M}$ be a subfactor of type II_1. Then we have* $\mathrm{Cnt}_{\mathrm{o}}(\mathscr{M}, \mathscr{N}) \subset \mathrm{Cnt}(\mathscr{M}, \mathscr{N})$. *Moreover if $\mathscr{N} \subset \mathscr{M}$ is strongly amenable, then we also have* $\mathrm{Cnt}_{\mathrm{o}}(\mathscr{M}, \mathscr{N}) = \mathrm{Cnt}(\mathscr{M}, \mathscr{N})$.

Here we give a proof of Theorem 2.10 for reader's convenience, which is different from one in [**30**]. First we prepare

Lemma 2.11. *Let $\mathscr{M} \supset \mathscr{N} \supset \cdots \supset \mathscr{N}_k \supset \cdots$ be a tunnel, and $\alpha \in \mathrm{Aut}(\mathscr{M}, \mathscr{N})$. If $\sup\{\|\alpha(u) - u\|_2 | u \in U(\mathscr{N}_k)\} < 1$ for some k, then α is non-strongly outer.*

Proof. Set $K := \overline{\mathrm{conv}}\{\alpha(u)u^* | u \in U(\mathscr{N}_k)\}$, where the closure is taken in the σ-weak topology. Then K is a σ-weakly compact set. By the lower semicontinuity of $\|\cdot\|_2$ in the σ-weak topology, there exists a unique $a \in K$ such that $\|a\|_2 := \min\{\|b\|_2 | b \in K\}$. Moreover by the assumption, $0 \notin K$, and hence $a \neq 0$. Since $\|\alpha(v)av^*\|_2 = \|a\|_2$, $\alpha(v)a = av$ holds for every $v \in U(\mathscr{N}_k)$ by the uniqueness of a. This implies that α is non-strongly outer. □

Proof of Theorem 2.10. First assume that $\sigma \in \mathrm{Aut}(\mathscr{M}, \mathscr{N})$ is non-strongly outer. Then we can find a non-zero $a \in \mathscr{M}_k$ such that $\sigma(x)a = ax$ holds for every $x \in \mathscr{M}$. Take $\{x_n\} \in \ell^\infty(\boldsymbol{N}, \mathscr{N})$ which is central in $\mathscr{M}$. Then $\|[x_n, a]\|_2 \to 0$ as n goes to infinity. Then $\{\sigma(x_n)a - x_n a\} = \{ax_n - x_n a\}$ converges to 0 strongly, and so does $\{\sigma(x_n)aa^* - x_n aa^*\}$. Since $aa^* \in \mathscr{M}' \cap \mathscr{M}_k$, $E_{\mathscr{M}}(aa^*)$ is a non-zero scalar, where $E_{\mathscr{M}}$ is the minimal conditional expectation. Hence we get $\lim_{n\to\infty} \sigma(x_n) - x_n = 0$ strongly.

Next we assume that $\mathscr{N} \subset \mathscr{M}$ is strongly amenable. We fix a tunnel $\cdots \subset \mathscr{N}_k \subset \cdots \subset \mathscr{N}_1 := \mathscr{N} \subset \mathscr{M}$ with the generating property.

Let σ be a strongly outer automorphism. For every k, we can choose $u_k \in U(\mathscr{N}_k)$ such that $\|\sigma(u_k) - u_k\|_2 \geq 1/2$ by Lemma 2.11. By the generating property, $\{u_k\}$ is central in $\mathscr{M}$. Then $\liminf \|\sigma(u_k) - u_k\|_2 \geq 1/2$, showing that σ is not centrally trivial. □

Corollary 2.12 ([**39**, Theorem 3.4]). *Let $\mathscr{N} \subset \mathscr{M}$ be a strongly amenable subfactor of type II_∞. Then* $\mathrm{Cnt}_{\mathrm{o}}(\mathscr{M}, \mathscr{N}) = \mathrm{Cnt}(\mathscr{M}, \mathscr{N})$.

2.4. Inclusions of factors of type III_λ, $\lambda \neq 0$.

Let $\mathscr{Q} \subset \mathscr{P}$ be an inclusion of factors of type III_λ, $\lambda \neq 0$. When $0 < \lambda < 1$, we assume that $\mathscr{Q} \subset \mathscr{P}$ has the common discrete decomposition, i.e., there exists an inclusion of factors $B \subset A$ of type II_∞, and an automorphism $\theta \in \mathrm{Aut}(A, B)$ with $\mathrm{mod}\,(\theta) = \lambda$ such that $\mathscr{Q} \subset \mathscr{P}$ is isomorphic to $B \rtimes_\theta \boldsymbol{Z} \subset A \rtimes_\theta \boldsymbol{Z}$. We refer to $\mathscr{G}_{\mathscr{P},\mathscr{Q}}$ (resp. $\mathscr{G}_{B,A}$) as the type III standard invariant (resp. type II standard invariant) for $\mathscr{Q} \subset \mathscr{P}$. We also use the notation $\widetilde{\mathscr{G}}_{\mathscr{P},\mathscr{Q}}$ to denote the type II standard invariant for $\mathscr{Q} \subset \mathscr{P}$. For subfactors of type III_1, we use similar notations by considering the common continuous decomposition.

We need to clarify the relationship between $\mathscr{G}_{\mathscr{P},\mathscr{Q}}$ and $\widetilde{\mathscr{G}}_{\mathscr{P},\mathscr{Q}}$.

Proposition 2.13. *We have $\mathscr{Q}' \cap \mathscr{P}_k = (B' \cap A_k)^\theta$ for every $k \geq 0$. Hence $\mathscr{G}_{\mathscr{P},\mathscr{Q}} = \mathscr{G}_{A,B}$ holds if and only if θ acts trivially on $B' \cap A_k$ for every $k \geq 0$.*

See [**24**, Proposition 3.1] for $0<\lambda<1$ case and [**23**, Corollary 6] for $\lambda=1$ case.

Remark. The type II standard invariant and the type III one coincide if and only if $\mathscr{Q}\subset\mathscr{P}$ has the same type II principal graph and type III principal graph.

The coincidence of the type II principal graph and type III one is characterized by Longo's canonical endomorphism γ for $\mathscr{Q}\subset\mathscr{P}$.

Theorem 2.14 ([**16**, Theorem 3.5]). *The type II principal graph and type III principal graph of $\mathscr{P}\subset\mathscr{P}_1$ coincide if and only if σ_t^φ never appears as an irreducible component of γ^k for any $k\geq 1$ and $t\notin T(\mathscr{P})$.*

By combining the above theorem with Theorem 2.5, we get

Corollary 2.15. *The type II principal graph and type III principal graph of $\mathscr{P}\subset\mathscr{P}_1$ coincide if and only if σ_t^φ is strongly outer for every $t\notin T(\mathscr{P})$.*

By Corollary 2.9 and Corollary 2.12, we get the following classification result for strongly amenable subfactors of type III$_\lambda$, $0<\lambda<1$.

Theorem 2.16 ([**24**, Theorem 6.1]). Let $\mathscr{Q}\subset\mathscr{P}$ be a strongly amenable subfactor of type III$_\lambda$, $0<\lambda<1$, with $\mathscr{G}_{\mathscr{P},\mathscr{Q}}=\widetilde{\mathscr{G}}_{\mathscr{P},\mathscr{Q}}$. Then $\mathscr{Q}\subset\mathscr{P}$ is isomorphic to $\mathscr{Q}^{\mathrm{st}}\otimes\mathscr{R}_\lambda\subset\mathscr{P}^{\mathrm{st}}\otimes\mathscr{R}_\lambda$.

We have the following characterization of $\overline{\mathrm{Int}}(\mathscr{P},\mathscr{Q})$ and $\mathrm{Cnt}(\mathscr{P},\mathscr{Q})$.

Theorem 2.17 ([**39**, Theorem 3.8], [**40**, Theorem 4.6]). *Let $\mathscr{Q}\subset\mathscr{P}$ be as in Theorem 2.16. Then we have $\overline{\mathrm{Int}}(\mathscr{P},\mathscr{Q})=\mathrm{Ker}\,\Phi\cap\mathrm{Ker}\,\mathrm{mod}$ and $\mathrm{Cnt}(\mathscr{P},\mathscr{Q})=\mathrm{Cnt}_{\mathrm{f}}(\mathscr{P},\mathscr{Q})$.*

2.5. Longo-Rehren construction.

The Longo-Rehren construction was introduced originally in [**26**], but we need the crossed product type approach worked out in [**27**].

Here we collect basic definitions. Define $\mathrm{Sect}\,(\mathscr{M}):=\mathrm{End}(\mathscr{M})/\sim$, where $\sim$ is the usual unitary equivalence. We denote by $[\rho]$ the equivalence class of ρ. The statistical dimension $d(\rho)$ is defined by $\sqrt{[\mathscr{M}:\rho(\mathscr{M})]}$. Then $d(\rho)$ is additive and multiplicative. For $\rho,\sigma\in\mathrm{End}(\mathscr{M})$, the intertwiner space (ρ,σ) is defined by $\{a\in\mathscr{M}|\sigma(x)a=a\rho(x),x\in\mathscr{M}\}$. If ρ is irreducible, i.e., $\rho(\mathscr{M})'\cap\mathscr{M}=\boldsymbol{C}1$, then (ρ,σ) has an inner product by $\langle v,w\rangle=w^*v1(\in\boldsymbol{C}1)$. Let $[\bar\rho]$ be the conjugate sector of $[\rho]$. Then there exist two isometries $v\in(\mathrm{id},\rho\bar\rho)$ and $\bar v\in(\mathrm{id},\bar\rho\rho)$ such that $v^*\rho(\bar v)=1/d(\rho)$ and $\bar v^*\bar\rho(v)=1/d(\rho)$. The standard left inverse ϕ_ρ of ρ is given by $\phi_\rho(x)=\bar v^*\bar\rho(x)\bar v$.

Let $\Delta=\{[\rho_i]\}_{i\in I}$ be a set of irreducible sectors of $\mathscr{M}$ closed under conjugation and irreducible decomposition of multiplication. We assume that I is at most countable. The index set I is a fusion algebra in the sense of [**14**, Definition 1.1]. Let $j_{\mathscr{M}}$ be the canonical conjugate linear isomorphism from $\mathscr{M}$ onto $\mathscr{M}^{\mathrm{opp}}$. Put $\hat\rho:=\rho\otimes j_{\mathscr{M}}\circ\rho\circ j_{\mathscr{M}}^{-1}\in\mathrm{End}(\mathscr{M}\otimes\mathscr{M}^{\mathrm{opp}})$.

Set $N_{i,j}^k=\dim(\rho_k,\rho_i\rho_j)$, and $d(i):=d(\rho_i)$. Let $\{v\binom{k}{i,j}^e\}_{e=1}^{N_{i,j}^k}\in(\rho_k,\rho_i\rho_j)$ be an orthonormal basis, and define the canonical intertwiner $\hat v_{i,j}^k\in(\hat\rho_k,\hat\rho_i\hat\rho_j)$ as follows.

$$\hat{v}^k_{i,j} := \sqrt{\frac{d(i)d(j)}{d(k)}} \sum_{e=1}^{N^k_{i,j}} v\binom{k}{i,j}^e \otimes j_{\mathscr{M}}\left(v\binom{k}{i,j}^e\right).$$

The canonical intertwiner $\hat{v}^k_{i,j}$ is independent from the choice of an orthonormal basis.

LEMMA 2.18. *The canonical intertwiners satisfy the following relations.*
(1) $\sum_m \hat{v}^m_{i,j}\hat{v}^l_{m,k} = \sum_m \hat{\rho}_i(\hat{v}^m_{j,k})\hat{v}^l_{i,m}$ $(\in (\hat{\rho}_l, \hat{\rho}_i\hat{\rho}_j\hat{\rho}_k))$.
(2) $\sum_m \hat{v}^m_{i,j}\hat{v}^{m*}_{k,l} = \sum_m \hat{\rho}_i(\hat{v}^{j*}_{m,l})\hat{v}^k_{i,m}$ $(\in (\hat{\rho}_k\hat{\rho}_l, \hat{\rho}_i\hat{\rho}_j))$.
(3) $\hat{v}^{k*}_{i,j}\hat{v}^l_{i,j} = \delta_{k,l}\frac{d(i)d(j)}{d(k)}N^k_{i,j}$.

See [**26**] for proof.

COROLLARY 2.19. *We have* $\hat{v}^{i*}_{k,l} = \hat{\rho}_i(\hat{v}^{0*}_{\bar{l},l})\hat{v}^k_{i,\bar{l}}$ *and* $\hat{v}^{l*}_{i,j} = \hat{v}^{0*}_{i,\bar{i}}\hat{\rho}_i(\hat{v}^j_{\bar{i},l})$.

PROOF. If we put $j = 0$ in Lemma 2.18(2), we get the first equation. In a similar way, we get the second one by putting $k = 0$. □

In what follows, we use Lemma 2.18 and Corollary 2.19 frequently.

Set $A := \mathscr{M} \otimes \mathscr{M}^{\mathrm{opp}}$, $H_j := L^2(A)$, $j \in I$, and define an action π of A and an operator V_i on $H := \bigoplus_{j\in I} H_j$ as follows.

$$(\pi(x)\xi)(i) : = \hat{\rho}_i(x)\xi(i),\ x \in A,$$

$$(V_i\xi)(j) : = \sum_k \hat{v}^k_{j,i}\xi(k).$$

By using Lemma 2.18 and Corollary 2.19, we get the following relations.

LEMMA 2.20. *We have the following relations.*
(1) $V_iV_j = \sum_k \pi(\hat{v}^k_{i,j})V_k$.
(2) $V_i\pi(a) = \pi(\hat{\rho}_i(a))V_i$, $a \in A$.
(3) $V_i^* = \pi(\hat{v}^{0*}_{\bar{i},i})V_{\bar{i}}$.

DEFINITION 2.21. Let $A(\Delta)$ be the von Neumann algebra generated by $\pi(A)$ and $\{V_i\}_{i\in I}$. We call $\pi(A) \subset A(\Delta)$ the Longo-Rehren inclusion associated with Δ.

In fact, $A(\Delta)$ is a factor. In what follows, we identify $\pi(A)$ with A, and often omit π. When Δ is arising from the irreducible decomposition of γ^n, $n \geq 1$, where γ is Longo's canonical endomorphism for a subfactor $\mathscr{N} \subset \mathscr{M}$, we say $A \subset A(\Delta)$ as the Longo-Rehren inclusion for $\mathscr{N} \subset \mathscr{M}$. By Lemma 2.20, $\{\sum \pi(a_i)V_i | a_i \in A,\ a_i = 0$ except for finitely many $i\}$ is a dense $*$-subalgebra in $A(\Delta)$.

Let P be the projection from H onto H_0, and define $\mathscr{E}_A(x) := PxP^*$ for $x \in A(\Delta)$. Then $\mathscr{E}_A$ is a conditional expectation from $A(\Delta)$ onto A as in the case of usual crossed product construction. Set $a_i := \mathscr{E}_A(aV_i^*)$, $a \in A(\Delta)$. Then we have the formal expansion formula $a = \sum_{i\in I} \pi(a_i)V_i$.

Next we discuss on an operator valued weight from $A \otimes B(\ell^2(I))$ to $A(\Delta)$. (Note $A(\Delta)$ is a subalgebra of $A \otimes B(\ell^2(I))$.) To do so, it is convenient to express $\pi(a)V_i$ in the "matrix form". Let $\{e_{ij}\}_{i,j\in I}$ be the matrix units for $B(\ell^2(I))$. Then it is easy to see $\pi(a)=\sum_{i\in I}\hat{\rho}_i(a)\otimes e_{ii}$, and $V_i=\sum_{j,k\in I}\hat{v}^k_{j,i}\otimes e_{j,k}$ by the definition of $\pi(a)$ and V_i. Then we have the expression $\pi(a)V_i=\sum_{j,k}\hat{\rho}_j(a)\hat{v}^k_{j,i}\otimes e_{jk}$.

Let $\mathfrak{M}$ be a $*$-subalgebra of $A\otimes B(\ell^2(I))$ generated by $\{x\otimes e_{ij}|x\in A\}$. Set $\bar{V}_i:=\sum_{j,k}\hat{v}^{k*}_{i,j}\otimes e_{k,j}$, and define $\mathscr{F}_{A(\Delta)}$ as follows.

$$\mathscr{F}_{A(\Delta)}(x):=\sum_i \bar{V}_i\hat{\rho}_i\otimes \mathrm{id}(x)\bar{V}_i^*, x\in\mathfrak{M}.$$

Then $\mathscr{F}_{A(\Delta)}$ is a normal operator valued weight from $A\otimes B(\ell^2(I))$ to $A(\Delta)$ with the domain $\mathfrak{M}$. For example, we have the following for $y\otimes e_{ij}\in\mathfrak{M}$.

$$\begin{aligned}
\mathscr{F}_{A(\Delta)}(y\otimes e_{ij}) &= \sum_k \bar{V}_k(\hat{\rho}_k(y)\otimes e_{ij})\bar{V}_k^*\\
&= \sum_{k,l,m} v^{l*}_{k,i}\hat{\rho}_k(y)v^m_{k,j}\otimes e_{lm}\\
&= \sum_{k,l,m} \rho_l(\hat{v}^{0*}_{\bar{l},i})v^k_{l,\bar{l}}\hat{\rho}_k(y)v^m_{k,j}\otimes e_{lm}\\
&= \sum_{l,m} \rho_l(\hat{v}^{0*}_{\bar{l},i})\rho_l\hat{\rho}_{\bar{l}}(y)\left(\sum_k v^k_{l,\bar{l}}v^m_{k,j}\right)\otimes e_{lm}\\
&= \sum_{l,m} \rho_l(\hat{v}^{0*}_{\bar{l},i})\rho_l\hat{\rho}_{\bar{l}}(y)\left(\sum_k \rho_l(\hat{v}^k_{\bar{l},j})\hat{v}^m_{lk}\right)\otimes e_{lm}\\
&= \sum_{k,l,m} \rho_l(\hat{v}^{0*}_{\bar{l},i}\hat{\rho}_{\bar{l}}(y)\hat{v}^k_{\bar{l},j})\hat{v}^m_{lk}\otimes e_{lm}\\
&= \sum_k \pi(\hat{v}^{0*}_{\bar{l},i}\hat{\rho}_{\bar{l}}(y)\hat{v}^k_{\bar{l},j})V_k.
\end{aligned}$$

It is not difficult to show $\mathscr{F}_{A(\Delta)}(a(y\otimes e_{ij})b)=a\mathscr{F}_{A(\Delta)}(y\otimes e_{ij})b$ for $a,b\in A(\Delta)$. The special case $y=1$, $i=j$ implies $\mathscr{F}_{A(\Delta)}(1\otimes e_{ii})=d(i)^2$. In fact $A\otimes B(\ell^2(I))$ is the basic construction for $\pi(A)\subset A(\Delta)$, and $\mathscr{F}_{A(\Delta)}$ is the dual operator valued weight for $\mathscr{E}_A$.

We will construct a common Jones projection $e_{\mathscr{N}}$ for $\mathscr{N}\subset\mathscr{M}$ and $\mathscr{N}^{\mathrm{opp}}\subset\mathscr{M}^{\mathrm{opp}}$ inside of $A(\Delta)$. Let ι be the inclusion map $\mathscr{N}\hookrightarrow\mathscr{M}$. Let $I_0:=\{i\in I|\rho_i\prec\gamma\}$, $\{a^e_i\}\subset(\iota,\rho_i\iota)$ an orthonormal bases, and $\tilde{a}_i:=\sum_e a^e_i\otimes j_{\mathscr{M}}(a^e_i)$. Then $e_{\mathscr{N}}$ is expressed as $e_{\mathscr{N}}=[\mathscr{M}:\mathscr{N}]^{-1}\sum_{i\in I_0}\sqrt{d(i)}\tilde{a}_i^*V_i$. See [**27**], or Appendix A.

Conversely, let $\{w^e_i\}\subset(\rho_i,\gamma)$ be an orthonormal basis, and set $\tilde{w}_i:=\sum_e w^e_i\otimes j_{\mathscr{M}}(w^e_i)$. Then we have $V_i=C\tilde{w}_i{}^*e_{\mathscr{N}}\tilde{w}_0$ for some $0\neq C\in\boldsymbol{C}$. These relations show $\mathrm{Alg}(\mathscr{M},\mathscr{M}^{\mathrm{opp}},\{V_i\})=\mathrm{Alg}(\mathscr{M},\mathscr{M}^{\mathrm{opp}},e_{\mathscr{N}})$.

Next we recall the extension of automorphisms of a subfactor to the Longo-Rehren inclusion discussed in [**28**]. Let $\alpha\in\operatorname{Ker}\Phi$. Then we can find a half braiding $u(\alpha,\rho)\in U(\mathscr{M})$ in the sense of [**17**]. Namely $u(\alpha,\rho)$ satisfies the following.

(i) $\alpha\circ\rho\circ\alpha^{-1}=\operatorname{Ad}u(\alpha,\rho)\rho$,

(ii) $u(\alpha\beta, \rho) = \alpha(u(\beta, \rho))u(\alpha, \rho)$,
(iii) For $v \in (\rho_3, \rho_1\rho_2)$, $u(\alpha, \rho_1)\rho_1(u(\alpha, \rho_2))v = \alpha(v)u(\alpha, \rho_3)$.

We briefly sketch how to find $u(\alpha, \rho)$. Fix isometries $R \in (\mathrm{id}_{\mathscr{M}}, \gamma)$ and $\bar{R} \in (\mathrm{id}_{\mathscr{N}}, \gamma|_{\mathscr{N}})$ such that $R^*\bar{R} = [\mathscr{M}, \mathscr{N}]^{-\frac{1}{2}}$ and $\bar{R}^*\gamma(R) = [\mathscr{M}, \mathscr{N}]^{-\frac{1}{2}}$. Next we fix $u \in U(\mathscr{N})$ such that $\alpha \circ \gamma \circ \alpha^{-1} = \mathrm{Ad}\, u\gamma$ and $u^*\alpha(R) = R$. Then $u^*\alpha(\bar{R}) = \bar{R}$ holds automatically. Set $u_k := u\gamma(u)\cdots\gamma^{k-1}(u)$. Then $\alpha \circ \gamma^k \circ \alpha^{-1} = \mathrm{Ad}\, u_k\gamma$ holds.

Take an isometry $v \in (\rho, \gamma^k)$, and define $u(\alpha, \rho)$ by $u(\alpha, \rho) = \alpha(v^*)u_k v$. Then $\alpha \in \mathrm{Ker}\, \Phi$ assures that $u(\alpha, \rho)$ is well-defined. We can verify $u(\alpha, \rho)$ satisfy the above conditions (i), (ii) and (iii) in a similar way as in [**28**].

We can extend $\alpha \otimes \mathrm{id} \in \mathrm{Aut}(\mathscr{M} \otimes \mathscr{M}^{\mathrm{opp}})$ to $\alpha \boxtimes \mathrm{id} \in \mathrm{Aut}(A(\varDelta))$ by setting

$$\alpha \boxtimes \mathrm{id}(V_i) = (u(\alpha, \rho_i) \otimes 1)V_i.$$

In fact, let $U_\alpha \in B(L^2(\mathscr{M}))$ be the standard implementing unitary for α, and define $V \in U(H)$ by

$$(V\xi)(i) := (u(\alpha, \rho_i)^* U_\alpha \otimes 1)\xi(i).$$

Then $\mathrm{Ad}\, V$ gives an automorphism of $A(\varDelta)$, and satisfies the above property.

In a similar way, we can extend $\alpha \otimes \alpha^{\mathrm{opp}} \in \mathrm{Aut}(\mathscr{M} \otimes \mathscr{M}^{\mathrm{opp}})$ to $\alpha \boxtimes \alpha^{\mathrm{opp}} \in \mathrm{Aut}(A(\varDelta))$ by

$$\alpha \boxtimes \alpha^{\mathrm{opp}}(V_i) = (u(\alpha, \rho_i) \otimes j_{\mathscr{M}}(u(\alpha, \rho_i)))V_i.$$

Both extensions fix the common Jones projection $e_{\mathscr{N}}$.

We mainly apply the above extension to the modular automorphism σ_t^φ. In this case, $u(\sigma_t^\varphi, \rho)$ is given by $d\rho^{-it}(D\varphi : D\varphi \circ \phi_\rho)_t$.

3. Torus actions on III$_\lambda$ subfactors.

In this section, we will discuss how to reduce our classification theorem to that of torus actions on strongly amenable subfactors of type III$_\lambda$.

PROPOSITION 3.1. *Let $\mathscr{N} \subset \mathscr{M}$ be a subfactor of type III$_1$ with $\mathscr{G}_{\mathscr{M},\mathscr{N}} = \widetilde{\mathscr{G}}_{\mathscr{M},\mathscr{N}}$. Fix $T > 0$ and set $\theta := \sigma_T^\varphi$. Then $\mathscr{Q} \subset \mathscr{P} := \mathscr{N} \rtimes_\theta \boldsymbol{Z} \subset \mathscr{M} \rtimes_\theta \boldsymbol{Z}$ is a subfactor of type III$_\lambda$, $T = -\frac{2\pi}{\log\lambda}$, with $\mathscr{G}_{\mathscr{M},\mathscr{N}} = \mathscr{G}_{\mathscr{P},\mathscr{Q}} = \widetilde{\mathscr{G}}_{\mathscr{P},\mathscr{Q}}$.*

PROOF. By [**6**, Lemma 1.1], $\mathscr{P}$ and $\mathscr{Q}$ are type III$_\lambda$ factors. Since $\mathscr{G}_{\mathscr{M},\mathscr{N}} = \widetilde{\mathscr{G}}_{\mathscr{M},\mathscr{N}}$, σ_T^φ is strongly outer for any non-zero $T \in \boldsymbol{R}$ by Corollary 2.15. We also have $\sigma_T^\varphi \in \mathrm{Ker}\, \Phi$. Thus $\mathscr{G}_{\mathscr{M},\mathscr{N}} = \mathscr{G}_{\mathscr{P},\mathscr{Q}}$ holds. Next we investigate $\widetilde{\mathscr{G}}_{\mathscr{P},\mathscr{Q}}$. By standard argument, we can identify $\widetilde{\mathscr{Q}} \subset \widetilde{\mathscr{P}}$ with $\widetilde{\mathscr{N}} \rtimes_{\widetilde{\theta}} \boldsymbol{Z} \subset \widetilde{\mathscr{M}} \rtimes_{\widetilde{\theta}} \boldsymbol{Z}$. In this case, $\widetilde{\theta}$ is inner. Hence

$$\widetilde{\mathscr{Q}} \subset \widetilde{\mathscr{P}} \cong \widetilde{\mathscr{N}} \rtimes_{\widetilde{\theta}} \boldsymbol{Z} \subset \widetilde{\mathscr{M}} \rtimes_{\widetilde{\theta}} \boldsymbol{Z} \cong \widetilde{\mathscr{N}} \otimes L^\infty(\boldsymbol{T}) \subset \widetilde{\mathscr{M}} \otimes L^\infty(\boldsymbol{T}),$$

showing $\mathscr{G}_{\mathscr{M},\mathscr{N}} = \mathscr{G}_{\mathscr{P},\mathscr{Q}} = \widetilde{\mathscr{G}}_{\mathscr{P},\mathscr{Q}}$. □

Proposition 3.2. (i) *Let $\mathscr{Q} \subset \mathscr{P}$ be a strongly amenable subfactor of type III$_\lambda$ with $\mathscr{G}_{\mathscr{P},\mathscr{Q}} = \widetilde{\mathscr{G}}_{\mathscr{P},\mathscr{Q}}$, and $\theta \in \mathrm{Aut}(\mathscr{P},\mathscr{Q})$ an automorphism such that $\mathscr{Q} \rtimes_\theta \boldsymbol{Z} \subset \mathscr{P} \rtimes_\theta \boldsymbol{Z}$ is a subfactor of type III$_\lambda$ with $\mathscr{G}_{\mathscr{P},\mathscr{Q}} = \mathscr{G}_{\mathscr{P}\rtimes_\theta \boldsymbol{Z},\mathscr{Q}\rtimes_\theta \boldsymbol{Z}} = \widetilde{\mathscr{G}}_{\mathscr{P}\rtimes_\theta \boldsymbol{Z},\mathscr{Q}\rtimes_\theta \boldsymbol{Z}}$. If the dual action $\hat{\theta}_t$ is strongly free for every $0 \neq t \in \boldsymbol{T}$, then θ is outer conjugate to $\mathrm{id}_{\mathscr{P}} \otimes \sigma$, where σ is an aperiodic automorphism of the injective type II$_1$ factor $\mathscr{R}_0$.*
(ii) *Let $\mathscr{Q} \subset \mathscr{P}$ be as in* (i), *and θ a strongly free action of $\boldsymbol{T}$ such that $\mathscr{Q} \rtimes_\theta \boldsymbol{T} \subset \mathscr{P} \rtimes_\theta \boldsymbol{T}$ is a subfactor of type III$_\lambda$ with $\mathscr{G}_{\mathscr{P},\mathscr{Q}} = \mathscr{G}_{\mathscr{P}\rtimes_\theta \boldsymbol{T},\mathscr{Q}\rtimes_\theta \boldsymbol{T}} = \widetilde{\mathscr{G}}_{\mathscr{P}\rtimes_\theta \boldsymbol{T},\mathscr{Q}\rtimes_\theta \boldsymbol{T}}$. Then θ is cocycle conjugate to $\mathrm{id}_{\mathscr{P}} \otimes \hat{\sigma}$.*

Proof. (i) If the approximate innerness and central freeness of the action $(\theta, \boldsymbol{Z})$ is known, then we can apply [**24**, Theorem 4.3(2)]. By Theorem 2.17, we have $\overline{\mathrm{Int}}(\mathscr{P},\mathscr{Q}) = \mathrm{Ker}\,\Phi \cap \mathrm{Ker}\,\mathrm{mod}$ and $\mathrm{Cnt_f}(\mathscr{P},\mathscr{Q}) = \mathrm{Cnt}(\mathscr{P},\mathscr{Q})$. Hence we only have to prove $\theta^n \notin \mathrm{Cnt_f}(\mathscr{P},\mathscr{Q})$ for $0 \neq n \in \boldsymbol{Z}$ and $\theta \in \mathrm{Ker}\,\Phi \cap \mathrm{Ker}\,\mathrm{mod}$. Since $\mathscr{Q} \rtimes_\theta \boldsymbol{Z}$ and $\mathscr{P} \rtimes_\theta \boldsymbol{Z}$ are type III$_\lambda$ factors, θ is an outer action of $\boldsymbol{Z}$, and $\mathrm{mod}\,\theta = \mathrm{id}$. (See [**37**, Lemma XVIII.4.18], for example.) Next we compute $\mathscr{Q}' \cap (\mathscr{P}_n \rtimes_\theta \boldsymbol{Z})$. Then it is elementary to see

$$\mathscr{Q}' \cap (\mathscr{P}_n \rtimes_\theta \boldsymbol{Z}) = \left\{ \sum_k a_k u^k | a_k \in \mathscr{P}_n,\, x a_k = a_k \theta^k(x) \text{ for every } x \in \mathscr{Q} \right\},$$

where u is the implementing unitary. Set $I_k^n := \{a \in \mathscr{P}_n | xa = a\theta^k(x) \text{ for every } x \in \mathscr{Q}\}$. Then I_k^n is a finite dimensional Hilbert space, and θ acts on I_k^n as a unitary operator in the natural way. We claim that the action of θ on I_k^n is trivial. Suppose the converse holds. Then we can find an eigenvalue $0 \neq t_0 \in \boldsymbol{T}$ of θ, and an eigenvector a for t_0. Hence $\theta(a) = e^{it_0}a$ holds. We will show that $\hat{\theta}_{t_0}$ is non-strongly outer. We have $\hat{\theta}_{t_0}(x)au^k = xau^k = a\theta^k(x)u^k = au^k x$ for $x \in \mathscr{Q}$. We also have $\hat{\theta}_{t_0}(u)au^k = e^{-it_0}uau^k = e^{-it_0}\theta(a)u^k u = au^k u$. Hence $\hat{\theta}_{t_0}(x)au^k = au^k x$ holds for every $x \in \mathscr{Q} \rtimes_\theta \boldsymbol{Z}$, and this implies that $\hat{\theta}_{t_0}$ is non-strongly outer, and hence non-strongly free by [**21**, Theorem 17]. This contradicts the assumption on $\hat{\theta}$. Hence θ acts on I_k^n trivially. This holds for every k and n, and hence this especially implies $\theta \in \mathrm{Ker}\,\Phi$ if we consider the case $k = 0$.

Next we compute $(\mathscr{Q} \rtimes_\theta \boldsymbol{Z})' \cap (\mathscr{P}_n \rtimes_\theta \boldsymbol{Z})$. Since θ acts on I_k^n trivially, $(\mathscr{Q} \rtimes_\theta \boldsymbol{Z})' \cap (\mathscr{P}_n \rtimes_\theta \boldsymbol{Z}) = \{\sum_k a_k u^k | a_k \in I_k^n\}$. Since we assume $\mathscr{G}_{\mathscr{P},\mathscr{Q}} = \mathscr{G}_{\mathscr{P}\rtimes_\theta \boldsymbol{Z},\mathscr{Q}\rtimes_\theta \boldsymbol{Z}}$, we have $I_k^n = \{0\}$ for every non-zero $k \in \boldsymbol{Z}$ and $n \geq 0$. Hence θ is a strongly outer action of $\boldsymbol{Z}$. In a similar way as above, we can show that θ is a strongly free action of $\boldsymbol{Z}$.

(ii) This can be shown by using (i) and the Takesaki duality [**36**]. □

Remark. In Proposition 3.2(i), we assume strong freeness of $\hat{\theta}_t$. If we do not have this condition, then Proposition 3.2(i) may fail. If θ is a non-strongly outer action such that $\gamma_h(\theta)^n \neq 1$ for all non-zero $n \in \boldsymbol{Z}$, where $\gamma_h(\theta)$ is the higher obstruction in [**19**], then the standard invariant does not change by taking the crossed product. However the author does not know whether such subfactors exist or not.

Let $\mathscr{N} \subset \mathscr{M}$ and $\mathscr{Q} \subset \mathscr{P}$ be as in Proposition 3.1. Let $\mathscr{L}$ be a type III$_1$ factor, and φ_0 a faithful normal state of $\mathscr{L}$. Set $\theta^{(0)} := \sigma_T^{\varphi_0}$ and $\bar{\mathscr{L}} := \mathscr{L} \rtimes_{\theta^{(0)}} \boldsymbol{Z}$. Then $\alpha_t := \hat{\theta}_t \otimes \hat{\theta}_{-t}^{(0)}$ is a strongly free action of $\boldsymbol{T}$ on $\mathscr{Q} \otimes \bar{\mathscr{L}} \subset \mathscr{P} \otimes \bar{\mathscr{L}}$ with $\mathrm{mod}\,\alpha_t = \mathrm{id}$.

Lemma 3.3. *The crossed product $(\mathscr{Q} \otimes \bar{\mathscr{L}}) \rtimes_\alpha \boldsymbol{T} \subset (\mathscr{P} \otimes \bar{\mathscr{L}}) \rtimes_\alpha \boldsymbol{T}$ is a subfactor of type III$_\lambda$, and the standard invariant of this subfactor is equal to that of $\mathscr{Q} \subset \mathscr{P}$.*

PROOF. As in the proof of [**6**, Lemma I.1], we can identify $(\mathscr{Q} \otimes \bar{\mathscr{L}}) \rtimes_\alpha \boldsymbol{T} \subset (\mathscr{P} \otimes \bar{\mathscr{L}}) \rtimes_\alpha \boldsymbol{T}$ and $(\mathscr{N} \otimes \mathscr{L}) \rtimes_{\sigma_T^{\varphi\otimes\varphi_0}} \boldsymbol{Z} \subset (\mathscr{M} \otimes \mathscr{L}) \rtimes_{\sigma_T^{\varphi\otimes\varphi_0}} \boldsymbol{Z}$, which is a subfactor of type III$_\lambda$. Moreover since $\mathscr{N} \otimes \mathscr{L} \subset \mathscr{M} \otimes \mathscr{L}$ has the same type III and type II standard invariants, so does $(\mathscr{N} \otimes \mathscr{L}) \rtimes_{\sigma_T^{\varphi\otimes\varphi_0}} \boldsymbol{Z} \subset (\mathscr{M} \otimes \mathscr{L}) \rtimes_{\sigma_T^{\varphi\otimes\varphi_0}} \boldsymbol{Z}$. (See Proposition 3.1) $\square$

When $\mathscr{N} \subset \mathscr{M}$ is strongly amenable and $\mathscr{L} = \mathscr{R}_\infty$ in Lemma 3.3, we can apply Proposition 3.2 since α_t is strongly free for $t \neq 0$. Hence α is cocycle conjugate to $\mathrm{id}_{\mathscr{P}} \otimes \hat{\sigma}$.

PROPOSITION 3.4. *Let $\mathscr{N} \subset \mathscr{M}$ be a strongly amenable subfactor of type III$_1$, and $\mathscr{Q} \subset \mathscr{P}$ as in Proposition* 3.1. *If $\hat{\theta}_t$ is cocycle conjugate to $\hat{\theta}_t \otimes \mathrm{id}_{\mathscr{R}_\lambda} \otimes \hat{\sigma}_t$, then $\mathscr{N} \subset \mathscr{M}$ is isomorphic to $\mathscr{N}^{\mathrm{st}} \otimes \mathscr{R}_\infty \subset \mathscr{M}^{\mathrm{st}} \otimes \mathscr{R}_\infty$.*

PROOF. By Theorem 2.16, $\mathscr{Q} \subset \mathscr{P}$ is isomorphic to $\mathscr{N}^{\mathrm{st}} \otimes \mathscr{R}_\lambda \subset \mathscr{M}^{\mathrm{st}} \otimes \mathscr{R}_\lambda$. On one hand, $\mathrm{id}_{\mathscr{R}_\lambda} \otimes \hat{\sigma}_t$ is cocycle conjugate to $\hat{\theta}^{(0)}_{-t} \otimes \hat{\theta}^{(0)}_t$. On the other hand $\hat{\theta}_t \otimes \hat{\theta}^{(0)}_{-t}$ is cocycle conjugate to $\mathrm{id}_{\mathscr{P}} \otimes \hat{\sigma}_t$, and hence cocycle conjugate to $\mathrm{id}_{\mathscr{M}^{\mathrm{st}}} \otimes \mathrm{id}_{\mathscr{R}_\lambda} \otimes \hat{\sigma}_t$. Therefore

$$\begin{aligned}\hat{\theta}_t &\sim \hat{\theta}_t \otimes \mathrm{id}_{\mathscr{R}_\lambda} \otimes \hat{\sigma}_t \\ &\sim \hat{\theta}_t \otimes \hat{\theta}^{(0)}_{-t} \otimes \hat{\theta}^{(0)}_t \\ &\sim \mathrm{id}_{\mathscr{M}^{\mathrm{st}}} \otimes \mathrm{id}_{\mathscr{R}_\lambda} \otimes \hat{\sigma}_t \otimes \hat{\theta}^{(0)}_t \\ &\sim \mathrm{id}_{\mathscr{M}^{\mathrm{st}}} \otimes \hat{\theta}^{(0)}_t\end{aligned}$$

holds. By the Takesaki duality, we get $\mathscr{N} \subset \mathscr{M} \cong \mathscr{N}^{\mathrm{st}} \otimes \mathscr{R}_\infty \subset \mathscr{M}^{\mathrm{st}} \otimes \mathscr{R}_\infty$. $\square$

Thanks to Proposition 3.4, all we need for classification of type III$_1$ subfactors is the cocycle conjugacy of $\hat{\theta}_t$ to $\hat{\theta}_t \otimes \mathrm{id}_{\mathscr{R}_\lambda} \otimes \hat{\sigma}_t$. However, this follows from the following two conditions by the same argument as in [**6**].

(1) $\mathscr{N} \subset \mathscr{M} \cong \mathscr{N} \otimes \mathscr{R}_\lambda \subset \mathscr{M} \otimes \mathscr{R}_\lambda$,

(2) $\sigma_T^\varphi \in \overline{\mathrm{Int}}(\mathscr{M}, \mathscr{N})$.

According to [**37**, Chapter XVIII], a subfactor satisfying the condition (1) is called a relatively λ-stable subfactor.

We sketch how to deduce $\hat{\theta}_t \cong \hat{\theta}_t \otimes \mathrm{id}_{\mathscr{R}_\lambda} \otimes \hat{\sigma}_t$ from the above conditions.

Let φ_1 be a periodic state on $\mathscr{R}_\lambda$. By (1), we may assume $\theta = \sigma_T^\varphi \otimes \sigma_T^{\varphi_1} = \sigma_T^\varphi \otimes \mathrm{id}_{\mathscr{R}_\lambda}$. This implies $\hat{\theta}_t \sim \hat{\theta}_t \otimes \mathrm{id}_{\mathscr{R}_\lambda}$. Also by (1), we have $\mathscr{N} \subset \mathscr{M} \cong \mathscr{N} \otimes \mathscr{R}_0 \subset \mathscr{M} \otimes \mathscr{R}_0$. Hence we can find a centralizing sequence $\{u_n\}$ such that $u_n^2 = 0$ and $u_n^* u_n + u_n u_n^* = 1$. Since u_n can be chosen in $\mathscr{R}_0$, we can assume $\sigma_t^\varphi(u_n) = u_n$. By (2), there exists $\{v_n\} \subset U(\mathscr{N})$ such that $\sigma_T^\varphi = \lim_{n\to\infty} \mathrm{Ad}\, v_n$. Let U be the implementing unitary in $(\mathscr{N} \otimes \mathscr{R}_0) \rtimes_\theta \boldsymbol{Z}$. Define $w_n := u_n(v_n^* \otimes 1)U \in (\mathscr{N} \otimes \mathscr{R}_0) \rtimes_\theta \boldsymbol{Z}$. Then $\{w_n\}$ is a centralizing sequence of $(\mathscr{M} \otimes \mathscr{R}_0) \rtimes_\theta \boldsymbol{Z}$. We have $w_n^2 = 0$, $w_n w_n^* + w_n^* w_n = 1$ and $\hat{\theta}_t(w_n) = e^{it} w_n$. Thus we get $\hat{\theta}_t \cong \hat{\theta}_t \otimes \hat{\sigma}_t$ in a similar way as in [**37**, Lemma XVIII.4.22].

Moreover the condition (1) follows from the condition (2). In [**6**, Theorem II.2], Connes gave the "local" characterization of property L'_λ [**1**]. It is not difficult to translate [**6**, Theorem II.2] to subfactor case. For readers' convenience we prepare a subfactor version of [**6**, Theorem II.2].

Theorem 3.5. *Let $\mathscr{N} \subset \mathscr{M}$ be a subfactor of type III. Then $\mathscr{N} \subset \mathscr{M}$ is relatively λ-stable if and only if the following holds. For any $\varepsilon > 0$, faithful normal states $\varphi_1, \cdots, \varphi_n$ on $\mathscr{M}$, there exists a non-zero $x \in \mathscr{N}$ such that*

$$\|(\Delta_{\varphi_j}^{\frac{1}{2}} - \lambda^{\frac{1}{2}})x\xi_j\|^2 \leq \varepsilon \sum \varphi_i(x^*x),$$

where $\xi_j \in L^2(\mathscr{M})_+$ is the representing vector for φ_j, i.e., $\varphi_j(x) = \langle x\xi_j, \xi_j\rangle$.

The arguments in [**6**, Theorem II.4] together with this result yield

Theorem 3.6. *Let $A \subset B$ be a subfactor of type III, $(\theta, \boldsymbol{Z})$ an approximately inner, outer action on $A \subset B$. If $A \rtimes_\theta \boldsymbol{Z} \subset B \rtimes_\theta \boldsymbol{Z}$ is relatively λ-stable, then so is $A \subset B$.*

In our case, $\mathscr{Q} \subset \mathscr{P} = \mathscr{N} \rtimes_\theta \boldsymbol{Z} \subset \mathscr{M} \rtimes_\theta \boldsymbol{Z}$ is obviously relatively λ-stable due to Theorem 2.16, thus so is $\mathscr{N} \subset \mathscr{M}$ under the condition (2). Therefore the rest of this paper is devoted to show the approximate innerness of modular automorphisms (i.e., the condition (2)).

4. Relative bicentralizer.

Let $\mathscr{N} \subset \mathscr{M}$ be a subfactor of type III_1 with finite index and $\mathscr{E}_0$ the minimal expectation.

As an analogue of [**11**], we introduce a notion of the relative bicentralizer for $\mathscr{N} \subset \mathscr{M}$.

Definition 4.1. (i) Let φ be a faithful normal state of $\mathscr{M}$ with $\varphi \circ \mathscr{E}_0 = \varphi$. Set $C(\varphi) := \{\{x_n\} \in l^\infty(\boldsymbol{N}, \mathscr{N}) | \lim_{n\to\infty} \|[x_n, \varphi]\| = 0\}$.
(ii) Define $B(\varphi)$ as the set of $a \in \mathscr{M}$ such that $x_n a - a x_n$ converges to zero σ-strongly for every $\{x_n\} \in C(\varphi)$. We call $B(\varphi)$ the relative bicentralizer of φ.

It is clear that $\mathscr{N}' \cap \mathscr{M} \subset B(\varphi)$. It is also easy to see $B(\varphi) \subset \mathscr{N}_\varphi' \cap \mathscr{M}$ since $\mathscr{N}_\varphi$ is embedded in $C(\varphi)$ as constant sequences. Hence if $\mathscr{N}' \cap \mathscr{M} = \mathscr{N}_\varphi' \cap \mathscr{M}$, then we have $B(\varphi) = \mathscr{N}' \cap \mathscr{M}$. This condition is satisfied as long as $\mathscr{N} \subset \mathscr{M}$ is the tensor product of a type II_1 subfactor and $\mathscr{R}_\infty$, for example.

We collect basic facts on the relative bicentralizer.

Set $C_\varphi(a, \delta) := \overline{\mathrm{conv}}\{u^* a u | u \in U(\mathscr{N}), \|[u, \varphi]\| \leq \delta\}$, where the closure is taken in the σ-weak topology.

Lemma 4.2. *For $a \in \mathscr{M}$, $a \in B(\varphi)$ if and only if $\{a\} = \bigcap_{\delta>0} C_\varphi(a, \delta)$.*

Proof. The same proof as in [**11**, Lemma 1.2] works with obvious changes. □

As in [**11**, Proposition 1.3], it is shown that $B(\varphi)$ is a von Neumann subalgebra in $\mathscr{M}$ by the above lemma.

Proposition 4.3. *Following four conditions are equivalent.*
(i) *$B(\varphi) = \mathscr{N}' \cap \mathscr{M}$.*
(ii) *$C_\varphi(a, \delta) \cap \mathscr{N}' \cap \mathscr{M} \neq \varnothing$ for every $a \in \mathscr{M}$ and every $\delta > 0$.*

(iii) $E^{\varphi}_{\mathscr{N}'\cap\mathscr{M}}(a) \in C_\varphi(a,\delta)$ *for every* $a \in \mathscr{M}$ *and* $\delta > 0$.
(iv) $\psi \circ E^{\varphi}_{\mathscr{N}'\cap\mathscr{M}} \in \overline{\mathrm{conv}}\{u\psi u^* | u \in U(\mathscr{N}), \|[u,\varphi]\| \leq \delta\}$ *for every* $\psi \in \mathscr{M}_*$ *and* $\delta > 0$.

Proof. (i)⇒(ii). Put $C_\varphi(a) := \bigcap_{\delta>0} C_\varphi(a,\delta)$. Then $C_\varphi(a)$ is a σ-weakly compact convex set, and $C_\varphi(a) \neq \varnothing$ since $a \in C_\varphi(a,\delta)$ for every δ. Let $\mathfrak{H}_\varphi$ be the completion of $\mathscr{M}$ with φ-norm. Then $C_\varphi(a)$ is a norm closed convex subset of $\mathfrak{H}_\varphi$. Hence there exists a unique element $b \in C_\varphi(a)$ with $\|b\xi_\varphi\| = \min\{\|x\xi_\varphi\| | x \in C_\varphi(a)\}$. If $a' \in C_\varphi(a,\delta)$, then $C_\varphi(a',\delta) \subset C_\varphi(a,2\delta)$. Hence $C_\varphi(b) = \bigcap_{\delta>0} C_\varphi(b,\delta) \subset \bigcap_{\delta>0} C_\varphi(a,2\delta) = C_\varphi(a)$. If $\|[u,\varphi]\| \leq \delta$, then $\|u^*bu\|^2_\varphi \leq \|b\|^2_\varphi + \delta\|b\|^2_\varphi$. By the lower semicontinuity of φ-norm in the σ-weak topology, $\|x\|^2_\varphi \leq \|b\|^2_\varphi + \delta\|b\|^2_\varphi$ for every $x \in C_\varphi(b,\delta)$. Hence $\|x\|^2_\varphi \leq \|b\|^2_\varphi$ holds for $x \in C_\varphi(b)$. Since $x \in C_\varphi(b) \subset C_\varphi(a)$, we get $x = b$, and hence $C_\varphi(b) = \{b\}$ holds. By Lemma 4.2 and the assumption, we have $b \in B(\varphi) = \mathscr{N}' \cap \mathscr{M}$.

(ii)⇒(iii). First note that $\mathscr{N}' \cap \mathscr{M}$ is finite dimensional. So the sets $C_\varphi(a,\delta) \cap \mathscr{N}' \cap \mathscr{M}$, $\delta > 0$, form a decreasing family of nonempty compact sets. Thus their intersection is also non-empty. Hence there exists $b \in C_\varphi(a) \cap \mathscr{N}' \cap \mathscr{M}$. For every $u \in U(\mathscr{N})$ with $\|[u,\varphi]\| \leq \delta$ and $x \in \mathscr{N}' \cap \mathscr{M}$, we have

$$\begin{aligned}|\varphi(u^*aux) - \varphi(ax)| &= |(u\varphi u^* - \varphi)(ax)| \\ &\leq \|u\varphi u^* - \varphi\|\|ax\| \\ &\leq \delta\|ax\|,\end{aligned}$$

and we get $|\varphi(bx) - \varphi(ax)| = 0$ for every $x \in \mathscr{N}' \cap \mathscr{M}$. Since $0 = \varphi(bx) - \varphi(ax) = \varphi((b - E^{\varphi}_{\mathscr{N}'\cap\mathscr{M}}(a))x)$ holds, we get $b = E^{\varphi}_{\mathscr{N}'\cap\mathscr{M}}(a)$, where $E^{\varphi}_{\mathscr{N}'\cap\mathscr{M}}$ is the φ-preserving conditional expectation on $\mathscr{N}' \cap \mathscr{M}$.

(iii)⇒(i). We have $a - E^{\varphi}_{\mathscr{N}'\cap\mathscr{M}}(a) \in \overline{\mathrm{conv}}\{a - u^*au | u \in U(\mathscr{N}), \|[u,\varphi]\| \leq \delta\}$. By the lower semicontinuity of φ-norm in the σ-weak topology, we get $\|a - E^{\varphi}_{\mathscr{N}'\cap\mathscr{M}}(a)\|_\varphi \leq \sup\{\|a - u^*au\|_\varphi | u \in U(\mathscr{N}), \|[u,\varphi]\| \leq \delta\}$.

Here if $a \in B(\varphi)$, then the right hand side of the above inequality converges to 0 as $\delta \to 0$. Hence we get $a = E^{\varphi}_{\mathscr{N}'\cap\mathscr{M}}(a)$ and $B(\varphi) \subset \mathscr{N}' \cap \mathscr{M}$.

(iii)⇔(iv). This follows from the duality argument. □

Corollary 4.4. *If* $B(\varphi) = \mathscr{N}' \cap \mathscr{M}$ *holds for some* φ *with* $\varphi \circ \mathscr{E}_0 = \varphi$, *then* $B(\psi) = \mathscr{N}' \cap \mathscr{M}$ *holds for every normal faithful state* ψ *with* $\psi \circ \mathscr{E}_0 = \psi$.

Proof. The same proof as in [**11**, Corollary 1.5] works by using Proposition 4.3 and the Connes-Størmer transitivity [**7**]. □

Proposition 4.5. *Let* $\mathscr{N} \subset \mathscr{M}$ *be a subfactor of type* III_1 *with* $[\mathscr{M} : \mathscr{N}] < \infty$, *and assume* $B(\varphi) = \mathscr{N}' \cap \mathscr{M}$. *Then for any* $\varepsilon > 0$, $\delta > 0$ *and any* $\psi_1, \cdots, \psi_n \in \mathscr{M}_*$, *there exist* $u_1, \cdots, u_m \in U(\mathscr{N})$ *and* $\lambda_1, \cdots, \lambda_m > 0$ *such that* $\sum_{i=1}^m \lambda_i = 1$, $\|[u_i,\varphi]\| < \delta$ *and* $\|\psi_j \circ E^{\varphi}_{\mathscr{N}'\cap\mathscr{M}} - \psi_j \circ P\| < \varepsilon$, $j = 1, \cdots, n$, *where* $P(x) = \sum_i \lambda_i u_i^* x u_i$.

Proof. We prove by induction. When $n = 1$, the proposition follows from Proposition 4.3(iv). Assume that the proposition holds for $n - 1$. Take $\psi_1, \cdots, \psi_n \in \mathscr{M}_*$. By the induction hypothesis, for any $\varepsilon' > 0$ and $\delta' > 0$, we can find $u_1, \cdots, u_m \in U(\mathscr{N})$ and $\lambda_1, \cdots, \lambda_m > 0$ satisfying the conclusion of proposition for ε', δ' and $\psi_1, \cdots, \psi_{n-1}$. Apply

Proposition 4.3(iv) to $\psi_n \circ P$. Then we can find $v_1, \cdots, v_l \in U(\mathscr{N})$ and $\mu_1, \cdots, \mu_l > 0$ such that $\sum \mu_i = 1$, $\|[v_i, \varphi]\| < \delta'$ and $\|\psi_n \circ P \circ E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} - \psi_n \circ P \circ Q\| < \varepsilon'$, where $Q(x) = \sum \mu_j v_j^* x v_j$. It is clear that $P \circ E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} = E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}}$ holds, and hence we get $\|\psi_n \circ E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} - \psi_n \circ P \circ Q\| < \varepsilon'$. By the choice of $\{u_i\}$ and $\{v_j\}$, $\|[u_i v_j, \varphi]\| < 2\delta'$ holds. Next we will estimate $\|\psi_i \circ E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} - \psi_i \circ P \circ Q\|$, $1 \leq i \leq n-1$. Since we have

$$\begin{aligned}
&\|\psi_i \circ E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} - \psi_i \circ P \circ Q\| \\
&\quad \leq \|\psi_i \circ (E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} - E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} \circ Q)\| + \|\psi_i \circ E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} \circ Q - \psi_i \circ P \circ Q\| \\
&\quad \leq \|\psi_i \circ (E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} - E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} \circ Q)\| + \varepsilon',
\end{aligned}$$

we have to estimate $\|E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} - E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} \circ Q\|$. Let $\|\cdot\|_1$ be the L^1 norm on $\mathscr{N}' \cap \mathscr{M}$ for $\varphi|_{\mathscr{N}' \cap \mathscr{M}}$. (Note that the restriction of φ on $\mathscr{N}' \cap \mathscr{M}$ is a tracial state.) Then we have

$$\sup\{|\varphi(yx)| | x \in \mathscr{N}' \cap \mathscr{M}, \|x\| \leq 1\} = \|y\|_1,\ y \in \mathscr{N}' \cap \mathscr{M}.$$

On the other hand, we have the following inequality.

$$\begin{aligned}
|\varphi((E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}}(y) - E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} \circ Q(y))x)| &= |\varphi(yx - Q(y)x)| \\
&= \left|\varphi\left(yx - \sum_i \mu_i v_i^* y x v_i\right)\right| \\
&\leq \sum_i \mu_i |\varphi(yx) - v_i \varphi v_i^*(yx)| \\
&= \sum_i \mu_i |(\varphi - v_i \varphi v_i^*)(yx)| \\
&\leq \delta' \|yx\|.
\end{aligned}$$

Combining these, we get $\|E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}}(y) - E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} \circ Q(y)\|_1 \leq \delta' \|y\|$. Since $\mathscr{N}' \cap \mathscr{M}$ is finite dimensional, there exists a constant $0 \neq C$ such that $\|\cdot\|_1 \geq C\|\cdot\|$. Hence we get $\|E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}}(y) - E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} \circ Q(y)\| \leq \delta'/C\|y\|$, and $\|E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} - E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} \circ Q\| \leq \delta'/C$. Finally we get $\|\psi_i \circ E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}} - \psi_i \circ P \circ Q\| < \max \|\psi_i\| \delta'/C + \varepsilon'$. If we take ε' and δ' in such a way that $2\delta' < \delta$ and $\max \|\psi_i\| \delta'/C + \varepsilon' < \varepsilon$, then $\{\lambda_i \mu_j\}$ and $\{u_i v_j\}$ satisfy the conclusion of the proposition for n. □

Let $\mathscr{N} \subset \mathscr{M}$ be an inclusion of AFD factors of type III$_1$ with $\mathscr{G}_{\mathscr{M},\mathscr{N}} = \widetilde{\mathscr{G}}_{\mathscr{M},\mathscr{N}}$, and $\mathscr{E}_k$ the minimal conditional expectation from $\mathscr{M}_k$ onto $\mathscr{M}_{k-1}$. We extend φ to $\mathscr{M}_k$ by $\varphi \circ \mathscr{E}_1 \circ \cdots \circ \mathscr{E}_k$, and denote by φ for simplicity.

Popa proved the following theorem in [**33**, Theorem 4.2].

Theorem 4.6. *Let $\mathscr{N} \subset \mathscr{M}$ be an inclusion of AFD type III$_1$ factors, and φ_0 be a dominant weight for $\mathscr{M}$ with $\varphi_0 \circ \mathscr{E}_0 = \varphi_0$. If we have $\mathscr{N}'_{\varphi_0} \cap \mathscr{M} = \mathscr{N}' \cap \mathscr{M}$, then there exists a faithful normal state φ such that $\varphi \circ \mathscr{E}_0 = \varphi$ and $\mathscr{N}_\varphi \cap \mathscr{M} = \mathscr{N}' \cap \mathscr{M}$.*

Here is an important corollary.

COROLLARY 4.7. *Let $\mathscr{N} \subset \mathscr{M}$ be as above. Then we have $\mathscr{N}' \cap \mathscr{M}_k = B(\varphi)$ for every k and every faithful normal state φ of $\mathscr{M}_k$ such that $\varphi = \varphi \circ \mathscr{E}_0 \circ \cdots \circ \mathscr{E}_k$.*

PROOF. Since $\mathscr{G}_{\mathscr{M},\mathscr{N}} = \widetilde{\mathscr{G}}_{\mathscr{M},\mathscr{N}}$, we have $\mathscr{N}' \cap \mathscr{M}_k = \mathscr{N}'_{\varphi_0} \cap (\mathscr{M}_k)_{\varphi_0}$ for every k and a dominant weight φ_0. By combining this with the relative commutant theorem $\mathscr{N}'_{\varphi_0} \cap \mathscr{M}_k = \mathscr{N}'_{\varphi_0} \cap (\mathscr{M}_k)_{\varphi_0}$ ([**33**, Theorem 4.3]), we get $\mathscr{N}'_{\varphi_0} \cap \mathscr{M}_k = \mathscr{N}' \cap \mathscr{M}_k$. By the above Popa's theorem, there exists a faithful normal state φ on $\mathscr{M}_k$ such that $\varphi = \varphi \circ \mathscr{E}_1 \circ \cdots \circ \mathscr{E}_k$ and $\mathscr{N}' \cap \mathscr{M}_k = \mathscr{N}'_{\varphi} \cap \mathscr{M}_k$. By Corollary 4.4, we get the conclusion. $\square$

Set $\mathscr{M}_\infty := (\bigcup_k \mathscr{M}_k)^-$, where the closure is taken in the GNS representation with respect to φ.

PROPOSITION 4.8. *We can find $\{u_n^k\}_{k=1}^{m_n} \subset U(\mathscr{N})$ and $\{\lambda_n^k\}_{k=1}^{m_n} \subset \boldsymbol{R}_+^*$ such that $\sum_{k=1}^{m_n} \lambda_n^k = 1$ for each n, $\sup_k \|[u_n^k, \varphi]\| \leq 1/n$ and $\psi \circ P_n \to \psi \circ E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}_\infty}$, $n \to \infty$, for every $\psi \in (\mathscr{M}_\infty)_*$, where $P_n(x) = \sum_{k=1}^{m_n} \lambda_n^k u_n^{k*} x u_n^k$.*

PROOF. Let E_k^∞ be a φ preserving conditional expectation from $\mathscr{M}_\infty$ onto $\mathscr{M}_k$. By the martingale convergence theorem [**4**, Lemma I.2], we have $\lim_{k\to\infty} \|\psi - \psi \circ E_k^\infty\| = 0$. Let V_k be a dense countable subset of $(\mathscr{M}_k)_*$. We regard $\varphi \in (\mathscr{M}_k)_*$ as the element of $(\mathscr{M}_\infty)_*$ via $\varphi \to \varphi \circ E_{k+1}^\infty$. Set $V := \bigcup_k V_k$. Then V is a dense countable subset of $(\mathscr{M}_\infty)_*$. We index V as $\{\psi_i\}_{i=1}^\infty$. By the definition of V, there exists $k > 0$ such that $\psi_i \circ E_k^\infty = \psi_i$, $1 \leq i \leq n$. We regard ψ_i as a normal functional on $\mathscr{M}_k$, and apply Proposition 4.5. Then we can find $\lambda_k^n \subset \boldsymbol{R}_+^*$, $\sum_k \lambda_k^n = 1$, and $\{u_k^n\} \subset U(\mathscr{N})$ such that $\|\psi_i \circ P_n - \psi_i \circ E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}_k}\| < 1/n$ and $\|[\psi_i, u_k^n]\| \leq 1/n$, where $P_n = \sum_k \lambda_k^n \mathrm{Ad}\, u_k^n$. Then we get $\|\psi_i \circ P_n - \psi_i \circ E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}_\infty}\| < 1/n$ since $E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}_k} \circ E_k^\infty = E_k^\infty \circ E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}_\infty}$ and $E_k^\infty \circ P_n = P_n \circ E_k^\infty$. These $\{\lambda_k^n\}$ and $\{u_k^n\}$ are desired ones. $\square$

By Proposition 4.8, $P_n(x)$ converges to $E^{\varphi}_{\mathscr{N}' \cap \mathscr{M}_k}(x)$ σ-strongly* for $x \in \mathscr{M}_k$. This fact is crucial in later sections.

5. Symmetric enveloping algebras and Longo-Rehren inclusions.

Now we discuss how to construct the symmetric enveloping algebra for a subfactor $\mathscr{N} \subset \mathscr{M}$ of type III_1, since details of construction in type III case is not presented in [**35**].

First we fix a tunnel $\mathscr{M} \supset \mathscr{N} =: \mathscr{M}_{-1} \supset \mathscr{M}_{-2} \supset \cdots$ for $\mathscr{N} \subset \mathscr{M}$. Let $e_k \in \mathscr{M}_k$ be the Jones projection for $\mathscr{M}_{k-1} \supset \mathscr{M}_{k-2}$, and $\mathscr{E}_k$ the minimal conditional expectation from $\mathscr{M}_k$ on $\mathscr{M}_{k-1}$. We assume $\mathscr{M}$ acts on $L^2(\mathscr{M})$, and identify $\mathscr{M}^{\mathrm{opp}}$ with $\mathscr{M}' = J\mathscr{M}J$ via $j_{\mathscr{M}}(x) \leftrightarrow JxJ$. Set $e_k := j_{\mathscr{M}}(e_{2-k})$ for $k \geq 2$. Then $\mathscr{M} \subset \mathscr{M}_1 \subset \cdots \subset \mathscr{M}_k := \mathrm{Alg}\langle \mathscr{M}_{k-1}, e_k\rangle \subset \mathscr{M}_{k+1} \subset \cdots$ is algebraically isomorphic to the Jones tower, and each $\mathscr{M}_k$ is σ-weakly closed in $B(L^2(\mathscr{M}))$. Set $B_0 := \bigcup_{k\geq 0} \mathscr{M}^{\mathrm{opp}} \mathscr{M}_k \mathscr{M}^{\mathrm{opp}} \subset B_1 := C^*(\mathscr{M}, e_{\mathscr{N}}, \mathscr{M}^{\mathrm{opp}}) \subset B(L^2(\mathscr{M}))$. Then B_0 is a dense $*$-subalgebra in B_1. See [**35**]. In fact, we have $B_0 = \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, e_{\mathscr{N}})$.

We recall the construction of the symmetric enveloping algebra for type II_1 case, so assume $\mathscr{N} \subset \mathscr{M}$ is of type II_1 for a moment. The relative Dixmier property [**34**] (also see [**35**, Appendix]) enable us to construct a conditional expectation E from B_1 on $\mathscr{M}^{\mathrm{opp}}$ by $xyz \in \mathscr{M}^{\mathrm{opp}} \mathscr{M}_k \mathscr{M}^{\mathrm{opp}} \to x E_{\mathscr{M}' \cap \mathscr{M}_k}(y) z \in \mathscr{M}^{\mathrm{opp}}$. Indeed assume $0 = \sum_i x_i y_i z_i \in \mathscr{M}^{\mathrm{opp}} \mathscr{M}_k \mathscr{M}^{\mathrm{opp}}$. Then we can choose $\{\lambda_n^k\}_{k=1}^{m_n} \subset \boldsymbol{R}_+^*$, $\sum_k \lambda_n^k = 1$, and $\{u_n^k\}_{k=1}^{m_k} \subset U(\mathscr{M})$

such that $P_n(y_i) := \sum_{k=1}^{m_n} \lambda_n^k {u_n^k}^* y_i u_n^k$ converges to $E_{\mathscr{M}'\cap\mathscr{M}_k}(y_i)$ in the norm topology for every i. (In fact, σ-strong convergence is enough in the following argument since $\mathscr{M}_k$ is σ-weakly closed.) It follows that

$$\begin{aligned}\sum_i x_i E_{\mathscr{M}'\cap\mathscr{M}_k}(y_i) z_i &= \lim_n \sum_i x_i P_n(y_i) z_i \\ &= \lim_n \sum_i x_i \Big(\sum_k \lambda_n^k {u_n^k}^* y_i u_n^k \Big) z_i \\ &= \lim_n \sum_k \lambda_n^k {u_n^k}^* \sum_i (x_i y_i z_i) u_n^k \\ &= 0,\end{aligned}$$

so E is well-defined. Then it is shown that $\mathrm{tr}_{\mathscr{M}^{\mathrm{opp}}} \circ E$ is a unique tracial state for B_1. Then by GNS construction, we get the symmetric enveloping algebra as $\mathscr{M} \underset{e_{\mathscr{N}}}{\boxtimes} \mathscr{M}^{\mathrm{opp}} := \pi_{\mathrm{tr}_{\mathscr{M}^{\mathrm{opp}}} \circ E}(B_1)''$, which is a II_1 factor due to the uniqueness of a tracial state.

In the type III case, it is not clear if we have such a projection E. However when $\mathscr{N} \subset \mathscr{M}$ is a strongly amenable subfactor of type III_1 with $\mathscr{G}_{\mathscr{M},\mathscr{N}} = \widetilde{\mathscr{G}}_{\mathscr{M},\mathscr{N}}$, we can use Proposition 4.8 instead of the relative Dixmier property. Namely, let P_n be as in Proposition 4.8. Then $xyz \to xE^{\varphi}_{\mathscr{M}'\cap\mathscr{M}_k}(y)z = \lim_{n\to\infty} xP_n(y)z$ is shown to be a well-defined map from B_0 onto $\mathscr{M}^{\mathrm{opp}}$ in a similar way as above. This map can be extended to a conditional expectation E from B_1 onto $\mathscr{M}^{\mathrm{opp}}$. Then we get a state ψ on B_1 by $\psi(x) := \langle E(x)\xi_\varphi, \xi_\varphi \rangle$, where $\xi_\varphi \in L^2(\mathscr{M})_+$ is the representing vector for φ.

Definition 5.1. Define the symmetric enveloping algebra for $\mathscr{N} \subset \mathscr{M}$ as $\mathscr{M} \underset{e_{\mathscr{N}}}{\boxtimes} \mathscr{M}^{\mathrm{opp}} := \pi_\psi(B_1)''$, where π_ψ is the GNS representation for ψ.

Though the symmetric enveloping algebra for a strongly amenable subfactor of type III_1 is defined, it is not clear that this construction produces indeed an factor, or independent on the choice of φ. Here we already have (another) similar construction, the Longo-Rehren construction. We compare these ones.

To do so, we recall the canonical implementation for endomorphisms of $\mathscr{M}$. Let W_ρ be the canonical implementing isometry for ρ defined as in [**10**, Appendix A]. Namely fix a faithful normal state $\varphi \in M_*$, and we define W_ρ by $W_\rho(x\xi_\varphi) = \rho(x)\xi_{\varphi\circ\phi_\rho}$ for $\rho \in \mathrm{End}(\mathscr{M})$. Indeed, W_ρ is an isometry. See [**10**, Appendix A] for more properties of W_ρ.

Denote W_{ρ_i} by W_i for simplicity for $i \in I$. Then $\{d(j)W_j\}_{j\in I}$ satisfy the same relations as $\{V_i\}$ in the Longo-Rehren construction by [**10**, Proposition A.4], i.e., $d(i)W_i d(j)W_j = \sum_k \hat{v}^k_{i,j} d(k) W_k$ and $W_i xJyJ = \rho_i(x)\rho^{\mathrm{opp}}(j_{\mathscr{M}}(y))W_\rho$ hold for $x, y \in \mathscr{M}$. (Of course, $\hat{v}^k_{i,j} = \sum_e \sqrt{\frac{d(i)d(j)}{d(k)}} v\binom{k}{i,j}^e J v\binom{k}{i,j}^e J$, in this case.) We can identify $d(i)W_i \in B(L^2(\mathscr{M}))$ and $V_i \in A(\Delta)$. Hence we have a *-homomorphism $\mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\}) \to B(L^2(\mathscr{M}))$ by $\pi(x \otimes j_{\mathscr{M}}(y))V_i \to xJyJd(i)W_i$, which we call the canonical map.

We also have the formula $e_{\mathscr{N}} = [\mathscr{M} : \mathscr{N}]^{-1} \sum_{i\in I_0} \sqrt{d(i)}\tilde{a}_i^* d(i) W_i$ for the common

Jones projection with the same notations as in §2.5. (Also see Appendix A.) As a consequence we have $B_0 = \mathrm{Alg}(\mathscr{M}, J\mathscr{M}J, e_{\mathscr{N}}) = \mathrm{Alg}(\mathscr{M}, J\mathscr{M}J, \{W_i\})$.

Let $(A :=)\mathscr{M} \otimes \mathscr{M}^{\mathrm{opp}} \subset A(\Delta)$ be the Longo-Rehren inclusion for $\mathscr{N} \subset \mathscr{M}$. As was explained §2.5, we can construct the common Jones projection $e_{\mathscr{N}}$ in $A(\Delta)$. Once we have the common Jones projection, we can construct a Jones tower for $\mathscr{N} \subset \mathscr{M}$ within $A(\Delta)$ as in the symmetric enveloping algebra case, and we have $\bigcup_k \mathscr{M}^{\mathrm{opp}}\mathscr{M}_k\mathscr{M}^{\mathrm{opp}} = \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$.

LEMMA 5.2. *We have the following.*
(1) $\varphi \otimes \varphi^{\mathrm{opp}} \circ \mathscr{E}_A(xyz) = \varphi^{\mathrm{opp}}(xE^{\varphi}_{\mathscr{M}'\cap\mathscr{M}_k}(y)z)$, $x, z \in \mathscr{M}^{\mathrm{opp}}$, $y \in \mathscr{M}_k(\subset A(\Delta))$.
(2) $\psi(\sum_i a_iW_i) = \varphi \otimes \varphi^{\mathrm{opp}}(a_0)$ *for* $\sum_i a_iW_i \in \mathrm{Alg}(\mathscr{M}, J\mathscr{M}j, \{W_i\})$.

PROOF. (1) Note that since $\mathscr{M}' \cap A(\Delta) = \mathscr{M}^{\mathrm{opp}}$, $E^{\varphi}_{\mathscr{M}'\cap\mathscr{M}_k}(y) \in \mathscr{M}^{\mathrm{opp}}$. Then we have

$$\begin{aligned}
&|\varphi \otimes \varphi^{\mathrm{opp}} \circ \mathscr{E}_A(xyz) - \varphi \otimes \varphi^{\mathrm{opp}} \circ \mathscr{E}_A(xP_n(y)z)| \\
&= \left|\varphi \otimes \varphi^{\mathrm{opp}} \circ \mathscr{E}_A(xyz) - \sum_k \lambda_n^k \varphi \otimes \varphi^{\mathrm{opp}} \circ \mathscr{E}_A(xu_n^{k*}yu_n^k z)\right| \\
&\leq \sum_k \lambda_n^k |\varphi \otimes \varphi^{\mathrm{opp}} \circ \mathscr{E}_A(xyz) - u_n^k\varphi u_n^{k*} \otimes \varphi^{\mathrm{opp}} \circ \mathscr{E}_A(xyz)| \\
&= \sum_k \lambda_n^k |(\varphi - u_n^k\varphi u_n^{k*}) \otimes \varphi^{\mathrm{opp}} \circ \mathscr{E}_A(xyz)| \\
&\leq \frac{1}{n}\|\mathscr{E}(xyz)\| \to 0
\end{aligned}$$

if $n \to \infty$. □

To prove (2), we begin with the following lemma. (In the following, we use notations in §2.5.)

LEMMA 5.3. *For every $x \in \mathscr{M}$ and $0 \neq j \in I_0$, we have $E^{\varphi}_{\mathscr{M}'\cap\mathscr{M}_1}(xw_j^{e*}e_{\mathscr{N}}w_0^f) = 0$.*

PROOF. We verify $\varphi \circ \mathscr{E}_1(yxw_j^{e*}e_{\mathscr{N}}w_0^f) = 0$ for every $y \in \mathscr{M}' \cap \mathscr{M}_1$, which implies the conclusion since $\varphi \circ \mathscr{E}_1(yxw_j^{e*}e_{\mathscr{N}}w_0^f) = \varphi \circ \mathscr{E}_1(yE^{\varphi}_{\mathscr{M}'\cap\mathscr{M}_1}(xw_j^{e*}e_{\mathscr{N}}w_0^f))$. First note that there exists a unique $z \in \mathscr{N}' \cap \mathscr{M}$ such that $ze_{\mathscr{N}} = ye_{\mathscr{N}}$ by the push-down lemma. Then we have

$$\begin{aligned}
\varphi \circ \mathscr{E}_1(yxw_j^{e*}e_{\mathscr{N}}w_0^f) &= \varphi \circ \mathscr{E}_1(xw_j^{e*}ye_{\mathscr{N}}w_0^f) \\
&= \varphi \circ \mathscr{E}_1(xw_j^{e*}ze_{\mathscr{N}}w_0^f) \\
&= [\mathscr{M} : \mathscr{N}]^{-1}\varphi(xw_j^{e*}zw_0^f).
\end{aligned}$$

It is easy to see $w_j^{e*}zw_0^f \in (\mathrm{id}, \rho_j)$. Hence $w_j^{e*}zw_0^f = 0$ for $j \neq 0$. □

Proof of Lemma 5.2(2). We compute $\psi(xJyJW_j)$ for $x, y \in \mathscr{M}$. As was explained in §2.5, we have

$$W_j = C\tilde{w}_j^* e_{\mathscr{N}} \tilde{w}_0 = C \sum_{e,f} w_j^{e*} J w_j^{e*} J e_{\mathscr{N}} w_0^f J w_0^f J$$

for some nonzero constant C.

Hence for $0 \neq j \in I_0$, we have

$$\begin{aligned}
\psi(xJyJW_j) &= C\psi\left(xJyJ \sum_{e,f} w_j^{e*} J w_j^{e*} J e_{\mathscr{N}} w_0^f J w_0^f J\right) \\
&= C \sum_{e,f} \psi(Jyw_j^{e*} J x w_j^{e*} e_{\mathscr{N}} w_0^f J w_0^f J) \\
&= C \sum_{e,f} \varphi^{\mathrm{opp}}(Jyw_j^{e*} J E^{\varphi}_{\mathscr{M}' \cap \mathscr{M}_1}(x w_j^{e*} e_{\mathscr{N}} w_0^f) J w_0^f J) \\
&= 0.
\end{aligned}$$

In a similar way, we can show $\psi(xJyJW_j) = 0$ for all $0 \neq j \in I$ by considering $\mathscr{M}_{-k} \subset \mathscr{M} \subset \mathscr{M}_k$ instead of $\mathscr{N} \subset \mathscr{M} \subset \mathscr{M}_1$. Note that $\psi(xJyJ) = \varphi(x)\varphi^{\mathrm{opp}}(JyJ) = \varphi \otimes_{\mathrm{alg}} \varphi^{\mathrm{opp}}(x \otimes JyJ)$.

Take $\sum_i x_i W_i \in \mathrm{Alg}(\mathscr{M}, J\mathscr{M}J, \{W_j\}_{j\in\Delta})$, $x_i \in \mathrm{Alg}(\mathscr{M}, J\mathscr{M}J)$, and assume $\sum_i x_i W_i = 0$. We have $(\sum_i x_i W_i)^* (\sum_j x_j W_j) = \sum_{i,j,k} \hat{v}^{0*}_{\bar{\imath},i} \rho_{\bar{\imath}}(x_i^* x_j) \hat{v}^k_{\bar{\imath},j} W_k$. By the above argument, $0 = \psi((\sum_i x_i W_i)^* \sum_j x_j W_j) = \varphi \otimes_{\mathrm{alg}} \varphi^{\mathrm{opp}}(\sum_i \hat{v}^{0*}_{\bar{\imath},i} \hat{\rho}_{\bar{\imath}}(x_i^* x_i) \hat{v}^0_{\bar{\imath},i})$. Thus we get $\hat{\rho}_{\bar{\imath}}(x_i)\hat{v}^0_{\bar{\imath},i} = 0$ for every i, and we get $x_i = x_i \hat{v}^{0*}_{i,\bar{\imath}} \hat{\rho}_i(\hat{v}^0_{\bar{\imath},i}) = \hat{v}^{0*}_{i,\bar{\imath}} \hat{\rho}_i(\hat{\rho}_{\bar{\imath}}(x_i)\hat{v}^0_{\bar{\imath},i}) = 0$. Hence the map $\mathscr{E}$

$$\mathscr{E} : \mathrm{Alg}(\mathscr{M}, J\mathscr{M}J, \{W_j\}) \ni \sum_i a_i W_i \to a_0 \in \mathrm{Alg}(\mathscr{M}, J\mathscr{M}J)$$

is well-defined. Thus we have

$$\psi\left(\sum a_i W_i\right) = \varphi \otimes \varphi^{\mathrm{opp}}(a_0) = \varphi \otimes \varphi^{\mathrm{opp}} \circ \mathscr{E}$$

on $\mathrm{Alg}(\mathscr{M}, J\mathscr{M}J, \{W_j\}_{j\in I}) = \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, e_{\mathscr{N}})$. □

Now we can state the main result in this section.

Theorem 5.4. *Let ψ be a state on $C^*(\mathscr{M}, e_{\mathscr{N}}, J\mathscr{M}J)$ defined as above. Let π_ψ be the GNS representation of $C^*(\mathscr{M}, e_{\mathscr{N}}, J\mathscr{M}J)$ via ψ. Then $\mathscr{M} \underset{e_{\mathscr{N}}}{\boxtimes} \mathscr{M}^{\mathrm{opp}} = \pi_\psi(C^*(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, e_{\mathscr{N}}))''$ is isomorphic to $A(\Delta)$ in the canonical way.*

Proof. By Lemma 5.2, the canonical map $\mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\}) \to \mathrm{Alg}(\mathscr{M}, J\mathscr{M}J, \{W_i\})$ can be extended to a unitary V between $L^2(A(\Delta), \varphi \otimes \varphi^{\mathrm{opp}} \circ \mathscr{E}_A)$ and $L^2(B_1, \psi)$. Then it is easy to see $A(\Delta) \cong \pi_\psi(B_1)''$ via V. □

6. Approximation property for Longo-Rehren inclusion and Effros-Lance type characterization.

Let $\mathscr{N} \subset \mathscr{M}$ be a strongly amenable subfactor of type III$_1$, and $\pi(A) \subset A(\Delta)$ the Longo-Rehren inclusion for $\mathscr{N} \subset \mathscr{M}$. (We write an isomorphism π from A into $A(\Delta)$ explicitly for a while, and freely use notations in §2.5.) We often write $x \otimes j_{\mathscr{M}}(y) \in \mathscr{M} \otimes \mathscr{M}^{\mathrm{opp}}$ as $x j_{\mathscr{M}}(y)$ for simplicity.

Lemma 6.1. *Let $\mathscr{M}$ be an injective factor of type III$_1$. Then there exist finite dimensional factors M_n, unital completely positive maps $S_n : \mathscr{M} \to M_n$ and $T_n : M_n \to \mathscr{M}$ such that $\lim_{n\to\infty} T_n \circ S_n(x) = x$ in strong* topology and $\lim_{n\to\infty} \|\psi \circ T_n \circ S_n - \psi\| = 0$ for every $\psi \in \mathscr{M}_*$.*

Proof. Fix $T > 0$, set $\theta := \sigma_T^{\varphi}$ for some faithful normal state φ on $\mathscr{M}$. Then $\mathscr{P} := \mathscr{M} \rtimes_\theta \boldsymbol{Z}$ is an injective type III$_\lambda$ factor, $(T = -2\pi/\log\lambda)$, and hence we can identify $\mathscr{P}$ with the infinite tensor product factor $\bigotimes_{k=1}^{\infty}(M_2(\boldsymbol{C}), \varphi_\lambda)$, where $\varphi_\lambda\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \frac{a}{1+\lambda} + \frac{\lambda d}{1+\lambda}$. Set $M_n := \bigotimes_{k=1}^{n} M_2(\boldsymbol{C})$, and let E_n be the $\bigotimes_{k=1}^{\infty}\varphi_\lambda$ preserving conditional expectation from $\mathscr{P}$ on M_n. Then $E_n(x)$ (resp $\psi \circ E_n$) converge to x (resp. ψ) in strong* (resp. norm) topology for $x \in \mathscr{P}$ (resp. $\psi \in \mathscr{P}_*$). Let $E_{\mathscr{M}}$ be the conditional expectation from $\mathscr{P}$ on $\mathscr{M}$. Define $S_n := E_n|_{\mathscr{M}}$ and $T_n := E_{\mathscr{M}}|_{M_n}$. Then these T_n and S_n are desired ones. Indeed we have

$$\lim_{n\to\infty} T_n \circ S_n(x) = \lim_{n\to\infty} E_{\mathscr{M}} \circ E_n(x) = E_{\mathscr{M}}(x) = x$$

in strong* topology and

$$\begin{aligned}\lim_{n\to\infty} \|\psi \circ T_n \circ S_n - \psi\| &= \lim_{n\to\infty} \|\psi \circ E_{\mathscr{M}} \circ E_n|_{\mathscr{M}} - \psi \circ E_{\mathscr{M}}|_{\mathscr{M}}\| \\ &\leq \lim_{n\to\infty} \|\psi \circ E_{\mathscr{M}} \circ E_n - \psi \circ E_{\mathscr{M}}\| = 0. \end{aligned}$$ □

To state the following lemmas, we need some definitions and results on amenable fusion algebras in [**14**]. Let μ be a measure on I given by $\mu(\{i\}) = d(i)^2$. We denote the pairing between $\ell^1(I,\mu)$ and $\ell^\infty(I)$ by $\langle\cdot,\cdot\rangle_\mu$.

A right convolution operator λ_i^r on $\ell^1(I,\mu)$ is defined by

$$\lambda_i^r(f)(j) = \sum_k \frac{d(k)}{d(i)d(j)} N_{j,i}^k f(k),\ f \in \ell^1(I,\mu).$$

In [**14**, Theorem 4.6], Hiai and Izumi proved that the amenability of $\mathscr{G}_{\mathscr{N}\subset\mathscr{M}}$ in the sense of Popa, i.e., the principal graph of $\mathscr{N} \subset \mathscr{M}$ satisfies the Følner type condition in [**31**, Definition 3.1], is equivalent to the existence of an almost invariant finite subset in I. Namely, for any finite set $J \subset I$ and $\varepsilon > 0$, there exists a finite subset $F \subset I$ such that

$$\|\lambda_i^r(\chi_F) - \chi_F\|_{1,\mu} < \varepsilon \|\chi_F\|_{1,\mu}$$

for any $i \in J$.

LEMMA 6.2. *Let $F \subset I$ be a finite set, and set $e_F := \sum_{i \in F} 1 \otimes e_{ii} \in A \otimes B(\ell^2(I))$. Define $S_F : A(\Delta) \to A \otimes M_{|F|}(\boldsymbol{C})$ by $S_F(x) = e_F x e_F$, and $T_F : A \otimes M_{|F|}(\boldsymbol{C}) \to A(\Delta)$ by $T_F(x) = \frac{1}{\|\chi_F\|_{1,\mu}} \mathscr{F}_{A(\Delta)}(x)$. Let Ψ be a completely positive map from A into itself. Then*

$$T_F \circ \Psi \otimes \mathrm{id} \circ S_F(\pi(a)V_i) = \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j,k \in F, l \in I} \pi(\hat{v}^{0*}_{\bar{j},j} \hat{\rho}_{\bar{j}}(\Psi(\hat{\rho}_j(a)\hat{v}^k_{j,i}))\hat{v}^l_{\bar{j},k})V_l$$

holds.

PROOF. First note $\|\chi_F\|_{1,\mu} = \mathscr{F}_{A(\Delta)}(e_F)$. Hence T_F is unital. The above lemma is shown by the following computation. (Recall matrix form representation in §2.)

$$\begin{aligned}
&T_F \circ (\Psi \otimes \mathrm{id}) \circ S_F(\pi(a)V_i) \\
&\quad = \sum_{j,k \in F} T_F \circ (\Psi \otimes \mathrm{id})(\hat{\rho}_j(a)\hat{v}^k_{j,i} \otimes e_{j,k}) \\
&\quad = \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j,k \in F, l \in I} \bar{V}_l(\hat{\rho}_l(\Psi(\hat{\rho}_j(a)\hat{v}^k_{j,i})) \otimes e_{j,k})\bar{V}_l^* \\
&\quad = \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j,k \in F, l,m,n \in I} \hat{v}^{m*}_{l,j} \hat{\rho}_l(\Psi(\hat{\rho}_j(a)\hat{v}^k_{j,i}))\hat{v}^n_{l,k} \otimes e_{m,n} \\
&\quad = \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j,k \in F, l,m,n \in I} \hat{\rho}_m(\hat{v}^{0*}_{\bar{j},j})\hat{v}^l_{m,\bar{j}} \hat{\rho}_l(\Psi(\hat{\rho}_j(a)\hat{v}^k_{j,i}))\hat{v}^n_{l,k} \otimes e_{m,n} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Corollary 2.19)} \\
&\quad = \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j,k \in F, l,m,n \in I} \hat{\rho}_m(\hat{v}^{0*}_{\bar{j},j})\hat{\rho}_m \hat{\rho}_{\bar{j}}(\Psi(\hat{\rho}_j(a)\hat{v}^k_{j,i}))\hat{v}^l_{m,\bar{j}} \hat{v}^n_{l,k} \otimes e_{m,n} \\
&\quad = \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j,k \in F, l,m,n \in I} \hat{\rho}_m(\hat{v}^{0*}_{\bar{j},j} \hat{\rho}_{\bar{j}}(\Psi(\hat{\rho}_j(a)\hat{v}^k_{j,i}))\hat{\rho}_m(\hat{v}^l_{\bar{j},k})\hat{v}^n_{m,l} \otimes e_{m,n} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.18(1))} \\
&\quad = \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j,k \in F, l \in I} \pi(\hat{v}^{0*}_{\bar{j},j} \hat{\rho}_{\bar{j}}(\Psi(\hat{\rho}_j(a)\hat{v}^k_{j,i}))\hat{v}^l_{\bar{j},k})V_l. \qquad \square
\end{aligned}$$

LEMMA 6.3. *There exist finite dimensional factors N_n, unital completely positive maps $S_n : A(\Delta) \to A \otimes N_n$ and $T_n : A \otimes N_n \to A(\Delta)$ such that $T_n \circ S_n(x) \to x$ σ-strongly* for $x \in \mathrm{Alg}(\pi(A), \{V_i\})$.*

PROOF. At first we compute $\langle \lambda^r_i(\chi_F), \chi_F \rangle_\mu$. Then we have

$$\begin{aligned}
\langle \lambda^r_i(\chi_F), \chi_F \rangle_\mu &= \sum_{j \in I} \lambda^r_i(\chi_F)(j)\chi_F(j)d(j)^2 = \sum_{j,k \in I} \frac{d(k)}{d(i)d(j)} N^k_{j,i}\chi_F(k)\chi_F(j)d(j)^2 \\
&= \sum_{j,k \in F} \frac{d(k)d(j)}{d(i)} N^k_{j,i}.
\end{aligned}$$

By Hiai-Izumi's Følner type condition, we can find a sequence $\{F_n\}$ of finite subsets in I such that

$$\lim_{n\to\infty} \frac{\|\lambda_i^r(\chi_{F_n}) - \chi_{F_n}\|_{1,\mu}}{\|\chi_{F_n}\|_{1,\mu}} = 0$$

for every $i \in I$. Set $N_n := M_{|F_n|}(\boldsymbol{C})$. Let $T_n := T_{F_n}$, $S_n := S_{F_n}$ be the unital completely positive maps constructed in Lemma 6.2.

By Lemma 6.2 and the above computation, we have

$$\begin{aligned}
T_n \circ S_n(\pi(a)V_i) &= \frac{1}{\|\chi_{F_n}\|_{1,\mu}} \sum_{j,k\in F_n, l\in I} \pi(\hat{v}^{0*}_{\bar{j},j}\hat{\rho}_{\bar{j}}(\hat{\rho}_j(a)\hat{v}^k_{j,i})\hat{v}^l_{\bar{j},k})V_l \\
&= \frac{1}{\|\chi_{F_n}\|_{1,\mu}} \sum_{j,k\in F_n, l\in I} \pi(a\hat{v}^{0*}_{\bar{j},j}\hat{\rho}_{\bar{j}}(\hat{v}^k_{j,i})\hat{v}^l_{\bar{j},k})V_l \\
&= \frac{1}{\|\chi_{F_n}\|_{1,\mu}} \sum_{j,k\in F_n, l\in I} \pi(a\hat{v}^{i*}_{\bar{j},k}\hat{v}^l_{\bar{j},k})V_l \\
&= \frac{1}{\|\chi_{F_n}\|_{1,\mu}} \sum_{j,k\in F_n} \frac{d(j)d(k)}{d(i)} N^i_{\bar{j},k}\pi(a)V_i \\
&= \frac{1}{\|\chi_{F_n}\|_{1,\mu}} \sum_{j,k\in F_n} \frac{d(j)d(k)}{d(i)} N^k_{j,i}\pi(a)V_i \\
&= \frac{\langle \lambda_i^r(\chi_{F_n}), \chi_{F_n}\rangle_\mu}{\|\chi_{F_n}\|_{1,\mu}}\pi(a)V_i.
\end{aligned}$$

Here

$$\begin{aligned}
\left| \frac{\langle \lambda_i^r(\chi_{F_n}), \chi_{F_n}\rangle_\mu}{\|\chi_{F_n}\|_{1,\mu}} - 1 \right| &= \left| \frac{\langle \lambda_i^r(\chi_{F_n}), \chi_{F_n}\rangle_\mu - \langle \chi_{F_n}, \chi_{F_n}\rangle_\mu}{\|\chi_{F_n}\|_{1,\mu}} \right| \\
&\le \frac{\|\lambda_i^r(\chi_{F_n}) - \chi_{F_n}\|_{1,\mu}\|\chi_{F_n}\|_\infty}{\|\chi_{F_n}\|_{1,\mu}} \\
&\to 0
\end{aligned}$$

as n goes to infinity. Hence $\lim_{n\to\infty} T_n \circ S_n(\pi(a)V_i) = \pi(a)V_i$ holds. $\square$

Now we can present the Effros-Lance type characterization on the amenability of $\mathscr{N} \subset \mathscr{M}$.

THEOREM 6.4. *Let (ϱ, K) be a representation of $\mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$ such that $\varrho|_{\mathscr{M}}$ and $\varrho|_{\mathscr{M}^{\mathrm{opp}}}$ are normal. Then $\|\varrho(x)\|_{B(K)} \le \|x\|_{A(\Delta)}$ for every $x \in \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$.*

PROOF. Take a unit vector $\xi \in K$. Let $T_n^{(1)}$, $S_n^{(1)}$ be as in Lemma 6.1, and $T_n^{(2)}$, $S_n^{(2)}$ as in Lemma 6.3. Set $\hat{T}_n^{(1)} = T_n^{(1)} \otimes T_n^{(1)\mathrm{opp}}$ and $\hat{S}_n^{(1)} = S_n^{(1)} \otimes S_n^{(1)\mathrm{opp}}$. Let ω_ξ be a vector state associated with ξ and define a normal state $\psi_{m,n}$ on $A(\Delta)$ by

$$\psi_{m,n} = \omega_\xi \circ \varrho \circ T_n^{(2)} \circ (\hat{T}_m^{(1)} \otimes \mathrm{id}_n) \circ (\hat{S}_m^{(1)} \otimes \mathrm{id}_n) \circ S_n^{(2)},$$

where id_n is the identity map on N_n.

Indeed, since $(\hat{S}_m^{(1)} \otimes \mathrm{id}_n) \circ S_n^{(2)}$ is a normal unital completely positive map from $A(\Delta)$ to $M_m \otimes M_m^{\mathrm{opp}} \otimes N_n$, and $\omega_\xi \circ \varrho \circ T_n^{(2)} \circ (\hat{T}_m^{(1)} \otimes \mathrm{id}_n)$ is a normal state on $M_m \otimes M_m^{\mathrm{opp}} \otimes N_n$, $\psi_{m,n}$ is a normal state on $A(\Delta)$. Note that $T_n^{(2)} \circ (\hat{T}_m^{(1)} \otimes \mathrm{id}_n)(M_m \otimes M_m^{\mathrm{opp}} \otimes N_n) \subset \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$, and hence $\varrho \circ T_n^{(2)} \circ (\hat{T}_m^{(1)} \otimes \mathrm{id}_n)$ is well defined. Hence we have $|\psi_{m,n}(x)| \le \|x\|_{A(\Delta)}$ for $x \in \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$.

We will verify $\lim_{n\to\infty} \lim_{m\to\infty} \psi_{m,n}(x) = \omega_\xi \circ \varrho(x)$ for $\pi(xj_{\mathscr{M}}(y))V_i \in \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$. By Lemma 6.2, we have

$$\begin{aligned}
&T_n^{(2)} \circ (\hat{T}_m^{(1)} \otimes \mathrm{id}_n) \circ (\hat{S}_m^{(1)} \otimes \mathrm{id}_n) \circ S_n^2(\pi(xj_{\mathscr{M}}(y))V_i)\\
&\quad = \frac{1}{\|F_n\|_{1,\mu}} \sum_{j,k\in F_n, l\in I} \pi(\hat{v}_{\bar{\jmath},j}^{0*} \hat{\rho}_{\bar{\jmath}}(\hat{T}_m^{(1)} \circ \hat{S}_m^{(1)}(\hat{\rho}_j(xj_{\mathscr{M}}(y))\hat{v}_{j,i}^k))\hat{v}_{\bar{\jmath},k}^l)V_l\\
&\quad = \frac{1}{\|F_n\|_{1,\mu}} \sum_{j,k\in F_n, l\in I} \pi(\hat{v}_{\bar{\jmath},j}^{0*})\pi(\rho_{\bar{\jmath}} T_m^{(1)} S_m^{(1)} \rho_j(x) \otimes j_{\mathscr{M}}(\rho_{\bar{\jmath}} T_m^{(1)} S_m^{(1)} \rho_j(y))\pi(\hat{v}_{j,i}^k \hat{v}_{\bar{\jmath},k}^l)V_l.
\end{aligned}$$

Then

$$\begin{aligned}
&\psi_{m,n}(xj_{\mathscr{M}}(y)V_i)\\
&\quad = \frac{1}{\|F_n\|_{1,\mu}} \sum_{j,k\in F_n, l\in I} \langle \varrho(\hat{v}_{\bar{\jmath},j}^{0*})\varrho(\rho_{\bar{\jmath}} T_m^{(1)} S_m^{(1)} \rho_j(x))\\
&\qquad\qquad \varrho(j_{\mathscr{M}}(\rho_{\bar{\jmath}} T_m^{(1)} S_m^{(1)} \rho_j(y)))\varrho(\hat{v}_{j,i}^k \hat{v}_{\bar{\jmath},k}^l)\varrho(V_l)\xi, \xi\rangle
\end{aligned}$$

holds. Since $\varrho|_{\mathscr{M}}$ and $\varrho|_{\mathscr{M}^{\mathrm{opp}}}$ are normal, and $\{T_m^{(1)} \circ S_m^{(1)}(a)\}_{m=1}^\infty$ converges to a strongly* for every $a \in \mathscr{M}$,

$$\begin{aligned}
\lim_{m\to\infty} \psi_{m,n}(x) &= \frac{1}{\|F_n\|_{1,\mu}} \sum_{j,k\in F_n, l\in I} \langle \varrho(\hat{v}_{\bar{\jmath},j}^{0*})\varrho(\rho_{\bar{\jmath}}\rho_j(x))\varrho(j_{\mathscr{M}}(\rho_{\bar{\jmath}}\rho_j(y)))\varrho(\hat{v}_{j,i}^k \hat{v}_{\bar{\jmath},k}^l)\varrho(V_l)\xi, \xi\rangle\\
&= \omega_\xi \circ \varrho \circ T_n^{(2)} \circ S_n^{(2)}(\pi(xj_{\mathscr{M}}(y))V_i)
\end{aligned}$$

for $\pi(xj_{\mathscr{M}}(y))V_i \in \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$. By letting $n \to \infty$, we have

$$\lim_{n\to\infty} \lim_{m\to\infty} \psi_{m,n}(\pi(xj_{\mathscr{M}}(y))V_i) = \omega_\xi \circ \varrho(\pi(xj_{\mathscr{M}}(y))V_i)$$

for $\pi(xj_{\mathscr{M}}(y))V_i \in \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$. Thus $\lim_{n\to\infty} \lim_{m\to\infty} \psi_{m,n}(x) = \omega_\xi \circ \varrho(x)$ for every $x \in \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$. It follows

$$\|\varrho(x)\xi\|^2 = \omega_\xi \circ \varrho(x^*x) \leq \|x^*x\|_{A(\Delta)} = \|x\|^2_{A(\Delta)}.$$

This holds for any unit vector ξ, and hence we have $\|\varrho(x)\|_{B(K)} \leq \|x\|_{A(\Delta)}$. □

As a corollary to Theorem 6.4, we have

COROLLARY 6.5. *We have $C^*(\mathscr{M}, J\mathscr{M}J, e_{\mathscr{N}}) \cong C^*_{\min}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, e_{\mathscr{N}})$ via the natural isomorphism, where $C^*_{\min}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, e_{\mathscr{N}})(= C^*_{\min}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\}))$ is the C^*-subalgebra in $A(\Delta)$ generated by $\mathscr{M}, \mathscr{M}^{\mathrm{opp}}$ and $e_{\mathscr{N}}$.*

PROOF. Let $W_l \in B(L^2(\mathscr{M}))$ be the canonical implementing isometry for ρ_l. Then the canonical map $\pi(x \otimes j_{\mathscr{M}}(y))V_l \in A(\Delta) \to xJyJd(l)W_l \in B(L^2(\mathscr{M}))$ is a representation of $\mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$, which is normal on $\mathscr{M}$ and $\mathscr{M}^{\mathrm{opp}}$. By Theorem 6.4, $\|\sum_i a_i d(i) W_i\|_{B(L^2(\mathscr{M}))} \leq \|\sum_i \pi(a_i)V_i\|_{A(\Delta)}$, $a_i \in \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}})$.

On the other hand, $\|\sum_i a_i d(i) W_i\|_{B(L^2(\mathscr{M}))} \geq \|\sum_i \pi(a_i)V_i\|_{A(\Delta)}$ by Theorem 5.4. Hence $\|\sum_i a_i d(i) W_i\|_{B(L^2(\mathscr{M}))} = \|\sum_i \pi(a_i)V_i\|_{A(\Delta)}$ holds. It follows that the canonical map is extended to the isomorphism between $C^*(\mathscr{M}, J\mathscr{M}J, \{W_i\})$ and $C^*_{\min}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$. □

In the above argument, we approximate $\omega_\xi \circ \varrho$ by a double sequence $\{\psi_{m,n}\}$. However we would like to approximate by a *sequence* of normal states (Proposition 6.7), which will be crucial in the proof of Lemma 7.6.

LEMMA 6.6. *Fix a faithful normal state φ on $\mathscr{M}$. Let $\xi \in L^2(\mathscr{M})_+$ be the representing vector for φ. For any finite set $\{x_k\}_{1\leq k\leq m}$, $\{y_k\}_{1\leq k\leq m} \subset \mathscr{M}$, finite subset $J \subset I$ and $\varepsilon > 0$, we can find a normal state ψ' on $A(\Delta)$ such that*
(1) $|\psi'(x_k j_{\mathscr{M}}(y_k)V_i) - \langle x_k J y_k J V_i \xi, \xi\rangle| < \varepsilon$, $1 \leq k \leq m$, $i \in J$,
(2) $\|\psi'|_{\mathscr{M}\otimes \boldsymbol{C}1} - \varphi\| < \varepsilon$,
(3) $\|\psi'|_{\boldsymbol{C}1\otimes\mathscr{M}^{\mathrm{opp}}} - \varphi^{\mathrm{opp}}\| < \varepsilon$.
(From now on, we omit π, and write aV_i instead of $\pi(a)V_i$.)

PROOF. The argument in the proof of Theorem 6.4 shows the existence of a state on $A(\Delta)$ satisfying (1). We see this more carefully. Set $X := \{x_k j_{\mathscr{M}}(y_k)V_i | 1 \leq k \leq m, i \in J\}$. Let $T_n^{(i)}$, $i = 1, 2$, be as in the proof of Theorem 6.4. Then we can find $n \in \boldsymbol{N}$ such that $|\langle x\xi, \xi\rangle - \langle T_n^{(2)} \circ S_n^{(2)}(x)\xi, \xi\rangle| < \varepsilon/2$ for $x \in X$.

By Lemma 6.1, we choose $T_m^{(1)}$, and $S_m^{(1)}$ such that

$$|\langle T_n^{(2)} \circ S_n^{(2)}(x)\xi, \xi\rangle - \langle T_n^{(2)} \circ (\hat{T}_m^{(1)} \circ \hat{S_m^{(1)}} \otimes \mathrm{id}_n) \circ S_n^{(2)}(x)\xi, \xi\rangle| < \varepsilon/2,\ x \in X$$
$$\|\varphi \circ \phi_i \circ T_m^{(1)} \circ S_m^{(1)} - \varphi \circ \phi_i\| < \varepsilon, i \in J,$$

where ϕ_i is the standard left inverse for ρ_i. Note ϕ_i is given by $\phi_i(x) = v\binom{0}{\bar{\imath},i}^* \rho_{\bar{\imath}}(x) v\binom{0}{\bar{\imath},i}$ in the notation in §2.

Set $\psi' := \psi_{m,n}$. Then ψ' satisfies the condition (1). Next we will verify (2). (In the following, we denote $T_n^{(i)}$ and $S_m^{(i)}$ by $T^{(i)}$ and $S^{(i)}$ for simplicity.)

By Lemma 6.2,

$$\begin{aligned}
&T^{(2)} \circ (\hat{T}^{(1)} \circ \hat{S}^{(1)} \otimes \mathrm{id}) \circ S^{(2)}(a \otimes 1) \\
&= \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j,k \in F, l \in I} \hat{v}^{0*}_{\bar{j},j} \hat{\rho}_{\bar{j}}(\hat{T}^{(1)} \circ \hat{S}^{(1)}((\rho_j(a) \otimes 1)\hat{v}^{k}_{\bar{j},0}))\hat{v}^{l}_{\bar{j},k} V_l \\
&= \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j \in F, l \in I} \hat{v}^{0*}_{\bar{j},j} (\rho_{\bar{j}} \circ T^{(1)} \circ S^{(1)} \circ \rho_j(a) \otimes 1)\hat{v}^{l}_{\bar{j},j} V_l \\
&= \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j \in F, l \in I, 1 \le e \le N^{l}_{\bar{j},j}} \frac{d(j)^2}{\sqrt{d(l)}} v\binom{0}{\bar{j},j}^* \rho_{\bar{j}} \circ T^{(1)} \circ S^{(1)} \circ \rho_j(a) v\binom{l}{\bar{j},j}^e \\
&\quad \otimes j_{\mathscr{M}}(v\binom{0}{\bar{j},j}^* v\binom{l}{\bar{j},j}^e) V_l \\
&= \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j \in F} d(j)^2 \phi_j \circ T^{(1)} \circ S^{(1)} \circ \rho_j(a) \otimes 1
\end{aligned}$$

holds. Hence we have

$$\psi'|_{\mathscr{M} \otimes \boldsymbol{C}} = \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j \in F} d(j)^2 \varphi \circ \phi_j \circ T^{(1)} \circ S^{(1)} \circ \rho_j.$$

By the choice of $T^{(1)}, S^{(1)}$,

$$\begin{aligned}
\|\psi'|_{\mathscr{M} \otimes \boldsymbol{C}} - \varphi\| &= \frac{1}{\|\chi_F\|_{1,\mu}} \left\| \sum_{j \in F} d(j)^2 (\varphi \circ \phi_j \circ T^{(1)} \circ S^{(1)} \circ \rho_j - \varphi \circ \phi_j \circ \rho_j) \right\| \\
&\le \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j \in F} d(j)^2 \|\varphi \circ \phi_j \circ T^{(1)} \circ S^{(1)} \circ \rho_j - \varphi \circ \phi_j \circ \rho_j\| \\
&\le \frac{1}{\|\chi_F\|_{1,\mu}} \sum_{j \in F} d(j)^2 \|\varphi \circ \phi_j \circ T^{(1)} \circ S^{(1)} - \varphi \circ \phi_j\| \\
&< \frac{\varepsilon}{\|\chi_F\|_{1,\mu}} \sum_{j \in F} d(j)^2 \\
&= \varepsilon.
\end{aligned}$$

Hence we have the condition (2). The condition (3) can be shown in a similar way. □

PROPOSITION 6.7. *Let φ and ξ be as in the previous lemma. Then there exists a sequence $\{\psi_n\}$ of normal states on $A(\Delta)$ such that* $\lim_{n\to\infty} \psi_n(a) = \langle a\xi, \xi \rangle$ *for every* $a \in \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$.

PROOF. First we prove $V_l^*(x \otimes 1)V_l = d(l)^2 \phi_l(x)$. This can be shown as follows.

$$\begin{aligned}
V_l^*(x\otimes 1)V_l &= \hat{v}_{\bar{l},l}^{0*}V_{\bar{l}}(x\otimes 1)V_l \\
&= \hat{v}_{\bar{l},l}^{0*}(\rho_{\bar{l}}(x)\otimes 1)V_{\bar{l}}V_l \\
&= \sum_m \hat{v}_{\bar{l},l}^{0*}(\rho_{\bar{l}}(x)\otimes 1)\hat{v}_{\bar{l},l}^{m}V_m \\
&= \sum_{m,1\le e\le N_{\bar{l},l}^m} \frac{d(l)^2}{\sqrt{d(m)}} v\binom{0}{\bar{l},l}^*\rho_{\bar{l}}(x)v\binom{m}{\bar{l},l}^e\otimes j_{\mathscr{M}}\left(v\binom{0}{\bar{l},l}^*v\binom{m}{\bar{l},l}^e\right)V_m \\
&= d(l)^2 v\binom{0}{\bar{l},l}^*\rho_{\bar{l}}(x)v\binom{0}{\bar{l},l}\otimes 1 \\
&= d(l)^2\phi_l(x)\otimes 1.
\end{aligned}$$

We also have $V_l^*(1\otimes j_{\mathscr{M}}(y))V_l = j_{\mathscr{M}}(\phi_l(y))$.

Let $\{x_i\}_{i=1}^\infty$ be a strongly dense countable subset in the unit ball $(\mathscr{M})_1$ of $\mathscr{M}$, $\{I_n\}$ an increasing sequence of finite subsets of I such that $\bigcup I_n = I$. Set $\psi(a)=\langle a\xi,\xi\rangle$. By Lemma 6.6, for each $n\in\boldsymbol{N}$, there exists a normal state ψ_n on $A(\Delta)$ such that
(1.n) $|\psi(x_i j_{\mathscr{M}}(x_j^*)V_l)-\psi_n(x_i j_{\mathscr{M}}(x_j^*)V_l)|<1/n$, $1\le i,j\le n$, $l\in I_n$,
(2.n) $\|\psi_n|_{\mathscr{M}}-\varphi\|<1/n$,
(3.n) $\|\psi_n|_{\mathscr{M}^{\mathrm{opp}}}-\varphi^{\mathrm{opp}}\|<1/n$.

We will prove $\lim_{n\to\infty}\psi_n(a)=\psi(a)$, $a\in \mathrm{Alg}(\mathscr{M},\mathscr{M}^{\mathrm{opp}},\{V_i\})$. Fix $x\in(\mathscr{M})_1$, $j\in\boldsymbol{N}$ and $l\in I$. First we verify $\lim_{n\to\infty}\psi_n(xj_{\mathscr{M}}(x_j^*)V_l)=\psi(xj_{\mathscr{M}}(x_j^*)V_l)$. For $\varepsilon>0$, choose x_i such that $d(l)\|x-x_i\|_{\varphi\circ\phi_l}<\varepsilon$. Choose $N\in\boldsymbol{N}$ such that $d(l)/\sqrt{N}<\varepsilon$, $l\in I_N$, $i,j\le N$. Then we have (1.n), (2.n) and (3.n) for every $n\ge N$. Then

$$\begin{aligned}
&|\psi_n(xj_{\mathscr{M}}(x_j^*)V_l)-\psi(xj_{\mathscr{M}}(x_j^*)V_l)| \\
&\quad\le |\psi_n((x-x_i)j_{\mathscr{M}}(x_j^*)V_l)|+|\psi_n(x_ij_{\mathscr{M}}(x_j^*)V_l)-\psi(x_ij_{\mathscr{M}}(x_j^*)V_l)| \\
&\qquad +|\psi((x-x_i)j_{\mathscr{M}}(x_j^*)V_l)| \\
&\quad\le \|j_{\mathscr{M}}(x_j)\|_\psi\|(x-x_i)V_l\|_\psi+\frac{1}{n}+\|j_{\mathscr{M}}(x_j)\|_{\psi_n}\|(x-x_i)V_l\|_{\psi_n} \\
&\quad\le \|(x-x_i)V_l\|_\psi+\frac{1}{n}+\|(x-x_i)V_l\|_{\psi_n}
\end{aligned}$$

holds. Here we have

$$\|(x-x_i)V_l\|_\psi^2=d(l)^2\|x-x_i\|_{\varphi\circ\phi_l}^2$$

and

$$\|(x-x_i)V_l\|_{\psi_n}^2=d(l)^2\|x-x_i\|_{\psi_n|_{\mathscr{M}}\circ\phi_l}^2$$

since $V_l(x^*-x_i)(x-x_i)V_l=d(l)^2\phi_l((x-x_i)^*(x-x_i))$. We also have that $\|x\|_{\varphi_1}\le\|x\|_{\varphi_2}+\sqrt{\delta}\|x\|$ if $\|\varphi_1-\varphi_2\|<\delta$ for $\varphi_1,\varphi_2\in(\mathscr{M}_*)_+$.

Hence we have

$$\begin{aligned}|\psi_n(xj_{\mathscr{M}}(x_j^*)V_l)-\psi(xj_{\mathscr{M}}(x_j^*)V_l)| &\leq \varepsilon+\frac{1}{n}+d(l)\|x-x_i\|_{\varphi\circ\phi_l}+\frac{2d(l)}{\sqrt{n}}\\ &\leq 5\varepsilon\end{aligned}$$

for any $n \geq N$. Hence we get $\lim_{n\to\infty}\psi_n(xj_{\mathscr{M}}(x_j^*)V_l)=\psi(xj_{\mathscr{M}}(x_j^*)V_l)$. In other words, for any $\varepsilon>0$, there exists an positive integer $N=N(x,\varepsilon,j,l)$ such that if $n\geq N(x,\varepsilon,j,l)$, then $|\psi_n(xj_{\mathscr{M}}(x_j^*)V_l)-\psi(xj_{\mathscr{M}}(x_j^*)V_l)|<\varepsilon$ holds.

Next we prove $\lim_{n\to\infty}\psi_n(xj_{\mathscr{M}}(y)V_l)=\psi(xj_{\mathscr{M}}(y)V_l)$ for $x,y\in(\mathscr{M})_1$ and $l\in I$. For $\varepsilon>0$, fix x_j such that $d(l)\|y^*-x_j\|_{\varphi\circ\phi_l}<\varepsilon$. Fix $N'\in\boldsymbol{N}$ such that $N'\geq N(x,\varepsilon,j,l)$, $d(l)/\sqrt{N'}<\varepsilon$.

If $n\geq N'$, then

$$\begin{aligned}&|\psi_n(xj_{\mathscr{M}}(y)V_l)-\psi(xj_{\mathscr{M}}(y)V_l)|\\ &\quad\leq|\psi_n(xj_{\mathscr{M}}(y-x_j^*)V_l)|+|\psi_n(xj_{\mathscr{M}}(x_j^*)V_l)-\psi(xj_{\mathscr{M}}(x_j^*)V_l)|+|\psi(xj_{\mathscr{M}}(y-x_j^*)V_l)|\\ &\quad\leq\|j_{\mathscr{M}}(y-x_j^*)V_l\|_{\psi}+\varepsilon+\|j_{\mathscr{M}}(y-x_j^*)V_l\|_{\psi_n}\\ &\quad\leq\varepsilon+d(l)\|j_{\mathscr{M}}(y-x_j^*)\|_{\varphi^{\mathrm{opp}}\circ\phi_l^{\mathrm{opp}}}+d(l)\|j_{\mathscr{M}}(y-x_j^*)\|_{\psi_n|_{\mathscr{M}^{\mathrm{opp}}}\circ\phi_l^{\mathrm{opp}}}\\ &\quad\leq\varepsilon+d(l)\|y^*-x_j\|_{\varphi\circ\phi_l}+d(l)\|j_{\mathscr{M}}(y-x_j^*)\|_{\varphi^{\mathrm{opp}}\circ\phi_l^{\mathrm{opp}}}+\frac{2d(l)}{\sqrt{n}}\\ &\quad\leq 5\varepsilon\end{aligned}$$

holds. Hence $\lim_{n\to\infty}\psi_n(a)=\psi(a)$ for all $a\in\mathrm{Alg}(\mathscr{M},\mathscr{M}^{\mathrm{opp}},\{V_l\})$. □

Remark. Let us consider the single factor case, i.e., $A(\Delta)=A=\mathscr{M}\otimes\mathscr{M}^{\mathrm{opp}}$. In [**6**, pp. 210], Connes claimed the existence of a normal state ψ' on $\mathscr{M}\otimes\mathscr{M}^{\mathrm{opp}}$ satisfying Lemma 6.6(1), $\psi'|_{\mathscr{M}}=\varphi$ and $\psi'|_{\mathscr{M}^{\mathrm{opp}}}=\varphi^{\mathrm{opp}}$ instead of Lemma 6.6(2), (3). The author cannot find a proof for this claim in the literature, and unable to prove this. However Lemma 6.6(2), (3) are enough for our purpose.

At the end of this section, we discuss the extension of automorphisms to the symmetric enveloping algebra. Let $\mathscr{N}\subset\mathscr{M}$ be a strongly amenable subfactor of type III_1. Take $\alpha\in\mathrm{Ker}\,\Phi$. Then we can extend $\alpha\otimes\mathrm{id}\in\mathrm{Aut}(\mathscr{M}\otimes\mathscr{M}^{\mathrm{opp}})$ to $\alpha\boxtimes\mathrm{id}\in\mathrm{Aut}(\mathscr{M}\underset{e_{\mathscr{N}}}{\boxtimes}\mathscr{M}^{\mathrm{opp}})$ such that $\alpha\boxtimes\mathrm{id}(e_{\mathscr{N}})=e_{\mathscr{N}}$ as explained in §2. It is clear that $\alpha\boxtimes\mathrm{id}$ is an automorphism of $C^*_{\min}(\mathscr{M},e_{\mathscr{N}},\mathscr{M}^{\mathrm{opp}})$. Hence by Theorem 6.4, we have $\tilde{\alpha}\in\mathrm{Aut}(C^*(\mathscr{M},e_{\mathscr{N}},J\mathscr{M}J))$ such that $\tilde{\alpha}=\alpha$ on $\mathscr{M}$, $\tilde{\alpha}=\mathrm{id}$ on $\mathscr{M}'$ and $\tilde{\alpha}(e_{\mathscr{N}})=e_{\mathscr{N}}$. Especially since $\sigma_t^{\varphi}\in\mathrm{Ker}\,\Phi$, we can apply the above argument for modular automorphisms. Here we remark that $\tilde{\alpha}(e_k)=e_k$ for $k\geq 1$ since $\tilde{\alpha}$ is trivial on $J\mathscr{M}J$. Hence $\tilde{\alpha}$ preserves $\mathscr{M}_k$, and this coincides with the usual extension of α to $\mathscr{M}_k$. So we denote $\tilde{\alpha}|_{\mathscr{M}_k}$ by α.

7. Approximate innerness of modular automorphisms.

As an analogue of [**6**, Theorem III.1], approximately inner automorphisms of subfactors can be characterized as follows. (Also see Appendix B.)

Theorem 7.1. *For $\theta\in\mathrm{Aut}(\mathscr{M},\mathscr{N})$, $\theta\in\overline{\mathrm{Int}}(\mathscr{M},\mathscr{N})$ if and only if for any*

$\varphi_1, \cdots, \varphi_n \in (\mathscr{M}_*)_+$ and $\varepsilon > 0$, we can find $0 \neq x \in \mathscr{N}$ such that

$$(*) \quad \|x\xi_{\varphi_j} - U_\theta(\xi_{\varphi_j})x\|^2 \leq \varepsilon^2 \sum_j \varphi_j(x^*x),$$

where $\xi_{\varphi_j} \in L^2(\mathscr{M})_+$ is the representing vector for φ_j, $U_\theta \in B(L^2(\mathscr{M}))$ the standard implementing unitary of θ.

By using Theorem 7.1, we will prove the following theorem, which implies Theorem 2.2 as remarked at the end of §3.

THEOREM 7.2. *For any $0 \neq T \in \boldsymbol{R}$ and any faithful normal state ψ with $\psi \circ \mathscr{E}_0 = \psi$, σ_T^ψ is approximately inner.*

PROOF. Fix $\varepsilon > 0$ and $\varphi_1, \cdots, \varphi_n \in (\mathscr{M}_*)_+$. We may assume φ_1 is faithful. Set $\varphi := [\mathscr{M} : \mathscr{N}] \sum_j \varphi_j \circ \mathscr{E}_0$. Then $\varphi \circ \mathscr{E}_0 = \varphi$ and $\varphi \geq \varphi_j$ hold. For the last inequality, we used the Pimsner-Popa inequality $\mathscr{E}_0(x) \geq [\mathscr{M} : \mathscr{N}]^{-1}x$ [**29**]. For simplicity, we denote ξ_{φ_j} by ξ_j, and U_θ by θ. Put $\alpha = \sigma_T^\varphi$. Since $\sigma_T^\psi = \mathrm{Ad}\,(D\psi : D\varphi)_T\alpha$, it suffices to show that α satisfies Theorem 7.1($*$). Define T_j as $T_j x\xi_\varphi = x\xi_j$. Then $\|T_j\| \leq 1$, and T_j belongs to $\mathscr{M}'$. Set $b_j := JT_j^*J \in \mathscr{M}$. Then $b_j^*\xi_\varphi = \xi_\varphi b_j = \xi_j$ holds by definition. We have $\|b_j\| \leq 1$. Set $A := C^*(\mathscr{M}, e_{\mathscr{N}}, J\mathscr{M}J)$. Define $X := e_{\mathscr{N}} \sum_j |Jb_j^*J - b_j^*|^2 e_{\mathscr{N}} \in A$.

Fix $f(x) \in C_c^\infty(\boldsymbol{R}_+^*)$ as in [**6**, p. 205]. Then $0 \leq 1 - e_{\mathscr{N}} + X + e_{\mathscr{N}}|f(\Delta_\varphi) - 1|^2 e_{\mathscr{N}} \leq 4n + 2$ holds. Here note that $e_{\mathscr{N}}\xi_\varphi = \xi_\varphi$, $X\xi_\varphi = 0$ and $e_{\mathscr{N}}|f(\Delta_\varphi) - 1|^2 e_{\mathscr{N}}\xi_\varphi = 0$. Hence we get $\|4n + 2 - (1 - e_{\mathscr{N}}) - X - e_{\mathscr{N}}|f(\Delta_\varphi) - 1|^2 e_{\mathscr{N}}\| = 4n + 2$.

Let $\tilde{\alpha}$ be the automorphism of A as in the end of the previous section. We actually have $\|4n+2-(1-e_{\mathscr{N}})-\tilde{\alpha}(X)-e_{\mathscr{N}}|f(\Delta_\varphi)-1|^2 e_{\mathscr{N}}\| = 4n+2$, whose proof is postponed (see Lemma 7.8). Then we can find a unit vector $\eta \in L^2(\mathscr{M})$ such that
(i) $\|(Jb_j^*J - \alpha(b_j)^*)e_{\mathscr{N}}\eta\| \leq \varepsilon$,
(ii) $\|(f(\Delta_\varphi) - 1)e_{\mathscr{N}}\eta\| \leq \varepsilon$,
(iii) $\|(1 - e_{\mathscr{N}})\eta\| \leq \varepsilon$.

Set $\eta_1 := e_{\mathscr{N}}\eta$. Then $\eta_1 \in e_{\mathscr{N}}L^2(\mathscr{M}) = L^2(\mathscr{N})$. By (iii), $1 \geq \|\eta_1\| \geq 1 - \varepsilon$. Next set $\eta_2 := f(\Delta_\varphi)\eta_1$. Then $\eta_2 \in \mathscr{D}(\Delta_\varphi^{\frac{1}{2}}) \cap L^2(\mathscr{N})$. By (ii), we have $\|\eta_2 - \eta_1\| \leq \varepsilon$, and hence we get $1 \geq \|\eta_2\| \geq 1 - 2\varepsilon$. By (i), we get $\|(Jb_j^*J - \alpha(b_j^*))\eta_2\| \leq 3\varepsilon$. By the choice of $f(x)$, $\|(\Delta_\varphi^{\frac{1}{2}} - 1)\eta_2\| \leq \varepsilon\|\eta_1\| \leq \varepsilon$. Since $\mathscr{N}\xi_\varphi$ is dense in $\mathscr{D}(\Delta_\varphi^{\frac{1}{2}}) \cap L^2(\mathscr{N})$ in the graph norm, we can choose $x\xi_\varphi \in \mathscr{N}\xi_\varphi$ such that $\|\eta_2 - x\xi_\varphi\| \leq \varepsilon$ and $\|\Delta_\varphi^{\frac{1}{2}}\eta_2 - \Delta_\varphi^{\frac{1}{2}}x\xi_\varphi\| \leq \varepsilon$. Then $\|x\xi_\varphi\| \geq \|\eta_2\| - \|\eta_2 - x\xi_\varphi\| \geq 1 - 3\varepsilon$ holds. Thus we get

$$\begin{aligned}
\|(Jb_j^*J - \alpha(b_j^*))x\xi_\varphi\| &\leq \|(Jb_j^*J - \alpha(b_j^*))\eta_2\| + \|(Jb_j^*J - \alpha(b_j))(\eta_2 - x\xi_\varphi)\| \\
&\leq 3\varepsilon + 2\varepsilon \\
&= 5\varepsilon \\
&\leq \frac{5\varepsilon}{1 - 3\varepsilon}\|x\xi_\varphi\|
\end{aligned}$$

and

$$\begin{aligned}\|x\xi_\varphi - \xi_\varphi x\| &= \|x\xi_\varphi - \Delta_\varphi^{\frac{1}{2}} x\xi_\varphi\| \\ &\le \|x\xi_\varphi - \eta_2\| + \|\eta_2 - \Delta_\varphi^{\frac{1}{2}}\eta_2\| + \|\Delta_\varphi^{\frac{1}{2}}\eta_2 - \Delta_\varphi^{\frac{1}{2}} x\xi_\varphi\| \\ &\le 3\varepsilon \\ &\le \frac{3\varepsilon}{1-3\varepsilon}\|x\xi_\varphi\|.\end{aligned}$$

Hence if we take a sufficiently small ε, we can assume $\|x\xi_\varphi - \xi_\varphi x\| \le 6\varepsilon\|x\xi_\varphi\|$ and $\|(Jb_j^* J - \alpha(b_j^*))x\xi_\varphi\| \le 6\varepsilon\|x\xi_\varphi\|$.

Finally we have

$$\begin{aligned}\|x\xi_j - \alpha(\xi_j)x\| &= \|x\xi_\varphi b_j - \alpha(b_j^*\xi_\varphi)x\| \\ &= \|Jb_j^* Jx\xi_\varphi - \alpha(b_j^*)\xi_\varphi x\| \\ &\le \|Jb_j^* Jx\xi_\varphi - \alpha(b_j^*)x\xi_\varphi\| + \|\alpha(b_j^*)(x\xi_\varphi - \xi_\varphi x)\| \\ &\le 12\varepsilon\|x\xi_\varphi\|.\end{aligned}$$

Hence we have $\|x\xi_j - \alpha(\xi_j)x\|^2 \le 144\varepsilon^2\|x\xi_\varphi\|^2 = \varepsilon^2\varphi(x^*x) = 144\varepsilon^2[\mathscr{M} : \mathscr{N}]\sum_j \varphi_j \circ \mathscr{E}_0(x^*x) = 144\varepsilon^2[\mathscr{M} : \mathscr{N}]\sum_j \varphi_j(x^*x)$. Hence α satisfies Theorem 7.1(∗), and σ_T^ψ is approximately inner. □

It remains to show $\|4n + 2 - (1 - e_{\mathscr{N}}) - \tilde{\alpha}(X) - e_{\mathscr{N}}|f(\Delta_\varphi) - 1|^2 e_{\mathscr{N}}\| = 4n + 2$. To do this, we use the Effros-Lance type characterization (Theorem 6.4). To construct the Jones tower in the symmetric enveloping algebra for $\mathscr{N} \subset \mathscr{M}$, hence we fix a tunnel for $\mathscr{M} \supset \mathscr{N} := \mathscr{M}_{-1} \supset \mathscr{M}_{-2} \supset \mathscr{M}_{-3} \supset \cdots$, and denote by $\mathscr{E}_{-k}$ the minimal conditional expectation from $\mathscr{M}_{-k}$ onto $\mathscr{M}_{-k-1}$. Note that σ_t^φ does not necessarily preserve $\mathscr{M}_{-k}$, however $\sigma_t^{\varphi^{(k)}}$ does, where $\varphi^{(k)} := \varphi \circ \mathscr{E}_{-k+1} \circ \cdots \circ \mathscr{E}_0$.

Lemma 7.3. *Let K be a compact metric space, μ a probability measure on K, $a(k)$, $c(k)$ bounded σ-strongly* continuous maps from K to $\mathscr{M}'$, and $b(k)$ a bounded σ-strongly* continuous map from K to $\mathscr{M}_n$, and consider $\int_K a(k)b(k)c(k)d\mu(k) \in B(L^2(\mathscr{M}))$.*
(i) *The set $B \subset B(L^2(\mathscr{M}))$ of all elements of the above form, (with K, μ, n varying), is a $*$-subalgebra of $B(L^2(\mathscr{M}))$.*
(ii) *B is invariant under* $\mathrm{Ad}\,\Delta_\varphi^{it}$.

Proof. (i) It is easy to see that B is closed under summation and $*$-operation. We verify that B is closed under multiplication. Let $a_i(k_i), c_i(k_i)$ be bounded σ-strongly* continuous maps from K_i to $\mathscr{M}^{\mathrm{opp}}$, and $b_i(k_i)$ a bounded σ-strongly* continuous map from K_i to $\mathscr{M}_n$, $i = 1, 2$, and consider $\int_{K_i} a_i(k_i)b_i(k_i)c_i(k_i)d\mu_i(k_i)$, $i = 1, 2$. Let $f_n \in \mathscr{M}^{\mathrm{opp}}$ be the Jones projection for $\mathscr{M}_{-2n}^{\mathrm{opp}} \subset \mathscr{M}_{-n}^{\mathrm{opp}} \subset \mathscr{M}^{\mathrm{opp}}$, and choose $v \in \mathscr{M}^{\mathrm{opp}}$ such that $v^* f_n v = 1$. Note $\mathscr{M}_n \vee \{f_n\} = \mathscr{M}_{2n}$. Then

$$\begin{aligned}c_1(k_1)a_2(k_2) &= c_1(k_1)a_2(k_2)v^* f_n v \\ &= [\mathscr{M} : \mathscr{N}]^2 \mathscr{E}_{\mathscr{M}_{-n}^{\mathrm{opp}}}(c_1(k_1)a_2(k_2)v^* f_n) f_n \mathscr{E}_{\mathscr{M}_{-n}^{\mathrm{opp}}}(f_n v)\end{aligned}$$

holds by the push-down lemma. Set $d(k_1,k_2) := [\mathscr{M}:\mathscr{N}]^2 \mathscr{E}_{\mathscr{M}_{-n}^{\text{opp}}}(c_1(k_1)a_2(k_2)v^* f_n)$, and $v' := \mathscr{E}_{\mathscr{M}_{-n}^{\text{opp}}}(f_n v)$. Then $d(k_1,k_2)$ is a σ-strongly* continuous map from $K_1 \times K_2$ to $\mathscr{M}_{-n}^{\text{opp}}$. Then we have

$$
\begin{aligned}
&\left(\int_{K_1} a_1(k_1)b_1(k_1)c_1(k_1)d\mu_1(k_1)\right)\left(\int_{K_2} a_2(k_2)b_2(k_2)c_2(k_2)d\mu_2(k_2)\right)\\
&\quad = \int_{K_1\times K_2} a_1(k_1)b_1(k_1)c_1(k_1)a_2(k_2)b_2(k_2)c_2(k_2)d(\mu_1\times\mu_2)(k_1,k_2)\\
&\quad = \int_{K_1\times K_2} a_1(k_1)b_1(k_1)d(k_1,k_2)f_n v' b_2(k_2)c_2(k_2)d(\mu_1\times\mu_2)(k_1,k_2)\\
&\quad = \int_{K_1\times K_2} a_1(k_1)d(k_1,k_2)b_1(k_1)f_n b_2(k_2)v' c_2(k_2)d(\mu_1\times\mu_2)(k_1,k_2).
\end{aligned}
$$

Here $a_1(k_1)d(k_1,k_2)$ and $v'c_2(k_2)$ are σ-strongly* continuous maps from $K_1 \times K_2$ to $\mathscr{M}^{\text{opp}}$, and $b_1(k_1)f_n b_2(k_2)$ is a σ-strongly* continuous map from $K_1 \times K_2$ to $\mathscr{M}_{2n}$. Hence the above operator is in B.

(ii) First note that we have $\operatorname{Ad}\Delta_{\varphi^{(n-1)}}^{it}\mathscr{M}_n = \mathscr{M}_n$ for every $n \geq 0$. We have $\Delta_\varphi^{it} = u_t^n J u_t^n J \Delta_{\varphi^{(n-1)}}^{it}$, where $u_t^n := (D\varphi : D\varphi^{(n-1)})_t$. Let $b(k)$ be a σ-strongly* continuous map from K to $\mathscr{M}_n$. Then we have $\operatorname{Ad}\Delta_\varphi^{it}(b(k)) = Ju_t^n J u_t^n \operatorname{Ad}\Delta_{\varphi_0}^{it}(b(k))u_t^* J u_t^* J$. Here $\operatorname{Ad} u_t^n \Delta_{\varphi^{(n-1)}}^{it}(b(k))$ is a σ-strongly$*$ continuous map from K to $\mathscr{M}_n$. Now it is easy to see that $\operatorname{Ad}\Delta_\varphi^{it}$ preserves B. □

Lemma 7.4. *Define* $\mathscr{A} := \{T \in B | t \to \Delta_\varphi^{it} T \Delta_\varphi^{-it}$ *is norm continuous*$\}^{-\|\cdot\|}$.
(i) $\mathscr{A}$ *is a* C^**-algebra.*
(ii) $\theta_t := \operatorname{Ad}\Delta_\varphi^{it}$ *is a pointwise norm continuous action of* $\boldsymbol{R}$ *on* $\mathscr{A}$.
(iii) *For* $a \in C^*(\mathscr{M}, e_{\mathscr{N}}, \mathscr{M}^{\text{opp}})$ *and* $f \in C_c(\boldsymbol{R})$, $\int_{\boldsymbol{R}} f(t)\Delta_\varphi^{it} a \Delta_\varphi^{-it} dt$ *is in* $\mathscr{A}$.

Proofs of (i) and (ii) are the same as those in [**6**, Lemma IV.3]. To prove (iii), we only have to prove it for abc with $a, c \in \mathscr{M}^{\text{opp}}$ and $b \in \mathscr{M}_n$. Here we use the notations in the proof of Lemma 7.3. We have

$$
\begin{aligned}
\operatorname{Ad}\Delta_\varphi^{it}(abc) &= \operatorname{Ad}\Delta_\varphi^{it}(a)\operatorname{Ad}\Delta_\varphi^{it}(b)\operatorname{Ad}\Delta_\varphi^{it}(c)\\
&= \operatorname{Ad}\Delta_\varphi^{it}(a) J u_t^n J u_t^n \operatorname{Ad}\Delta_{\varphi^{(n-1)}}^{it}(b) u_t^{n*} J u_t^{n*} J \operatorname{Ad}\Delta_\varphi^{it}(c).
\end{aligned}
$$

Here $\operatorname{Ad}\Delta_\varphi^{it}(a)Ju_t^n J, Ju_t^{n*}J\operatorname{Ad}\Delta_\varphi^{it}(c) \in \mathscr{M}^{\text{opp}}$ and $\operatorname{Ad} u_t^n \Delta_{\varphi^{(n-1)}}^{it}(b) \in \mathscr{M}_n$ are σ-strongly* continuous maps. Hence $\int_{\boldsymbol{R}} f(t)\operatorname{Ad}\Delta_\varphi^{it}(abc)dt \in B$. It is clear that this operator is in $\mathscr{A}$. □

By Lemma 7.4, $(\mathscr{A}, \theta)$ is a C^*-dynamical system and $(\mathscr{A}, \Delta_\varphi^{it})$ is a C^*-covariant representation for $(\mathscr{A}, \theta)$, say π. Set $\mathscr{B} := \pi(\mathscr{A} \rtimes_\theta \boldsymbol{R})$. Then any element of $\mathscr{B}$ is the norm limit of operators of the form $\int a(s)\Delta_\varphi^{is} ds$, where $a(s)$ is a norm continuous map from $\boldsymbol{R}$ to $\mathscr{A}$ with a compact support. Our next purpose is to construct $\beta \in \operatorname{Aut}(\mathscr{B})$ such that $\beta(\int a(s)\Delta_\varphi^{is} ds) = \int \tilde{\alpha}(a(s))\Delta_\varphi^{is} ds$.

The next observation will be used in the proof of Lemma 7.6.

LEMMA 7.5. *For a given sequence* $\{a^i_{nm}\}_{(i,n,m)\in \boldsymbol{N}\times\boldsymbol{N}\times\boldsymbol{Z}} \subset \boldsymbol{C}$ *with* $\lim_{n\to\infty} a^i_{nm} = a_i$, *one can find an increasing sequence* $\{N_n\} \subset \boldsymbol{N}$ *satisfying* $\lim_{n\to\infty} N_n = \infty$ *and*

$$\lim_{n\to\infty} \frac{1}{2N_n} \sum_{m=-N_n+1}^{N_n} a^i_{nm} = a_i \quad (\textit{for each } i).$$

PROOF. By subtracting a_i from each a^i_{nm}, we may assume $a_i = 0$. We construct a strictly increasing sequence $\{M_k\}_{k=0}^{\infty}$ inductively as follows: We set $M_0 := 0$ and assume M_{k-1} as been chosen. Since $\lim_{n\to\infty} a^i_{mn} = 0$ for each (i,m), we can certainly choose $M_k > M_{k-1}$ in such a way that as long as $n > M_k$ we have $|a^i_{nm}| < k^{-1}$ for $i = 1, 2, \cdots, k$, and $m = -k+1, -k+2, \cdots, k$. For each n we set $N_n = k$ with the index k satisfying $M_k < n \le M_{k+1}$. We claim that these N_i's do the job. First we have $N_n \nearrow \infty$ from the construction. Secondly for each i we choose n satisfying $i \le N_n (= k)$. Note $M_k < n \le M_{k+1}$ by definition of N_n and consequently $|a^i_{nm}| < 1/k$ for ($i \le k$ and) each $m = -k+1, -k+2, \cdots, k$. We thus have

$$\left|\frac{1}{2N_n} \sum_{m=-N_n+1}^{N_n} a^i_{nm}\right| = \left|\frac{1}{2k} \sum_{m=-k+1}^{k} a^i_{nm}\right| < \frac{1}{k} = \frac{1}{N_n}.$$

Since $N_n^{-1} \searrow 0$, we are done. □

Let $\mathfrak{H}$ be the standard Hilbert space for $\mathscr{M} \underset{e_{\mathscr{N}}}{\boxtimes} \mathscr{M}^{\mathrm{opp}}$. Note that $L^2(\mathscr{M} \otimes \mathscr{M}^{\mathrm{opp}}) = L^2(\mathscr{M}) \otimes L^2(\mathscr{M}^{\mathrm{opp}})$ is a subspace of $\mathfrak{H}$, and $\varphi \otimes \varphi^{\mathrm{opp}} \circ \mathscr{E}_A$ is given by a vector state for $\xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}$. As was explained in §2, we have the extensions $\sigma^\varphi_t \boxtimes \mathrm{id}$ and $\sigma^\varphi_t \boxtimes \sigma^{\varphi^{\mathrm{opp}}}_{-t}$. Let $\Delta^{it}_\varphi \boxtimes \mathrm{id}$ and $\Delta^{it}_\varphi \boxtimes \Delta^{-it}_{\varphi^{\mathrm{opp}}}$ be the standard implementing unitaries for $\sigma^\varphi_t \boxtimes \mathrm{id}$ and $\sigma^\varphi_t \boxtimes \sigma^{\varphi^{\mathrm{opp}}}_{-t}$ respectively.

LEMMA 7.6. (A) *There exists a sequence* $\{\xi_\nu\} \subset \mathfrak{H}$ *such that* (1) $\|\Delta^{it}_\varphi \boxtimes \Delta^{-it}_{\varphi^{\mathrm{opp}}} \xi_\nu - \xi_\nu\| \to 0$ *for any* $t \in \boldsymbol{R}$ *and* (2) $\langle abc\xi_\nu, \xi_\nu\rangle \to \langle abc\xi_\varphi, \xi_\varphi\rangle$ *for any* $a, c \in \mathscr{M}^{\mathrm{opp}}$ *and* $b \in \mathscr{M}_k$.
(B) *There exists a sequence* $\{\Psi_\nu\}$ *of normal states on* $B(L^2(\mathscr{M}))$ *such that* (1) $\Psi_\nu(\Delta^{it}_\varphi) \to 1$ *for any* $t \in \boldsymbol{R}$ *and* (2) $\Psi_\nu(abc) \to \langle abc\xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}, \xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}\rangle$ *for any* $a, c \in \mathscr{M}^{\mathrm{opp}}$ *and* $b \in \mathscr{M}_k$.

PROOF. (A) Since $C^*(\mathscr{M}, e_{\mathscr{N}}, J\mathscr{M}J) \cong C^*_{\min}(\mathscr{M}, e_{\mathscr{N}}, \mathscr{M}^{\mathrm{opp}})$, $\Psi(abc) := \langle abc\xi_\varphi, \xi_\varphi\rangle$ can be viewed as a state of $C^*_{\min}(\mathscr{M}, e_{\mathscr{N}}, \mathscr{M}^{\mathrm{opp}})$, and can be extended to a state on $B(\mathfrak{H})$ by the Hahn-Banach extension Theorem. By Lemma 6.7, we can find a sequence $\{\Psi_n\}$ of normal states on $A(\Delta)$ such that
(*a*) $\lim_{n\to\infty} \Psi_n(a) = \Psi(a)$ for every $\mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$,
(*b*) $\|\Psi_n|_{\mathscr{M}} - \varphi\| < 1/n$,
(*c*) $\|\Psi_n|_{\mathscr{M}^{\mathrm{opp}}} - \varphi^{\mathrm{opp}}\| < 1/n$.
We extend Ψ_n to a normal state on $B(\mathfrak{H})$.

Let $\{x_i\}_{i=1}^{\infty}$ be a countable strongly dense subset in the unit ball $(\mathscr{M})_1$. For simplicity we denote $\Delta^{it}_\varphi \boxtimes \Delta^{-it}_{\varphi^{\mathrm{opp}}}$ by v_t. Since Ψ is invariant under $\mathrm{Ad}\, v_t$, $\lim_{n\to\infty}(\mathrm{Ad}\, v_t \Psi_n)(x) = \Psi(x)$ for every $x \in \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\})$ and $t \in \boldsymbol{R}$. Define

$$a_{nm}^{i,j,l} = \int_{m-1}^{m} v_t \Psi_n v_t^*(x_i j_{\mathscr{M}}(x_j^*)V_l)dt.$$

By the Lebesgue convergence theorem, $\lim_{n\to\infty} a_{nm}^{i,j,l} = \Psi(x_i j_{\mathscr{M}}(x_j^*)V_l)$ holds. We choose a sequence $\{N_n\}$ for $\{a_{nm}^{i,j,l}\}$ as in Lemma 7.5. We replace Ψ_n by

$$\Psi_n' := \frac{1}{2N_n} \int_{-N_n}^{N_n} v_t \Psi_n v_t^* dt.$$

Hence

$$\lim_{n\to\infty} \frac{1}{2N_n} \sum_{m=-N_n+1}^{N_n} a_{nm}^{i,j,l} = \lim_{n\to\infty} \frac{1}{2N_n} \int_{-N_n}^{N_n} v_t \Psi_n v_t^*(x_i j_{\mathscr{M}}(x_j^*)V_l)dt = \Psi(x_i j_{\mathscr{M}}(x_j^*)V_l)$$

holds by Lemma 7.5. This means that $\lim_{n\to\infty} \Psi_n'(x_i j_{\mathscr{M}}(x_j^*)V_l) = \Psi(x_i j_{\mathscr{M}}(x_j^*)V_l)$. Since $\Psi_n'|_{\mathscr{M}} = \frac{1}{2N_n}\int_{-N_n}^{N_n} \Psi_n|_{\mathscr{M}} \circ \sigma_{-t}^{\varphi}$,

$$\|\Psi_n'|_{\mathscr{M}} - \varphi\| \le \frac{1}{2N_n} \int_{-N_n}^{N_n} \|(\Psi_n|_{\mathscr{M}} - \varphi) \circ \sigma_{-t}^{\varphi}\| < \frac{1}{n}$$

holds. In a similar way, we can show $\|\Psi_n'|_{\mathscr{M}^{\mathrm{opp}}} - \varphi^{\mathrm{opp}}\| < 1/n$. Then the same argument as in Proposition 6.7 works in this case, and $\lim_{n\to\infty} \Psi_n'(x) = \Psi(x)$ holds for every $x \in \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, \{V_i\}) (= \mathrm{Alg}(\mathscr{M}, \mathscr{M}^{\mathrm{opp}}, e_{\mathscr{N}}) = \bigcup_k \mathscr{M}^{\mathrm{opp}} \mathscr{M}_k \mathscr{M}^{\mathrm{opp}})$.

Since $\lim_{n\to\infty} N_n = \infty$ we have $\|\Psi_n' - v_t \Psi_n' v_t^*\| \to 0$ for every $t \in \boldsymbol{R}$. Let $\xi_n \in \mathfrak{H}$ be the representing vector for $\Psi_n'|_{\mathscr{M} \underset{e_{\mathscr{N}}}{\boxtimes} \mathscr{M}^{\mathrm{opp}}}$. Then $\{\xi_n\}$ satisfies (1) and (2).

(B) By Proposition 4.8, we can find $\{\lambda_n^k\}_{k=1}^{m_n} \subset \boldsymbol{R}_+^*$ and $\{u_n^k\}_{k=1}^{m_n} \subset U(\mathscr{M})$ such that $\sum_k \lambda_n^k = 1$, $\|[u_n^k, \varphi]\| < 1/n$ and $\sum_{k=1}^{m_n} \lambda_n^k u_n^{k*} x u_n^k \to E_{\mathscr{M}' \cap \mathscr{M}_k}^{\varphi}(x)$ σ-strongly* for $x \in \mathscr{M}_k$. Define a state Ψ_n on $B(L^2(\mathscr{M}))$ by $\Psi_n(x) := \sum_k \lambda_n^k \langle x u_n^k \xi_\varphi, u_n^k \xi_\varphi \rangle$. Since $\|[u_n^k, \varphi]\| < 1/n$, $\Psi_n(\Delta_\varphi^{it}) \to 1$ for any $t \in \boldsymbol{R}$. (For example, see [**37**, Lemma.XVIII.4.13].) For $a, c \in \mathscr{M}^{\mathrm{opp}}$ and $b \in \mathscr{M}_k$, $\Psi_n(abc) \to \langle a E_{\mathscr{M}' \cap \mathscr{M}_k}^{\varphi}(b) c \xi_\varphi, \xi_\varphi \rangle = \langle abc \xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}, \xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}} \rangle$. (See §5 for the last equation.) □

Lemma 7.7. (1) *With the notations in Lemma* 7.3, *there exists an automorphism* $\tilde{\alpha} \in \mathrm{Aut}(\mathscr{A})$ *such that*

$$\tilde{\alpha}\Big(\int_K a(s)b(s)c(s)d\mu(s)\Big) = \int_K a(s)\alpha(b(s))c(s)d\mu(s).$$

(2) *There exists an automorphism* $\beta \in \mathrm{Aut}(\mathscr{B})$ *such that*

$$\beta\Big(\int a(s)\Delta_\varphi^{is} ds\Big) = \int \tilde{\alpha}(a(s))\Delta_\varphi^{is} ds.$$

Proof. (1) Let $\mathscr{A}_1$ be the subalgebra of $B(\mathfrak{H})$ formed by the operators $\int_K a(k)b(k)c(k)d\mu(k)$ with the notations in Lemma 7.3. To distinguish elements in B and $\mathscr{A}_1$, we use the notations $\int_{K,B} a(k)b(k)c(k)d\mu(k)$ and $\int_{K,\mathscr{A}_1} a(k)b(k)c(k)d\mu(k)$. Define ϱ by

$$\varrho\Big(\int_{K,\mathscr{A}_1} a(k)b(k)c(k)d\mu(k)\Big) = \int_{K,B} a(k)b(k)c(k)d\mu(k).$$

By Lemma 7.6(A) and the Lebesgue convergence theorem,

$$\Big\langle \int_{K,B} a(k)b(k)c(k)d\mu(k)\xi_\varphi, \xi_\varphi \Big\rangle = \lim_{\nu\to\infty} \Big\langle \int_{K,\mathscr{A}_1} a(k)b(k)c(k)d\mu(k)\xi_\nu, \xi_\nu \Big\rangle.$$

Hence $|\langle \varrho(T)\xi_\varphi, \xi_\varphi\rangle| \leq \|T\|$ holds for $T \in \mathscr{A}_1$. Since ξ_φ is cyclic for B, ϱ is a well-defined by [**6**, Lemma IV.5].

By Lemma 7.6(B) and the Lebesgue convergence theorem, we have

$$\Big\langle \int_{K,\mathscr{A}_1} a(k)b(k)c(k)d\mu(k)\xi_{\varphi\otimes\varphi^{\mathrm{opp}}}, \xi_{\varphi\otimes\varphi^{\mathrm{opp}}} \Big\rangle = \lim_{\nu\to\infty} \Psi_\nu\Big(\int_{K,B} a(k)b(k)c(k)d\mu(k)\Big).$$

Hence we get $|\langle T\xi_{\varphi\otimes\varphi^{\mathrm{opp}}}, \xi_{\varphi\otimes\varphi^{\mathrm{opp}}}\rangle| \leq \|\varrho(T)\|$ for $T \in \mathscr{A}_1$. Since $\xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}$ is cyclic for $\mathscr{A}_1$, ϱ is an isometry by [**6**, Lemma IV.5].

Then

$$\begin{aligned}
\Big\|\int_{K,B} a(k)\alpha(b(k))c(k)d\mu(k)\Big\| &= \Big\|\varrho\Big(\int_{K,\mathscr{A}_1} a(k)\alpha(b(k))c(k)d\mu(k)\Big)\Big\| \\
&= \Big\|\int_{K,\mathscr{A}_1} a(k)\alpha(b(k))c(k)d\mu(k)\Big\| \\
&= \Big\|(\Delta_\varphi^{iT} \boxtimes \mathrm{id})\int_{K,\mathscr{A}_1} a(k)b(k)c(k)d\mu(k)(\Delta_\varphi^{-iT} \boxtimes \mathrm{id})\Big\| \\
&= \Big\|\int_{K,\mathscr{A}_1} a(k)b(k)c(k)d\mu(k)\Big\| \\
&= \Big\|\int_{K,B} a(k)b(k)c(k)d\mu(k)\Big\|.
\end{aligned}$$

Since $\tilde{\alpha}$ and θ_t commutes, $\tilde{\alpha} \in \mathrm{Aut}(\mathscr{A})$.

(2) By (1), there exists an isomorphism η from $\mathscr{A}$ into $B(\mathfrak{H})$ such that

$$\eta\Big(\int_{K,B} a(k)b(k)c(k)d\mu(k)\Big) = \int_{K,\mathscr{A}_1} a(k)b(k)c(k)d\mu(k).$$

Then $\eta(\Delta_\varphi^{it} X \Delta_\varphi^{-it}) = \Delta_\varphi^{it} \boxtimes \Delta_{\varphi^{\mathrm{opp}}}^{-it} \eta(X) \Delta_\varphi^{-it} \boxtimes \Delta_{\varphi^{\mathrm{opp}}}^{it}$ holds. Let $\mathscr{B}_1 \subset B(L^2(\mathscr{M}))$ be the $*$-algebra of elements of the form $T = \int_{\boldsymbol{R}} a(s)\Delta_\varphi^{is} ds$, $a(s) \in C_c(\boldsymbol{R}, \mathscr{A})$. Define η_1 by

$$\eta_1\left(\int_{\boldsymbol{R}} a(s)\Delta_\varphi^{is} ds\right) = \int_{\boldsymbol{R}} \eta(a(s))\Delta_\varphi^{is} \boxtimes \Delta_{\varphi^{\mathrm{opp}}}^{-is} ds.$$

Then by Lemma 7.6(B), $\lim_{\nu\to\infty} \Psi_\nu(T) = \langle \eta_1(T)\xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}, \xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}\rangle$. Indeed we can verify this as follows. First note that $\lim_{\nu\to\infty} \Psi_\nu(a) = \langle \eta(a)\xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}, \xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}\rangle$ for $a \in \mathscr{A}$. We have

$$\begin{aligned}
\Psi_\nu(T) &= \Psi_\nu\left(\int_{\boldsymbol{R}} Y(s)\Delta_\varphi^{is} ds\right) \\
&= \Psi_\nu\left(\int_{\boldsymbol{R}} Y(s)(\Delta_\varphi^{is} - 1) ds\right) + \Psi_\nu\left(\int_{\boldsymbol{R}} Y(s) ds\right) \\
&= \int_{\boldsymbol{R}} \Psi_\nu(Y(s)(\Delta_\varphi^{is} - 1)) ds + \int_{\boldsymbol{R}} \Psi_\nu(Y(s)) ds.
\end{aligned}$$

Here

$$\begin{aligned}
|\Psi_\nu(Y(s)(\Delta_\varphi^{is} - 1))| &\le \Psi_\nu(Y(s)Y(s)^*)\Psi_\nu((\Delta_\varphi^{is} - 1)^*(\Delta_\varphi^{is} - 1)) \\
&= \Psi_\nu(Y(s)Y(s)^*)\Psi_\nu(2 - \Delta_\varphi^{is} - \Delta_\varphi^{-is}) \\
&\to 0
\end{aligned}$$

as ν goes to infinity. Hence

$$\begin{aligned}
\lim_{\nu\to\infty} \Psi_\nu(T) &= \lim_{\nu\to\infty} \int_{\boldsymbol{R}} \Psi_\nu(Y(s)) ds \\
&= \int_{\boldsymbol{R}} \langle \eta(Y(s))\xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}, \xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}\rangle ds \\
&= \left\langle \int_{\boldsymbol{R}} \eta(Y(s))\Delta_\varphi^{is} \boxtimes \Delta_{\varphi^{\mathrm{opp}}}^{-is} \xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}, \xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}} \right\rangle
\end{aligned}$$

holds.

Hence we get

$$|\langle \eta_1(T)\xi_{\varphi\otimes\varphi^{\mathrm{opp}}}, \xi_{\varphi\otimes\varphi^{\mathrm{opp}}}\rangle| \le \|T\|.$$

Since $\xi_\varphi \otimes \xi_{\varphi^{\mathrm{opp}}}$ is cyclic for $\eta_1(\mathscr{B}_1)$, η_1 is a well-defined homomorphism by [**6**, Lemma IV.5]. In a similar way, we have

$$\left\langle \int_{\boldsymbol{R}} a(s)\Delta_\varphi^{is} ds \xi_\varphi, \xi_\varphi \right\rangle = \lim_{\nu\to\infty} \left\langle \int_{\boldsymbol{R}} \eta(a(s))\Delta_\varphi^{is} \boxtimes \Delta_{\varphi^{\mathrm{opp}}}^{-is} ds \xi_\nu, \xi_\nu \right\rangle$$

by using Lemma 7.6(A). So $|\langle T\xi_\varphi, \xi_\varphi\rangle| \le \|\eta_1(T)\|$ holds. Since ξ_φ is cyclic for $\mathscr{B}_1$, η_1 is an isometry by [**6**, Lemma IV.5]. Then

$$\begin{aligned}
\left\| \int_{\boldsymbol{R}} \tilde{\alpha}(a(s))\Delta_\varphi^{is} ds \right\| &= \left\| \eta_1\left(\int_{\boldsymbol{R}} \tilde{\alpha}(a(s))\Delta_\varphi^{is} ds \right) \right\| \\
&= \left\| \int_{\boldsymbol{R}} \eta(\tilde{\alpha}(a(s)))\Delta_\varphi^{is} \boxtimes \Delta_{\varphi^{\mathrm{opp}}}^{-is} ds \right\| \\
&= \left\| \int_{\boldsymbol{R}} (\Delta_\varphi^{iT} \boxtimes \mathrm{id})\eta(a(s))(\Delta_\varphi^{-iT} \boxtimes \mathrm{id})\Delta_\varphi^{is} \boxtimes \Delta_{\varphi^{\mathrm{opp}}}^{-is} ds \right\| \\
&= \left\| (\Delta_\varphi^{iT} \boxtimes \mathrm{id})\left(\int_{\boldsymbol{R}} \eta(a(s))\Delta_\varphi^{is} \boxtimes \Delta_{\varphi^{\mathrm{opp}}}^{-is} ds \right)(\Delta_\varphi^{-iT} \boxtimes \mathrm{id}) \right\| \\
&= \left\| \int_{\boldsymbol{R}} \eta(a(s))\Delta_\varphi^{is} \boxtimes \Delta_{\varphi^{\mathrm{opp}}}^{-is} ds \right\| \\
&= \left\| \int_{\boldsymbol{R}} a(s)\Delta_\varphi^{is} ds \right\|.
\end{aligned}$$

Hence we get a desired isomorphism. □

Lemma 7.8. *With the notations in the proof of Theorem* 7.2, *we have*

$$\|4n + 2 - (1 - e_{\mathscr{N}}) - \tilde{\alpha}(X) - e_{\mathscr{N}}|f(\Delta_\varphi) - 1|^2 e_{\mathscr{N}}\| = 4n + 2.$$

Proof. Let $h(t) \in C_c^\infty(\boldsymbol{R})$ be a positive function with $\int_{\boldsymbol{R}} h(t)dt = 1$. Then we have $\int_{\boldsymbol{R}} \Delta_\varphi^{it} X \Delta_\varphi^{-it} h(t)dt \in \mathscr{A}$. Also

$$Z := \left(\int_{\boldsymbol{R}} \Delta_\varphi^{it}(4n+2-1+e_{\mathscr{N}}+X+e_{\mathscr{N}}|f(\Delta_\varphi)-1|^2 e_{\mathscr{N}})\Delta_\varphi^{-it}h(t)dt \right) \int_{\boldsymbol{R}} \Delta_\varphi^{it}h(t)dt \in \mathscr{B}.$$

By Lemma 7.7,

$$\beta(Z) = \left(\int_{\boldsymbol{R}} \Delta_\varphi^{it}(4n+2-1+e_{\mathscr{N}}+\tilde{\alpha}(X)+e_{\mathscr{N}}|f(\Delta_\varphi)-1|^2 e_{\mathscr{N}})\Delta_\varphi^{-it}h(t)dt \right) \int_{\boldsymbol{R}} \Delta_\varphi^{it}h(t)dt,$$

and $\|\beta(Z)\| = \|Z\|$. Here $\int_{\boldsymbol{R}} h(t)\Delta_\varphi^{it} dt = \hat{h}(\Delta_\varphi)$, and $\|\hat{h}(\Delta_\varphi)\| \le 1$, where $\hat{h}(\lambda) = \int \lambda^{it}h(t)dt$. Hence we have

$$\|\beta(Z)\| \le \|4n + 2 - (1 - e_{\mathscr{N}}) + \tilde{\alpha}(X) + e_{\mathscr{N}}|f(\Delta_\varphi) - 1|^2 e_{\mathscr{N}}\|.$$

Since ξ_φ is an eigenvector of Z for an eigenvalue $4n + 2$, $\|Z\| \ge 4n + 2$. Hence we get $\|4n + 2 - (1 - e_{\mathscr{N}}) + \tilde{\alpha}(X) + e_{\mathscr{N}}|f(\Delta_\varphi) - 1|^2 e_{\mathscr{N}}\| = 4n + 2$. □

A. Common Jones projection in Longo-Rehren inclusion.

In [**27**], we constructed the common Jones projection in the Longo-Rehren inclusion for a type II_1 subfactor. The proof in [**27**] is based on the computation of a biunitary connection in paragroup theory. In the first appendix, we present a direct proof of the existence of the common Jones projection in the Longo-Rehren inclusion for a subfactor of type III.

Let $N \subset M$ be a subfactor of type III with finite index, E the minimal conditional expectation, ι the inclusion map, and $\iota\bar{\iota} \cong \oplus_i N_i \rho_i$ the irreducible decomposition. (Of course, $\iota\bar{\iota}$ is the canonical endomorphism for $N \subset M$.) Let $\{a_i^e\}_{e=1}^{N_i} \subset (\iota, \rho_i \iota)$ be an orthonormal basis.

We assume $\mathscr{M}$ acts standardly on $L^2(\mathscr{M})$. Let W_i be the standard implementing isometry for ρ_i. Set $\tilde{a}_i := \sum_e a_i^e J a_i^e J$, which is independent on the choice of an orthonormal basis.

Theorem A.1. *Let e_N be the Jones projection for $N \subset M$. Then*

$$e_N = [M : N]^{-1} \sum_i \sqrt{d(i)} d(i) \tilde{a}_i^* W_i$$

holds.

Lemma A.2. *Let $\sigma \cong \oplus_i \sigma_i$ be the irreducible decomposition for $\sigma \in \mathrm{End}(\mathscr{M})$, and fix an isometry $w_i \in (\sigma_i, \sigma)$ with $w_i^* w_j = \delta_{i,j}$. Let ϕ_σ be the standard left inverse for σ. Then*

$$d(\sigma)\phi_\sigma(x) = \sum_i d(\sigma_i)\phi_{\sigma_i}(w_i^* x w_i)$$

holds.

Proof. Fix an isometry $\bar{w}_i \in (\bar{\sigma}_i, \bar{\sigma})$ with $\bar{w}_i^* \bar{w}_j = \delta_{i,j}$. Let $R_i \in (\mathrm{id}, \sigma_i \bar{\sigma}_i)$ and $\bar{R}_i \in (\mathrm{id}, \bar{\sigma}_i \sigma)$ be isometries such that $R_i^* \sigma(\bar{R}_i) = d(\sigma_i)^{-1}$ and $\bar{R}_i^* \bar{\sigma}_i(R_i) = d(\sigma_i)^{-1}$. Define $R := \sum_k \sqrt{\frac{d(\sigma_k)}{d(\sigma)}} w_k \sigma_k(\bar{w}_k) R_k$, and $\bar{R} := \sum_k \sqrt{\frac{d(\sigma_k)}{d(\sigma)}} \bar{w}_k \bar{\sigma}_k(w_k) \bar{R}_k$. It is easy to see that R and $\bar{R}$ are isometries such that $R \in (\mathrm{id}, \sigma\bar{\sigma})$, $\bar{R} \in (\mathrm{id}, \bar{\sigma}\sigma)$, $R^* \sigma(\bar{R}) = \bar{R}^* \bar{\sigma}(R) = d(\sigma)^{-1}$. Then $\phi_\sigma(x)$ is given by $\bar{R}^* \bar{\sigma}(x) \bar{R}$. Hence we have

$$\begin{aligned}
\phi_\sigma(x) &= \bar{R}^* \bar{\sigma}(x) \bar{R} \\
&= \sum_{k,l} \frac{\sqrt{d(\sigma_k) d(\sigma_l)}}{d(\sigma)} \bar{R}_k^* \bar{\sigma}_k(w_k^*) \bar{w}_k^* \bar{\sigma}(x) \bar{w}_l \bar{\sigma}_l(w_l) \bar{R}_l \\
&= \sum_k \frac{d(\sigma_k)}{d(\sigma)} \bar{R}_k^* \bar{\sigma}_k(w_k^*) \bar{\sigma}_k(x) \bar{\sigma}_k(w_k) \bar{R}_k \\
&= \sum_k \frac{d(\sigma_k)}{d(\sigma)} \phi_{\sigma_k}(w_k^* x w_k).
\end{aligned}$$

□

COROLLARY A.3. *Let σ, σ_i, w_i be as above. Let φ be a faithful normal state of M. Then $\sqrt{d(\sigma)}\xi_{\varphi\circ\phi_\sigma}w_i = \sqrt{d(\sigma_i)}w_i\xi_{\varphi\circ\phi_{\sigma_i}}$ holds.*

PROOF. By Lemma A.2, we get $d(\sigma)\phi_\sigma(w_ixw_i^*) = d(\sigma_i)\phi_{\sigma_i}(x)$. Hence $d(\sigma)\varphi\circ\phi_\sigma(w_ixw_i^*) = d(\sigma_i)\varphi\circ\phi_{\sigma_i}(x)$ holds. If we replace x by xw_i, we get $d(\sigma)\varphi\circ\phi_\sigma(w_ixw_iw_i^*) = d(\sigma_i)\varphi\circ\phi_{\sigma_i}(xw_i)$. Since $w_iw_i^* \in \sigma(M)'\cap M$, and $\phi_\sigma\circ E_\sigma = \phi_\sigma$, $w_iw_i^*$ is in the centralizer of $\varphi\circ\phi_\sigma$. Hence we get $d(\sigma)\varphi\circ\phi_\sigma(w_ix) = d(\sigma_i)\varphi\circ\phi_{\sigma_i}(xw_i)$. This implies the conclusion. □

PROOF OF THEOREM A.1. Let φ be a faithful normal state of M such that $\varphi\circ E = \varphi$. Then e_N is given by $e_N(x\xi_\varphi) = E(x)\xi_\varphi$. Fix $v \in (\mathrm{id}, \iota\bar{\iota})$ and $\bar{v} \in (\mathrm{id}, \bar{\iota}\iota)$ such that $v^*\iota(\bar{v}) = \bar{v}^*\bar{\iota}(v) = [M : N]^{-\frac{1}{2}}$. Let $\{w_i^e\} \subset (\rho_i, \iota\bar{\iota})$ be an orthonormal basis. By the Frobenius reciprocity, we may assume $a_i^e = \sqrt{[M : N]/d(i)}w_i^{e*}\iota(\bar{v})$. We have

$$\begin{aligned}
[M : N]^{-1}\sum_i \sqrt{d(i)}d(i)\tilde{a}_i^*W_i(x\xi_\varphi) &= [M : N]^{-1}\sum_i \sqrt{d(i)}d(i)\tilde{a}_i^*\rho_i(x)\xi_{\varphi\circ\phi_{\rho_i}} \\
&= [M : N]^{-1}\sum_{i,e} \sqrt{d(i)}d(i)a_i^{e*}\rho_i(x)\xi_{\varphi\circ\phi_{\rho_i}}a_i^e \\
&= \sum_{i,e} \sqrt{d(i)}\iota(\bar{v}^*)w_i^e\rho_i(x)\xi_{\varphi\circ\phi_{\rho_i}}w_i^{e*}\iota(\bar{v}) \\
&= \sum_{i,e} \sqrt{d(i)}\iota(\bar{v}^*)\iota\bar{\iota}(x)w_i^e\xi_{\varphi\circ\phi_{\rho_i}}w_i^{e^*}\iota(\bar{v}).
\end{aligned}$$

By Corollary A.3, we have $\sqrt{d(i)}w_i^e\xi_{\varphi\circ\phi_{\rho_i}} = \sqrt{[M : N]}\xi_{\varphi\circ\phi_{\iota\bar{\iota}}}w_i^e$.
Hence we get

$$\begin{aligned}
\sum_{i,e} \sqrt{d(i)}\iota(\bar{v}^*)\iota\bar{\iota}(x)w_i^e\xi_{\varphi\circ\phi_{\rho_i}}w_i^{e^*}\iota(\bar{v}) &= \sum_{i,e} \sqrt{[M : N]}\iota(\bar{v}^*)\iota\bar{\iota}(x)\xi_{\varphi\circ\phi_{\iota\bar{\iota}}}w_i^ew_i^{e*}\iota(\bar{v}) \\
&= \sqrt{[M : N]}\iota(\bar{v}^*)\iota\bar{\iota}(x)\xi_{\varphi\circ\phi_{\iota\bar{\iota}}}\iota(\bar{v}).
\end{aligned}$$

Here

$$\begin{aligned}
E\circ\phi_{\iota\bar{\iota}}(x) &= \iota(\bar{v}^*\bar{\iota}(\phi_{\iota\bar{\iota}}(x))\bar{v}) \\
&= \iota(\bar{v}^*\bar{\iota}(v^*)\bar{\iota}\iota(\bar{v}^*)\bar{\iota}\iota\bar{\iota}(x)\bar{\iota}\iota(\bar{v})\bar{\iota}(v)\bar{v}) \\
&= \iota(\phi_{\iota\bar{\iota}\iota}(x))
\end{aligned}$$

holds. So

$$\begin{aligned}
[M : N]\varphi\circ\phi_{\iota\bar{\iota}}(\iota(\bar{v})x) &= [M : N]\varphi\circ\iota(\phi_{\iota\bar{\iota}\iota}(\iota(\bar{v})x)) \\
&= [M : N]\varphi\circ\iota(\phi_{\bar{\iota}\iota}(\bar{v}\phi_\iota(x))) \\
&= \varphi\circ\iota(\phi_\iota(x)\bar{v}) \\
&= \varphi(x\iota(\bar{v}))
\end{aligned}$$

holds, and we have $\sqrt{[M:N]}\xi_{\varphi\circ\phi_{\iota\bar{\iota}}}\iota(\bar{v}) = \iota(\bar{v})\xi_\varphi$.

Finally we get $\sqrt{[M:N]}\iota(\bar{v}^*)\iota\bar{\iota}(x)\xi_{\varphi\circ\phi_{\iota\bar{\iota}}}\iota(\bar{v}) = \iota(\bar{v}^*)\iota\bar{\iota}(x)\iota(\bar{v})\xi_\varphi = E(x)\xi_\varphi$. □

B. Exhaustion trick.

In the second appendix we give a proof of a subfactor-version of [**6**, Lemma III.4], since the proof was omitted in [**6**]. This lemma is a key for the proof of Theorem 7.1. As mentioned in [**6**], idea is similar to that of [**6**, Theorem II.2].

Lemma B.1. *Let $\mathscr{N} \subset \mathscr{M}$, ε, φ_j and θ be as in the "if part" of Theorem* 7.1*, and $\xi_j \in L^2(\mathscr{M})_+$ be the representing vector for φ_j. Let c be as in* [**6**, III.Lemma 2]*. Then there exist a projection $E \in \mathscr{N}$ and non-zero $y \in \mathscr{N}$ such that,*
(1) $\|y\| \le 1$, $y = \theta(E)yE$,
(2) $\sum \|y\xi_j\|^2 \ge 2^{-6}c^{-1}\sum \|E\xi_j\|^2$,
(3) $\|[E,\xi_j]\|^2 \le \varepsilon^2 \sum \|E\xi_j\|^2$,
(4) $\|y\xi_j - \theta(\xi_j)y\|^2 \le \varepsilon^2 \sum \|y\xi_j\|^2$.

A proof of this lemma is same as that of [**6**, III.Lemma 3]. Here note that since $x \in \mathscr{N}$ in Theorem 7.1, E and y can be chosen in $\mathscr{N}$ by construction.

Lemma B.2. *Let $\mathscr{N} \subset \mathscr{M}$, and θ be as in the "if part" of Theorem* 7.1*. Then there exists a bounded sequence $\{y_n\} \subset \mathscr{N}$ such that $\{y_n\}$ does not converge to 0 strongly and*

$$\|y_n\varphi - \theta(\varphi)y_n\| \to 0,\ \varphi \in \mathscr{M}_*.$$

Proof. Fix $\varphi_1, \cdots, \varphi_n \in \mathscr{M}_*^+$. We may and do assume that φ_1 is faithful. Let $\varepsilon > 0$ be such that $2^{-6}c^{-1} - n\varepsilon^2 \ge 2^{-7}c^{-1}$, where c is a constant defined in [**6**, Lemma III.3]. Let $\xi_j \in L^2(\mathscr{M})_+$ be the representing vector for φ_j, i.e., $\varphi_j(x) = \langle x\xi_j, \xi_j\rangle$. Let R be the set of $r = (E, x, \alpha_1, \cdots, \alpha_n) \in \mathrm{Proj}\,(\mathscr{N}) \times \mathscr{N} \times L^2(\mathscr{M})^n$ satisfying the following conditions.
(i) $\theta(E)xE = x$, $\|x\| \le 1$.
(ii) $E\alpha_j = \alpha_j$, $\eta_j := \xi_j - \alpha_j - J\alpha_j \in L^2(\mathscr{M})_+$ and $[E, \eta_j] = 0$.
(iii) $\|\alpha_j\|^2 \le \varepsilon^2 \sum_j \|E\xi_j\|^2$.
(iv) $\sum_j \|x\eta_j\|^2 \ge 2^{-7}c^{-1}\sum_j \|E\eta_j\|^2$.
(v) $\|x\eta_j - \theta(\eta_j)x\|^2 \le \varepsilon^2 \sum_j \|x\xi_j\|^2$.

We define a partial ordering $r = (E, x, \alpha_1, \cdots, \alpha_n) \le r' = (E', x', \alpha_1', \cdots, \alpha_n')$ as follows.
(α) $E \le E'$.
(β) $\theta(E)x'E = x$.
(γ) $E(\alpha_j' - \alpha_j) = 0$.
(δ) $\|\alpha_j' - \alpha_j\|^2 \le \varepsilon^2 \sum_j \|(E' - E)\xi_j\|^2$.

We claim that this is indeed an order.

(1) It is trivial that $r \le r$.

(2) Assume $r \le r'$ and $r' \le r$. By (α), $E = E'$. By (β) and (i), $x = \theta(E)x'E = \theta(E')x'E' = x'$. By ($\delta$), $\alpha_j = \alpha_j'$. Hence we get $r = r'$.

(3) Assume $r \leq r'$ and $r' \leq r''$. It is clear that $E \leq E''$. Next $\theta(E)x''E = \theta(E)\theta(E')x''E'E = \theta(E)x'E = x$, so (β) holds. $E(\alpha_j'' - \alpha_j) = E(\alpha_j'' - \alpha_j') + E(\alpha_j' - \alpha_j) = E'(\alpha_j'' - \alpha_j') = 0$. Hence (γ) holds. To prove (δ), first note $\langle \alpha_j'' - \alpha_j', \alpha_j' - \alpha_j \rangle = \langle E'(\alpha_j'' - \alpha_j'), \alpha_j' - \alpha_j \rangle = 0$. Then $\|\alpha_j'' - \alpha_j\|^2 = \|\alpha_j'' - \alpha_j'\|^2 + \|\alpha_j' - \alpha_j\|^2 \leq \varepsilon^2 \sum_j \|(E'' - E')\xi_j\|^2 + \varepsilon^2 \sum_j \|(E' - E)\xi_j\|^2 = \varepsilon^2 \sum_j \|(E'' - E)\xi_j\|^2$.

Next we prove that R is an inductively ordered set. Let $\{r_i\}_{i\in I}$ be a totally ordered subset. Set $r_i = (E_i, x_i, \alpha_1^i, \cdots, \alpha_n^i)$. Then we can see that $r_i \to \varphi_1(E_i) \in \boldsymbol{R}$ is a faithful map. Hence there exists a cofinal sequence $\{r_{i_k}\}$. So we may assume that $\{r_k\}$ is an increasing sequence. Then $\{E_k\}$ is an increasing sequence of projections, and hence $E = \lim_k E_k$ exists in the strong topology. By a similar reason, $x = \lim_k x_k$ also exists. By (δ), $\|\alpha_j^k - \alpha_j^l\|^2 \leq \varepsilon^2 \sum_j \|(E_k - E_l)\xi_j\|^2$. By letting $k, l \to \infty$, we know that $\{\alpha_j^k\}$ is a Cauchy sequence. Hence a limit $\alpha_j = \lim_k \alpha_j^k$ exists. Then $r = (E, x, \alpha_1, \cdots, \alpha_n)$ is in R by continuity. By construction r is an upper bound of $\{r_k\}$. Hence R is inductive.

By Zorn's lemma, there exists a maximal element $r = (E, x, \alpha_1, \cdots, \alpha_n) \in R$. Actually we have $E = 1$. We assume $E = 1$ for a moment, and it will be proved at the end of the proof. Then we get $x \in \mathscr{N}$, $\alpha_j \in L^2(\mathscr{M})$ satisfying the following conditions.
(1) $\|x\| \leq 1$.
(2) $\|\alpha_j\|^2 \leq \varepsilon^2 \sum_j \|\xi_j\|^2$.
(3) With $\eta_j = \xi_j - \alpha_j - J\alpha_j$, $\sum_j \|x\eta_j\| \geq 2^{-7}c^{-1} \sum_j \|\eta_j\|^2$.
(4) $\|x\eta_j - \theta(\eta_j)x\|^2 \leq \varepsilon^2 \sum_j \|\xi_j\|^2$.

Set $a := \sqrt{\sum_j \|\xi_j\|^2}$. Then $\|\alpha_j\| \leq \varepsilon a$ holds. By (4), we get

$$\begin{aligned}\|x\xi_j - \theta(\xi_j)x\| &= \|x\eta_j - \theta(\eta_j)x + x(\alpha_j + J\alpha_j) + \theta(\alpha_j + J\alpha_j)x\| \\ &\leq 5\varepsilon a,\end{aligned}$$

and

$$\begin{aligned}\|x\eta_j\| &= \|x(\xi_j - \alpha_j - J\alpha_j)\| \\ &\leq \|x\xi_j\| + 2\|\alpha_j\| \\ &\leq \|x\xi_j\| + 2\varepsilon a.\end{aligned}$$

Hence we have the following.

$$\begin{aligned}\sum_j \|x\eta_j\|^2 &\leq \sum_j (\|x\xi_j\| + 2\varepsilon a)^2 \\ &= \sum_j (\|x\xi_j\|^2 + 4\varepsilon a\|x\xi_j\| + 4\varepsilon^2 a^2) \\ &\leq \sum_j \|x\xi_j\|^2 + 4\varepsilon n^{\frac{1}{2}} \left(\sum_j \|x\xi_j\|^2\right)^{\frac{1}{2}} a + 4\varepsilon^2 n a^2 \\ &= \left(\left(\sum_j \|x\xi_j\|^2\right)^{\frac{1}{2}} + 2\varepsilon a n^{\frac{1}{2}}\right)^2.\end{aligned}$$

On the other hand, we have the following.

$$
\begin{aligned}
\sum_j \|\eta_j\|^2 &= \sum_j \|\xi_j - \alpha_j - J\alpha_j\|^2 \\
&\geq \sum_j (\|\xi_j\| - 2\varepsilon a)^2 \\
&= \sum_j (\|\xi_j\|^2 - 4\varepsilon a\|\xi_j\| + 4\varepsilon^2 a^2) \\
&\geq a^2 - 4\varepsilon n^{\frac{1}{2}} a^2 + 4\varepsilon^2 a^2 n \\
&= a^2(1 - 2\varepsilon n^{\frac{1}{2}})^2.
\end{aligned}
$$

So by (3), $((\sum_j \|x\xi_j\|^2)^{\frac{1}{2}} + 2\varepsilon a n^{\frac{1}{2}})^2 \geq 2^{-7}c^{-1}a^2(1 - 2\varepsilon n^{\frac{1}{2}})^2$ holds. If we take sufficient small ε, then we get $x \in \mathscr{N}$, $\|x\| \leq 1$ such that $\|x\xi_j - \theta(\xi_j)x\| \leq \varepsilon a$ and $\sum_j \|x\xi_j\|^2 \geq d'a^2$ for some constant $0 \neq d'$.

Since $\mathfrak{H}$ is separable, we can construct a bounded sequence $\{y_n\} \subset \mathscr{N}$ such that $\{y_n\}$ does not converge to 0 strongly and $\lim_{n\to\infty} \|y_n\varphi - \theta(\varphi)y_n\| = 0$ for every $\varphi \in \mathscr{M}_*$.

Now it remains to show $E = 1$. To do so by contradiction, we assume $E \neq 1$. Let $u \in \mathscr{N}$ be a unitary such that $u\theta(1-E)u^* = 1 - E$, and consider $\bar{\theta} := \operatorname{Ad} u\theta|_{\mathscr{M}_{(1-E)}}$. Since $E\alpha_j = \alpha_j$, we have $(1-E)\alpha_j = 0$ and $(J\alpha_j)(1-E) = J(1-E)\alpha_j = 0$, so that $(1-E)\xi_j(1-E) = (1-E)\eta_j(1-E)(=: \zeta_j)$. By applying the above lemma to $\mathscr{M}_{(1-E)}$, $\bar{\theta}$ and $\{\zeta_j\}$, we can find $0 \neq y \in \mathscr{N}_{(1-E)}$ and $F \in \operatorname{Proj}(\mathscr{N}_{(1-E)})$ such that
(1) $\bar{\theta}(F)yF = y$, $\|y\| \leq 1$,
(2) $\sum_j \|y\zeta_j\|^2 \geq 2^{-6}c^{-1}\sum_j \|F\zeta_j\|^2$,
(3) $\|[F, \zeta_j]\|^2 \leq \varepsilon^2 \sum_j \|F\zeta_j\|^2$,
(4) $\|y\zeta_j - \bar{\theta}(\zeta_j)y\|^2 \leq \varepsilon^2 \sum_j \|y\zeta_j\|^2$.

Set $x' = x + u^*y$, $E' = E + F$, $\alpha'_j = \alpha_j + F\eta_j(1-F)$. We claim that $r' = (E', x', \alpha'_1, \cdots, \alpha'_n) \in R$ and r' majorizes r. Note that $F\eta_j = F\zeta_j$ and $\eta_j F = \zeta_j F$ since $1 - E \geq F$ and $(1-E)$ commutes with η_j.

(i) First note that $\bar{\theta}(F)y = y$ implies $\theta(F)u^*y = u^*y$. Then $\|x'\| \leq 1$ is clear from the facts $\theta(E)xE = x$, $\theta(F)u^*yF = u^*y$.

(ii) $E'\alpha'_j = (E+F)(\alpha_j + F\eta_j(1-F)) = E\alpha_j + F\eta_j(1-F) = \alpha'_j$. Set $\eta'_j = \xi_j - \alpha'_j - J\alpha'_j$. Since

$$
\begin{aligned}
\eta'_j &= \xi_j - \alpha'_j - J\alpha'_j \\
&= \xi_j - \alpha_j - F\eta_j(1-F) - J\alpha_j - (1-F)\eta_j F \\
&= \eta_j - F\eta_j(1-F) - (1-F)\eta_j F \\
&= F\eta_j F + (1-F)\eta_j(1-F),
\end{aligned}
$$

we have $\eta'_j \in L^2(\mathscr{M})_+$.

Next we verify $[\eta', E'] = 0$. Note that $E\eta_j = \eta_j E = E\eta_j E$ holds due to $[E, \eta_j] = 0$. Then we get the following.

$$
\begin{aligned}
[E', \eta_j'] &= (E+F)(F\eta_j F + (1-F)\eta_j(1-F)) - (F\eta_j F + (1-F)\eta_j(1-F))(E+F) \\
&= F\eta_j F + E\eta_j(1-F) - F\eta_j F - (1-F)\eta_j E \\
&= E\eta_j(1-F) - (1-F)\eta_j E \\
&= E\eta_j E(1-F) - (1-F)E\eta_j E \\
&= 0.
\end{aligned}
$$

(iii) Since $E\alpha_j = \alpha_j$, α_j and $F\eta_j(1-F)$ are orthogonal. Then

$$
\begin{aligned}
\|\alpha_j'\|^2 &= \|\alpha_j\|^2 + \|F\eta_j(1-F)\|^2 \\
&\leq \varepsilon^2 \sum_j \|E\xi_j\|^2 + \|F\zeta_j(1-F)\|^2 \\
&= \varepsilon^2 \sum_j \|E\xi_j\|^2 + \|F[F,\zeta_j]\|^2 \\
&\leq \varepsilon^2 \sum_j \|E\xi_j\|^2 + \varepsilon^2 \sum_j \|F\zeta_j\|^2 \\
&= \varepsilon^2 \sum_j \|E\xi_j\|^2 + \varepsilon^2 \sum_j \|F\xi_j(1-E)\|^2 \\
&\leq \varepsilon^2 \sum_j \|E\xi_j\|^2 + \varepsilon^2 \sum_j \|F\xi_j\|^2 \\
&= \varepsilon^2 \sum_j \|(E+F)\xi_j\|^2 \\
&= \varepsilon^2 \sum_j \|E'\xi_j\|^2.
\end{aligned}
$$

(iv) First we compute $x'\eta_j'$.

$$
\begin{aligned}
x'\eta_j' &= (x+u^*y)(F\eta_j F + (1-F)\eta_j(1-F)) \\
&= x\eta_j(1-F) + u^*y\eta_j F \\
&= x\eta_j + u^*y\zeta_j F.
\end{aligned}
$$

Hence $\|x'\eta_j'\|^2 = \|x\eta_j\|^2 + \|u^*y\zeta_j F\|^2$ holds. We also have

$$
\begin{aligned}
\|y\zeta_j\|^2 &= \|y\zeta_j F\|^2 + \|y\zeta_j(1-F)\|^2 \\
&\leq \|y\zeta_j F\|^2 + \|F\zeta_j(1-F)\|^2 \\
&= \|y\zeta_j F\|^2 + \|F[F,\zeta_j]\|^2 \\
&\leq \|y\zeta_j F\|^2 + \varepsilon^2 \sum_j \|F\zeta_j\|^2.
\end{aligned}
$$

Hence $\sum_j \|y\zeta_j F\| \geq \sum_j \|y\zeta_j\|^2 - \varepsilon^2 n \sum_j \|F\zeta_j\|^2$ holds.
We will estimate $\sum_j \|x'\eta'_j\|$.

$$\begin{aligned}
\sum_{j=1}^{n} \|x'\eta'_j\|^2 &\geq \sum_j \|x\eta_j\|^2 + \sum_j \|y\zeta_j\|^2 - \varepsilon^2 n \sum_j \|F\zeta_j\|^2 \\
&\geq 2^{-7}c^{-1} \sum_j \|E\eta_j\|^2 + (2^{-6}c^{-1} - n\varepsilon^2) \sum_j \|F\zeta_j\|^2 \\
&\geq 2^{-7}c^{-1} \sum_j \|E\eta_j\|^2 + 2^{-7}c^{-1} \sum_j \|F\zeta_j\|^2 \\
&\geq 2^{-7}c^{-1} \sum_j \|E(1-F)\eta_j(1-F)\|^2 + 2^{-7}c^{-1} \sum_j \|F\zeta_j F\|^2 \\
&= 2^{-7}c^{-1} \sum_j \|E(1-F)\eta_j(1-F) + F\zeta_j F\|^2 \\
&= 2^{-7}c^{-1} \sum_j \|(E+F)((1-F)\eta_j(1-F) + F\eta_j F)\|^2 \\
&= 2^{-7}c^{-1} \sum_j \|E'\eta'_j\|^2.
\end{aligned}$$

(v) We compute $\theta(\eta'_j)x'$.

$$\begin{aligned}
\theta(\eta'_j)x' &= \theta(F\eta_j F + (1-F)\eta_j(1-F))(x + u^*y) \\
&= \theta(F\eta_j)u^*y + \theta((1-F)\eta_j)x \\
&= u^*\bar{\theta}(F\zeta_j)y + \theta((1-F)\eta_j)x \\
&= u^*\bar{\theta}(F\zeta_j)y + \theta((1-F)\eta_j E)x \\
&= u^*\bar{\theta}(F\zeta_j)y + \theta((1-F)E\eta_j)x \\
&= u^*\bar{\theta}(F\zeta_j)y + \theta(E\eta_j)x \\
&= u^*\bar{\theta}(F\zeta_j)y + \theta(\eta_j)x.
\end{aligned}$$

Then we have the following estimate.

$$\begin{aligned}
\|x'\eta'_j - \theta(\eta'_j)x'\|^2 &= \|x\eta_j + u^*y\zeta_j F - \theta(\eta_j)x - u^*\bar{\theta}(F\zeta_j)y\|^2 \\
&= \|x\eta_j - \theta(\eta_j)x + u^*y\zeta_j F - u^*\bar{\theta}(F)\bar{\theta}(\zeta_j)y\|^2 \\
&= \|x\eta_j - \theta(\eta_j)x\|^2 + \|u^*\bar{\theta}(F)y\zeta_j F - u^*\bar{\theta}(F)\bar{\theta}(\zeta_j)y\|^2 \\
&\leq \varepsilon^2 \sum_j \|x\xi_j\|^2 + \|y\zeta_j F - \bar{\theta}(\zeta_j)yF\|^2
\end{aligned}$$

$$\leq \varepsilon^2 \sum_j \|x\xi_j\|^2 + \|y\zeta_j - \bar{\theta}(\zeta_j)y\|^2$$
$$\leq \varepsilon^2 \sum_j \|x\xi_j\|^2 + \varepsilon^2 \sum_j \|y\zeta_j\|^2$$
$$= \varepsilon^2 \sum_j \|x\xi_j\|^2 + \varepsilon^2 \sum_j \|y(1-E)\xi_j(1-E)\|^2$$
$$\leq \varepsilon^2 \sum_j \|x\xi_j\|^2 + \varepsilon^2 \sum_j \|u^* y\xi_j\|^2$$
$$= \varepsilon^2 \sum_j \|(x+u^*y)\xi_j\|^2 \quad (x\xi_j \text{ and } u^*y\xi_j \text{ are orthogonal,})$$
$$= \varepsilon^2 \sum_j \|x'\xi_j\|^2.$$

Hence r' is in R. Next we verify $r' > r$. (α) is obvious. Since $\theta(E)(x+u^*y)E = \theta(E)xE + \theta(E)\theta(F)u^*yFE = x$, so (β) holds. Next we verify (γ). Then $E(\alpha'_j - \alpha_j) = E(F\eta_j(1-F) = 0)$. Finally $\|\alpha'_j - \alpha_j\|^2 = \|F\eta_j(1-F)\|^2 = \|F([F,\eta_j])\|^2 \leq \varepsilon^2 \sum_j \|F\zeta\|^2 = \varepsilon^2 \sum_j \|F\xi(1-E)\|^2 \leq \|(E'-E)\xi\|^2$, so (δ) holds. This contradicts the maximality of r, so we get $E = 1$. $\square$

References

[1] H. Araki, Asymptotic ratio set and property L'_λ, Publ. Res. Inst. Math. Sci., **6** (1970), 443–460.

[2] D. Bisch, On the existence of central sequences in subfactors, Trans. Amer. Math. Soc., **321** (1990), 117–128.

[3] M. Choda and H. Kosaki, Strongly outer actions for an inclusion of factors, J. Funct. Anal., **122** (1994), 315–332.

[4] A. Connes, On hyperfinite factors of type III_0 and Krieger's factors, J. Funct. Anal., **18** (1975), 318–327.

[5] A. Connes, Classification of injective factors, Ann. of Math., **104** (1976), 73–115.

[6] A. Connes, Type III_1 factors, property L'_λ and closure of inner automorphisms, J. Operator Theory, **14** (1985), 189–211.

[7] A. Connes and E. Størmer, Homogeneity of the state spaces of factors of type III_1, J. Funct. Anal., **28** (1978), 187–196.

[8] E. G. Effros and E. C. Lance, Tensor products of operator algebras, Adv. Math., **25** (1977), 1–34.

[9] D. E. Evans and Y. Kawahigashi, Quantum symmetries on operator algebras, Oxford Press, 1998.

[10] D. Guido and R. Longo, The conformal spin and statistics theorem, Comm. Math. Phys., **181** (1996), 11–35.

[11] U. Haagerup, Connes' bicentralizer problem and uniqueness of the injective factor of type III_1, Acta Math., **158** (1987), 95–148.

[12] U. Haagerup and E. Størmer, Equivalence of normal states on von Neumann algebras and the flow of weights, Adv. Math., **83** (1990), 180–262.

[13] F. Hiai, Minimalizing indices of conditional expectations onto a subfactor, Publ. Res. Inst. Math. Sci., **24** (1988), 673–678.

[14] F. Hiai and M. Izumi, Amenability and strong amenability for fusion algebras with applications to subfactor theory, Internat. J. Math., **9** (1998), 669–722.

[15] M. Izumi, Application of fusion rules to classification of subfactors, Publ. Res. Inst. Math. Sci., **27** (1991), 953–994.

[16] M. Izumi, On type II and type III principal graphs for subfactors, Math. Scand., **73** (1993), 307–319.
[17] M. Izumi, The structure of sectors associated with Longo-Rehren inclusions, I, General theory, Comm. Math. Phys., **213** (2000), 127–179.
[18] V. F. R. Jones, Index for subfactors, Invent. Math., **72** (1983), 1–25.
[19] Y. Kawahigashi, Classification of approximately inner automorphisms of subfactors, Math. Ann., **138** (1997), 425–438.
[20] H. Kosaki, Automorphisms in the irreducible decomposition of sectors, In: Quantum and Non-commutative Analysis (eds. H. Araki, et al.), Kluwer Academic Publishers, (1993), 305–316.
[21] H. Kosaki, Sector theory and automorphisms for factor-subfactor pairs, J. Math. Soc. Japan, **48** (1996), 427–454.
[22] H. Kosaki, Type III factors and index theory, Res. Inst. of Math., Lecture Notes Series, Seoul Nat. Univ., **43** (1998).
[23] H. Kosaki and R. Longo, A remark on the minimal index of subfactors, J. Funct. Anal., **107** (1992), 458–470.
[24] P. Loi, On automorphisms of subfactors, J. Funct. Anal., **141** (1996), 275–293.
[25] R. Longo, Index of subfactors and statistics of quantum fields II, Comm. Math. Phys., **130** (1990), 285–309.
[26] R. Longo and K.-H. Rehren, Nets of subfactors, Rev. Math. Phys., **7** (1995), 567–597.
[27] T. Masuda, Generalization of Longo-Rehren construction to subfactors of infinite depth and amenability of fusion algebras, J. Funct. Anal., **171** (2000), 53–77.
[28] T. Masuda, T. Extension of automorphisms of a subfactor to the symmetric enveloping algebra, Internat. J. Math., **12** (2001), 637–659.
[29] M. Pimsner and S. Popa, Entropy and index for subfactors, Ann. Sci. École Norm., **19** (1986), 57–106.
[30] S. Popa, Classification actions of discrete amenable groups on subfactors of type II, preprint, 1992.
[31] S. Popa, Approximate innerness and central freeness for subfactors: A classification result, In: Subfactors (eds. H. Araki, et al.), World Scientific, 1994, 274–293.
[32] S. Popa, Classification of amenable subfactor of type II, Acta Math., **172** (1994), 163–255.
[33] S. Popa, Classification of subfactors and their endomorphisms, Regional Conference Series in Mathematics, **86** (1995).
[34] S. Popa, The relative Dixmier property for inclusions of von Neumann algebras of finite index, Ann. Sci. École Norm., **32** (1999), 743–767.
[35] S. Popa, Some properties of the symmetric enveloping algebra of a subfactor, with applications to amenability and property T, Doc. Math., **4** (1999), 665–744.
[36] M. Takesaki, Duality for crossed products and the structure of von Neumann algebras of type III, Acta Math., **131** (1973), 249–310.
[37] M. Takesaki, Theory of Operator Algebras, I, II, III, Springer, Berlin-Heidelberg-New York, 2002.
[38] S. Wassermann, Injective W^*-algebras, Math. Proc. Cambridge Philos. Soc., **82** (1977), 39–47.
[39] C. Winsløw, Strongly free actions on subfactors, Internat. J. Math., **4** (1993), 675–688.
[40] C. Winsløw, Approximately inner automorphisms on inclusions of type III$_\lambda$ factors, Pacific J. Math., **166** (1994), 385–400.
[41] S. Wright, Uniqueness of the injective III$_1$ factor, **1413**, Springer, Berlin, 1989.

Toshihiko Masuda
Graduate School of Mathematics
Kyushu University
6-10-1 Hakozaki
Fukuoka, 812-5250
Japan