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On framed simple Lie groups

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Abstract. For a compact simple Lie group G, we show that the element $[G,\mathcal{L}] \in \pi_*^S(S^0)$ represented by the pair $(G,\mathcal{L})$ is zero, where $\mathcal{L}$ denotes the left invariant framing of G. The proof relies on the method of E. Ossa [Topology, **21** (1982), 315–323].

1. Introduction.

A compact connected Lie group G of dimension d, together with its left invariant framing $\mathcal{L}$, defines an element $[G,\mathcal{L}]$ in π_d^S via the Thom–Pontrjagin construction. In [**7**] E. Ossa proved that if G is semi-simple, then there holds

$$72[G,\mathcal{L}] = 0 \quad \text{or} \quad 24[G,\mathcal{L}] = 0 \tag{1}$$

according as G is or is not locally isomorphic to a product of E_6, E_7, E_8. In this note we show that when G is restricted to a simple Lie group, the method of [**7**] allows us to obtain a more conclusive result by altering the expression of a certain specific element. The result is the following:

Theorem 1.

(i) $[SU(n),\mathcal{L}] = [Spin(n),\mathcal{L}] = [SO(n),\mathcal{L}] = 0 \quad (n \geq 8)$; $\quad [Sp(n),\mathcal{L}] = 0 \quad (n \geq 4)$,
(ii) $[F_4,\mathcal{L}] = [E_6,\mathcal{L}] = [E_7,\mathcal{L}] = [E_8,\mathcal{L}] = 0$.

This gives an affirmative partial answer to the conjecture due to J. C. Becker and R. E. Schultz [**3**] that $[G,\mathcal{L}] = 0$ for all compact Lie groups with rank $\geq r_0$ where r_0 is a constant smaller than 10 or so. We provide here a proof of the theorem only for the 2- and 3-component cases since (1) tells us that $[G,\mathcal{L}]_{(p)} = 0$ for any prime $p \geq 5$. Here $-_{(p)}$ denotes the localization at p. However, before proceeding to the proof of the theorem, we want to gather together some of the results obtained in other studies [**3**], [**4**], [**5**], [**6**], [**10**] and [**12**] relevant to the present work, since Theorem 1 lacks partially the description of simple Lie groups of low rank. We list them in referring to Table 1 of [**7**] with [**11**].

$$[SU(2),\mathcal{L}] = \nu \in \pi_3^S \cong \mathbb{Z}_{24} \cdot \nu,$$

$$[SU(3),\mathcal{L}] = \bar{\nu} \in \pi_8^S \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

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$$[SU(4),\mathcal{L}] = \kappa\eta \in \pi_{15}^S \cong \mathbb{Z}_{480} \oplus \mathbb{Z}_2 \cdot \kappa\eta,$$
$$[Sp(2),\mathcal{L}] = \beta_1 \in \pi_{10}^S \cong \mathbb{Z}_6,$$
$$[Sp(3),\mathcal{L}] = \sigma^3 + \bar{\kappa}\eta \in \pi_{21}^S \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$
$$[SO(3),\mathcal{L}] = 2\nu,$$
$$[SO(5),\mathcal{L}] = -\beta_1,$$
$$[SO(4),\mathcal{L}] = [SO(6),\mathcal{L}] = [SO(7),\mathcal{L}] = 0,$$
$$[G_2,\mathcal{L}] = \kappa \in \pi_{14}^S \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cdot \kappa.$$

There are isomorphisms

$$Spin(3) \cong SU(2) \cong Sp(1), \quad Spin(4) \cong Sp(1) \times Sp(1),$$
$$Spin(5) \cong Sp(2), \quad Spin(6) \cong SU(4),$$

so that from the above list we find that simple Lie groups which remain to be discussed are $SU(5)$, $SU(6)$, $SU(7)$ and perhaps $SO(5)$. Regarding these groups we have

$$[SU(5),\mathcal{L}] = [SU(6),\mathcal{L}] = [SU(7),\mathcal{L}] = 0, \quad [SO(5),\mathcal{L}]_{(2)} = 0$$

in a manner similar to Theorem 1 and the proofs are given in Appendix. The first equality $[SU(5),\mathcal{L}] = 0$ is consistent with the previously known result due to H. U. Schön [**9**]. If the last equality holds then, since β_1 is of order 3, we have $[SO(5),\mathcal{L}] = -\beta_1$ by combining this with $\beta_1 = [Sp(2),\mathcal{L}]$ mentioned above and $4[Sp(2),\mathcal{L}] = 2[SO(5),\mathcal{L}]$ of [**8**].

We now turn to the proof of Theorem 1. We begin with a brief review of the method used to prove (1). Let S be a circle subgroup of G with isomorphism $t : S \cong S^1$. Define a complex line bundle L over G/S as the quotient space of $G \times \mathbb{C}$ obtained. To simplify notations, using the same symbol as above we write $t = [L]$ for the isomorphism class of L. Let $\tilde{J}(\mu t) \in \pi_S^0(S^1 \wedge G/S^+) = \pi_S^{-1}(G/S^+)$ be the image of $\mu t \in \tilde{K}^{-1}(S^1 \wedge G/S^+)$ by the (complex) J-homomorphism where μ denotes the Bott element. Let $[G/S] \in \pi_{d-1}^S(G/S^+)$ be the framed bordism fundamental class of G/S with the framing induced by $\mathcal{L}$. Then [**7**] establishes the following formula:

$$[G,\mathcal{L}] = -\langle \tilde{J}(\mu t), [G/S] \rangle \tag{2}$$

where $\langle -, - \rangle$ denotes the Kronecker product in the stable homotopy theory. This shows that the order of $[G,\mathcal{L}]$ is subordinate to that of $\tilde{J}(\mu t)$. In fact, (1) is obtained by evaluating $\tilde{J}(\mu t)$ for a well chosen $S \subset G$ using the solution of the Adams conjecture for elements $\tilde{J}(\mu t^\ell)$ ($\ell \in \mathbb{Z}$). Since $\psi^k(\mu t^\ell) = k\mu t^{k\ell}$ we find that the solution of this conjecture is given by

$$k\tilde{J}(\mu t^{k\ell})_{(p)} = \tilde{J}(\mu t^\ell)_{(p)} \tag{3}$$

for a prime p such that $(p,k)=1$. In particular, $\tilde{J}(\mu t^{-i})_{(p)} = -\tilde{J}(\mu t^{i})_{(p)}$ holds for $i \geq 0$; hence $2\tilde{J}(\mu)_{(p)}=0$, which are used here freely.

In the next section we introduce an alternative formulation of μt and by showing that there is a difference between the actions of ψ^k on this formulation and μt itself we attempt to fill the gap between (1) and Theorem 1. In Sections 3 and 4 we discuss the classical case and the exceptional case is discussed in Section 5.

2. The alternative formulation of μt.

But first we give a formula for computing the action of the Adams operations ψ^k $(k=3,4,5)$ on the new formulation of μt explicitly.

Given a continuous map $f : X \to U(n)$ on a compact pointed space X, we can associate to it an element of $\tilde{K}^{-1}(X)$ in the following way. Let SX denote the suspension of X. Viewing it as the union of two reduced cones CX_+ and CX_- over X, we have a bundle E_f over SX obtained by gluing together two trivial bundles $CX_+ \times \mathbb{C}^n$ and $CX_- \times \mathbb{C}^n$ along $X = CX_+ \cap CX_-$ by f. Here if f is a constant map, then this bundle becomes homotopic to the trivial bundle of dimension n, for which we write $\underline{n}$. We set $\beta(f) = [E_f] - [\underline{n}]$ in $\tilde{K}(SX) = \tilde{K}^{-1}(X)$ where $[F]$ denotes the isomorphism class of a vector bundle F.

For all $i \geq 1$ we define the i-th exterior power $\lambda^i f$ of f by $(\lambda^i f)(x) = \lambda^i(f(x))$ for $x \in X$, then we have the following formula for $k=3,4,5$:

$$\psi^k\beta(f) = k(\Sigma_{j=1}^{k}(-1)^{j-1}\binom{n+k-j-1}{k-j}\beta(\lambda^j f)). \tag{4}$$

This follows directly from the equalties

$$m_3 = e_1^3 - 3e_1e_2 + 3e_3, \quad m_4 = e_1^4 - 4e_1^2e_2 + 4e_1e_2 + 2e_2^2,$$
$$m_5 = e_1^5 - 5e_1^3e_2 + 5e_1e_2^2 + 5e_1^2e_3 - 5e_2e_3 - 5e_1e_4 + 5e_5$$

where m_k and e_k denote the kth power sum $t_1^k + \cdots + t_n^k$ and kth elementary symmetric function respectively. If we write $m_k = Q_k(e_1,\ldots,e_k)$ for these equalities, then $\psi^k[E_f] = Q_k([E_f],\ldots,[E_{\lambda^k f}])$ according to the definition. Substituting $[E_{\lambda^i f}] = \beta(\lambda^i f) + [\underline{\binom{n}{i}}]$ into this formula and using the relations $\beta(fg) = m\beta(f) + n\beta(g)$ and $\beta(f)\beta(g) = 0$, we see that it can be transformed into the desired form. Here g is another continuous map from X to $U(m)$ and fg denotes the product of f and g, which arises from the tensor product of matrices. (Note that in fact (4) holds true for all $k \geq 2$, which can be verified using formulas (1) and (2) on page 178 of [**2**].)

We now give an alternative formulation of μt of the above. Let G, S and t be as in the previous section and let $\varrho : G \to U(n)$ be a complex n-dimensional representation of G. We construct an element $\tilde{K}^{-1}(S^1 \wedge G/S^+)$ associated with this ϱ. The restriction of ϱ to S can be written as a direct sum

$$\varrho|_S = t^{d_1} \oplus \cdots \oplus t^{d_n}$$

for some $d_i \in \mathbb{Z}$. We know that the image group $\varrho(S)$ is conjugate in $U(n)$ to a subgroup of the standard maximal torus $S^1 \times \cdots \times S^1$ of $U(n)$. So we assume here that the above expression on the right-hand side indicates that the value of ϱ itself at every $s \in S$ is a diagonal matrix with $t^{d_1}(s), \dots, t^{d_n}(s)$ on the diagonal in that order.

For each $1 \le i \le n$ we define a map $\varrho^{(d_i)} : S^1 \wedge G/S^+ \to U(n)$ by setting

$$\varrho^{(d_i)}(z \wedge gS) = \varrho(g)\mathrm{diag}(1, \dots, 1, \overset{\overset{i}{\vee}}{z}, 1, \dots, 1)\varrho(g)^{-1} \tag{5}$$

with $z \in S^1$ and $g \in G$ where $\mathrm{diag}(z_1, \dots, z_n)$ denotes the diagonal matrix with diagonal entries $z_1, \dots, z_n$ in that order. If ι denotes the identity map on $S^1 = U(1)$, then $\beta(\iota)$ is just the Bott element $\mu \in \tilde{K}(S^2) = \tilde{K}^{-1}(S^1)$. Looking at the definition of each element, it can be seen that $\beta(\varrho^{(d_i)}) = \mu[L^{d_i}]$ in $\tilde{K}^{-1}(S^1 \wedge G/S^+)$, hence we have

$$\tilde{J}(\beta(\varrho^{(d_i)})) = \tilde{J}(\mu t^{d_i}) \quad \text{in } \pi_S^{-1}(G/S^+).$$

This provides two different ways of applying the solution of the Adams conjecture to $\tilde{J}(\mu t^{d_i})$. In the following, together with (3), we use the equation

$$\tilde{J}(\psi^k \beta(\varrho^{(d_i)}))_{(p)} = \tilde{J}(\beta(\varrho^{(d_i)}))_{(p)} \tag{6}$$

with respect to $t_{(p)}^{d_i}$ where $k = 3, 4, 5$ such that $(p, k) = 1$.

In order to prove Theorem 1 we need one further simple lemma.

Lemma 2. *For $i = 1, 2$, let S_i be a circle subgroup of a Lie group G_i as specified above and L_i the complex line bundle over G_i/S_i associated with the principal S_i-bundle $G_i \to G_i/S_i$. Suppose there is given a homomorphism $f : G_1 \to G_2$ such that the image of S_1 by f coincides with S_2. Then $\tilde{J}(\mu t_2)_{(p)} = 0$ implies that $\tilde{J}(\mu t_1)_{(p)} = 0$, so we have $[G_1, \mathcal{L}]_{(p)} = 0$, where $t_i = [L_i]$.*

Proof. This is immediate from the assumption given and formula (2), since there holds $\tilde{f}^* L_2 \cong L_1$, where $\tilde{f}$ indicates a map $G_1/S_1 \to G_2/S_2$ induced by f. □

Finally we note that in the following for simplicity we use the abbreviations

$$t_{(p)}^i = \tilde{J}(\mu t^i)_{(p)} \quad \text{and} \quad c_j(\varrho^{(d_\ell)})_{(p)} = \tilde{J}(\beta(\lambda^j(\varrho^{(d_\ell)})))_{(p)} \quad (i \in \mathbb{Z},\ j,\ \ell \ge 1)$$

and sometimes use t_G instead of t in order to avoid confusion.

3. Proof for classical 3-components.

The proof of Theorem 1 breaks up into two parts, the classical and exceptional cases, and that of each of them is also subdivided into the 2- and 3-component cases. We begin with the classical 3-component case, particularly $[SU(n), \mathcal{L}]_{(3)}$ from which the results for the other classical groups follow easily. The proof of the other three cases proceeds along lines similar to this case.

Put $G = SU(n)$ and choose as $S \subset G$, mentioned above, the circle subgroup consisting of elements of the form

$$\mathrm{diag}(z, \bar{z}, z, \bar{z}, z^2, \bar{z}^2, z^3, \bar{z}^3, 1, \ldots, 1)$$

with $z \in S^1$ and then take for $t : S \cong S^1$ the isomorphism sending each such element to z. As above, let L denote the line bundle over G/S associated with the principal bundle $G \to G/S$ via t and use also t to denote the isomorphism class of L.

We now proceed in three steps. First we give an estimation of the order of $t_{(3)} = \tilde{J}(\mu t)_{(3)}$ based on the original method. Let $\rho : G \hookrightarrow U(n)$ be the identity representation of G. Then

$$\rho|_S = 2t \oplus 2t^{-1} \oplus t^2 \oplus t^{-2} \oplus t^3 \oplus t^{-3} \oplus (n-8).$$

This induces an isomorphism $2L \oplus 2\bar{L} \oplus L^2 \oplus \bar{L}^2 \oplus L^3 \oplus \bar{L}^3 \oplus \underline{n-8} \cong \underline{n}$ of vector bundles over G/S, so we have $2\mu t + 2\mu t^{-1} + \mu t^2 + \mu t^{-2} + \mu t^3 + \mu t^{-3} = 8\mu \cdot 1$ in $\tilde{K}^{-1}(S(G/S^+))$. Multiplying this from the right by t, t^2 and t^6 and operating $\tilde{J}(-)_{(3)}$ on the equalities thereby obtained we have

$$\begin{aligned}
2\mu t^2_{(3)} + 2\mu \cdot 1_{(3)} + \mu t^3_{(3)} + \mu t^{-1}_{(3)} + \mu t^4_{(3)} + \mu t^{-2}_{(3)} &= 8\mu t_{(3)}, \\
2\mu t^3_{(3)} + 2\mu t_{(3)} + \mu t^4_{(3)} + \mu \cdot 1_{(3)} + \mu t^5_{(3)} + \mu t^{-1}_{(3)} &= 8\mu t^2_{(3)}, \\
2\mu t^7_{(3)} + 2\mu t^5_{(3)} + \mu t^8_{(3)} + \mu t^4_{(3)} + \mu t^9_{(3)} + \mu t^3_{(3)} &= 8\mu t^6_{(3)}
\end{aligned}$$

in $\pi_S^{-1}(G/S^+)$. On the other hand, we have $2t^2_{(3)} = 4t^4_{(3)} = 5t^5_{(3)} = 7t^7_{(3)} = t_{(3)}$ from formula (3). Applying these to the above two equalities we get

$$9t_{(3)} = 0, \quad t^3_{(3)} = 6t_{(3)}, \quad t^9_{(3)} = 0.$$

Next, choose $\varrho = 5\rho \oplus 2$ and assume that $n = 3^s$ $(s \geq 2)$. Then the use of these relations allows us to solve the equation $\tilde{J}(\psi^4\beta(\varrho^{(1)}))_{(3)} = \tilde{J}(\beta(\varrho^{(1)}))_{(3)}$ considered in (6). However for simplicity of calculation we perform this with reduction mod $(3t_{(3)})$. Using the relations obtained above we deduce from formula (3) that

$$t^{3i+2}_{(3)} \equiv 2t_{(3)}, \quad t^{3i+1}_{(3)} \equiv t_{(3)}, \quad t^{3i}_{(3)} \equiv 0 \mod (3t_{(3)}) \qquad (0 \leq i \leq 3).$$

Taking account of these relations, a glance at definition (5) shows that every $c_i(\varrho^{(2)})_{(3)}$, $1 \leq i \leq 4$, has the form $n_i t_{(3)}$ where $n_i \in \mathbb{Z}$ and hence from formula (4) it follows that the above reduction can be written in the form

$$c_1(\varrho^{(1)})_{(3)} \equiv 4(c_1(\varrho^{(1)})_{(3)} + 2c_3(\varrho^{(1)})_{(3)} - c_4(\varrho^{(1)})_{(3)})) \mod (3t_{(3)}).$$

By performing the calculation of its terms in more detail we have

$$c_1(\varrho^{(1)})_{(3)} = t_{(3)}, \quad c_3(\varrho^{(1)})_{(3)} \equiv 0, \quad c_4(\varrho^{(1)})_{(3)} \equiv 2t_{(3)} \mod (3t_{(3)})$$

where the first equality is already verified in the formula preceding (6). Substitution of these into the above equality yields $t_{(3)} \equiv 0 \bmod (3t_{(3)})$ immediately. But since $9t_{(3)} = 0$, this means that $t_{(3)} = 0$. Thus we have

Proposition 3. *Let $N = 3^s$ $(s \geq 3)$. Then $\tilde{J}(\mu t)_{(3)} = 0$ in $\pi_S^{-1}(SU(N)/S^+)_{(3)}$.*

Finally we must show that the equality $\tilde{J}(\mu t_{SU(n)})_{(3)} = 0$ holds for any $SU(n)$ with $n \geq 8$. However this follows immediately from Lemma 2 and Proposition 3, because such a group $SU(n)$ has a circle subgroup S such that its image under the standard inclusion $i : SU(n) \hookrightarrow SU(N)$ coincides with S in $SU(N)$ above for large s. That is, we have $\tilde{J}(\mu t_{SU(n)})_{(3)} = \tilde{i}^* \tilde{J}(\mu t_{SU(N)})_{(3)} = 0$ and hence by (2) we can conclude that $[SU(n), \mathcal{L}]_{(3)} = 0$ for $n \geq 8$.

For the other two types of classical groups, we can proceed quite similarly as in the case of $SU(n)$. In the case of $Sp(n)$, let $n \geq 4$ and take as S the circle subgroup consisting of elements of the form $\mathrm{diag}(z, z, z^2, z^3, 1, \ldots, 1)$ with $z \in S^1$. Then it is clear that the complexification $c : Sp(n) \to SU(2n)$ sends this S identically to S in $SU(2n)$. From the fact that $\tilde{J}(\mu t_{SU(2n)})_{(3)} = 0$, we have $\tilde{J}(\mu t_{Sp(n)})_{(3)} = 0$, that is, $[Sp(n), \mathcal{L}]_{(3)} = 0$ for $n \geq 4$.

In the case of real groups, let $n \geq 8$ and first we consider about $SO(n)$. Let S be the circle subgroup consisting of elements of the form $\mathrm{diag}(D(z), D(z), D(z^2), D(z^3), 1, \ldots, 1)$ with $z \in S^1$, where $D(z)$ denotes the realification of the one-dimensional matrix (z). Let $c : SO(n) \to SU(n)$ be the complexification of $SO(n)$. We know that this c can be transformed by conjugation by an element of $SU(n)$ so that the image of S coincides with S in $SU(n)$, which proves that $[SO(n), \mathcal{L}]_{(3)} = 0$ for $n \geq 8$.

In order to complete the proof in the present case we need to prove that this holds true for the double covering group $Spin(n)$ of $SO(n)$. For this we choose as the circle subgroup $\tilde{S}$ of $Spin(n)$ the double covering $Spin(2)$ of the standard rotation subgroup $S = SO(2) \subset SO(n)$, this S being used as that itself for $SO(n)$. Let $\tilde{L}$ be the complex line bundle over $Spin(n)/\tilde{S}$ associated to the principal $\tilde{S}$-bundle $Spin(n) \to Spin(n)/\tilde{S}$. Then the complex line bundle L over $SO(n)/S$ can be identified with $\tilde{L}^{\otimes 2}$ through the homeomorphism $SO(n)/S \approx Spin(n)/\tilde{S}$, that is, it holds that $t_{SO(n)} = t_{Spin(n)}^2$. Hence by the solution of the Adams conjecture we have $2\tilde{J}(\mu t_{SO(n)})_{(3)} = \tilde{J}(\mu t_{Spin(n)})_{(3)}$, so $2[SO(n), \mathcal{L}]_{(3)} = [Spin(n), \mathcal{L}]_{(3)}$. Thus we get $[Spin(n), \mathcal{L}]_{(3)} = 0$ for $n \geq 8$. This completes the proof of (i) for 3-components.

4. Proof for classical 2-components.

Similarly to the above we first consider the case $G = SU(n)$. Let $\rho : SU(n) \hookrightarrow U(n)$ be the identity representation of $SU(n)$ and S denote the same circle subgroup of $SU(n)$ as above. Then from the restriction formula

$$\rho|_S = 2t \oplus 2t^{-1} \oplus t^2 \oplus t^{-2} \oplus t^3 \oplus t^{-3} \oplus (n-8),$$

arguing in a similar way we get

$$16t_{(2)} = 0, \quad t_{(2)}^2 = 10t_{(2)} + 1_{(2)}, \quad t_{(2)}^4 = 4t_{(2)} + 1_{(2)}, \quad t_{(2)}^8 = 1_{(2)}.$$

In this case we choose $\varrho = 3\rho \oplus 2$ and assume $n = 2^s$ $(s \geq 3)$, and attempt to calculate the equation $\tilde{J}(\psi^5\beta(\varrho^{(3)}))_{(2)} = \tilde{J}(\beta(\varrho^{(3)}))_{(2)}$ under the relations above, though in fact we consider its reduction mod $(2t_{(2)})$ for the same reason as above. Then as in the case above we see that we can write

$$\tilde{J}(\psi^5\beta(\varrho^{(3)}))_{(2)} \equiv t_{(2)}^3 + c_3(\varrho^{(3)})_{(2)} + c_5(\varrho^{(3)})_{(2)} \mod (2t_{(2)})$$

and we have

$$c_3(\varrho^{(3)})_{(2)} \equiv 0, \quad c_5(\varrho^{(3)})_{(2)} \equiv t_{(2)} \mod (2t_{(2)}).$$

These results show immediately that the above reduction $\tilde{J}(\psi^5\beta(\varrho^{(3)}))_{(2)} \equiv \tilde{J}(\beta(\varrho^{(3)}))_{(2)}$ mod $(2t_{(2)})$ has a unique solution $t_{(2)} \equiv 0$ mod $(2t_{(2)})$ as desired, hence we have $t_{(2)} = 0$ due to the relation $16t_{(2)} = 0$. Consequently we get

PROPOSITION 4. *Let $N = 2^s$ where $s \geq 4$. Then $\tilde{J}(\mu t)_{(2)} = 0$ in $\pi_S^{-1}(SU(N)/S^+)_{(2)}$.*

By applying the same reasoning as the previous case used, we see that this completes the proof of (i) for 2-components except proving the case of $Spin(n)$.

This case needs to be considered separately. Let $\pi : Spin(n) \to SO(n)$ be the covering homomorphism where $n = 2^s$ for a fixed $s \geq 4$ and let $\tilde{S}$ be the two-fold covering group of $S \subset SO(n)$ consisting of elements of the form $\mathrm{diag}(D(z^2), D(z^2), D(z^4), D(z^6), 1, \ldots, 1)$ with $z \in S^1$. Then $t_{SO(n)}$ represents the isomorphism $S \cong S^1$ sending each such element to z and when putting $t = t_{Spin(n)} : \tilde{S} \cong S^1$, it satisfies the identity $t^2 = t_{SO(n)} \circ (\pi|\tilde{S})$. With the identification $Spin(n)/\tilde{S} \approx SO(n)/S$ induced by π we prove

PROPOSITION 5. *For n as above there holds $\tilde{J}(\mu t)_{(2)} = 0$ in $\pi_S^{-1}(Spin(n)/\tilde{S}^+)_{(2)}$.*

This means that $[Spin(n), \mathcal{L}]_{(2)} = 0$ holds for $n \geq 8$.

PROOF OF PROPOSITION 5. Let $\tilde{\rho}$ be the composite of π with the identity representation $\rho : SO(n) \hookrightarrow SU(n)$. Then we have

$$\tilde{\rho}|_{\tilde{S}} = 2t^2 \oplus 2t^{-2} \oplus t^4 \oplus t^{-4} \oplus t^6 \oplus t^{-6} \oplus (n-8).$$

From this formula and the result for $SO(n)$ above it follows that

$$8t_{(2)} = 0, \quad t_{(2)}^2 = 0, \quad t_{(2)}^4 = t_{(2)}^8 = 1_{(2)}.$$

If we set $\varrho = \Delta \oplus 3\tilde{\rho} \oplus 1$, where Δ denotes the spin representation of $Spin(n)$, then we see that we can define $\varrho^{(1)}$, which we owe to the choice of S. Analogously to $t_{SO(n)}$, using the relations above, we find that

$$c_i(\varrho^{(1)})_{(2)} \equiv t_{(2)} \quad (i = 2, 4, 5), \quad c_3(\varrho^{(1)})_{(2)} \equiv 0 \mod (2t_{(2)}).$$

Inserting these into the equation $\tilde{J}(\psi^5\beta(\varrho^{(1)}))_{(2)} \equiv \tilde{J}(\beta(\varrho^{(1)}))_{(2)} \bmod (2t_{(2)})$ we obtain $t_{(2)} \equiv 0 \bmod (2t_{(2)})$. Since $8t_{(2)} = 0$ it follows that $t_{(2)} = 0$, which proves the proposition. □

5. Proof for exceptional groups.

For the proof of the exceptional case we employ the following well-known chain of inclusions:

$$Spin(9) \subset F_4 \subset E_6 \subset E_7 \subset E_8$$

and choose a circle subgroup S of $Spin(9)$, together with isomorphism $t : S \cong S^1$, which is viewed as a common circle subgroup of these four exceptinal groups through the inclusions. We use the letter t to denote the isomorphism class of the complex line bundle L over E_8/S associated with the principal S-bundle $E_8 \to E_8/S$, as before.

Proposition 6. *If S is nicely chosen, then $\tilde{J}(\mu t)_{(p)} \in \pi_S^{-1}(E_8/S^+)_{(p)}$ becomes zero for $p = 3, 2$.*

Combining this proposition and Lemma 2, we have

$$[F_4, \mathcal{L}]_{(p)} = [E_6, \mathcal{L}]_{(p)} = [E_7, \mathcal{L}]_{(p)} = [E_8, \mathcal{L}]_{(p)} = 0.$$

Proof of Proposition 6. Let $e_1, \ldots, e_9$ be the standard basis of the Clifford algebra $Cl_{0,9}(\mathbb{R})$ and put $\omega_{r,s}(\theta) = \cos\theta + e_r e_s \sin\theta$ for simplicity. We consider the homomorphism $\alpha : Spin(2) \to Spin(9)$ given by

$$\alpha(\omega_{1,2}(\theta)) = \omega_{1,2}(\theta)\omega_{1,3}(\theta)\omega_{1,4}(\theta),$$

which is clearly injective. Set $S = \operatorname{Im}\alpha \subset E_8$ and choose $\rho = \mathrm{Ad}_{E_8}$, the (comlexified) adjoint representation of E_8 whose dimension is 248. Then $t : S \cong S^1$ can be given by $t(\alpha(\omega_{1,2}(\theta))) = e^{2i\theta}$. From the formula on page 52 of [**1**] we see that the restriction of ρ to $Spin(10) \subset E_8$ satisfies

$$\rho|_{Spin(10)} = \lambda_{10}^2 \oplus 6\lambda_{10}^1 \oplus 4\Delta \oplus 15$$

where λ_{10}^2 denotes the 2nd exterior of λ_{10}^1, λ_{10}^1 being the composite $Spin(10) \to SO(10) \subset U(10)$ of the covering homomorphism with the inclusion homomorphism and Δ the spin representation of $Spin(10)$. Consider the restriction of this to $S \subset Spin(9) \subset Spin(10)$, then we can write as

$$\rho|_S = 48t \oplus 48t^{-1} \oplus 30t^2 \oplus 30t^{-2} \oplus 16t^3 \oplus 16t^{-3} \oplus 3t^4 \oplus 3t^{-4} \oplus 54.$$

The argument here is based on the relations obtained from this restriction formula in a similar manner as the previous cases. We proceed with these relations.

In the case $p = 3$, we have the relations

$$9t_{(3)} = 3t_{(3)}^3 = 0.$$

Using these relations we solve $\tilde{J}(\psi^5\beta(\varrho^{(1)}))_{(3)} = \tilde{J}(\beta(\varrho^{(1)}))_{(3)}$ for $\varrho = \rho \oplus 4$. But for easy of calculation we consider its reduction mod $(3t_{(3)})$ as before. In a similar way to the above cases we have from (4) and (5) that

$$\tilde{J}(\psi^5\beta(\varrho^{(1)}))_{(3)} \equiv 2c_5(\varrho^{(1)})_{(3)} \quad \text{and} \quad c_5(\varrho^{(1)})_{(3)} \equiv 0 \mod (3t_{(3)}).$$

Hence we see that the above reduction equation has a unique solution $t_{(3)} \equiv 0 \bmod (3t_{(3)})$. This means that $t_{(3)} = 0$ since $9t_{(3)} = 0$.

Finally we consider the case $p = 2$. Then we have the relations

$$16t_{(2)} = 16t_{(2)}^2 = 0, \quad 2t_{(2)}^4 = -4t_{(2)}^2, \quad t_{(2)}^8 = -4t_{(2)}^2 + 1_{(2)}, \quad t_{(2)}^{16} = 8t_{(2)}^2 + 1_{(2)}.$$

However the calculation for obtaining these relations is somewhat lengthy than in the previous case. In order to proceed to the next step we need one more relation. We take $\varrho = \rho$ and consider the equation $\tilde{J}(\psi^3\beta(\varrho^{(2)}))_{(2)} = \tilde{J}(\beta(\varrho^{(2)}))_{(2)}$. Similarly as above we then have

$$c_1(\varrho^{(2)})_{(2)} = t_{(2)}, \quad c_2(\varrho^{(2)})_{(2)} = -3t_{(2)}^4 + 4t_{(2)}^2, \quad c_3(\varrho^{(2)})_{(2)} = -3t_{(2)}^2.$$

Substituting these into the above equation it follows that $2t_{(2)}^2 = 0$. This yields a refinement of the above relations such that

$$16t_{(2)} = 0, \quad 2t_{(2)}^2 = 2t_{(2)}^4 = 0, \quad t_{(2)}^8 = t_{(2)}^{16} = 1_{(2)},$$

which allow us to calculate $\tilde{J}(\psi^5\beta(\varrho^{(3)}))_{(2)} \equiv \tilde{J}(\beta(\varrho^{(3)}))_{(2)} \bmod (2t_{(2)})$. In fact, we find that $\tilde{J}(\psi^5\beta(\varrho^{(3)}))_{(2)} \equiv c_5(\varrho^{(3)})_{(2)} \bmod (2t_{(2)})$ and also its term $c_5(\varrho^{(3)})_{(2)}$ is zero mod $(2t_{(2)})$. Consequently we have that $t_{(2)}^3 \equiv 0 \bmod (2t_{(2)})$ which is equivalent to $t_{(2)} \equiv 0 \bmod (2t_{(2)})$. Since $16t_{(2)} = 0$, this means that $t_{(2)} = 0$, which proves the proposition and so completes the proof of Theorem 1. □

Appendix.

Here we prove $[SU(5), \mathcal{L}] = [SU(6), \mathcal{L}] = [SU(7), \mathcal{L}] = 0$ and $[SO(5), \mathcal{L}]_{(2)} = 0$, announced in Introduction, following the same procedure as given in the proof of Theorem 1. Then we use the same notations S, t, $t_{(p)}$ and ϱ as those used in the above.

First we consider the case of $G = SU(7)$. Let $\rho : G \hookrightarrow U(7)$ be the identity representation of G and choose as $S \subset G$ the circle subgroup consisting of elements of the form $\mathrm{diag}(z,z,z,z,z^{-4},1,1)$ with $z \in S^1$. Then $\rho|_S = 4t \oplus t^{-4} \oplus 2$. This yields

$$3t_{(3)} = 0, \quad t_{(3)}^3 = 0$$

and

$$8t_{(2)} = 0, \quad 2t_{(2)}^2 = 4t_{(2)}, \quad t_{(2)}^4 = 4t_{(2)} + 1_{(2)}, \quad t_{(2)}^8 = 1_{(2)}.$$

Set $\varrho = \rho \oplus \lambda^2\rho \oplus (p-1)$ for the cases of $p = 3, 2$, respectively. Then the relations above allow us to solve the equation $\tilde{J}(\psi^5\beta(\varrho^{(1)}))_{(p)} = \tilde{J}(\beta(\varrho^{(1)}))_{(p)} \bmod (pt_{(p)})$ for $p = 3, 2$ and gain $t_{(3)} = 0$ and $t_{(2)} = 0$. Thus we have $[SU(7), \mathcal{L}] = 0$.

There is obtained a chain of inclusions $S \subset SU(5) \subset SU(6) \subset SU(7)$ by viewing the S above as a subgroup of $SU(5)$. By applying Lemma 2 to this chain, we find that the results for $SU(5)$ and $SU(6)$ follow immediately from that for $SU(7)$.

Finally we consider the 2-component case of $SO(5)$. Let $Spin(5) = Sp(2)$ and $\tilde{S}$ be the circle subgroup of $Spin(5)$ which covers the circle subgroup $S \subset SO(5)$ consisting of elements of the form $\mathrm{diag}(D(z),1,1,1)$ with $z \in S^1$. Let $\rho : SO(5) \hookrightarrow U(5)$ be the identity representation of $SO(5)$ and $\Delta : Spin(5) \to U(4)$ be the spin representations of $Spin(5)$. Then $\tilde{\rho}|_{\tilde{S}} = t^2 \oplus t^{-2} \oplus 3$ and $\Delta|_{\tilde{S}} = 2t \oplus 2t^{-1}$ where $\tilde{\rho}$ is the composition of ρ and the covering homomorphism $Spin(5) \to SO(5)$. From these two restriction formulas we obtain the same relations $8t_{(2)} = 0$, $2t_{(2)}^2 = 4t_{(2)}$, $t_{(2)}^4 = 4t_{(2)} + 1_{(2)}$, $t_{(2)}^8 = 1_{(2)}$ as above. Let $\varrho = \tilde{\rho} \otimes \Delta \oplus \Delta \oplus 2$ and calculate $\tilde{J}(\psi^3\beta(\varrho^{(2)}))_{(2)} \equiv \tilde{J}(\beta(\varrho^{(2)}))_{(2)} \bmod (2t_{(2)})$ using the relations above, then it follows that $t_{(2)}^2 \equiv 0 \bmod (2t_{(2)})$. On the other hand, in a similar manner from $\tilde{J}(\psi^3\beta(\varrho^{(1)}))_{(2)} \equiv \tilde{J}(\beta(\varrho^{(1)}))_{(2)} \bmod (2t_{(2)})$ we get $t_{(2)} \equiv 0 \bmod (2t_{(2)})$, which implies $t_{(2)} = 0$ since $8t_{(2)} = 0$, so we have $t_{(2)}^2 = 0$ which concludes that $[SO(5), \mathcal{L}]_{(2)} = 0$.

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