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Ore-Rees rings which are maximal orders

By Monika R. Helmi, Hidetoshi Marubayashi and Akira Ueda

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Abstract. Let R be a Noetherian prime ring with an automorphism σ and a left σ-derivation δ, and let X be an invertible ideal of R with $\sigma(X) = X$. We define an Ore-Rees ring $S = R[Xt; \sigma, \delta]$ which is a subring of an Ore extension $R[t; \sigma, \delta]$, where t is an indeterminate. It is shown that if R is a maximal order, then so is S. In case $\sigma = 1$, we define the concepts of $(\delta; X)$-stable ideals of R and of $(\delta; X)$-maximal orders and prove that S is a maximal order if and only if R is a $(\delta; X)$-maximal order. Furthermore we give a complete description of v-S-ideals, which is used to characterize S to be a generalized Asano ring. In case $\delta = 0$, we define the concepts of $(\sigma; X)$-invariant ideals of R and of $(\sigma; X)$-maximal orders in order to show that S is a maximal order if and only if R is a $(\sigma; X)$-maximal order. We also give examples R such that either R is a $(\delta; X)$-maximal order or is a $(\sigma; X)$-maximal order but they are not maximal orders.

1. Introduction.

Throughout this paper, R denotes a Noetherian prime ring with quotient ring Q otherwise stated (in other word, R is a Noetherian prime order in a simple Artinian ring Q), σ is an automorphism of R, δ is a left σ-derivation on R and X is an invertible ideal of R. A subset $S = R[Xt; \sigma, \delta] = R \oplus Xt \oplus \cdots \oplus X^n t^n \oplus \cdots$ of the Ore extension $R[t; \sigma, \delta]$ in an indeterminate t is called an *Ore-Rees* ring if S is a ring (see Lemma 2.2).

Generalized Rees rings were studied in [**4**] and [**19**] under PI condition and in the book [**20**], they summarized them from torsion theoretical view-point under PI condition. In this paper, we do not assume that Ore-Rees rings satisfy PI conditions.

The aim of this paper is to study the order theoretical properties of S.

The paper is organized as follows:

In Section 2, first we show that if R is a maximal order, then so is S (Theorem 2.4).

Secondly, we define the concepts of $(\sigma, \delta; X)$-stable ideals of R, of $(\sigma, \delta; X)$-maximal orders and study some properties of prime ideals of S which are derived from the based ring R and from the Ore extension $Q[t; \sigma, \delta]$ (Propositions 2.12 and 2.14). If R is a $(\sigma, \delta; X)$-maximal order, then the set of $(\sigma, \delta; X)$-stable v-R-ideals is an Abelian group generated by maximal $(\sigma, \delta; X)$-stable v-ideals of R (Proposition 2.17).

These results are used to obtain more detailed properties of S in case either $\sigma = 1$ or $\delta = 0$ in Sections 3, and 4, respectively.

In case $\sigma = 1$, we write $S = R[Xt; \delta]$ for $R[Xt; 1, \delta]$. We just say $(\delta; X)$-stable ideals for $(1, \delta; X)$-stable ideals and $(\delta; X)$-maximal orders for $(1, \delta; X)$-maximal orders.

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In Section 3, we show that $S = R[Xt;\delta]$ is a maximal order if and only if R is a $(\delta;X)$-maximal order (Theorem 3.5). Furthermore if R is a $(\delta;X)$-maximal order, then we give a complete description of v-S-ideal A as follows; $A = w\mathfrak{a}[Xt;\delta]$ for some $(\delta;X)$-stable v-R-ideal $\mathfrak{a}$ and $w \in \mathbb{Z}(Q(T))$, the center of $Q(T)$ which is the quotient ring of $T = Q[t;\delta]$, the differential polynomial ring over Q (Proposition 3.6).

Proposition 3.6 is applied to get a characterization of a generalized Asano ring S (Corollary 3.7).

In case $\delta = 0$, we write $S = R[Xt;\sigma]$ for $R[Xt;\sigma,0]$. We say $(\sigma;X)$-invariant ideals for $(\sigma,0;X)$-stable ideals and $(\sigma;X)$-maximal orders for $(\sigma,0;X)$-maximal orders.

In Section 4, we show that $S = R[Xt;\sigma]$ is a maximal order if and only if R is a $(\sigma;X)$-maximal order (Thereom 4.4). If R is a $(\sigma;X)$-maximal order, then any v-S-ideal is of the form $t^n w\mathfrak{a}[Xt;\sigma]$, where $\mathfrak{a}$ is a $(\sigma;X)$-invariant v-R-ideal, $w \in \mathbb{Z}(Q(T))$ ($T = Q[t;\sigma]$, the skew polynomial ring over Q) and n is an integer.

In Section 5, we provide examples of orders such that either $(\delta;X)$-maximal orders or $(\sigma;X)$-maximal orders but not maximal orders. Furthermore we give an example R such that $S = R[Xt;\sigma]$ is a maximal order but the skew polynomial ring $R[t;\sigma]$ is not a maximal order.

We refer the readers to the books [**16**] and [**17**] for some elementary properties and some definitions of order theory which are not mentioned in the paper.

2. Ore-Rees rings.

Let σ be an automorphism of R, δ be a left σ-derivation on R. σ is naturally extended to an automorphism σ of Q by $\sigma(rc^{-1}) = \sigma(r)\sigma(c)^{-1}$, where $r,\ c \in R$ and c is regular, and δ is extended to a left σ-derivation on Q by $\delta(c^{-1}) = -\sigma(c^{-1})\delta(c)c^{-1}$.

Let $R[t;\sigma,\delta]$ be an Ore extention of R in an indeterminate t, that is $tr = \sigma(r)t + \delta(r)$ for any $r \in R$ and put $T = Q[t;\sigma,\delta]$ throughout the paper. For symmetric argument, it is sometimes convenient to write the coefficients of a polynomial in T the right hand side. In this case $T = Q[t;\sigma',\delta']$, where $\sigma' = \sigma^{-1}$ and $\delta' = -\delta\sigma^{-1}$, a right σ'-derivation on Q.

Let X be an invertible ideal of R. We need the following lemmas for symmetric argument.

Lemma 2.1. *Let X be an invertible ideal of R with $\sigma(X) = X$. Then for any natural numbers l and n,*

(1) $\delta(X^l) \subseteq X^{l-1}$ $(X^0 = R)$.
(2) $X^l t^n \subseteq \sum_{i=0}^n t^{n-i}X^{l-i}$ *and* $t^n X^l \subseteq \sum_{i=0}^n X^{l-i}t^{n-i}$*, where we put* $X^{l-i} = R$ *if* $n-i > 0$ *and* $l-i \le 0$*, and* $X^{l-i} = \delta(R)$ *if* $n-i = 0$ *and* $l-i \le 0$.

Proof. (1) is clear by induction on l.

(2) $Xt \subseteq t\sigma'(X) + \delta'(X) \subseteq tX + \delta(R)$, $X^2t = X(Xt) \subseteq X(tX+R) \subseteq (tX+R)X + XR = tX^2 + X$. So it inductively follows that $X^l t \subseteq tX^l + X^{l-1}$. By induction on n, we may assume that $X^l t^n \subseteq \sum_{i=0}^n t^{n-i}X^{l-i}$. Then the following formula is proved by checking in case $l \ge n$ and $l < n$ separately:

$$X^l t^{n+1} = (X^l t^n)t \subseteq \left(\sum_{i=0}^{n} t^{n-i} X^{l-i}\right) t \subseteq \sum_{i=0}^{n} \left\{ t^{n-i}(tX^{l-i} + \delta'(X^{l-i})) \right\}$$
$$\subseteq \sum_{i=0}^{n+1} t^{n+1-i} X^{l-i}.$$

The second statement follows by symmetric argument. □

Now let X be a fixed invertible ideal of R. Put

$$S = R[Xt; \sigma, \delta] = R \bigoplus Xt \bigoplus X^2t^2 \bigoplus \cdots \bigoplus X^n t^n \bigoplus \cdots$$

and

$$S_1 = R \bigoplus tX \bigoplus t^2X^2 \bigoplus \cdots \bigoplus t^n X^n \bigoplus \cdots,$$

which are both subsets of T. If S is a ring, then it is called an *Ore-Rees ring* associated to X. In this case S and $R[t; \sigma, \delta]$ have the same quotient ring $Q(S) = Q(R[t; \sigma, \delta])$ which is a simple Artinian ring.

Lemma 2.2. *S is a ring if and only if $\sigma(X) = X$ if and only if $\sigma'(X) = X$ if and only if S_1 is a ring. In this case, $S = S_1$ and is Noetherian.*

Proof. If S is a ring, then for any $x, y \in X$, we have

$$xtyt = x\big(\sigma(y)t + \delta(y)\big)t = x\sigma(y)t^2 + x\delta(y)t \in X^2t^2 + Xt,$$

that is $\sigma(X) \subseteq X$ and so $\sigma(X) = X$, because R is Noetherian. Conversely if $\sigma(X) = X$, then, for any natural numbers l, n,

$$X^n t^n X^l t^l \subseteq X^n \left(\sum_{i=0}^{n} X^{l-i} t^{n-i}\right) t^l \subseteq \sum_{i=0}^{n} X^{n+l-i} t^{n+l-i} \subseteq S$$

by Lemma 2.1. Hence S is a ring. If S is a ring, then S_1 is a ring and $S = S_1$ by Lemma 2.1. That S is Noetherian is proved in the similar way as [**9**, Proposition 2.1]. □

In the remainder of this paper, we assume that $\sigma(X) = X$. First we show that if R is a maximal order, then so is S by using following lemma.

Lemma 2.3. *If A is an ideal of S, then AT is an ideal of T.*

Proof. We first prove that $c^{-1}AT = AT$ for any regular elememt c of R and $X^{-1}AT = AT$. Since $cAT \subseteq AT$, $AT \subseteq c^{-1}AT \subseteq c^{-2}AT \subseteq \cdots \subseteq T$. Hence $c^{-n}AT = c^{-(n+1)}AT$ for some n because T is Noetherian, in fact, it is a principal ideal ring ([**16**, Corollary 2.3.7]) and so $AT = c^{-1}AT$. Similarly $AT = X^{-1}AT$ holds.

Next, for any $q(t) \in T$, there exists a regular element $c \in R$ such that $cq(t) = r(t) \in$

$R[t;\sigma,\delta]$. If $\deg q(t)=n$, then

$$\begin{aligned} q(t)AT &= c^{-1}r(t)AT \subseteq c^{-1}Rr(t)AT = c^{-1}X^{-n}X^{n}r(t)AT \subseteq c^{-1}X^{-n}AT \\ &= c^{-1}AT = AT \end{aligned}$$

because $X^n r(t) \subseteq S$. Thus AT is an ideal of T. $\square$

Theorem 2.4. *If R is a maximal order, then so is the Ore-Rees ring $S = R[Xt;\sigma,\delta]$.*

Proof. For any ideal A of S, let

$$C_n(A) = \{a \in R \mid \exists\ h(t) = at^n + \cdots + a_0 \in A\} \cup \{0\}.$$

Then $C_n(A)$ is an ideal of R (note: $C_n(A) \subseteq X^n$). For $a \in C_n(A)$, there is some $h(t) = at^n + a_{n-1}t^{n-1} + \cdots + a_0 \in A$. Then $(Xt)h(t) = X\sigma(a)t^{n+1} +$ (the lower degree parts) $\subseteq A$ and so $X\sigma(a) \subseteq C_{n+1}(A)$ holds. Hence $X\sigma(C_n(A)) \subseteq C_{n+1}(A)$, that is $C_n(A) \subseteq X^{-1}\sigma^{-1}(C_{n+1}(A))$ for any n. Thus we have a following chain of right ideals of R,

$$C_0(A) \subseteq X^{-1}\sigma^{-1}(C_1(A)) \subseteq X^{-2}\sigma^{-2}(C_2(A)) \subseteq \cdots \subseteq X^{-n}\sigma^{-n}(C_n(A)) \subseteq \cdots \subseteq R.$$

Because R is Noetherian, $X^{-m}\sigma^{-m}(C_m(A)) = X^{-(m+k)}\sigma^{-(m+k)}(C_{m+k}(A))$ for some m and for any $k \geq 1$. Thus we have

$$X^k\sigma^k(C_m(A)) = C_{m+k}(A)$$

for any $k \geq 1$.

Now let $f \in Q(S)$ such that $fA \subseteq A$, where $Q(S)$ is the quotient ring of S. Then $fAT \subseteq AT$ and AT is an ideal of T by Lemma 2.3. Since T is a maximal order, $f \in O_l(AT) = T$ and so $f = f_kt^k + \cdots + f_0$, where $f_i \in Q$. Let $a \in C_m(A)$ and $h = at^m + a_{m-1}t^{m-1} + \cdots + a_0 \in A$. Then

$$fh = f_k\sigma^k(a)t^{m+k} + \text{ (the lower degree parts) } \in A$$

and so $f_k\sigma^k(a) \in C_{m+k}(A)$. Hence $f_k\sigma^k(C_m(A)) \subseteq C_{m+k}(A)$ holds and

$$\begin{aligned} C_{m+k}(A) \supseteq f_k\sigma^k(C_m(A)) &= f_kR\sigma^k(C_m(A)) = f_kX^{-k}X^k\sigma^k(C_m(A)) \\ &= f_kX^{-k}C_{m+k}(A). \end{aligned}$$

Thus $f_kX^{-k} \subseteq O_l(C_{m+k}(A)) = R$ because R is a maximal order and so we have $f_k \in X^k$. Hence $f_kt^k \in S \subseteq T$. Then $f - f_kt^k = f_{k-1}t^{k-1} +$ (the lower degree parts) $\in T$ and

$$(f - f_kt^k)A \subseteq fA - f_kt^kA \subseteq A,$$

and we obtain $f_{k-1} \in X^{k-1}$ in the similar way. Continuing this process, we have $f \in S$ and so $O_l(A) = S$. The symmetric argument shows that $O_r(A) = S$ (Lemma 2.2) and hence S is a maximal order. □

Second we study ideals of R and S which are induced by the properties of $(\sigma, \delta; X)$, some of which are used in Sections 3 and 4 to give a necessary and sufficient conditions for S to be a maximal order and to describe the complete structure of v-ideals in $Q(S)$ in case either $\sigma = 1$ or $\delta = 0$.

Lemma 2.5. *Let $\mathfrak{a}$ be an ideal of R. Then $A = \mathfrak{a}[Xt; \sigma, \delta]$ is an ideal of S if and only if $X\sigma(\mathfrak{a}) = \mathfrak{a}X$ and $X\delta(\mathfrak{a}) \subseteq \mathfrak{a}$.*

Proof. Suppose A is an ideal of S. For any $x \in X$ and $a \in \mathfrak{a}$, we have $xta = x\sigma(a)t + x\delta(a)$. So $X\sigma(\mathfrak{a}) \subseteq \mathfrak{a}X$ and $X\delta(\mathfrak{a}) \subseteq \mathfrak{a}$. Since $X\mathfrak{a} \subseteq \sigma^{-1}(\mathfrak{a})X$, we have $\mathfrak{a} \subseteq X^{-1}\sigma^{-1}(\mathfrak{a})X$ which gives $X^{-1}\sigma^{-1}(\mathfrak{a})X \subseteq X^{-2}\sigma^{-2}(\mathfrak{a})X^2$. Thus inductively we have

$$\mathfrak{a} \subseteq X^{-1}\sigma^{-1}(\mathfrak{a})X \subseteq X^{-2}\sigma^{-2}(\mathfrak{a})X^2 \subseteq \cdots \subseteq X^{-n}\sigma^{-n}(\mathfrak{a})X^n \subseteq \cdots \subseteq R.$$

There is an n such that $X^{-n}\sigma^{-n}(\mathfrak{a})X^n = X^{-(n+1)}\sigma^{-(n+1)}(\mathfrak{a})X^{n+1}$ and so $\sigma(\mathfrak{a}) = X^{-1}\mathfrak{a}X$, that is $X\sigma(\mathfrak{a}) = \mathfrak{a}X$. Conversely suppose $X\sigma(\mathfrak{a}) = \mathfrak{a}X$ and $X\delta(\mathfrak{a}) \subseteq \mathfrak{a}$. To prove that A is an ideal of S, it is enough to show that A is a left ideal. Since $\sigma(X) = X$, we have

$$tX\mathfrak{a} \subseteq \sigma(X\mathfrak{a})t + \delta(X\mathfrak{a}) \subseteq \mathfrak{a}Xt + \mathfrak{a} \subseteq A.$$

Inductively assume that $t^nX^n\mathfrak{a} \subseteq A$. Then, by Lemma 2.1,

$$t^{n+1}X^{n+1}\mathfrak{a} = t^n(tX^{n+1}\mathfrak{a}) \subseteq t^n(X^{n+1}\sigma(\mathfrak{a})t + \delta(X^{n+1}\mathfrak{a})) \subseteq t^n(X^n\mathfrak{a}Xt + X^n\mathfrak{a}) \subseteq A.$$

Thus for any n and l, $t^nX^n\mathfrak{a}X^lt^l \subseteq AX^lt^l \subseteq A$ and hence A is a left ideal. □

An $(R,\ R)$-bimodule $\mathfrak{a}$ in Q is called *$(\sigma, \delta; X)$-stable* if $X\sigma(\mathfrak{a}) = \mathfrak{a}X$ and $X\delta(\mathfrak{a}) \subseteq \mathfrak{a}$. We can see from Lemma 2.5 that the concept of $(\sigma, \delta; X)$-stable is natural to study Ore-Rees rings.

Lemma 2.6. *Let $\mathfrak{a}$ be an R-ideal in Q. Then $\mathfrak{a}$ is $(\sigma, \delta; X)$-stable if and only if it is $(\sigma', \delta'; X)$-stable, that is $X\mathfrak{a} = \sigma'(\mathfrak{a})X$ and $\delta'(\mathfrak{a})X \subseteq \mathfrak{a}$.*

Proof. It is clear that $X\sigma(\mathfrak{a}) = \mathfrak{a}X$ if and only if $X\mathfrak{a} = \sigma'(\mathfrak{a})X$. If $X\delta(\mathfrak{a}) \subseteq \mathfrak{a}$, then for any $a \in \mathfrak{a}$ and $x \in X$, we have $\sigma^{-1}(a)x \in X\mathfrak{a}$. So $\delta(\sigma^{-1}(a)x) \in \delta(X\mathfrak{a}) \subseteq X\delta(\mathfrak{a}) + \delta(X)\mathfrak{a} \subseteq \mathfrak{a}$, that is $\delta'(a)x \in \mathfrak{a}$ since $\delta(\sigma^{-1}(a)x) = a\delta(x) + \delta\sigma^{-1}(a)x$. Hence $\delta'(\mathfrak{a})X \subseteq \mathfrak{a}$ follows. If $\delta'(\mathfrak{a})X \subseteq \mathfrak{a}$, then we have $X\delta(\mathfrak{a}) \subseteq \mathfrak{a}$ similarly. □

Lemma 2.7. *If $\mathfrak{a}$ is a $(\sigma, \delta; X)$-stable ideal of R, then*

$$A = \mathfrak{a}[Xt; \sigma, \delta] = \mathfrak{a}[tX; \sigma', \delta'] = S_1\mathfrak{a}.$$

Proof. By the right versions of Lemmas 2.5 and 2.6, $S_1\mathfrak{a}$ is an ideal of $S_1 = S$. Hence $A = \mathfrak{a}S \supseteq S\mathfrak{a} = S_1\mathfrak{a} \supseteq \mathfrak{a}S_1 = \mathfrak{a}S$. Hence $A = S_1\mathfrak{a}$. □

Corollary 2.8. *$XS = X[Xt; \sigma, \delta]$ is an ideal of S with $XS = SX$.*

Proof. This is clear because X is $(\sigma, \delta; X)$-stable. □

Lemma 2.9. *(1) $P_1 = XtS$ is an ideal of S which is equal to $P_1^* = X\delta(R) \oplus \sum_{i=1}^{\infty} X^n t^n$.*

(2) $P_2 = StX$ is an ideal of S which is equal to $P_2^* = \delta(R)X \oplus \sum_{i=1}^{\infty} t^n X^n$.

Proof. (1) By using Lemma 2.1, we have $t^l X^l X\delta(R) \subseteq X^l t^l + \cdots + Xt + X\delta(R)$ and so it is easily proved that P_1^* is a left ideal of S. Thus, to prove P_1 is an ideal, it is enough to prove $P_1 = P_1^*$. Note $P_1 = XtS_1 = \sum_{n=1}^{\infty} Xt^n X^{n-1}$ $(X^0 = R)$. It follows that $XtR \subseteq X(Rt + \delta(R)) \subseteq Xt + X\delta(R) \subseteq P_1^*$. We may inductively assume that $Xt^n X^{n-1} \subseteq X^n t^n + \cdots + Xt + X\delta(R) \subseteq P_1^*$. Then

$$\begin{aligned} Xt^{n+1}X^n = Xt^n tXX^{n-1} &\subseteq Xt^n(Xt + \delta(X))X^{n-1} \subseteq Xt^n XtX^{n-1} + Xt^n X^{n-1} \\ &\subseteq Xt^n X(X^{n-1}t + \delta(X^{n-1})) + Xt^n X^{n-1} \subseteq Xt^n X^{n-1}Xt + Xt^n X^{n-1} \\ &\subseteq (X^n t^n + \cdots + Xt + X\delta(R))Xt + Xt^n X^{n-1}, \end{aligned}$$

and for each $i \geq 1$,

$$X^i t^i Xt \subseteq X^i(Xt^i + Rt^{i-1} + \cdots + Rt + \delta(R))t \subseteq P_1^*$$

by Lemma 2.1. Therefore we have $Xt^{n+1}X^n \subseteq P_1^*$ and $P_1 \subseteq P_1^*$ follows.

To prove the converse inclusion, let $x \in X$ and $r \in R$. Then $P_1 \ni xtr = x(\sigma(r)t + \delta(r)) = x\sigma(r)t + x\delta(r)$ and so $x\delta(r) \in P_1$. Thus $X\delta(R) \subseteq P_1$ Since $Xt \subseteq P_1$, we may assume that $X^n t^n \subseteq P_1$ for a natural number $n \geq 1$. Then, by Lemma 2.1,

$$\begin{aligned} X^{n+1}t^{n+1} = XX^n t^{n+1} &\subseteq X(t^{n+1}X^n + \cdots + tR + \delta(R)) \\ &\subseteq Xt^{n+1}X^n + \cdots + XtR + X\delta(R) \subseteq P_1. \end{aligned}$$

Hence $P_1^* = P_1$ follows.

(2) Similar to the proof of (1). □

We now introduce some notation and terminology in a prime Goldie ring R with its quotient ring Q: For any fractional right R-ideal I and left R-ideal J, let

$$(R : I)_l = \{q \in Q \mid qI \subseteq R\} \text{ and } (R : J)_r = \{q \in Q \mid Jq \subseteq R\},$$

which is a left (right) R-ideal, respectively and

$$I_v = (R : (R : I)_l)_r \text{ and } {}_vJ = (R : (R : J)_r)_l,$$

which is a right (left) R-ideal containing $I(J)$. $I(J)$ is called a *right* (*left*) *v-ideal* if $I_v = I$ (${}_vJ = J$). In case I is a two-sided R-ideal, it is said to be a *v-ideal* if $I_v = I = {}_vI$, and if $I \subseteq R$, we just say I is a *v-ideal of* R. An R-ideal A is said to be *v-invertible* if ${}_v((R : A)_l A) = R = (A(R : A)_r)_v$. The following properties are well known and we use them without reference:
Let A be an R-ideal and I be a right R-ideal. Then

(1) If A is v-invertible, then $O_r(A) = R = O_l(A)$ and $(R : A)_l = A^{-1} = (R : A)_r$, where $A^{-1} = \{q \in Q \mid AqA \subseteq A\}$ (e.g. [**13**]).
(2) $(IA_v)_v = (IA)_v$. If A is v-invertible, then $(I_vA_v)_v = (IA)_v$ (e.g. [**13**]).

Lemma 2.10. *Let $P_1 = XtS$. Then $P_{1v} = S = {}_vP_1$ if $\delta \neq 0$.*

Proof. By Lemma 2.9, $P_1 \cap R = X\delta(R) \neq (0)$ which is an ideal since so is P_1. So we have $P_1T = T$. Let $\alpha \in (S : P_1)_l$. Then $\alpha \in \alpha T = \alpha P_1 T \subseteq T$. Write $\alpha = q_nt^n + \cdots + q_1t + q_0$, where $q_i \in Q$. It follows that, for any $x \in X$,

$$S \supseteq \alpha Xt \ni (q_nt^n + \cdots + q_1t + q_0)xt = q_nx't^{n+1} + \text{ (the lower degree parts)}$$

for some $x' \in X$. Thus $q_nX \subseteq X^{n+1}$ and $q_n \in X^n$, that is $q_nt^n \in S$. Put $\beta = q_nt^n - \alpha$. Then $\beta P_1 = (q_nt^n - \alpha)P_1 \subseteq S$ and inductively we get $\alpha \in S$, that is, $(S : P_1)_r = S$. Hence $P_{1v} = (S : (S : P_1)_l)_r = (S : S)_r = S$. Similarly we have ${}_vP_1 = S$. $\square$

Lemma 2.11. *Let I be a right S-ideal and J be a left S-ideal. Then*

(1) $(T : IT)_l = T(S : I)_l$ *and* $(T : TJ)_r = (S : J)_rT$.
(2) $(IT)_v = I_vT$ *and* ${}_v(TJ) = T_vJ$.
(3) *If I' is a right ideal of T, then $I' = (I' \cap S)T$. If I' is an essential right ideal, then $(I' \cap S)_v = I' \cap S$.*

Proof. (1) It is clear that $(T : IT)_l \supseteq T(S : I)_l$. Let $q \in (T : IT)_l$ and $I = \sum_{i=1}^n a_iS$, where $a_i \in Q(S)$. Then $qa_i = q_i(t) \in T$. Write $q_i(t) = \sum_j q_{ij}t^j$ for some $q_{ij} \in Q$, there exists a regular element $c \in R$ such that $cq_{ij} \in R$ and so $cqa_i \in R[t; \sigma, \delta]$. Let $l = \max\limits_{1 \leq i \leq n}\{\deg q_i(t)\}$. Then

$$X^lcqa_i = X^l\left(\sum_j cq_{ij}t^j\right) \subseteq \sum_j X^jX^{l-j}cq_{ij}t^j \subseteq S.$$

Thus $X^lcqI \subseteq S$ and so $X^lcq \in (S : I)_l$ which implies $q \in c^{-1}X^{-l}(S : I)_l \subseteq T(S : I)_l$. Hence $(T : IT)_l = T(S : I)_l$ follows and similarly $(T : TJ)_r = (S : J)_rT$.

(2) By (1) we have

$$IT = (IT)_v = (T : (T : IT)_l)_r = (T : T(S : I)_l)_r = (S : (S : I)_l)_rT = I_vT.$$

Similarly $TJ = T_vJ$.

(3) It is clear that $(I' \cap S)T \subseteq I'T = I'$. Since T is a principal ideal ring, $I' = q(t)T$, for some $q(t) \in T$ with $n = \deg q(t)$. There exists a regular element $c \in R$ such that $q(t)c \in R[t;\sigma,\delta]$ and $q(t)cX^n \subseteq I' \cap S$ which gives $q(t) \in q(t)T = q(t)cX^nT \subseteq (I' \cap S)T$. Thus, $I' = (I' \cap S)T$. If I' is an essential right ideal, then $I' = I'_v = ((I' \cap S)T)_v = (I' \cap S)_v T$ and so $I' \cap S = (I' \cap S)_v$ follows. $\square$

Proposition 2.12. *There is a $(1-1)$-correspondence between*

$$\mathrm{Spec}_0(S) = \{P : \text{ prime ideal of } S \mid P \cap R = (0)\} \quad \text{and} \quad \mathrm{Spec}(T)$$

via $P \longmapsto PT$ and $P' \longmapsto P' \cap S$. In particular, P is a v-ideal.

Proof. Let $P \in \mathrm{Spec}_0(S)$. Then $P' = PT = TP$, a proper ideal of T by Lemma 2.3 and its right version. Put $P = \sum p_i(t)S$ and $TP \cap S = \sum Sq_j(t)$ where $p_i(t) \in P$ and $q_j(t) \in TP \cap S$. Since $PT = \sum p_i(t)T$, we have $q_j(t) = \sum p_i(t)u_{ij}(t)$, for some $u_{ij}(t) \in T$. Then there exist a regular element c in R and $n \geq 1$ such that $u_{ij}(t)cX^n \subseteq S_1 = S$. It follows that $(TP \cap S)cX^n = \sum Sq_j(t)cX^n \subseteq P$. Since $P \cap R = (0)$, we have $TP \cap S \subseteq P$ and $P = TP \cap S$ follows. Now it is clear from Lemma 2.11 (3) that $P' = PT = TP$ is a prime ideal of T.

Conversely, let $P' \in \mathrm{Spec}(T)$ and $P = P' \cap S$. It is easy to check that $P \in \mathrm{Spec}_0(S)$. The last statement is clear from Lemma 2.11 (3) and its left version. $\square$

Lemma 2.13. *Let P be a prime ideal of S such that $P \not\supseteq Xt$ and $P \not\supseteq X$. Then $XP = PX$.*

Proof. If $X^2t \subseteq P$, then $P \supseteq SX^2t = SXXt$ and SX is an ideal of S by Corollary 2.8, which is impossible by the assumption. So $X^2t \not\subseteq P$ and then $XPX^{-1}X^2t \subseteq P$ implies $XPX^{-1} \subseteq P$ and hence $XP = PX$ follows. $\square$

Proposition 2.14. *Let P be a prime ideal of S such that $\mathfrak{p} = P \cap R$ is $(\sigma,\delta;X)$-stable. Then $P_0 = \mathfrak{p}[Xt;\sigma,\delta]$ is a prime ideal. Furthermore, if P_0 is a v-invertible ideal and $P = P_v$, then $P = P_0$ (see Lemmas 3.3 and 4.2).*

Proof. We may assume that $\mathfrak{p} \neq (0)$. On the contrary assume that P_0 is not a prime ideal. Then there are ideals A, B of S such that $AB \subseteq P_0$, $A \supset P_0$ and $B \supset P_0$. We may assume that $A = (P_0 : B)_l \cap S$, where $(P_0 : B)_l = \{q \in Q(S) \mid qB \subseteq P_0\}$. Let $a(t) = a_lt^l + \cdots + a_0 \in A \backslash P_0$ and $l = \deg(a(t))$ is minimal for this property, where $a_l \in X^l$ and $a_l \notin \mathfrak{p}X^l$. Then we claim that

$$X^{-l}\sigma^{-l}(a_l) \subseteq A \cap R = \mathfrak{a} \text{ and } X^{-l}\sigma^{-l}(a_l) \not\subseteq \mathfrak{p}.$$

It is easy to see that $X^{-l}\sigma^{-l}(a_l) \not\subseteq \mathfrak{p}$ because $\sigma^l(\mathfrak{p}) = X^{-l}\mathfrak{p}X^l$.

Consider $a(t)^{-1}P_0 = \{b(t) \in S \mid a(t)b(t) \in P_0\} \supseteq B$. If we prove $X^{-l}\sigma^{-l}(a_l)(a(t)^{-1}P_0) \subseteq P_0$, then $P_0 \supseteq X^{-l}\sigma^{-l}(a_l)B$. Hence $X^{-l}\sigma^{-l}(a_l) \subseteq A \cap R$.

Assume that $X^{-l}\sigma^{-l}(a_l)(a(t)^{-1}P_0) \not\subseteq P_0$. Then there exists $b(t) = b_mt^m + \cdots + b_0 \in a(t)^{-1}P_0$ with $X^{-l}\sigma^{-l}(a_l)b(t) \not\subseteq P_0$. We may assume that $\deg b(t) = m$ is minimal for this property. Since $P_0 \ni a(t)b(t) = a_l\sigma^l(b_m)t^{l+m} +$ (the lower degree parts), we have $a_l\sigma^l(b_m) \in \mathfrak{p}X^{l+m}$. This shows $a_l\sigma^l(b_m)X^{-m} \subseteq \mathfrak{p}X^l$. On the other hand, $A \supseteq a(t)b_mX^{-m}$ and $\deg a(t)b_mX^{-m} \leq l$ with $a_l\sigma^l(b_m)X^{-m} \subseteq \mathfrak{p}X^l$. So, by the choice of $a(t)$, $a(t)b_mX^{-m} \subseteq P_0$ and so $a(t)b_mt^m \in a(t)b_mX^{-m}X^mt^m \subseteq P_0$. Thus $a(t)(b(t) - b_mt^m) \in P_0$, that is, $b(t) - b_mt^m \in a(t)^{-1}P_0$ with $\deg(b(t) - b_mt^m) < m$. Hence, by the choice of $b(t)$,

$$X^{-l}\sigma^{-l}(a_l)(b(t) - b_mt^m) \subseteq P_0. \qquad (*)$$

Again $a_l\sigma^l(b_m) \in \mathfrak{p}X^{l+m}$ implies $X^{-l}a_l\sigma^l(b_m) \subseteq X^{-l}\mathfrak{p}X^{l+m} = \sigma^l(\mathfrak{p})X^m$. So $X^{-l}\sigma^{-l}(a_l)b_m \subseteq \mathfrak{p}X^m$ and $X^{-l}\sigma^{-l}(a_l)b_mt^m \subseteq P_0$. Hence, by $(*)$, $X^{-l}\sigma^{-l}(a_l)b(t) \subseteq P_0$ which is a contradiction. Thus $X^{-l}\sigma^{-l}(a_l) \subseteq A \cap R = \mathfrak{a}$, that is $\mathfrak{a} \supset \mathfrak{p}$. The symmetric argument shows $\mathfrak{b} = B \cap R \supset \mathfrak{p}$.

Now since $AB \subseteq P_0 \subseteq P$, we have either $A \subseteq P$ or $B \subseteq P$ and so either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, a contradiction. Hence P_0 is a prime ideal. Assume that P_0 is v-invertible and $P = P_v$. To prove $P = P_0$, suppose on the contrary, $P \supset P_0$. Since $(S : P)_l \subseteq (S : P_0)_l = P_0^{-1}$, we have $P_0(S : P)_l \subseteq S$ and $P_0(S : P)_lP \subseteq P_0$. So $P_0(S : P)_l \subseteq P_0$ and hence $(S : P)_l \subseteq O_r(P_0) = S$. It follows that $P_v = S$, a contradiction and $P = P_0$ follows. □

Lemma 2.15. *Let $\mathfrak{a}$ be a right R-ideal and $\mathfrak{b}$ be a left R-ideal. Then*

$$(S : \mathfrak{a}[Xt;\sigma;\delta])_l = S(R : \mathfrak{a})_l \quad \textit{and} \quad (S : S\mathfrak{b})_r = (R : \mathfrak{b})_rS.$$

In particular, $(\mathfrak{a}[Xt;\sigma,\delta])_v = \mathfrak{a}_v[Xt;\sigma,\delta]$ *and* ${}_v(S\mathfrak{b}) = S_v\mathfrak{b}$.

Proof. It is clear that $S(R : \mathfrak{a})_l \subseteq (S : \mathfrak{a}[Xt;\sigma,\delta])_l$. To prove the converse inclusion, let $q \in (S : \mathfrak{a}[Xt;\sigma,\delta])_l$. Then $q \in T$, because $\mathfrak{a}[Xt;\sigma,\delta]T = T$. Write $q = q_0 + tq_1 + \cdots + t^nq_n$ and $q\mathfrak{a} \subseteq S = S_1$ entails $t^iq_i\mathfrak{a} \subseteq t^iX^i$. Thus $X^{-i}q_i \subseteq (R : \mathfrak{a})_l$, that is $q_i \in X^i(R : \mathfrak{a})_l$ and $t^iq_i \in t^iX^i(R : \mathfrak{a})_l \subseteq S(R : \mathfrak{a})_l$. Hence $q \in S(R : \mathfrak{a})_l$, showing $(S : \mathfrak{a}[Xt;\sigma,\delta])_l = S(R : \mathfrak{a})_l$. Similarly $(S : S\mathfrak{b})_r = S(R : \mathfrak{b})_r$. Hence $(\mathfrak{a}[Xt;\sigma,\delta])_v = \mathfrak{a}_v[Xt;\sigma,\delta]$ and ${}_v(S\mathfrak{b}) = S_v\mathfrak{b}$. □

Lemma 2.16. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be R-ideals which are $(\sigma,\delta;X)$-stable. Then the following are all $(\sigma,\delta;X)$-stable;*

(1) $\mathfrak{a}\mathfrak{b}$ *and* $\mathfrak{a} \cap \mathfrak{b}$.
(2) $(R : \mathfrak{a})_l$ *and* $(R : \mathfrak{a})_r$.
(3) $\mathfrak{c} = \{r \in R \mid r\mathfrak{a} \subseteq R\}$ *and* $\mathfrak{d} = \{r \in R \mid \mathfrak{a}r \subseteq R\}$.

Proof. (1) It is easy to see that $\mathfrak{a}\mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ are $(\sigma,\delta;X)$-stable.

(2) To prove that $(R : \mathfrak{a})_l$ is $(\sigma,\delta;X)$-stable, first note that $\sigma(\mathfrak{a}) = X^{-1}\mathfrak{a}X$. So

$$\sigma((R : \mathfrak{a})_l) = (R : \sigma(\mathfrak{a}))_l = (R : X^{-1}\sigma(\mathfrak{a})X)_l = X^{-1}(R : \mathfrak{a})_lX$$

and $X\sigma((R : \mathfrak{a})_l) = (R : \mathfrak{a})_lX$ follows. To prove $X\delta((R : \mathfrak{a})_l) \subseteq (R : \mathfrak{a})_l$, let $x \in X$, $q \in$

$(R:\mathfrak{a})_l$ and $a \in \mathfrak{a}$. Then

$$R \ni x\delta(qa) = x(\sigma(q)\delta(a) + \delta(q)a) = x\sigma(q)\delta(a) + x\delta(q)a$$

and

$$x\sigma(q)\delta(a) \in X\sigma((R:\mathfrak{a})_l)X^{-1}X\delta(\mathfrak{a}) \subseteq (R:\mathfrak{a})_l\mathfrak{a} \subseteq R.$$

Thus $x\delta(q)a \in R$ and $X\delta((R:\mathfrak{a})_l)\mathfrak{a} \subseteq R$, that is $X\delta((R:\mathfrak{a})_l) \subseteq (R:\mathfrak{a})_l$. Hence $(R:\mathfrak{a})_l$ is $(\sigma,\delta;X)$-stable. Similarly $(R:\mathfrak{a})_r$ is $(\sigma,\delta;X)$-stable.

(3) This follows from (2), because X is flat, $\mathfrak{c} = (R:\mathfrak{a})_l \cap S$ and $\mathfrak{d} = (R:\mathfrak{a})_r \cap S$. □

R is called a *$(\sigma,\delta;X)$-maximal order* in Q if $O_l(\mathfrak{a}) = R = O_r(\mathfrak{a})$ for any $(\sigma,\delta;X)$-stable ideal $\mathfrak{a}$ of R. If R is a $(\sigma,\delta;X)$-maximal order in Q, then for any $(\sigma,\delta;X)$-stable R-ideal $\mathfrak{a}$, we have $O_l(\mathfrak{a}) = R = O_r(\mathfrak{a})$ by using Lemma 2.16. Hence $(R:\mathfrak{a})_l = \mathfrak{a}^{-1} = (R:\mathfrak{a})_r$, where $\mathfrak{a}^{-1} = \{q \in Q \mid \mathfrak{a}q\mathfrak{a} \subseteq \mathfrak{a}\}$ and $\mathfrak{a}_v = \mathfrak{a}^{-1-1} = {}_v\mathfrak{a}$ follows.

Let $D_{\sigma,\delta;X}(R)$ be the set of all $(\sigma,\delta;X)$-stable v-ideals. For any $\mathfrak{a}$, $\mathfrak{b} \in D_{\sigma,\delta;X}(R)$, we define $\mathfrak{a} \circ \mathfrak{b} = (\mathfrak{a}\mathfrak{b})_v$. Then we have the following.

Proposition 2.17. *Let R be a $(\sigma,\delta;X)$-maximal order in Q. Then $D_{\sigma,\delta;X}(R)$ is an Abelian group generated by maximal $(\sigma,\delta;X)$-stable v-ideals of R.*

Proof. This is proved in a standard way by using Lemma 2.16 (cf. [**16**, Theorem 2.1.2]). □

3. Differential Rees rings which are maximal orders.

In case $\sigma = 1$ and $\delta \neq 0$, we write $S = R[Xt;\delta]$ for $R[Xt;1,\delta]$, which is called a *differential Rees ring*. We just say $(\delta;X)$-stable ideals for $(1,\delta;X)$-stable ideals and $(\delta;X)$-maximal orders for $(1,\delta;X)$-maximal orders. Let R be a $(\delta;X)$-maximal order. Then we write $D_{\delta;X}(R)$ for $D_{1,\delta;X}(R)$.

In this section, we will prove that the differential Rees ring $S = R[Xt;\delta]$ is a maximal order if and only if R is a $(\delta;X)$-maximal order. Furthermore, we describe the structure of v-ideals of S in case R is a $(\delta;X)$-maximal order by using some properties prepared in Section 2. Note that δ is naturally extended to a derivation δ on $Q[t;\delta]$ by $\delta(q(t)) = \sum_{i=0}^{l}\delta(q_i)t^i$, where $q(t) = \sum_{i=0}^{l} q_it^i \in Q[t;\delta]$.

Lemma 3.1. *Let A be an ideal of S with $XA = AX$. Then $\mathfrak{a} = A \cap R$ and A are both $(\delta;X)$-stable, that is $X\mathfrak{a} = \mathfrak{a}X$, $X\delta(\mathfrak{a}) \subseteq \mathfrak{a}$ and $X\delta(A) \subseteq A$.*

Proof. Since $A = X^{-1}AX$, we have $\mathfrak{a} = X^{-1}AX \cap R \supseteq X^{-1}(A\cap R)X = X^{-1}\mathfrak{a}X$. Hence $\mathfrak{a}X = X\mathfrak{a}$. For any $a \in \mathfrak{a}$ and $x \in X$, $xta = xat + x\delta(a)$ and $xat \in X\mathfrak{a}t = \mathfrak{a}Xt \subseteq A$. Hence $x\delta(a) \in A$, that is $X\delta(\mathfrak{a}) \subseteq \mathfrak{a}$ follows. To prove A is $(\delta;X)$-stable, let $x \in X$ and $a(t) = \sum_{i=0}^{l} a_it^i \in A$. Then

$$xta(t) = x\Big(\sum_{i=0}^{l} ta_i t^i\Big) = x\Big(\sum_{i=0}^{l} \big(a_i t + \delta(a_i)\big)t^i\Big)$$
$$= x\Big\{\Big(\sum_{i=0}^{l} a_i t^i\Big)t + \Big(\sum_{i=0}^{l} \delta(a_i)t^i\Big)\Big\} = xa(t)t + x\delta(a(t)).$$

By assumption, $xa(t)t \in XAt = AXt \subseteq A$ and so $x\delta(a(t)) \in A$. Hence $X\delta(A) \subseteq A$. $\square$

Lemma 3.2. *Let P be a prime ideal of S such that $P \not\supseteq Xt$ and $P \supseteq X$. Then $\mathfrak{p} = P \cap R$ is $(\delta; X)$-stable.*

Proof. For any $x \in X$ and $p \in \mathfrak{p}$, we have $xpt = txp - \delta(xp) = txp - \delta(x)p - x\delta(p) \in P$ since $P \supseteq X$. So $X\mathfrak{p}t \subseteq P$ and $X\mathfrak{p}X^{-1}P_1 = X\mathfrak{p}X^{-1}XtS = X\mathfrak{p}tS \subseteq P$. Thus $X\mathfrak{p}X^{-1} \subseteq P$ by Lemma 2.9 and $X\mathfrak{p} = \mathfrak{p}X$ follows. Hence $\mathfrak{p}$ is $(\delta; X)$-stable since $X\delta(\mathfrak{p}) \subseteq P \cap R = \mathfrak{p}$. $\square$

Lemma 3.3. *Suppose R is a $(\delta; X)$-maximal order in Q. Let A be an ideal of S with $A = A_v$ and $\mathfrak{a} = A \cap R \neq (0)$. Then $A = \mathfrak{a}[Xt; \delta]$ and $\mathfrak{a} \in D_{\delta;X}(R)$. In particular, A is v-invertible.*

Proof. First assume that A is maximal in the set $\mathcal{B} = \{B :\ \text{ideal}\ \mid B_v = B\}$. Then A is a prime ideal and $A \not\supseteq Xt$ by lemma 2.10. So, by Lemmas 2.13, 3.1 and 3.2, $\mathfrak{a}$ is $(\delta; X)$-stable and $\mathfrak{a}_v = \mathfrak{a}$ by Lemma 2.15, that is $\mathfrak{a} \in D_{\delta;X}(R)$. It follows that $A_0 = \mathfrak{a}[Xt; \delta]$ is v-invertible. Hence $A = \mathfrak{a}[Xt; \delta]$ by Proposition 2.14.

If there is an A in $\mathcal{B}$ such that $A \supset \mathfrak{a}[Xt; \delta]$, then there exists maximal P with $P \supset A$ and $P = \mathfrak{p}[Xt; \delta]$, where $\mathfrak{p} = P \cap R \in D_{\delta;X}(R)$. We assume that A is maximal for this property. Then $S \supseteq AP^{-1} \supseteq A$. If $AP^{-1} = A$, then $AP^{-1}S_P = AS_P$, where S_P is a localization of S at P wihch is a local Dedekind prime ring with $J(S_P) = PS_P$ by [**13**, Lemma 2.1], and since $AP^{-1}S_P = AS_PP^{-1}S_P$ and AS_P is an ideal of S_P, we have $P^{-1}S_P \subseteq O_r(AS_P) = S_P$ and $S_P = P^{-1}S_P$, a contradiction. Hence $(AP^{-1})_v \supset A$ and so $(AP^{-1})_v = \mathfrak{b}[Xt; \delta]$ for some $\mathfrak{b} \in D_{\delta;X}(R)$. It follows that $A = (AP^{-1}P)_v = ((AP^{-1})_vP)_v = (\mathfrak{b}[Xt; \delta]\mathfrak{p}[Xt; \delta])_v = (\mathfrak{b}\mathfrak{p})_v[Xt; \delta]$ by Lemma 2.15, where $(\mathfrak{b}\mathfrak{p})_v \in D_{\delta;X}(R)$, which is a contradiction. This completes the proof. $\square$

Lemma 3.4. *Suppsoe R is a $(\delta; X)$-maximal order. Let A be an ideal of S such that $A = A_v$ and $A \cap R = (0)$. Then A is v-invertible.*

Proof. By Lemma 2.3, AT is an ideal of T. Thus, by Lemma 2.11, $T = AT(T : TA)_r = A(S : A)_rT$, which implies $A(S : A)_r \cap R \neq (0)$. So $(A(S : A)_r)_v = \mathfrak{a}[Xt; \delta]$ for some $\mathfrak{a} \in D_{\delta;X}(R)$ by Lemma 3.3. Thus $(A(S : A)_r \cdot \mathfrak{a}^{-1}[Xt; \delta])_v = ((A(S : A)_r)_v\mathfrak{a}^{-1}[Xt; \delta])_v = S$ by Lemma 2.15 and $(S : A)_r\mathfrak{a}^{-1}[Xt; \delta] = (S : A)_r$. Hence $(A(S : A)_r)_v = S$. Similarly $S = {}_v((S : A)_lA)$ and hence A is v-invertible with ${}_vA = A$. $\square$

We are now in position to prove the main theorem of this section:

Theorem 3.5. *Let R be a Noetherian prime ring with a non-zero derivation δ and*

X be an invertible ideal. Then R is a $(\delta;X)$-maximal order if and only if the differential Rees ring $S=R[Xt;\delta]$ is a maximal order.

Proof. Let R be a $(\delta;X)$-maximal order and A be an ideal of S. Since $S\subseteq O_l(A)\subseteq O_l(A_v)$, it suffices to prove $O_l(A_v)=S$ in order to prove $O_l(A)=S$. By Lemmas 3.3 and 3.4, A_v is v-invertible and so $O_l(A_v)=S$. By left versions of Lemmas 3.3 and 3.4, $O_r(A)=S$. Hence S is a maximal order.

Conversely, let S be a maximal order and $\mathfrak{a}$ be a $(\delta;X)$-stable ideal. To prove that R is a $(\delta;X)$-maximal order, we may assume that $\mathfrak{a}_v=\mathfrak{a}$. Put $A=\mathfrak{a}[Xt;\delta]$. Then $A=A_v$ by Lemma 2.15 and so A is a v-ideal. Since $(S:A)_l=(R:\mathfrak{a})_l[Xt;\delta]$, we have

$$S={}_v((S:A)_lA)={}_v((R:\mathfrak{a})_l[Xt;\delta]\ \mathfrak{a}[Xt;\delta])={}_v((R:\mathfrak{a})_l\mathfrak{a})[Xt;\delta].$$

Hence $R={}_v((R:\mathfrak{a})_l\mathfrak{a})$ and similarly $(\mathfrak{a}(R:\mathfrak{a})_l)_v=R$. Hence $\mathfrak{a}$ is v-invertible and so $O_l(\mathfrak{a})=R=O_r(\mathfrak{a})$. Hence R is a $(\delta;X)$-maximal order. □

Now we explicitly give the structure of all v-ideals in $Q(S)$ in case R is a $(\delta;X)$-maximal order as follows:

Proposition 3.6. *Suppose R is a $(\delta;X)$-maximal order. Let A be a v-ideal in $Q(S)$. Then $A=\mathfrak{a}[Xt;\delta]w$ for some $\mathfrak{a}\in D_{\delta;X}(R)$ and $w\in\mathbb{Z}(Q(T))$, the center of $Q(T)$.*

Proof. Since S is a maximal order, it is well known that the set of all v-ideals in $Q(S)$ is an Abelian group generated by maximal v-ideals of S and that a v-ideal of S is a maximal v-ideal if and only if it is a prime v-ideal. Thus, by Proposition 2.12 and Lemma 3.3, any maximal v-ideal is of the form either $P=\mathfrak{p}[Xt;\delta]$ with $\mathfrak{p}\in D_{\delta;X}(R)$ or B, a v-ideal such that BT is a maximal ideal of T.

Let A be a v-ideal in $Q(S)$. If $A\subseteq S$, then AT is an ideal of T and so $AT=wT$ for some $w\in\mathbb{Z}(T)$, the center of T by [**5**, Corollary 6.2.11] (also, see [**16**, Corollary 2.3.11]). Then $w^{-1}AT=T$ and $w^{-1}A$ is a v-ideal in $Q(S)$ and so $w^{-1}A=(P_1^{e_1}\cdots P_r^{e_r}B_1^{f_1}\cdots B_s^{f_s})_v$, where $P_i=\mathfrak{p}_i[Xt;\delta]$ are maximal v-ideals with $\mathfrak{p}_i\in D_{\delta;X}(R)$, B_j are maximal v-ideals such that $B_j\cap R=(0)$ and B_jT are maximal ideals of T and $e_i,\ f_j$ are integers. It follows that

$$T=w^{-1}AT=(P_1^{e_1}\cdots P_r^{e_r}B_1^{f_1}\cdots B_s^{f_s})_vT=(P_1^{e_1}\cdots P_r^{e_r}B_1^{f_1}\cdots B_s^{f_s})T=B_1^{f_1}\cdots B_s^{f_s}T.$$

Hence $f_1=\cdots=f_s=0$, that is $w^{-1}A=(P_1^{e_1}\cdots P_r^{e_r})_v=\mathfrak{a}[Xt;\delta]$, where $\mathfrak{a}=(\mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_r^{e_r})_v\in D_{\delta;X}(R)$ and thus $A=\mathfrak{a}[Xt;\delta]w$ as desired.

If A is a fractional v-ideal, then $CA\subseteq S$ for an ideal C of S. So $C_v=\mathfrak{c}_v[Xt;\delta]w_1$ for some $\mathfrak{c}\in D_{\delta;X}(R)$, $w_1\in\mathbb{Z}(T)$ and $(CA)_v=\mathfrak{b}_v[Xt;\delta]w_2$ for some $\mathfrak{b}\in D_{\delta;X}(R)$ and $w_2\in\mathbb{Z}(T)$. Hence

$$A=(C^{-1}CA)_v=(\mathfrak{c}^{-1}[Xt;\delta]w_1^{-1}\mathfrak{b}[Xt;\delta]w_2)_v=(\mathfrak{c}^{-1}\mathfrak{b})_v[Xt;\delta]w_1^{-1}w_2,$$

where $(\mathfrak{c}^{-1}\mathfrak{b})_v\in D_{\delta;X}(R)$ and $w_1^{-1}w_2\in\mathbb{Z}(Q(T))$. This completes the proof. □

We recall that a ring is *Asano* if any non-zero ideal is invertible. Any Asano ring is a maximal order. We say that a ring is a *generalized Asano* ring if it is a maximal order and any v-ideal is invetible. Furthermore, a ring is called a *generalized* $(\delta; X)$*-Asano* ring if it is a $(\delta; X)$-maximal order and any $(\delta; X)$-stable v-ideal is invertible.

From Theorem 3.5 and Proposition 3.6, we have

COROLLARY 3.7. *R is a generalized $(\delta; X)$-Asano ring if and only if $S = R[Xt; \delta]$ is a generalized Asano ring.*

4. Skew Rees rings which are maximal orders.

In case $\delta = 0$, as in Section 3 we write $S = R[Xt; \sigma]$ for $R[Xt; \sigma, 0]$, which is called a *skew Rees ring.* A $(\sigma, 0; X)$-stable ideal $\mathfrak{a}$ is called a $(\sigma; X)$*-invariant* ideal, because $X0(\mathfrak{a}) \subseteq \mathfrak{a}$ is always satisfied and a $(\sigma; 0; X)$-maximal order is called a $(\sigma; X)$*-maximal* order. If R is a $(\sigma; X)$-maximal order, then we write $D_{\sigma;X}(R)$ for $D_{\sigma,0;X}(R)$.

In this section, we will prove that a skew Rees ring $S = R[Xt; \sigma]$ is a maximal order if and only if R is a $(\sigma; X)$-maximal order.

LEMMA 4.1. *Let P be a prime ideal of S.*

(1) *If $P \not\supseteq Xt$, then $\mathfrak{p} = P \cap R$ is $(\sigma; X)$-invariant (we do not assume $\mathfrak{p} \neq 0$).*
(2) *If $P \supseteq Xt$ with $P = P_v$ then $P = XtS$ and is invertible.*

PROOF. (1) First we will prove that P is $(\sigma; X)$-invariant, that is $X\sigma(P) = PX$. Consider $XtP(Xt)^{-1}Xt \subseteq P$. Then we have $P \supseteq XtP(Xt)^{-1} = X\sigma(P)X^{-1}$. Hence $X\sigma(P) \subseteq PX$. To prove the converse inclusion, consider $P \supseteq tX(tX)^{-1}PtX$ then we have $P \supseteq (tX)^{-1}PtX = X^{-1}\sigma^{-1}(P)X$ and $PX \subseteq X\sigma(P)$. Hence $X\sigma(P) = PX$ and P is $(\sigma; X)$-invariant. $\sigma(P) = X^{-1}PX$ entails that $\sigma(\mathfrak{p}) = \sigma(P) \cap R = X^{-1}PX \cap R = X^{-1}(P \cap R)X = X^{-1}\mathfrak{p}X$ and hence $\mathfrak{p}$ is $(\sigma; X)$-invariant.

(2) It is enough to prove that $P_v = S$ if $P \supset XtS$. Suppose $P \supset XtS$. Then $P = \mathfrak{p} \oplus Xt \oplus X^2t^2 \oplus \cdots \oplus X^nt^n \oplus \cdots$ for some non zero ideal $\mathfrak{p}$ of R. Let $q \in (S : P)_l$. Then $q = q_nt^n + \cdots + q_1t + q_0 \in T$ since $PT = T$. It follows that $qXt \subseteq qP \subseteq S$ and so for each i, $q_iXt^{i+1} = q_it^iXt \subseteq X^{i+1}t^{i+1}$, which implies $q_i \in X^i$ and thus $q \in S$. Hence $(S : P)_l = S$ and $P_v = S$. □

Suppose R is a $(\sigma; X)$-maximal order. Let P be an ideal of S which is maximal in the set $\mathcal{B} = \{B :$ ideal of $S \mid B = B_v\}$ and $\mathfrak{p} = P \cap R \neq (0)$. Then P is a prime ideal and $\mathfrak{p}$ is a $(\sigma; X)$-invariant v-ideal by Lemmas 2.15 and 4.1. Thus $P = \mathfrak{p}[Xt; \sigma]$ by Proposition 2.14, v-invertible and $\mathfrak{p} \in D_{\sigma;X}(R)$. So the following lemmas 4.2 and 4.3 are obtained in similar ways as one in Lemmas 3.3 and 3.4.

LEMMA 4.2. *Suppose R is a $(\sigma; X)$-maximal order in Q. Let A be an ideal of S with $A = A_v$ and $\mathfrak{a} = A \cap R \neq (0)$. Then $A = \mathfrak{a}[Xt; \sigma]$ and $\mathfrak{a} \in D_{\sigma;X}(R)$.*

LEMMA 4.3. *Suppose R is a $(\sigma; X)$-maximal order in Q. Let A be an ideal of S such that $A = A_v$ and $A \cap R = (0)$. Then A is v-invertible.*

Now we obtain a necessary and sufficient conditions for $S = R[Xt; \sigma]$ to be a maximal order by using Lemmas 4.2 and 4.3, whose proof is similar to one in Theorem 3.5.

Theorem 4.4. *Let R be a Noetherian prime ring with its quotient ring Q, σ be an automorphism of R and $S = R[Xt;\sigma]$ be a skew Rees ring associated to X, where X is an invertible ideal with $\sigma(X) = X$. Then R is a $(\sigma;X)$-maximal order if and only if S is a maximal order in $Q(R)$.*

It is well known that any ideal of $T = Q[t;\sigma]$ is of the form $t^n wT$, where n is a non-negative integer and $w \in \mathbb{Z}(T)$ (see [**5**, Corollary 6.2.11] or [**16**, Corollary 2.3.11]). Hence we have the following proposition whose proof is similar to one in Proposition 3.6.

Proposition 4.5. *Suppose R is a $(\sigma;X)$-maximal order and let A be a v-ideal in $Q(S)$. Then $A = t^n w\mathfrak{a}[Xt;\sigma]$ for some $\mathfrak{a} \in D_{\sigma;X}(R), w \in \mathbb{Z}(Q(T))$ and n is an integer.*

As in case $\sigma = 1$ and $\delta \neq 0$, we can define the concept of a generalized $(\sigma;X)$-Asano ring, that is it is a $(\sigma;X)$-maximal order and any $(\sigma;X)$-invariant v-ideal is invertible.

From Theorem 4.4 and Proposition 4.5, we have

Corollary 4.6. *R is a generalized $(\sigma;X)$-Asano ring if and only if $S = R[Xt;\sigma]$ is a generalized Asano ring.*

5. Examples.

In this section, we provide examples of $(\delta;X)$-maximal orders and $(\sigma;X)$-maximal orders but not maximal orders. Furthermore we provide examples R with invertible ideals X satisfying; $R[t;\sigma]$ is a maximal order but $R[Xt;\sigma]$ is not a maximal order, and $R[Xt;\sigma]$ is a maximal order but $R[t;\sigma]$ is not a maximal order.

Let D be an HNP ring satisfying the following conditions:

(a) There is a cycle $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ $(n \geq 2)$ so that $\mathfrak{p}_0 = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n$ is an invertible ideal.
(b) Any maximal ideal different from $\mathfrak{m}_i$ $(1 \leq i \leq n)$ is invertible.

See [**1**] and [**10**] for examples of HNP rings satisfying the conditions (a) and (b). It follows from [**8**, Theorem 14] and [**7**, Proposition 2.8] that

(i) $\mathfrak{p}_0\mathfrak{m}_1\mathfrak{p}_0^{-1} = \mathfrak{m}_2, \ldots, \mathfrak{p}_0\mathfrak{m}_n\mathfrak{p}_0^{-1} = \mathfrak{m}_1$ and
(ii) $\mathfrak{p}_0\mathfrak{n}\mathfrak{p}_0^{-1} = \mathfrak{n}$ for all maximal ideals $\mathfrak{n}$ with $\mathfrak{n} \neq \mathfrak{m}_i$ $(1 \leq i \leq n)$.

Let $R = D[x]$, a polynomial ring over D in an indeterminate x. It is shown in [**13**] that R is a v-HC ordre with enough v-invertible ideals since D has enough invertible ideals ("v-HC orders" is a Krull type generalization of HNP rings. See [**12**] and [**13**] for the definition of v-HC orders and some ideal theoretical properties of v-HC orders).

We define a derivation δ on R as follows; $\delta(x) = 1$ and $\delta(a) = 0$ for all $a \in D$ and put $X = \mathfrak{p}_0[x]$, an invertible ideal of R. We will show that the differential Rees ring $S = R[Xt;\delta]$ is a maximal order in case char $D = 0$.

Recall some properties of v-ideals of $R = D[x]$ as follows.

(iii) For any ideal $\mathfrak{a}$ of R, if $\mathfrak{a} = \mathfrak{a}_v$ (or $\mathfrak{a} = {}_v\mathfrak{a}$), then it is a v-ideal ([**12**, Lemma 1.2]).

(iv) $\{\mathfrak{n}[x],\ X = \mathfrak{p}_0[x],\ \mathfrak{a} \mid \mathfrak{n}$ are maximal ideals different from $\mathfrak{m}_i$ $(1 \le i \le n)$, and $\mathfrak{a}$ is a v-ideal of R such that $\mathfrak{a}Q(D)[x]$ is a maximal ideal of $Q(D)[x]\}$ is the set of maximal v-invertible ideals of R ([**13**]). Since $\operatorname{gl.dim} R \le 1$, any v-ideal is projective and so v-invertible R-ideals in $Q(R)$ are invertible. Hence $D(R)$, the set of all invertible R-ideals in $Q(R)$ is a free Abelian group generated by maximal invertible ideals of R ([**12**, Theorem 1.13]).

(v) Let $\mathfrak{m}$ be a maximal v-ideal of R with $\mathfrak{m}_0 = \mathfrak{m} \cap D \ne (0)$. Then either $\mathfrak{m} = \mathfrak{m}_i[x]$ for some i or $\mathfrak{m} = \mathfrak{n}[x]$ for some maximal ideal $\mathfrak{n}$ different from $\mathfrak{m}_i$.

Proof. Since $\mathfrak{m}$ is a prime ideal, it follows that $\mathfrak{m}_0$ is a prime ideal. Thus either $\mathfrak{m}_0 = \mathfrak{m}_i$ for some i or $\mathfrak{m}_0 = \mathfrak{n}$ and so either $\mathfrak{m} \supseteq \mathfrak{m}_i[x]$ or $\mathfrak{m} \supseteq \mathfrak{n}[x]$. Hence either $\mathfrak{m} = \mathfrak{m}_i[x]$ or $\mathfrak{m} = \mathfrak{n}[x]$ since $\mathfrak{m}_i[x]$ and $\mathfrak{n}[x]$ are both maximal v-ideals. $\square$

A v-ideal $\mathfrak{a}$ of R is called *v-idempotent* if $\mathfrak{a} = (\mathfrak{a}^2)_v$. It is called *eventually v-idempotent* if $(\mathfrak{a}^n)_v$ is v-idempotent for some $n \ge 1$.

(vi) Let $\mathfrak{a}$ be eventually v-idempotent. Then there are $\mathfrak{m}_{i_1}, \dots, \mathfrak{m}_{i_r}$ $(i_1 < \cdots < i_r,\ r < n)$ which are the full set of maximal v-ideals containing $\mathfrak{a}$ and $(\mathfrak{a}^r)_v = \big((\mathfrak{m}_{i_1}[x] \cap \cdots \cap \mathfrak{m}_{i_r}[x])^r\big)_v$. This follows from [**13**, Proposition 1.4], (iv) and (v).

(vii) Let $\mathfrak{a}$ be a v-ideal of R. Then $\mathfrak{a} = (\mathfrak{b}\mathfrak{c})_v$ for a v-invertible ideal $\mathfrak{b}$ of R and eventually v-idempotent $\mathfrak{c}$ ([**14**, Proposition 3]).

Lemma 5.1. *Let $\mathfrak{a}$ be a v-ideal of $R = D[x]$. Then*

(1) *If $\mathfrak{a}$ is eventually v-idempotent, then $X\mathfrak{a}X^{-1} \ne \mathfrak{a}$ and $\mathfrak{a}$ is not $(\delta; X)$-stable.*
(2) *If char $D = 0$ and $\mathfrak{a} \cap D = (0)$, then $\mathfrak{a}$ is not $(\delta; X)$-stable.*

Proof. (1) Let $\mathfrak{m}_{i_1}[x], \dots, \mathfrak{m}_{i_r}[x]$ be the full set of maximal v-ideals containing $\mathfrak{a}$. By (i), $\mathfrak{m}_{i_1+1}[x], \dots, \mathfrak{m}_{i_r+1}[x]$ is the full set of maximal v-ideals containing $X\mathfrak{a}X^{-1}$ $(i_r + 1 = 1$ if $i_r = n)$. Hence $X\mathfrak{a}X^{-1} \ne \mathfrak{a}$.

(2) Let $f(x) = a_l x^l + \cdots + a_0$ be a non-zero element in $\mathfrak{a}$ such that l is minimal. $\delta(f(x)) = l a_l x^{l-1} + \cdots + a_1 \notin \mathfrak{a}$ and $X\delta(f(x)) \not\subseteq \mathfrak{a}$, because $X\delta(f(x))$ contains a non-zero polynomial whose degree is $l-1$. Hence $\mathfrak{a}$ is not $(\delta; X)$-stable. $\square$

Example 5.2. Let D be an HNP ring satisfying the conditions (a) and (b) with char $D = 0$. Let $R = D[x]$, $X = \mathfrak{p}_0[x]$ and δ be a derivation on R such that $\delta(x) = 1$ and $\delta(a) = 0$ for all $a \in D$. Then the differential Rees ring $S = R[Xt; \delta]$ is a maximal order but R is not a maximal order.

Proof. It is clear that R is not a maximal order. To prove that S is a maximal order, it is enough to prove that any $(\delta; X)$-stable v-ideal of R is invertible by Theorem 3.5. Let $\mathfrak{a}$ be a v-ideal of R. Then $\mathfrak{a} = (\mathfrak{b}\mathfrak{c})_v = \mathfrak{b}\mathfrak{c}$ for some invertible ideal $\mathfrak{b}$ and some eventually v-idempotent $\mathfrak{c}$ by (vii). Suppose $\mathfrak{a}$ is $(\delta; X)$-stable. Then $\mathfrak{b}\mathfrak{c}X = \mathfrak{a}X = X\mathfrak{a} = X\mathfrak{b}\mathfrak{c} = \mathfrak{b}X\mathfrak{c}$ by (iv) and $\mathfrak{c}X = X\mathfrak{c}$ follows. Thus $\mathfrak{c} = R$ by Lemma 5.1. Hence R is a $(\delta; X)$-maximal order. To describe $(\delta; X)$-stable invertible ideals, let $\mathfrak{b} = \mathfrak{b}_1\mathfrak{b}_2$, where $\mathfrak{b}_1 = X^e \mathfrak{n}_1^{e_1}[x] \cdots \mathfrak{n}_r^{e_r}[x]$, where $\mathfrak{n}_j$ are maximal invertible ideals different from $\mathfrak{m}_i$ $(1 \le i \le n)$, e, e_j are non-negative integers, $\mathfrak{b}_2 = \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_s^{f_s}$, where $\mathfrak{p}_i$ are maximal invertible ideals such that $\mathfrak{p}_i Q(D)[x]$ are maximal ideals of $Q(D)[x]$ and f_j are non-negative integers. If $\mathfrak{b}_2 \neq R$, then $d(\mathfrak{b}_2) = l > 1$ and $d(\delta(\mathfrak{b}_2)) = l - 1$, where $d(\mathfrak{s}) = \min\{n : \text{natural number} \mid 0 \neq f(x) = a_n x^n + \cdots + a_0 \in \mathfrak{s}\}$ for a subset $\mathfrak{s}$ of R. Since

$$d(X\delta(\mathfrak{b}_1\mathfrak{b}_2)) = d(\delta(\mathfrak{b}_1\mathfrak{b}_2)) = d(\delta(\mathfrak{b}_2)) < d(\mathfrak{b}_2) = d(\mathfrak{b}_1\mathfrak{b}_2),$$

it follows that $\mathfrak{b} = \mathfrak{b}_1\mathfrak{b}_2$ is not $(\delta; X)$-stable and so $\mathfrak{b}_2 = R$. Since $\mathfrak{b}_1 = X^e \cap \mathfrak{n}_1^{e_1}[x] \cap \cdots \cap \mathfrak{n}_r^{e_r}[x]$, we have

$$X\delta(\mathfrak{b}_1) \subseteq X(\delta(X^e) \cap \delta(\mathfrak{n}_1^{e_1}[x]) \cap \cdots \cap \delta(\mathfrak{n}_r^{e_r}[x]) \subseteq X^e \cap \mathfrak{n}_1^{e_1}[x] \cap \cdots \cap \mathfrak{n}_r^{e_r}[x] = \mathfrak{b}_1,$$

which implies that $\mathfrak{b}_1$ is $(\delta; X)$-stable by (iv). Hence $\{X^e \mathfrak{n}_1^{e_1}[x] \cdots \mathfrak{n}_r^{e_r}[x] \mid \mathfrak{n}_j$ are maximal invertible ideals different from $\mathfrak{m}_i$ $(1 \leq i \leq n)$ and e, e_j are non-negative integers$\}$ is the set of $(\delta; X)$-stable ideals of R. □

In order to obtain an example of a $(\sigma; X)$-maximal order but not a maximal order, suppose that $\mathfrak{p}_0$ is principal, say $\mathfrak{p}_0 = aD = Da$ for some $a \in \mathfrak{p}_0$. Define an automorphism σ of D by $\sigma(r) = ara^{-1}$ for all $r \in D$. Then we have the following examples:

Example 5.3. (1) Put $X = \mathfrak{n}_1^{e_1} \cdots \mathfrak{n}_s^{e_s}$, where $\mathfrak{n}_j$ are maximal ideals different from $\mathfrak{m}_i$ $(1 \leq i \leq n)$. Then D is a $(\sigma; X)$-maximal order which is not a maximal order. So the skew Rees ring $S = D[Xt; \sigma]$ is a maximal order.

(2) Put $X = \mathfrak{p}_0$. Then

(i) If $n = 2l$, an even number, then D is not a $(\sigma; X)$-maximal order so that $S = D[Xt; \sigma]$ is not a maximal order.

(ii) If $n = 2l + 1$, an odd number, then D is a $(\sigma; X)$-maximal order so that $S = D[Xt; \sigma]$ is a maximal order.

Proof. (1) Since the set of invertible D-ideals is an Abelian group generated by maximal invertible ideals, say $\mathfrak{p}_0$ and $\mathfrak{n}$, we have, for any invertible ideal $\mathfrak{a}$, $X\mathfrak{a} = \mathfrak{a}X$ and $\mathfrak{a}$ is σ-invariant, that is $\sigma(\mathfrak{a}) = \mathfrak{a}$. Hence $\mathfrak{a}$ is $(\sigma; X)$-invariant. Let $\mathfrak{a}$ be an ideal such that it is not invertible and $(\sigma; X)$-invariant. Then $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$, where $\mathfrak{b}$ is invertible and $\mathfrak{c}$ is eventually idempotent ([**7**, Theorem 4.2]). Hence $\mathfrak{c}$ is also $(\sigma; X)$-invariatnt. As in Example 5.2, let $\mathfrak{m}_{i_1}, \ldots, \mathfrak{m}_{i_r}$ be the full set of maximal ideals containing $\mathfrak{c}$. Then $\sigma(\mathfrak{m}_{i_1}), \ldots, \sigma(\mathfrak{m}_{i_r})$ is the set of maximal ideals containing $\sigma(\mathfrak{c}) = X^{-1}\mathfrak{c}X = \mathfrak{c}$ (the last equality follows from [**7**, Proposition 2.8]), which is a contradiction. Hence an ideal is $(\sigma; X)$-invariant if and only if it is invertible. Therefore D is a $(\sigma; X)$-maximal order.

(2) Let $\mathfrak{a}$ be eventually idempotent which is $(\sigma; X)$-invariant. Then $\sigma(\mathfrak{a}) = X^{-1}\mathfrak{a}X = \sigma^{-1}(\mathfrak{a})$, that is $\sigma^2(\mathfrak{a}) = \mathfrak{a}$.

(i) Put $\mathfrak{a} = \mathfrak{m}_1 \cap \mathfrak{m}_3 \cap \cdots \cap \mathfrak{m}_{2l-1}$. Then $\sigma^2(\mathfrak{a}) = \mathfrak{m}_3 \cap \cdots \mathfrak{m}_{2l-1} \cap \mathfrak{m}_1 = \mathfrak{a}$. Hence $\mathfrak{a}$ is $(\sigma; X)$-invariant. Suppose $O_r(\mathfrak{a}) = D = O_l(\mathfrak{a})$. Then it is easy to see that $\mathfrak{a}$ is invertible, which is a contradiction. Hence D is not a $(\sigma; X)$-maximal order so that $S = D[Xt; \sigma]$ is not a maximal order.

(ii) Let $\mathfrak{a}$ be eventually idempotent which is $(\sigma; X)$-invariant and $\mathfrak{s} = \{\mathfrak{m}_{i_1}, \ldots, \mathfrak{m}_{i_r}\}$ be the set of maximal ideals containing $\mathfrak{a}$. We may assume that $\mathfrak{m}_{i_1} = \mathfrak{m}_1$. Since $\sigma^2(\mathfrak{a}) =$

$\mathfrak{a}$, $\{\mathfrak{m}_1,\ \mathfrak{m}_3,\dots,\mathfrak{m}_{2l+1}\}\subseteq\mathfrak{s}$ and so $\sigma^2(\mathfrak{m}_{2l+1})=\mathfrak{m}_2$. Thus we have $\mathfrak{s}=\{\mathfrak{m}_1,\dots,\mathfrak{m}_{2l+1}\}$, a contradiction. Thus a $(\sigma;X)$-invariant ideal must be invertible. Hence D is a $(\sigma;X)$-maximal order and $S=D[Xt;\sigma]$ is a maximal order. $\square$

REMARK 5.4. (1) In case (1) and (2) (ii) in Example 5.3, as it is seen from the proofs, D is a generalized $(\sigma;X)$-Asano ring and $S=D[Xt;\sigma]$ is a generalized Asano ring.

(2) Suppose $\mathfrak{p}_0=aD=Da$. As in Example 5.2, put $R=D[x]$. Then $\mathfrak{p}_0[x]=aR=Ra$ and let σ be an automorphism induced by a. We have the following, by using the properties of (iii) $\sim$ (vii), whose proofs are similar to one in Example 5.3:

(1) Put $X=\mathfrak{n}_1^{e_1}[x]\cdots\mathfrak{n}_s^{e_s}[x]$, an invertible ideal. Then R is a $(\sigma;X)$-maximal order but not a maximal order and $S=R[Xt;\sigma]$ is a maximal order.
(2) Put $X=\mathfrak{p}_0[x]$. Then
   (i) If $n=2l$, an even number, then R is not a $(\sigma;X)$-maximal order so that $S=R[Xt;\sigma]$ is not a maximal order.
   (ii) If $n=2l+1$, an odd number, then R is a $(\sigma;X)$-maximal order and $S=R[Xt;\sigma]$ is a maximal order.

We are also interested in relations between $R[Xt;\sigma]$ and the skew polynomial ring $R[t;\sigma]$ from order theoretical view-point. It is known that $R[t;\sigma]$ is a maximal order if and only if R is a σ-maximal order, that is $O_l(\mathfrak{a})=R=O_r(\mathfrak{a})$ for any σ-invariant ideal $\mathfrak{a}$ of R (see, e.g., [**16**, Theorem 2.3.19]). It is easy to see, from our observation in Example 5.3, that D is a σ-maximal order so that $D[t;\sigma]$ is a maximal order. However, as we have already shown, in case (2) (i), $D[Xt;\sigma]$ is not a maximal order and in case either (1) or (2) (ii), $D[Xt;\sigma]$ is a maximal order.

We finally give examples of rings which are $(\sigma;X)$-maximal orders but not σ-maximal orders.

Let k be a field with automorphism σ and let $K=\left(\begin{smallmatrix}k&k\\k&k\end{smallmatrix}\right)$, the ring of 2×2 matrices over k. Then we can extend σ to an automorphism of K by $\sigma(q)=\left(\begin{smallmatrix}\sigma(a)&\sigma(b)\\\sigma(c)&\sigma(d)\end{smallmatrix}\right)$, where $q=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. Let $U=K[x;\sigma]$ and $I=eK+xU$, where $e=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. Then I is a σ-invariant maximal right ideal of U with $UI=U$. We consider $R=\{u\in U\mid uI\subseteq I\}$, the idealizer of I. By [**17**, Theorem 5.5.10], R is an HNP ring and I is an idempotent maximal ideal of R. We note that $R=K(1-e)+eK+xU$ and $\sigma(R)=R$. R has another idempotent maximal ideal $J=K(1-e)+xU$, which is a σ-invariant maximal left ideal of U with $JU=U$. Put $X=I\cap J=eK(1-e)+xU$. Since $O_r(I)=U=O_l(J)$ and $O_r(J)=x^{-1}(eK(1-e))+R=O_l(I)$, $\{I,\ J\}$ is a cycle and X is an invertible ideal of R by [**7**, Proposition 2.5].

EXAMPLE 5.5. Under the same notation and assumptions,

(1) R is not a σ-maximal order and $R[t;\sigma]$ is not a maximal order.
(2) R is a $(\sigma;X)$-maximal order and $S=R[Xt;\sigma]$ is a maximal order. In fact, it is a generalized Asano ring.

Furthermore

(i) If σ is of infinite order, then XS and XtS are only maximal v-ideals of S.
(ii) If σ is of finite order, say n, then there are infinite number of maximal v-ideals of S.

Proof. (1) I is σ-invariant ideal of R, and $O_r(I) = U \supset R$. Hence R is not a σ-maximal order and $R[t;\sigma]$ is not a maximal order.

(2) First we note that X is σ-invariant and so X is $(\sigma; X)$-invariant. Next we have

$$IX = eK(1-e) + x(eK + K(1-e)) + x^2U = XJ = IJ \text{ and } XI = JX = JI = xU$$

and $IX \neq XI$ follows. Since I is σ-invariant, $X\sigma(I) = XI \neq IX$. Hence I is not $(\sigma; X)$-invariant. Similarly J is not $(\sigma; X)$-invariant, either. As U is a principal ideal ring, each ideal of U is invertible and I contains non-zero prime ideal xU of U. Then $\{I, J\}$ is the full set of idempotent maximal ideals of R by [**17**, Theorem 5.6.11]. Other maximal ideals of R are invertible by [**7**, Proposition 2.2].

Let $\mathfrak{a}$ be any ideal of R. Then, $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$ for an eventually idempotent ideal $\mathfrak{b}$ and an invertible ideal $\mathfrak{c}$. But there are no idempotent maximal ideals of R different from I and J, and $I \cap J = X$ is invertible. Hence $\mathfrak{b} = I$ or $\mathfrak{b} = J$ by [**7**, Proposition 4.5] and so $\mathfrak{a}$ is invertible or of the form $I\mathfrak{c}$ or $J\mathfrak{c}$. If $\mathfrak{a} = I\mathfrak{c}$, then $\mathfrak{a}X = I\mathfrak{c}X = IX\mathfrak{c} = XJ\mathfrak{c}$. On the other hand, $X\sigma(\mathfrak{a}) = X\sigma(I\mathfrak{c}) = XI\sigma(\mathfrak{c})$. Thus, if $\mathfrak{a}$ is $(\sigma; X)-$invariant, we have $XJ\mathfrak{c} = XI\sigma(\mathfrak{c})$ and $J\mathfrak{c} = I\sigma(\mathfrak{c})$ follows. Since $UI = U$ and $UJ = J$,

$$I\sigma(\mathfrak{c}) = J\mathfrak{c} = UJ\mathfrak{c} = UI\sigma(\mathfrak{c}) = U\sigma(\mathfrak{c})$$

and we obtain $I = U$, a contradiction. Hence $\mathfrak{a} = I\mathfrak{c}$ is not $(\sigma; X)$-invariant. Similarly $J\mathfrak{c}$ is not $(\sigma; X)$-invariant. Thus $(\sigma; X)$-invariant ideals of R are all invertible. Hence R is a $(\sigma; X)$-maximal order and so S is a maximal order. In fact, S is a generalized Asano ring, because R is a generalized $(\sigma; X)$-Asano ring.

(i) If σ is of infinite order, then xU is the unique maximal ideal of U by [**11**, Theorem 2]. Thus I and J are only maximal ideals of R by [**17**, Theorem 5.6.11] and $D_{\sigma;X}(R) = \{X^n \mid n \in \mathbb{Z}\}$. Let P be a maximal v-ideal of S with $\mathfrak{p} = P \cap R \neq (0)$. Then $P = \mathfrak{p}[Xt; \sigma]$ with $\mathfrak{p} \in D_{\sigma;X}(R)$ by Lemma 4.2 and so $\mathfrak{p} = X$. Furthermore $T = Q(R)[t;\sigma]$ has the unique maximal ideal $tT \cap S = XtS$ by Lemma 4.1 and Poroposition 2.12, because $tT \cap S \supseteq Xt$. Hence XS and XtS are only maximal v-ideals of S.

(ii) If σ is of finite order, say n, then $\mathbb{Z}(U) = k_\sigma[x^n]$, where $k_\sigma = \{a \in k \mid \sigma(a) = a\}$, because $U \cong \begin{pmatrix} k[x;\sigma] & k[x;\sigma] \\ k[x;\sigma] & k[x;\sigma] \end{pmatrix}$. Let P be a maximal ideal of U different from xU. Then $P = wU$ for some $w \in k_\sigma[x^n]$, an irreducible element, by [**3**, Lemma 2.3] and $\mathfrak{p} = P \cap R$ is invertible and prime by [**17**, Theorem 5.6.11]. Furthermore, since $\sigma(w) = w$, $\sigma(\mathfrak{p}) = \mathfrak{p}$ and $X\sigma(\mathfrak{p}) = X\mathfrak{p} = \mathfrak{p}X$. So $\mathfrak{p}$ is $(\sigma; X)$-invariant. It follows that $\{\mathfrak{p},\ X \mid \mathfrak{p} = P \cap R, \text{where } P = wU\}$ is the set of maximal $(\sigma; X)$-invariant invertible ideals and that $\mathfrak{p}[Xt;\sigma]$ are all maximal v-ideals. Therefore there are infinite number of maximal v-ideals of S, because there are infinite number of irreducible elements w in $k_\sigma[x^n]$. This completes the proof. □

References

[1] G. Q. Abbasi, S. Kobayashi, H. Marubayashi and A. Ueda, Non commutative unique factorization rings, Comm. Algebra, **19** (1991), 167–198.

[2] E. Akalan, On generalized Dedekind prime rings, J. Algebra, **320** (2008), 2907–2916.

[3] A. K. Amir, H. Marubayashi and Y. Wang, Prime factor rings of skew polynomial rings over a commutative Dedekind domain, Rocky Mountain J. Math., **42** (2012), 2055–2073.

[4] L. Le Bruyn and F. Van Oystaeyen, Generalized Rees rings and relative maximal orders satisfying polynomial identities, J. Algebra, **83** (1983), 409–436.

[5] G. Cauchon, Les T-anneaux et les anneaux a identites polynomiales Noetheriens, thesis, 1977.

[6] M. Chamarie, Anneaux de Krull non commutatifs, thesis 1981.

[7] D. Eisenbud and J. C. Robson, Hereditary Noetherian prime rings, J. Algebra, **16** (1970), 86–104.

[8] K. R. Goodearl and R. B. Warfield Jr., Simple modules over hereditary Noetherian prime rings, J. Algebras, **57** (1979), 82–100.

[9] A. J. Gray, A note on the invertible ideal theorem, Glasgow Math. J., **25** (1984), 27–30.

[10] M. Harada, Structure of hereditary orders over local rings, J. Math. Osaka City Univ., **14** (1963), 1–22.

[11] N. Jacobson, Pseudo-linear transformations, Annals of Math., **38** (1937), 484–507.

[12] H. Marubayashi, A Krull type generalization of HNP rings with enough invertible ideals, Comm. Algebra, **11** (1983), 469–499.

[13] H. Marubayashi, A skew polynomial ring over a v-HC order with enough v-invertible ideals, Comm. Algebra, **12** (1984), 1567–1593.

[14] H. Marubayashi, On v-ideals in a VHC order, Proc. Japan Acad. Ser. A Math. Sci., **59** (1983), 339-342.

[15] H. Marubayashi, I. Muchtadi-Alamsyah and A. Ueda, Skew polynomial rings which are generalized Asano prime rings, J. Algebra Appl., **12**(7), 2013.

[16] H. Marubayashi and F. Van Oystaeyen, Prime Divisors and Noncommutative Valuation Theory, Lecture Notes in Math., **2059**, Springer, 2012.

[17] J. C. McConnel and J. C. Robson, Noncommutative Noetherian Rings, Wiley, 1987.

[18] B. Stenstrőm, Rings of Quotients, Springer-Verlag, 1975.

[19] F. Van Oystaeyen, Generalized Rees rings and arithmetical graded rings, J. Algebra, **82** (1983), 185–193.

[20] F. Van Oystaeyen and A. Verschoren, Relative invariants of rings, The noncommutative theory, Pure and Applied Math., Maecel Dekker, Inc., **86**, 1984.

Monika R. Helmi
Department of Mathematics
Andalas University
Kampus Unand Limau Manis
Padang 25163, Indonesia
E-mail: monika@fmipa.unand.ac.id

Hidetoshi Marubayashi
Faculty of Sciences and Engineering
Tokushima Bunri University
Sanuki
Kagawa 769-2193, Japan
E-mail: marubaya@naruto-u.ac.jp

Akira Ueda
Department of Mathematics
Shimane University
Matsue
Shimane 690-8504, Japan
E-mail: ueda@riko.shimane-u.ac.jp