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A class of almost $C_0(\mathcal{K})$-C*-algebras

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Abstract. We consider in this paper the family of exponential Lie groups $G_{n,\mu}$, whose Lie algebra is an extension of the Heisenberg Lie algebra by the reals and whose quotient group by the centre of the Heisenberg group is an $ax+b$-like group. The C*-algebras of the groups $G_{n,\mu}$ give new examples of almost $C_0(\mathcal{K})$-C*-algebras.

1. Introduction and notations.

Let $\mathcal{A}$ be a C*-algebra and $\widehat{\mathcal{A}}$ be its unitary spectrum. The C*-algebra $l^\infty(\widehat{\mathcal{A}})$ of all bounded operator fields defined over $\widehat{\mathcal{A}}$ is given by

$$l^\infty(\widehat{\mathcal{A}}) := \Big\{A = (A(\pi) \in \mathcal{B}(\mathcal{H}_\pi))_{\pi\in\widehat{\mathcal{A}}}; \|A\|_\infty := \sup_\pi \|A(\pi)\|_{\text{op}} < \infty\Big\},$$

where $\mathcal{H}_\pi$ is the Hilbert space on which π acts. Let $\mathcal{F}$ be the Fourier transform of $\mathcal{A}$, i.e.,

$$\mathcal{F}(a) := \hat{a} := (\pi(a))_{\pi\in\widehat{\mathcal{A}}} \quad \text{for} \quad a \in \mathcal{A}.$$

It is an injective, hence isometric, homomorphism from $\mathcal{A}$ into $l^\infty(\widehat{\mathcal{A}})$. Hence one can analyze the C*-algebra $\mathcal{A}$ by recognizing the elements of $\mathcal{F}(\mathcal{A})$ inside the (big) C*-algebra $l^\infty(\widehat{\mathcal{A}})$.

We know that the unitary spectrum $\widehat{C^*(G)}$ of the C*-algebra $C^*(G)$ of a locally compact group G can be identified with the unitary dual $\widehat{G}$ of G. If G is an *exponential* Lie group, i.e., if the exponential mapping $\exp : \mathfrak{g} \to G$ from the Lie algebra $\mathfrak{g}$ to its Lie group G is a diffeomorphism, then the Kirillov-Bernat-Vergne-Pukanszky-Ludwig-Leptin theory shows that there is a canonical homeomorphism $K : \mathfrak{g}^*/G \to \widehat{G}$ from the space of coadjoint orbits of G in the linear dual space $\mathfrak{g}^*$ onto the unitary dual space $\widehat{G}$ of G (see [**LepLud**] for details and references). In this case, one can therefore identify the unitary spectrum $\widehat{C^*(G)}$ of the C*-algebra of an exponential Lie group with the space $\mathfrak{g}^*/G$ of coadjoint orbits of the group G.

The C*-algebra of an $ax+b$-like group was characterised in [**LinLud**] and the C*-algebras of the Heisenberg group and of the threadlike groups were described in [**LuTu**] as algebras of operator fields defined on the dual spaces of the groups. The method of

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describing group C*-algebras as algebras of operator fields defined on the dual spaces was first used in [**Fell**] and [**Lee**].

In this paper, we consider the exponential solvable Lie group $G_{n,\mu}$, whose Lie algebra is an extension of the Heisenberg Lie algebra $\mathfrak{h}_n$ by the reals, which means that $\mathbb{R}$ acts on $\mathfrak{h}_n$ by a diagonal matrix with real eigenvalues. The quotient group of $G_{n,\mu}$ by the centre of the Heisenberg group is then an $ax+b$-like group, whose C*-algebra has been determined in [**LinLud**]. Since the orbit structure of exponential groups is well understood (see for instance [**ArLuSc**]), we can write down the spectrum of the group $G_{n,\mu}$ explicitly and determine its topology.

In [**ILL**] the example of the group $N_{6,28}$ motivated the introduction of a special class of C*-algebras which we called *almost* $C_0(\mathcal{K})$*-C*-algebra*, where $\mathcal{K}$ is the algebra of all compact operators on some Hilbert space. In Section 2, we recall the definition and the properties of almost $C_0(\mathcal{K})$-C*-algebras. In Section 3 we introduce the family of the $G_{n,\mu}$ groups and describe the space of coadjoint orbits $\mathfrak{g}^*_{n,\mu}/G_{n,\mu}$. We show that the spectrum $\widehat{G_{n,\mu}}$ of $G_{n,\mu}$ is a disjoint union of the sets $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$, where Γ_0 is the set of the characters of $G_{n,\mu}$, Γ_1 and Γ_2 are the sets of the representations corresponding to the two-dimensional coadjoint orbits of $G_{n,\mu}$, and Γ_3 is the union of the two generic irreducible representations π_+, π_- which correspond to the two open orbits. Note that each of the sets Γ_i needs a special treatment. The sets Γ_1 and Γ_2 have been treated in the paper [**LinLud**]. In Subsection 4.2, we discover the almost $C_0(\mathcal{K})$ conditions for Γ_3. This is the most intricate part of the paper and the treatment is inspired by the study of the boundary condition for a class of 4-dimensional orbits in [**ILL**, Subsection 6.3]. At the end (Subsection 4.4), we describe the actual C*-algebra of $G_{n,\mu}$ as an algebra of operator fields and we see that this C*-algebra has the structure of an almost $C_0(\mathcal{K})$-C*-algebra.

2. Almost $C_0(\mathcal{K})$-C*-algebras.

The following definitions were given in [**ILL**]; for completeness, we recall them here.

Definition 2.1. Let A be a C*-algebra and $\widehat{A}$ be the spectrum of A.

(1) Suppose there exists a finite increasing family $S_0 \subset S_1 \subset \cdots \subset S_d = \widehat{A}$ of subsets of $\widehat{A}$ such that for $i = 1, \ldots, d$, the subsets $\Gamma_0 = S_0$ and $\Gamma_i := S_i \setminus S_{i-1}$ are Hausdorff in their relative topologies. Furthermore we assume that for every $i \in \{0, \ldots, d\}$ there exists a Hilbert space $\mathcal{H}_i$ and a concrete realization $(\pi_\gamma, \mathcal{H}_i)$ of γ on the Hilbert space $\mathcal{H}_i$ for every $\gamma \in \Gamma_i$. Note that the set S_0 is the collection $\mathfrak{X}$ of all characters of A.

(2) For a subset $S \subset \widehat{A}$, denote by $CB(S)$ the *-algebra of all uniformly bounded operator fields $(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_i))_{\gamma \in S \cap \Gamma_i, i=1,\ldots,d}$, which are operator norm continuous on the subsets $\Gamma_i \cap S$ for every $i \in \{1, \ldots, d\}$ for which $\Gamma_i \cap S \neq \emptyset$. We provide the *-algebra $CB(S)$ with the infinity-norm:

$$\|\psi\|_S := \sup_{\gamma \in S} \|\psi(\gamma)\|_{\mathrm{op}}.$$

Definition 2.2. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K} := \mathcal{K}(\mathcal{H})$ be the algebra of all compact operators defined on $\mathcal{H}$. A C*-algebra A is said to be *almost* $C_0(\mathcal{K})$ if for every

$a \in A$:

(1) The mappings $\gamma \mapsto \mathcal{F}(a)(\gamma)$ are norm continuous on the different sets Γ_i, where $\mathcal{F} : A \to l^\infty(\widehat{A})$ is the Fourier transform given by

$$\mathcal{F}(a)(\gamma) = \widehat{a}(\gamma) := \pi_\gamma(a) \quad \text{for} \quad \gamma \in \widehat{A} \text{ and } a \in A.$$

(2) For each $i = 1, \dots, d$, we have a sequence $(\sigma_{i,k} : CB(S_{i-1}) \to CB(S_i))_k$ of linear mappings which are uniformly bounded in k (and independent of a) such that

$$\lim_{k\to\infty} \operatorname{dis}\big((\sigma_{i,k}(\mathcal{F}(a)_{|S_{i-1}}) - \mathcal{F}(a)_{|\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))\big) = 0,$$

and

$$\lim_{k\to\infty} \operatorname{dis}\big((\sigma_{i,k}(\mathcal{F}(a)^*_{|S_{i-1}}) - \mathcal{F}(a^*)_{|\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))\big) = 0,$$

where $C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))$ is the space of all continuous mappings $\varphi : \Gamma_i \to \mathcal{K}(\mathcal{H}_i)$ vanishing at infinity.

Definition 2.3. Let $D^*(A)$ be the set of all operator fields φ defined over $\widehat{A}$ such that

(1) The field φ is uniformly bounded, i.e., we have that $\|\varphi\| := \sup_{\gamma\in\widehat{A}} \|\varphi(\gamma)\|_{\mathrm{op}} < \infty$.
(2) $\varphi_{|\Gamma_i} \in CB(\Gamma_i)$ for every $i = 0, 1, \dots, d$.
(3) For every sequence $(\gamma_k)_{k\in\mathbb{N}}$ going to infinity in $\widehat{A}$, we have that $\lim_{k\to\infty} \|\varphi(\gamma_k)\|_{\mathrm{op}} = 0$.
(4) For each $i = 1, 2, \dots, d$,

$$\lim_{k\to\infty} \operatorname{dis}\big((\sigma_{i,k}(\varphi_{|S_{i-1}}) - \varphi_{|\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))\big) = 0$$

and

$$\lim_{k\to\infty} \operatorname{dis}\big((\sigma_{i,k}(\varphi^*_{|S_{i-1}}) - (\varphi_{|\Gamma_i})^*), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))\big) = 0.$$

We see immediately that if A is almost $C_0(\mathcal{K})$, then for every $a \in A$, the operator field $\mathcal{F}(a)$ is contained in the set $D^*(A)$. In fact it turns out that $D^*(A)$ is a C*-subalgebra of $l^\infty(\widehat{A})$ and that A is isomorphic to $D^*(A)$.

Theorem 2.4 ([**ILL**, Theorem 2.6]). *Let A be a separable C^*-algebra which is almost $C_0(\mathcal{K})$. Then the subset $D^*(A)$ of the C^*-algebra $l^\infty(\widehat{A})$ is a C^*-subalgebra which is isomorphic to A under the Fourier transform.*

3. The groups $G_{n,\mu}$.

Let $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $V_n = \mathbb{R}^{2n}$ and denote by ω_n the canonical non-degenerate skew-symmetric bilinear form on V_n. Let

$$\mathfrak{h}_n := V_n \oplus \mathbb{R}.$$

Choose a symplectic basis $\mathcal{B} := \{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ of V_n. Let

$$\mathfrak{g}_{n,\mu} := \mathbb{R} \times \mathfrak{h}_n \ \text{ and } \ A = (1, 0_{V_n}, 0),\ \ Z = (0, 0_{V_n}, 1) \in \mathfrak{g}_{n,\mu}.$$

Then $\{A, X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ is a basis of $\mathfrak{g}_{n,\mu}$. For

$$\mu := \{\lambda_1, \lambda_1', \ldots, \lambda_n, \lambda_n'\} \subset \mathbb{R}$$

with $\lambda_i + \lambda_i' = 2$ for all $i = 1, \ldots, n$, we define the brackets

$$[A, X_i] = \lambda_i X_i,\ [A, Y_i] = \lambda_i' Y_i,\ [A, Z] = 2Z \ \text{ for all } \ i = 1, \ldots, n,$$

and

$$[X_i, Y_j] = \delta_{i,j} Z \quad \text{for} \quad i, j = 1, \ldots, n.$$

Eventually by exchanging X_j and Y_j and replacing X_j by $-X_j$ we can assume that $\lambda_j' \geq 0$ for all j. We then obtain a structure of an exponential solvable Lie algebra on $\mathfrak{g}_{n,\mu}$, and its subalgebra $\mathfrak{h}_n$ is the Heisenberg Lie algebra.

Define the diagonal operator $l_\mu : V_n \to V_n$ by

$$l_\mu(v) := \sum_i \lambda_i v_i X_i + \lambda_i' v_i' Y_i \quad \text{ for } \quad v = \sum_{i=1}^n v_i X_i + \sum_{i=1}^n v_i' Y_i \in V_n.$$

For $v = \sum_{i=1}^n v_i X_i + v_i' Y_i \in V_n$ and $a \in \mathbb{R}$, we write

$$a \cdot v := \sum_{i=1}^n e^{a\lambda_i} v_i X_i + e^{a\lambda_i'} v_i' Y_i.$$

The corresponding simply connected Lie group $G_{n,\mu}$, which is exponential solvable, can be identified with the space $\mathbb{R} \times V_n \times \mathbb{R}$ equipped with the multiplication

$$(a, v, c) \cdot (a', v', c') := (a + a', (-a') \cdot v + v', e^{-2a'} c + c' + \frac{1}{2}\omega_n((-a') \cdot v, v')). \tag{3.0.1}$$

The inner automorphism $\mathrm{Ad}(a, u)$ on $\mathfrak{h}_n$ is given by

$$\begin{aligned}
\mathrm{Ad}(a, u)(0, v, z) &= (a, u, 0)(0, v, z)(-a, -(a \cdot u), 0) \\
&= (a, 0, 0)(0, u, 0)(0, v, z)(0, -u, 0)(-a, 0, 0) \\
&= (a, 0, 0)(0, v, z + \omega_n(u, v))(-a, 0, 0) \\
&= (0, a \cdot v, e^{2a} z + e^{2a} \omega_n(u, v)) \quad \text{for} \quad (v, z) \in \mathfrak{h}_n.
\end{aligned}$$

The centre $\mathcal{Z}$ of the normal subgroup $H_n := \{0\} \times V_n \times \mathbb{R}$ of $G_{n,\mu}$ is the subset $\mathcal{Z} = \exp(\mathbb{R}Z) = \{0\} \times \{0_{V_n}\} \times \mathbb{R}$. Denote by G_{V_n} the quotient group $G_{n,\mu}/\mathcal{Z}$ which can be identified with $\mathbb{R} \times V_n$ equipped with the multiplication

$$(s, v) \cdot (t, w) := (s + t, (-t) \cdot v + w).$$

We write $V_n = V_0 \oplus V_+ \oplus V_- = V_0 \oplus V_1$, where

$$\begin{aligned} V_+ &:= \operatorname{span}\{X_j, Y_k; \lambda_j > 0, \lambda'_k > 0\}, \\ V_- &:= \operatorname{span}\{X_j; \lambda_j < 0\}, \\ V_0 &:= \operatorname{span}\{X_j, Y_k; \lambda_j = 0, \lambda'_k = 0\}, \end{aligned}$$

and $V_1 := V_+ \oplus V_-$. Let

$$\mu_+ := \mu \cap \mathbb{R}^*_+,\ \mu_- := \mu \cap \mathbb{R}^*_-,\ \mu_0 := \mu \cap \{0\},$$

then we can write

$$V_+ = \sum_{\lambda \in \mu_+} V_{+,\lambda} \quad \text{and} \quad V_- = \sum_{\lambda \in \mu_-} V_{-,\lambda},$$

where $V_{+,\lambda}$ and $V_{-,\lambda}$ are the respective eigenspaces of the operator l_μ.

We can also identify $\mathfrak{g}^*_{n,\mu}$ with $\mathbb{R}A^* \oplus V_n^* \oplus \mathbb{R}Z^* \simeq \mathbb{R} \times V_n \times \mathbb{R}$, and then

$$\begin{aligned} \langle \mathrm{Ad}^*(a, u)(a^*, v^*, \lambda^*), (0, v, z) \rangle &= \langle (a^*, v^*, \lambda^*), \mathrm{Ad}((a, u)^{-1})(0, v, z) \rangle \\ &= \langle (a^*, v^*, \lambda^*), (0, (-a) \cdot v, e^{-2a} z + e^{-2a} \omega_n(-(a \cdot u), v)) \rangle \\ &= \langle 0, v^*, (-a) \cdot v \rangle + \lambda^* e^{-2a} z + \lambda^* e^{-2a} \omega_n(-(a \cdot u), v). \end{aligned}$$

Hence

$$\mathrm{Ad}^*(a, u)(a^*, v^*, \lambda^*)_{|\mathfrak{h}_n} = (a^*, (-a) \cdot v^* - \lambda^* e^{-2a}(a \cdot u) \times \omega_n, \lambda^* e^{-2a}).$$

Here we denote by $u \times \omega_n$ the linear functional on V_n as

$$u \times \omega_n(v) := \omega_n(u, v) \quad \text{for all} \quad v \in V_n.$$

The coadjoint orbit Ω_ℓ of an element $\ell = (a^*, v^*, \lambda^*) \in \mathfrak{g}^*_{n,\mu}$ is given by

$$\Omega_\ell = \{(a^* + v^*([A, u]) + 2z\lambda^*, (-a) \cdot v^* - \lambda^* e^{-2a}(a \cdot u) \times \omega_n, \lambda^* e^{-2a}) : a, z \in \mathbb{R}, u \in V_n\}.$$

Hence if $\lambda^* \neq 0$ then the corresponding coadjoint orbit is the subset

$$\Omega_{\lambda^*} = \mathbb{R} \times V_n^* \times \mathbb{R}^*_+ \lambda^*,$$

where V_n^* is the linear dual space of V_n. Therefore we have two open coadjoint orbits

$$\Omega_\varepsilon := \mathrm{Ad}^*(G_{n,\mu})\ell_\varepsilon = \mathbb{R} \times V_n^* \times \mathbb{R}_\varepsilon^* \quad \text{for} \quad \varepsilon \in \{+,-\}, \tag{3.0.2}$$

where $\ell_\varepsilon = \varepsilon Z^*$. The other orbits are contained in $Z^\perp$ with the form

$$\Omega_{v^*} = \mathbb{R}A^* + \mathbb{R} \cdot v^* \ \text{ for } \ v^* \in V_n^* \setminus V_0^*,$$

or the one point orbits

$$\{a^* A^* + v^*\} \quad \text{for} \quad a^* \in \mathbb{R}, v^* \in V_0^*.$$

We can decompose the linear dual space V_n^* of V_n into

$$\begin{aligned} V_+^* &:= \{f \in V_n^* : f(V_- \cup V_0) = \{0\}\}, \\ V_-^* &:= \{f \in V_n^* : f(V_+ \cup V_0) = \{0\}\}, \\ V_0^* &:= \{f \in V_n^* : f(V_+ \cup V_-) = \{0\}\}. \end{aligned}$$

The following definition was given in [**LinLud2**].

Definition 3.1. Denote by $\|\cdot\|$ the norm on V_n^* coming from the scalar product defined by the basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$. For $f_+ = \sum_{\lambda \in \mu_+} f_\lambda \in V_+^*$ and $f_- = \sum_{\lambda \in \mu_-} f_\lambda \in V_-^*$, let

$$|f_+|_\mu = |f_+| := \max_{\lambda_j \in \mu_+} \|f_{\lambda_j}\|^{1/\lambda_j} \ \text{ and } \ |f_-|_\mu = |f_-| := \max_{\lambda_j \in \mu_-} \|f_{\lambda_j}\|^{-1/\lambda_j}.$$

Then for $t \in \mathbb{R}$, we have the relation

$$|t \cdot f_+| = e^t |f_+| \ \text{ and } \ |t \cdot f_-| = e^{-t}|f_-| \ \text{ for } \ f_+ \in V_+^*, f_- \in V_-^*. \tag{3.0.3}$$

On V_0^* we shall use the norm coming from the scalar product. This gives us a global gauge on V_n^*:

$$|(f_0, f_+, f_-)| := \max\{\|f_0\|, |f_+|, |f_-|\}.$$

We denote by V_{gen}^* the open subset of V_n^* consisting of all the $f = (f_0, f_+, f_-) \in V_0^* \times V_+^* \times V_-^*$ for which $f_+ \neq 0$ and $f_- \neq 0$. The subset V_{sin}^* consists of all the $f = (f_0, f_+, f_-)$ for which either $f_+ \neq 0, f_- = 0$ or $f_+ = 0, f_- \neq 0$. We see that for every $f = (f_0, f_+, f_-) \in V_{gen}^*$ there exists exactly one element $f' = (f_0, f'_+, f'_-)$ in its $G_{n,\mu}$-orbit such that $|f'_+| = |f'_-|$. In the same way, for $f = (f_0, f_+, 0)$ (resp. $f = (f_0, 0, f_-)) \in V_{sin}^*$, there exists exactly one element $f' = (f_0, f'_+, 0)$ (resp. $f' = (f_0, 0, f'_-)$) in its $G_{n,\mu}$-orbit for which $|f'_+| = 1$ (resp. $|f'_-| = 1$).

For $f_+ \in V_+^* \setminus \{0\}$, let us denote by $r(f_+)$ the unique real number for which the vector $r(f_+) \cdot f_+$ in V_+^* has gauge 1. This means that

$$r(f_+) := -\ln(|f_+|).$$

Similarly, for $f_- \in V_-^* \setminus \{0\}$ we define the number $q(f_-)$ by

$$q(f_-) := \ln(|f_-|)$$

such that $|q(f_-) \cdot f_-| = 1$. Let

$$\mathcal{D} = \{(f_0, f_+, f_-) : |f_+| = |f_-| \neq 0\},$$
$$\mathcal{S}_+ = \{(f_0, f_+, 0) : |f_+| = 1\}, \quad \mathcal{S}_- = \{(f_0, 0, f_-) : |f_-| = 1\}, \text{ and}$$
$$\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-.$$

The orbit space $\mathfrak{g}_{n,\mu}^*/G_{n,\mu}$ can then be written as the disjoint union Γ of the sets

$$\Gamma_0 = \mathbb{R} \times V_0^*, \text{ corresponding to the unitary characters of } G_{n,\mu},$$
$$\Gamma_1 = \mathcal{S} \simeq V_{sin}^*/G_{n,\mu},$$
$$\Gamma_2 = \mathcal{D} \simeq V_{gen}^*/G_{n,\mu},$$
$$\Gamma_3 = \{+,-\} \simeq \{\Omega_+, \Omega_-\}/G_{n,\mu},$$

in the case where $V_{gen}^* \neq \emptyset$, i.e., $\mu_+ \neq \emptyset$ and $\mu_- \neq \emptyset$. In case $V_{gen}^* = \emptyset$, we have Γ as the union of

$$\Gamma_0 = \mathbb{R} \times V_0^*, \text{ corresponding to the unitary characters of } G_{n,\mu},$$
$$\Gamma_1 = \mathcal{S} \simeq V_{sin}^*/G_{n,\mu},$$
$$\Gamma_2 = \{+,-\} \simeq \{\Omega_+, \Omega_-\}/G_{n,\mu}.$$

In order to simplify notations, we shall treat only the first case in the following, i.e., we shall assume that V_{gen}^* is nonempty. The other case is similar and easier.

The topology of the orbit space $\mathfrak{g}_{V_n}^*/G_{V_n}$ of the quotient group $G_{n,\mu}/\mathcal{Z}$ has been described in [**LinLud**]. We recall that a sequence $y = (y_k)_k$ is called properly converging if y has limit points and if every cluster point of the sequence is a limit point, i.e., the set of limit points of any subsequence is always the same, indeed, it equals to the set of all limit points of the sequence y.

THEOREM 3.2 ([**LinLud**, Theorem 2.3]).

(1) *A properly converging sequence $(\Omega_{f_k})_k$ with $f_k = (f_{k,0}, f_{k_+}, f_{k_-}) \in \mathcal{D}$ has either a unique limit point Ω_f for some $f \in \mathcal{D}$ and then $f = \lim_k f_k$, or $\lim_k (f_{k_+}, f_{k_-}) = 0$ and then the limit set L of the sequence is given by*

$$L = \{\Omega_{(f_0, f_+, 0)}, \Omega_{(f_0, 0, f_-)}, \mathbb{R}\},$$

where $f_0 = \lim_k f_{k,0}$, $f_+ = \lim_k r(f_{k_+}) \cdot f_{k_+} \in \mathcal{S}_+$ and $f_- = \lim_k q(f_{k_-}) \cdot f_{k_-} \in \mathcal{S}_-$.

(2) *A properly converging sequence* (Ω_{f_k}) *with* $f_k = (f_{k,0}, f_{k_+}, f_{k_-}) \in \mathcal{S}$ *has the limit set*

$$L = \{\Omega_f, \mathbb{R}\},$$

where $f = \lim_k f_k \in \mathcal{S}$.

Corollary 3.3. *The orbit* Ω_f *for* $f \in \mathcal{D}$ *is closed in* $\mathfrak{g}^*_{n,\mu}$. *The closure of the orbit* Ω_f *for* $f \in \mathcal{S}$ *is the set* $\{\Omega_f, \mathbb{R}\}$.

From the description (3.0.2) of the open orbits Ω_ε, $\varepsilon = \pm$, we have the boundary of Ω_ε as the following.

Corollary 3.4. *For* $\varepsilon \in \{+, -\}$, *the boundary of the open orbit* Ω_ε *is the subset* $\mathbb{R} \times V_n^* \times \{0\} = Z^\perp \simeq \mathfrak{g}^*_{V_n}$.

On the other hand, for every coadjoint orbit we can write down a corresponding irreducible representation as an induced representation by using Kirillov's orbit theory.

(1) Let $P_n = \exp(\sum_{j=1}^n \mathbb{R}Y_j + \mathbb{R}Z)$. This is a closed connected normal abelian subgroup of $G_{n,\mu}$. Let also $\mathfrak{x}_n := \sum_{j=1}^n \mathbb{R}X_j$ and $\mathfrak{y}_n := \sum_{j=1}^n \mathbb{R}Y_j \subset V_n$ (an abelian subalgebra of $\mathfrak{g}_{n,\mu}$), then $\mathcal{X}_n := \exp(\mathfrak{x}_n)$ and $\mathcal{Y}_n = \exp(\mathfrak{y}_n)$ are closed connected abelian subgroups of $G_{n,\mu}$. We have

$$G_{n,\mu} = \exp(\mathbb{R}A) \cdot \mathcal{X}_n \cdot P_n = S_n \cdot P_n,$$

where $S_n := \exp(\mathbb{R}A) \cdot \mathcal{X}_n$ is a subgroup of $G_{n,\mu}$. The irreducible representations $\pi_\varepsilon, \varepsilon = \pm$, corresponding to the orbits Ω_ε are of the form

$$\pi_\varepsilon := \mathrm{ind}_{P_n}^{G_{n,\mu}} \chi_{\varepsilon Z^*}.$$

The Hilbert space of π_ε is the L^2-space $L^2(G_{n,\mu}/P_n, \chi_\varepsilon) \simeq L^2(S_n)$, where $\chi_\varepsilon(y,z) := e^{-i2\pi\varepsilon z}$ for $(y,z) \in P_n$. The elements of this space are the measurable functions $\xi : G_{n,\mu} \to \mathbb{C}$ satisfying the relations

$$\xi(gp) = \chi_\varepsilon(p^{-1})\xi(g) \text{ for } g \in G_{n,\mu}, p \in P_n, \text{ and}$$
$$\int_{G_{n,\mu}/P_n} |\xi(g)|^2 d\dot{g} < \infty,$$

where $d\dot{g}$ is the left invariant measure on $G_{n,\mu}/P_n$. For $F \in L^1(G_{n,\mu})$ and $\xi \in L^2(G_{n,\mu}/P_n)$, we have

$$\begin{aligned}\pi_\varepsilon(F)\xi(s') &= \int_{S_n P_n} F(sp)\xi(p^{-1}s^{-1}s')dsdp \\ &= \int_{S_n P_n} F(s'sp)\xi(p^{-1}s^{-1})dsdp\end{aligned}$$

$$= \int_{S_nP_n} F(s's^{-1}p)\Delta_{S_n}(s^{-1})\xi(p^{-1}s)dsdp$$
$$= \int_{S_nP_n} F(s's^{-1}p)\Delta_{S_n}(s^{-1})\xi(s(s^{-1}p^{-1}s))dsdp$$
$$= \int_{S_nP_n} F(s's^{-1}p)\Delta_{S_n}(s^{-1})\chi_\varepsilon(s^{-1}ps)\xi(s)dsdp$$
$$= \int_{S_nP_n} F(s's^{-1}p)\Delta_{S_n}(s^{-1})e^{-i2\pi \operatorname{Ad}^*(s)\ell_\varepsilon(\log(p))}\xi(s)dsdp$$
$$= \int_{S_n} \widehat{F}^{\mathfrak{p}_n}(s's^{-1}; \operatorname{Ad}^*(s)l_\varepsilon)\xi(s)\Delta_{S_n}(s^{-1})ds.$$

Here $\widehat{F}^{\mathfrak{p}_n}$ is the partial Fourier transform of F in the direction P_n given by

$$\widehat{F}^{\mathfrak{p}_n}(s;\ell) := \int_{P_n} F(sp)e^{-i2\pi\langle \ell, \log(p)\rangle}dp \ \text{ for } \ s \in S_n, \ell \in \mathfrak{p}_n^*.$$

Hence the operator $\pi_\varepsilon(F)$ is given by the kernel function

$$F_\varepsilon((a',x'),(a,x)) = \widehat{F}^{\mathfrak{p}_n}(a'-a, a\cdot(x'-x); (-\varepsilon e^{-2a}(a\cdot x)\times\omega_n, \varepsilon e^{-2a}))e^{|\lambda|a},$$

where $|\lambda| := \sum_{j=1}^n \lambda_j$. In fact the linear functional $\varepsilon e^{-2a}(a\cdot x)\times\omega_n$ is given by

$$\varepsilon e^{-2a}(a\cdot x)\times\omega_n = \varepsilon\bigg(\sum_{j=1}^n e^{(\lambda_j-2)a}x_jY_j^*\bigg) \ \text{ for } \ a \in \mathbb{R}, x \in \mathcal{X}_n.$$

Therefore,

$$F_\varepsilon((a',x'),(a,x)) = \widehat{F}^{\mathfrak{p}_n}\bigg(a'-a, a\cdot(x'-x); \bigg(-\varepsilon\bigg(\sum_{j=1}^n e^{(\lambda_j-2)a}x_jY_j^*\bigg), \varepsilon e^{-2a}\bigg)\bigg)e^{|\lambda|a}.$$

(2) For $v^* \in V_n^*$, we have the irreducible representation π_{v^*} on $L^2(\mathbb{R})$ defined by

$$\pi_{v^*} := \operatorname{ind}_{H_n}^{G_{n,\mu}} \chi_{v^*},$$

where $H_n := \exp(\mathfrak{h}_n)$. The kernel function F_{v^*} of the operator $\pi_{v^*}(F), F \in L^1(G_{n,\mu})$, is given by

$$F_{v^*}(a,b) = \widehat{F}^{\mathfrak{h}_n}(a-b, b\cdot v^*, 0) \quad \text{for} \quad a,b \in \mathbb{R}. \tag{3.0.4}$$

(3) Finally, for $(a^*, v_0^*) \in \mathbb{R}\times V_0^*$ we have the unitary characters

$$\chi_{(a^*,v_0^*)(a,v_0,v,c)} := e^{-2\pi i(a^*a+v_0^*(v_0))} \ \text{ for } \ a,c \in \mathbb{R}, v_0 \in V_0, v \in V_1.$$

Definition 3.5. We denote by $l^\infty(\Gamma)$ the C*-algebra

$$l^\infty(\Gamma) = \Big\{(\phi(\gamma) \in \mathcal{B}(\mathcal{H}_\gamma))_{\gamma\in\Gamma}; \|\phi\| := \sup_{\gamma\in\Gamma} \|\phi(\gamma)\|_{\mathrm{op}} < \infty\Big\}.$$

The Fourier transform $\mathcal{F}_{n,\mu} : C^*(G_{n,\mu}) \to l^\infty(\Gamma)$ for $C^*(G_{n,\mu})$ is given by

$$\begin{aligned}
\mathcal{F}_{n,\mu}(a)(\varepsilon) = \widehat{a}(\varepsilon) &:= \pi_\varepsilon(a) \ \text{ for } \ \varepsilon \in \{+,-\},\\
\mathcal{F}_{n,\mu}(a)(f) = \widehat{a}(f) &:= \pi_f(a) \ \text{ for } \ f \in \mathcal{D} \cup \mathcal{S},\\
\mathcal{F}_{n,\mu}(a)(a^*, v_0^*) &:= \chi_{(a^*,v_0^*)}(a) \ \text{ for } \ (a^*, v_0^*) \in \mathbb{R} \times V_0^*,\\
&\quad \Big(= \int_{\mathbb{R}\times V_0\times V\times\mathbb{R}} F(s, v_0, v_1, z) e^{-i2\pi a^* s} e^{-i2\pi v_0^*(v_0)} ds dv_0 dv_1 dz\\
&\quad\quad \text{for } F \in L^1(G_{n,\mu})\Big).
\end{aligned}$$

4. The C*-conditions.

4.1. The continuity and infinity conditions.

Theorem 4.1. *For every $a \in C^*(G_{n,\mu})$, the mapping*

$$\mathcal{S} \cup \mathcal{D} \mapsto \mathcal{B}(L^2(\mathbb{R})) : f \mapsto \widehat{a}(f),$$

is norm continuous. We also have that

$$\lim_{\substack{|f|\to\infty\\ f\in\mathcal{D}}} \|\pi_f(a)\|_{\mathrm{op}} = 0.$$

Proof. See [**LinLud**, Proposition 4.2]. □

4.2. The condition for the open orbits $\boldsymbol{\Omega_\varepsilon}$.

To understand the case of open orbits, we have to take into account the boundary points of such an orbit. It is well known that for $a \in C^*(G)$ the operator $\pi_\varepsilon(a)$ is compact if and only if $\pi(a) = 0$ for every π in the boundary of the representation π_ε, i.e., if $\pi_\gamma(a) = 0$ for every $\gamma \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. In this subsection we shall give a description of the algebra of operators $\pi_\varepsilon(C^*(G_{n,\mu}))$.

Definition 4.2. For $k \in \mathbb{Z}$ and $r \in \mathbb{R}$, let $I_{r,k}$ be the half-open interval:

$$I_{r,k} := [kr, kr + r[\subset \mathbb{R}.$$

(1) Let $S_{\delta,1} := \{(a, x) \in \mathbb{R} \times \mathcal{X}_n; e^{-a} > \delta^3\}$.
(2) Let $\delta \mapsto r_\delta \in \mathbb{R}_+$ be such that $\lim_{\delta\to 0} r_\delta = +\infty$ and $\lim_{\delta\to 0} e^{m r_\delta}\delta^{1/2} = 0$, where $1 \le m := \max_j(2 - \lambda_j)$.

(3) For constants $D = (D_1, \dots, D_n) \in (\mathbb{R}_+^*)^n$ and $\underline{k} = (k_0, k_1, \dots, k_n) \in \mathbb{Z}^{n+1}$, let

$$S_{\delta,D,\underline{k},2} := \{(a, x_1, \dots, x_n) \in \mathbb{R} \times \mathcal{X}_n; e^{-a} \leq \delta^3,\\ a \in I_{r_\delta,k_0}, x_j \in I_{D_j\delta^2 e^{r_\delta(2-\lambda_j)k_0},k_j}, j = 1, \dots, n\}.$$

Proposition 4.3. *For every compact subset $K \subseteq \mathbb{R} \times \mathcal{X}_n$ and $\delta > 0$ small enough, we have that*

$$KS_{\delta,D,\underline{k},2} \subset \bigcup_{\substack{j_0 \in \mathbb{Z}\\ |j_0| \leq 1}} S_{\delta,D_{\delta,j_0},\underline{k},2} =: R_{\delta,D,\underline{k},2},$$

where $D_{\delta,j_0} = (D_1 e^{-r_\delta(2-\lambda_1)(j_0)}, \dots, D_n e^{-r_\delta(2-\lambda_n)(j_0)}) \in (\mathbb{R}_+^*)^n$.

Proof. Indeed, there is an $M > 0$ such that $K \subset [-M, M]^{n+1} \subset \mathbb{R}^{n+1}$. Let $r_\delta > M$. For $(s, u) \in K$ and $(a, x) \in S_{\delta,D,\underline{k},2}$, it follows that

$$\zeta := (s, u) \cdot (a, x) = (s + a, (-a) \cdot u + x),$$

and $(k_0+j_0)r_\delta \leq s+a < (k_0+j_0+1)r_\delta$ for some $k_0 \in \mathbb{Z}$ and $j_0 \in \{-1, 0, 1\}$. Furthermore

$$\begin{aligned}|e^{-a\lambda_j} u_j| &= |u_j| e^{-2a} e^{(2-\lambda_j)a}\\ &\leq M e^{-2a} e^{r_\delta(2-\lambda_j)(k_0+1)}\\ &\leq D_j \delta^2 e^{-r_\delta(2-\lambda_j)j_0} e^{r_\delta(2-\lambda_j)(k_0+j_0)},\end{aligned}$$

since for δ small enough $M e^{-2a} e^{r_\delta(2-\lambda_j)} \leq M\delta^6 e^{r_\delta(2-\lambda_j)} < D_j\delta^2$ for every j. Hence

$$\begin{aligned}x_j + e^{-a\lambda_j} u_j &< (k_j + 1) D_j e^{r_\delta(2-\lambda_j)(-j_0)} \delta^2 e^{r_\delta(2-\lambda_j)(k_0+j_0)} + e^{-a\lambda_j} u_j\\ &< (k_j + 2) D_j e^{r_\delta(2-\lambda_j)(-j_0)} \delta^2 e^{r_\delta(2-\lambda_j)(k_0+j_0)},\end{aligned}$$

and also

$$\begin{aligned}x_j + e^{-a\lambda_j} u_j &\geq k_j D_j e^{r_\delta(2-\lambda_j)(-j_0)} \delta^2 e^{r_\delta(2-\lambda_j)(k_0+j_0)} - e^{-a\lambda_j}|u_j|\\ &\geq (k_j - 1) D_j e^{r_\delta(2-\lambda_j)(-j_0)} \delta^2 e^{r_\delta(2-\lambda_j)(k_0+j_0)}.\end{aligned}$$

Therefore ζ is contained in the set $R_{\delta,D,\underline{k},2}$. □

Remark 4.4.

(1) The family of sets $\{S_{\delta,1}, S_{\delta,D,\underline{k},2}; \delta > 0, \underline{k} \in \mathbb{Z}^{n+1}\}$ forms a partition of $\mathbb{R}^{n+1}$.
(2) Denote by $M_{\delta,1}$ the multiplication operator in $L^2(\mathbb{R}^{n+1}) \simeq L^2(G_{n,\mu}/P_n, \chi_\varepsilon)$ with the characteristic function of the set $S_{\delta,1}$. Similarly let $M_{\delta,D,\underline{k},2}$ be the multiplication operator on $L^2(G_{n,\mu}/P_n, \chi_\varepsilon)$ with the characteristic function of the set $S_{\delta,D,\underline{k},2}$. These

multiplication operators are pairwise disjoint orthogonal projections and the sum of them is the identity operator.

Let $N_{\delta,D,\underline{k},2}$ be the multiplication operator with the characteristic function of the set $R_{\delta,D,\underline{k},2}$ for $\delta > 0$ and $\underline{k} \in \mathbb{Z}^{n+1}$. We have the following property of the operator $N_{\delta,D,\underline{k},2}$.

Proposition 4.5. *There exists a constant $C > 0$ such that for any bounded linear operator L on the Hilbert space $L^2(G_{n,\mu}/P_n, \chi_\varepsilon)$, we have that*

$$\Big\| \sum_{\underline{k}\in\mathbb{Z}^{n+1}} N_{\delta,D,\underline{k},2} \circ L \circ M_{\delta,D,\underline{k},2} \Big\|_{\mathrm{op}} \leq C \sup_{\underline{k}} \| N_{\delta,D,\underline{k},2} \circ L \circ M_{\delta,D,\underline{k},2} \|_{\mathrm{op}}.$$

Proof. See Propositions 6.2 and 6.18 in [**ILL**]. □

Definition 4.6. For $\underline{k} \in \mathbb{Z}^{n+1}$ and $\delta > 0$, let

$$\ell_{\underline{k},\delta} = -\varepsilon \sum_{j=1}^{n} D_j \delta^2 e^{r_\delta (2-\lambda_j) k_0} k_j Y_j^* \in \mathfrak{h}_n^*.$$

Let $\sigma_{\underline{k},\delta} := \operatorname{ind}_{P_n}^{G_{n,\mu}} \chi_{\ell_{\underline{k},\delta}}$. The Hilbert space of this representation is the space

$$\mathcal{H}_{\underline{k},\delta} = L^2(G_{n,\mu}/P_n, \chi_{\ell_{\underline{k},\delta}})$$

and for $F \in L^1(G_{n,\mu}), \xi \in \mathcal{H}_{\underline{k},\delta}$ we have that

$$\sigma_{\underline{k},\delta}(F)\xi(a',x') = \int_S \widehat{F}^{\mathfrak{p}_n}(s's^{-1}; \mathrm{Ad}^*(s)\ell_{\underline{k},\delta})\xi(s)\Delta_S(s^{-1})ds.$$

Hence this operator has a kernel function given by

$$F_{\underline{k},\delta}((a',x'),(a,x)) = \widehat{F}^{\mathfrak{p}_n}(a'-a, a\cdot(x'-x); ((-a)\cdot\ell_{\underline{k},\delta}, 0))e^{|\lambda|a}.$$

Moreover, the representation $\sigma_{\underline{k},\delta}$ is equivalent to the representation

$$\overline{\sigma}_{n,\ell_{\underline{k},\delta}} := \int_{\mathfrak{p}_n^\perp \subset V_n^*}^{\oplus} \pi_{f+\ell_{\underline{k},\delta}} df,$$

and an equivalence is given by

$$U_{n,\ell_{\underline{k},\delta}} : L^2(\mathbb{R}\times\mathcal{X}) \equiv L^2(G_{n,\mu}/P_n, \chi_{\ell_{\underline{k},\delta}}) \to \int_{\mathfrak{p}_n^\perp}^{\oplus} L^2(G_{n,\mu}/H_n, \chi_{f+\ell_{\underline{k},\delta}})df,$$

$$U_{n,\ell_{\underline{k},\delta}}(\xi)(f)(g) := \int_{H_n/P_n} \chi_{f+\ell_{\underline{k},\delta}}(h_n)\xi(gh_n)d\dot{h}_n \ \text{ for } \ g\in G, f\in\mathfrak{p}_n^\perp. \tag{4.2.1}$$

Let $C_{\mathcal{S}\cup\mathcal{D}}$ be the C*-algebra of all uniformly bounded continuous mappings from $\mathcal{S}\cup\mathcal{D}$ into $\mathcal{B}(L^2(\mathbb{R}))$. It follows from Theorem 4.1 that for every $a \in C^*(G_{n,\mu})$ we have that $\widehat{a}_{|\mathcal{S}\cup\mathcal{D}}$ is contained in $C_{\mathcal{S}\cup\mathcal{D}}$.

For each $f = (f_0, f_+, f_-) \in V_n^*$, we denote by f_1 the unique element in its coadjoint orbit Ω_f contained in $\mathcal{S}\cup\mathcal{D}$. Let $U_{n,\underline{k},\delta}(f) : L^2(G_{n,\mu}/H_n, \chi_{f+\ell_{\underline{k},\delta}}) \to L^2(G_{n,\mu}/H_n, \chi_{(f+\ell_{\underline{k},\delta})_1})$ be the canonical intertwining operator of $\pi_{f+\ell_{\underline{k},\delta}}$ and $\pi_{(f+\ell_{\underline{k},\delta})_1}$. Formula (4.2.1) allows us to define a representation of the algebra $C_{\mathcal{S}\cup\mathcal{D}}$ on the space $L^2(G_{n,\mu}/P_n)$ by

$$\tau_{n,\ell_{\underline{k},\delta}}(\phi) := U_{n,\ell_{\underline{k},\delta}}^{-1} \circ \int_{\mathfrak{p}_n^\perp} U_{n,\underline{k},\delta}(f)^* \circ \phi((f+\ell_{\underline{k},\delta})_1) \circ U_{n,\underline{k},\delta}(f) df \circ U_{n,\ell_{\underline{k},\delta}}.$$

We have that

$$\overline{\sigma}_{n,\ell_{\underline{k},\delta}}(a) = \tau_{n,\ell_{\underline{k},\delta}}(\widehat{a}_{|\mathcal{S}}) \text{ for all } a \in C^*(G_{n,\mu}). \tag{4.2.2}$$

Definition 4.7. For $\delta > 0$, $\underline{k} \in \mathbb{Z}^{n+1}$ and $a \in C^*(G_{n,\mu})$, let

$$\sigma_{n,\underline{k},\delta}(a) := \overline{\sigma}_{n,\ell_{\underline{k},\delta}}(a) \circ M_{\delta,D,\underline{k},2},$$
$$\sigma_{n,\delta}(a) := \sum_{\underline{k}\in\mathbb{Z}^{n+1}} N_{\delta,D,\underline{k},2} \circ \sigma_{n,\underline{k},\delta}(a).$$

Proposition 4.8. *Let $a \in C^*(G_{n,\mu})$ and $\varepsilon \in \{+,-\}$. Then*

$$\lim_{\delta\to 0} \operatorname{dis}((\pi_\varepsilon(a) - \sigma_{n,\delta}(a)), \mathcal{K}(L^2(\mathbb{R}\times\mathcal{X}))) = 0.$$

Proof. Let L_c^1 be the space of all $F \in L^1(G_{n,\mu})$ for which the partial Fourier transform $\widehat{F}^{\mathfrak{p}_n}((a,x),(v^*,s))$ is a C^∞-function with compact support on $S_n \times \mathfrak{p}_n^*$. Take $F \in L_c^1$ and choose $C > 0$ such that $\widehat{F}^{\mathfrak{p}_n}((a,x),(v^*,s)) = 0$, whenever $|a| + \|x\| > C$ or $\|v^*\| + |s| > C$. By Proposition 4.3, for $\delta > 0$ small enough, we have that

$$\pi_\varepsilon(F) \circ M_{\delta,D,\underline{k},2} = N_{\delta,D,\underline{k},2} \circ \pi_\varepsilon(F) \circ M_{\delta,D,\underline{k},2}$$

for every $\underline{k}$ and hence

$$\begin{aligned}\pi_\varepsilon(F) \circ (\mathbb{I} - M_{\delta,1}) - \sigma_{n,\delta}(F) &= \pi_\varepsilon(F) \circ \left(\sum_{\underline{k}} M_{\delta,\underline{k},2}\right) - \sigma_{n,\delta}(F) \\ &= \sum_{\underline{k}\in\mathbb{Z}^{n+1}} N_{\delta,D,\underline{k},2} \circ \left(\pi_\varepsilon(F) - \overline{\sigma}_{n,\ell_{\underline{k},\delta}}(F)\right) \circ M_{\delta,D,\underline{k},2},\end{aligned}$$

and the kernel function $F_{\delta,\underline{k}}$ of the operator $a_{F,\delta,\underline{k}} := N_{\delta,D,\underline{k},2} \circ (\pi_\varepsilon(F) - \overline{\sigma}_{n,\ell_{\underline{k},\delta}}(F)) \circ M_{\delta,D,\underline{k},2}$ is therefore given by

$$F_{\delta,\underline{k}}((a',x'),(a,x)) = \Bigg(\widehat{F}^{\mathfrak{p}_n}\Bigg(a'-a, a\cdot(x'-x); \Bigg(-\varepsilon\Bigg(\sum_{j=1}^{n} e^{(\lambda_j-2)a}x_j Y_j^*\Bigg), \varepsilon e^{-2a}\Bigg)\Bigg)$$

$$- \widehat{F}^{\mathfrak{p}_n}(a'-a, a\cdot(x'-x); (-\varepsilon(-a)\cdot\ell_{\underline{k},\delta}, 0))\Bigg)$$

$$\times e^{|\lambda|a} 1_{S_{\delta,D,\underline{k},2}}(a,x) 1_{R_{\delta,D,\underline{k},2}}(a',x') \quad \text{for} \quad a,a'\in\mathbb{R}, x,x'\in V_n.$$

We see that

$$e^{(\lambda_j-2)a}x_j - e^{-\lambda_j' a}D_j\delta^2 e^{r_\delta(2-\lambda_j)k_0}k_j = e^{-\lambda_j' a}(x_j - D_j\delta^2 e^{r_\delta(2-\lambda_j)k_0}k_j).$$

Hence,

$$\begin{aligned}
&|e^{(\lambda_j-2)a}x_j - e^{-\lambda_j' a}D_j\delta^2 e^{r_\delta(2-\lambda_j)k_0}k_j| \\
&\quad \le e^{-\lambda_j' a}D_j\delta^2 e^{r_\delta(2-\lambda_j)k_0} \\
&\quad = D_j\delta^2 e^{(2-\lambda_j)(r_\delta k_0 - a)} \\
&\quad \le e^{r_\delta(2-\lambda_j)}D_j\delta^2 \\
&\quad \le e^{r_\delta m}D_j\delta^2 \\
&\quad \le \delta.
\end{aligned} \tag{4.2.3}$$

Since $F\in L_c^1$, there exists a continuous function $\varphi: S_n \to \mathbb{R}_+$ with compact support such that

$$|\widehat{F}^{\mathfrak{p}_n}(s;\ell) - \widehat{F}^{\mathfrak{p}_n}(s;\ell')| \le \varphi(s)\|\ell-\ell'\| \quad \text{for} \ \ \ell,\ell'\in\mathfrak{p}_n^*, s\in S_n.$$

Whence for any $(a,x),(a',x')\in S_n$ and any $\delta>0$ small enough,

$$\begin{aligned}
&|F_{\delta,\underline{k}}((a',x'),(a,x))| \\
&= \Bigg|\widehat{F}^{\mathfrak{p}_n}\Bigg(a'-a, a\cdot(x'-x); \Bigg(-\varepsilon\Bigg(\sum_{j=1}^{n} e^{(\lambda_j-2)a}x_j Y_j^*\Bigg), \varepsilon e^{-2a}\Bigg)\Bigg) \\
&\quad - \widehat{F}^{\mathfrak{p}_n}(a'-a, a\cdot(x'-x); (-\varepsilon(-a)\cdot\ell_{\underline{k},\delta}, 0))\Bigg| e^{|\lambda|a} 1_{S_{\delta,D,\underline{k},2}}(a,x) 1_{R_{\delta,D,\underline{k},2}}(a',x') \\
&\le \varphi(a'-a, a\cdot(x'-x))\Bigg\|\Bigg(-\varepsilon\Bigg(\sum_{j=1}^{n} e^{(\lambda_j-2)a}x_j Y_j^*\Bigg), \varepsilon e^{-2a}\Bigg) + (\varepsilon(-a)\cdot\ell_{\underline{k},\delta}, 0)\Bigg\| \\
&\quad \times e^{|\lambda|a} 1_{S_{\delta,D,\underline{k},2}}(a,x) 1_{R_{\delta,D,\underline{k},2}}(a',x')
\end{aligned}$$

$$\leq \varphi(a'-a, a\cdot(x'-x))\Big\|\Big(\sum_{j=1}^{n}(e^{(\lambda_j-2)a}x_j - e^{-\lambda'_j a}D_j\delta^2 e^{r_\delta(2-\lambda_j)k_0}k_j)Y_j^*, \varepsilon e^{-2a}\Big)\Big\|$$
$$\times\, e^{|\lambda|a}1_{S_{\delta,D,\underline{k},2}}(a,x)1_{R_{\delta,D,\underline{k},2}}(a',x')$$
$$\leq C\delta\varphi(a'-a, a\cdot(x'-x))e^{|\lambda|a}$$

for some constant $C>0$ independent of δ by (4.2.3). Therefore by Young's inequality we have that

$$\|a_{F,\delta,\underline{k}}\|_{\mathrm{op}} \leq C\delta \quad \text{for } \underline{k}\in\mathbb{Z}^{n+1},$$

and finally

$$\|\pi_\varepsilon(F)\circ(\mathbb{I}-M_{\delta,1}) - \sigma_{n,\delta}(F)\|_{\mathrm{op}} \leq C'\delta$$

for a new constant C', by Proposition 4.5.

On the other hand, the operator $\pi_\varepsilon(F)\circ M_{\delta,1}$ is compact since

$$\|\pi_\varepsilon(F)\circ M_{\delta,1}\|^2_{H-S}$$
$$= \int_{\mathbb{R}}\int_{\{e^{-a}>\delta^3\}}\int_{(\mathcal{X}_n\times\mathcal{X}_n)}\Big|\widehat{F}^{\mathfrak{p}_n}\Big(a'-a, a\cdot(x'-x); \Big(-\varepsilon\Big(\sum_{j=1}^{n}e^{(\lambda_j-2)a}x_jY_j^*\Big), \varepsilon e^{-2a}\Big)\Big)\Big|^2$$
$$\times\, e^{2|\lambda|a}\,dada'dxdx'$$
$$= \int_{\mathbb{R}}\int_{\{e^{-a}>\delta^3\}}\int_{(\mathcal{X}_n\times\mathcal{X}_n)}\Big|\widehat{F}^{\mathfrak{p}_n}\Big(a', x'; \Big(-\varepsilon\Big(\sum_{j=1}^{n}x_jY_j^*\Big), \varepsilon e^{-2a}\Big)\Big)\Big|^2 e^{2na}\,dada'dxdx'$$
$$<\infty.$$

Therefore,

$$\begin{aligned}
&\operatorname{dis}((\pi_\varepsilon(F)-\sigma_{n,\delta}(F)), \mathcal{K}(L^2(\mathbb{R}\times\mathcal{X})))\\
&\quad\leq \|\pi_\varepsilon(F)\circ(\mathbb{I}-M_{\delta,1}) - \sigma_{n,\delta}(F)\|_{\mathrm{op}}\\
&\quad\to 0 \quad \text{as } \delta\to 0.
\end{aligned}$$

The Proposition follows, since L^1_c is dense in $C^*(G_{n,\mu})$. □

4.3. The two-dimensional orbits $\mathbf{\Omega}_{v^*}$ and the characters.

The C*-algebras of the groups $G_{V_n} = G_{n,\mu}/\mathcal{Z}$ have been determined as algebras of operator fields in [**LinLud**]. We adapt this result to our present setting of almost $C_0(\mathcal{K})$-C*-algebras.

Definition 4.9. For $a\in C^*(G_{n,\mu})$, let $\Phi(a)$ be the element of $C^*(\mathbb{R}\times V_0)$ defined by $\widehat{\Phi(a)}(\theta) := \langle\chi_\theta, a\rangle$ for all $\theta\in\mathbb{R}\times V_0^*$. The mapping $\Phi: C^*(G_{n,\mu})\to C^*(\mathbb{R}\times V_0)$ is a surjective homomorphism. Let the kernel of Φ be denoted by $I_{\mathfrak{X}}$, then $C^*(G_{n,\mu})/I_{\mathfrak{X}} \simeq C^*(\mathbb{R}\times V_0)$. For $\eta \in C_c(G_{n,\mu})$, the element $\Phi(\eta) \in C^*(\mathbb{R}\times V_0)$ is the continuous function with compact support given by

$$\Phi(\eta)(t, v_0) = \int_{V_1\times\mathbb{R}} \eta(t, v_0, v, s)dvds \quad \text{for} \quad t \in \mathbb{R},\ v_0 \in V_0.$$

Choose $\zeta \in C_c(V_1 \times \mathbb{R})$ with $\zeta \geq 0$ and $\int_{V_1\times\mathbb{R}} \zeta(v, s)dvds = 1$, define the mapping $\beta : C_c(\mathbb{R} \times V_0) \to C_c(G_{n,\mu}) \subset C^*(G_{n,\mu})$ by

$$\beta(\varphi)(a, v_0, v, s) = \varphi(a, v_0)\zeta(v, s) \quad \text{for} \quad \varphi \in C_c(\mathbb{R} \times V_0), s \in \mathbb{R} \text{ and } v \in V_1.$$

It has been shown in [**LinLud**] that β can be extended to a linear mapping bounded by 1 from $C^*(\mathbb{R} \times V_0)$ into $C^*(G_{n,\mu})$, such that for every $\varphi \in C^*(\mathbb{R} \times V_0)$ we have $\Phi(\beta(\varphi)) = \varphi$.

Definition 4.10. Let $(\Omega_{f_k})_k$ ($f_k = (f_{k_+}, f_{k_-}) \in \mathcal{D}$ for all k) be a properly converging sequence in $\widehat{G_{n,\mu}}$, whose limit set contains the orbits $\Omega_{(f_+,0)}$ and $\Omega_{(0,f_-)}$. Let $r_k, q_k \in \mathbb{R}$ be such that $|r_k \cdot f_{k_+}| = 1$ and $|q_k \cdot f_{k_-}| = 1$ for $k \in \mathbb{N}$. Then $\lim_k r_k = -\infty$ and $\lim_k q_k = +\infty$. Choose two positive sequences $(\rho_k)_k, (\kappa_k)_k$ such that $\kappa_k > q_k, -r_k < \rho_k$ for all $k \in \mathbb{N}$, $\lim_{k\to\infty} \kappa_k - q_k = \infty, \lim_{k\to\infty} \rho_k + r_k = \infty$ and $\lim_{k\to\infty}((\kappa_k - q_k)/r_k) = 0$, $\lim_{k\to\infty}((\rho_k + r_k)/q_k) = 0$. We say that the sequences $(\rho_k, \kappa_k)_k$ are *adapted* to the sequence $(f_k)_k$.

For $r \in \mathbb{R}$, let $U(r)$ be the unitary operator on $L^2(\mathbb{R})$ defined by

$$U(r)\xi(s) := \xi(s + r) \quad \text{for all } \xi \in L^2(\mathbb{R}) \text{ and } s \in \mathbb{R}.$$

Definition 4.11. Let $A = (A(f) \in \mathcal{B}, f \in \Gamma)$ be a field of bounded operators. We say that A satisfies the *generic condition* if for every properly converging sequence $(\pi_{f_k})_k \subset \widehat{G_{n,\mu}}$ with $f_k \in \mathcal{D}$ for every $k \in \mathbb{N}$, which admits limit points $\pi_{(f_0,0,f_-)}, \pi_{(f_0,f_+,0)}$ and for every pair of sequences $(\rho_k, \kappa_k)_k$ adapted to the sequence $(f_k)_k$ we have that

$$\lim_{k\to\infty} \|U(r_k) \circ A(f_k) \circ U(-r_k) \circ M_{(\rho_k,+\infty)} - A(f_0, f_+, 0) \circ M_{(\rho_k,+\infty)}\|_{\mathrm{op}} = 0, \tag{1}$$

$$\lim_{k\to\infty} \|U(q_k) \circ A(f_k) \circ U(-q_k) \circ M_{(-\infty,\kappa_k)} - A(f_0, 0, f_-) \circ M_{(-\infty,\kappa_k)}\|_{\mathrm{op}} = 0. \tag{2}$$

The following proposition had been proved in [**LinLud**, Proposition 5.2].

Proposition 4.12. *For every $a \in C^*(G_{n,\mu})$, the operator field $\mathcal{F}(a)$ satisfies the generic condition.*

We must show that on $\mathcal{D}$, our C*-algebra satisfies the almost $C_0(\mathcal{K})$ conditions given in Definition 2.2. For $a \in C^*(G_{n,\mu})$ and $f = (f_0, f_+, f_-) \in V^*_{gen}$, we define the operator

$$\sigma_f(a) := U(-r(f)) \circ \pi_{(f_0,f_+,0)}(a) \circ U(r(f)) \circ M_{]-\infty,\kappa(f)+r(f)]} \\ + U(-q(f)) \circ \pi_{(f_0,0,f_-)}(a) \circ U(q(f)) \circ M_{[q(f)-\rho(f),+\infty[},$$

where

$$r(f) = -\ln(|f_+|), \qquad q(f) = \ln(|f_-|),$$
$$\rho(f) = q(f)^{1/3} - r(f), \qquad \kappa(f) = q(f) - r(f)^{1/3}.$$

We have the following proposition.

PROPOSITION 4.13. *For all $f \in \mathcal{D}$, the operator field*

$$f \mapsto \sigma_{\mathcal{D}}(f)(a) := \pi_f(a) - \sigma_f(a) \quad (a \in C^*(G_{n,\mu}))$$

is contained in $C_0(\mathcal{D}, \mathcal{K}(L^2(\mathbb{R})))$.

PROOF. Let $a \in C^*(G_{n,\mu})$. We know that $\pi_f(a)$ is a compact operator for any $f \in V^*_{gen}$, that the mapping $f \mapsto \pi_f(a)$ is norm continuous and that $\lim_{f\to\infty} \pi_f(a) = 0$ by Corollary 3.2 and Proposition 4.2 in [**LinLud**]. If $F \in L^1_c$, then the kernel function F_{f_0,f_+} of the operator $\pi_{(f_0,f_+,0)} \circ M_{[\rho(f),\infty[}$ is given by

$$F_{f_0,f_+}(s,t) = \widehat{F}^{\mathfrak{h}_n}(s-t, t\cdot f_+)1_{[\rho(f),\infty[}(t).$$

The function F_{f_0,f_+} is of compact support and ρ is continuous. Hence the mapping $f \mapsto \pi_{(f_0,f_+,0)} \circ M_{[\rho(f),\infty[}$ is norm continuous on $\mathcal{D}$ and for every $f \in \mathcal{D}$, the operator $\pi_{(f_0,f_+,0)} \circ M_{[\rho(f),\infty[}$ is compact. Since

$$\rho(f) = \ln(|f_-|)^{1/3} + \ln(|f_+|) \\ = \ln(|f_+|)^{1/3} + \ln(|f_+|)$$

goes to infinity as $\|f\|$ goes to infinity, it follows that $\pi_{(f_0,f_+,0)} \circ M_{[\rho(f),\infty[} = 0$ if $\|f\|$ is big enough. Similar properties hold for the mapping $f \mapsto \pi_{(f_0,0,f_-)} \circ M_{]-\infty,\kappa(f)]}$ on $\mathcal{D}$.

Since the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ is the set $\mathcal{S} \cup \mathbb{R}$, the generic condition tells us that $\lim_{f\to\partial\mathcal{D}} \|\sigma_{\mathcal{D}}(f)(a)\| = 0$. Hence the mapping $f \mapsto \sigma_{\mathcal{D}}(f)(F)$ is contained in $C_0(\mathcal{D}, \mathcal{K}(L^2(\mathbb{R})))$. The proposition follows from the density of L^1_c in $C^*(G_{n,\mu})$. □

4.4. The C*-algebras of the groups $G_{n,\mu}$.

Let $\Gamma_i \subseteq \mathfrak{g}^*_{n,\mu}/G_{n,\mu}$ be given as in Section 3 and $\Gamma = \bigcup \Gamma_i$.

DEFINITION 4.14. (1) For $f \in \mathcal{D}$ and $\phi \in l^\infty(\Gamma)$, let

$$\sigma_f(\phi) := U(-r(f)) \circ \phi(f_0,f_+,0) \circ U(r(f)) \circ M_{]-\infty,\kappa(f)+r(f)]} \\ + U(-q(f)) \circ \phi(f_0,0,f_-) \circ U(q(f)) \circ M_{[q(f)-\rho(f),+\infty[}.$$

(2) Let $\varphi = (\varphi(f) \in \mathcal{B}, f \in \Gamma)$ be a field of bounded operators such that the restriction of the field φ to the set of characters Γ_0 is contained in $C_0(\Gamma_0)$. We get the element $\varphi(0) \in C^*(\mathbb{R} \times V_0)$ determined as in Definition 4.9 by the condition $\gamma(\varphi(0)) = \varphi(\gamma)$ for $\gamma \in \Gamma_0$. We can then define as in Definition 4.9 that

$$\sigma_f(\varphi) := \beta(\varphi(0)) \in \mathcal{B}(L^2(\mathbb{R})) \text{ for } f \in \mathcal{S}.$$

Definition 4.15. Let $D^*(G_{n,\mu})$ be the subset of $l^\infty(\widehat{G_{n,\mu}})$ defined as a set of all the operator fields ϕ defined over $\widehat{G_{n,\mu}}$ such that the mappings $\gamma \mapsto \phi(\gamma)$ are norm continuous and vanish at infinity on the sets Γ_0 and Γ_2 and such that $\phi(f) \in \mathcal{K}(L^2(\mathbb{R}))$ for all $f \in \mathcal{D}$. Moreover, each ϕ must fulfills the following conditions:

(1) For $\varepsilon \in \{+,-\}$,

$$\lim_{\delta\to 0} \operatorname{dis}((\phi(\varepsilon) - \sigma_{n,\delta}(\phi)), \mathcal{K}(L^2(\mathbb{R}\times\mathcal{X}))) = 0, \quad \text{and}$$
$$\lim_{\delta\to 0} \operatorname{dis}((\phi^*(\varepsilon) - \sigma_{n,\delta}(\phi^*)), \mathcal{K}(L^2(\mathbb{R}\times\mathcal{X}))) = 0.$$

(2) The mappings

$$\mathcal{D} \ni f \mapsto (\phi(f) - \sigma_f(\phi)) \quad \text{and} \quad \mathcal{D} \ni f \mapsto (\phi(f)^* - \sigma_f(\phi^*))$$

are contained in $C_0(\mathcal{D}, \mathcal{K}(L^2(\mathbb{R})))$.

(3) The mappings

$$\mathcal{S} \ni f \mapsto (\phi(f) - \sigma_f(\phi)) \quad \text{and} \quad \mathcal{S} \ni f \mapsto (\phi(f)^* - \sigma_f(\phi^*))$$

are contained in $C_0(\mathcal{S}, \mathcal{K}(L^2(\mathbb{R})))$.

Theorem 4.16. *The C^*-algebra of $G_{n,\mu}$ is an almost $C_0(\mathcal{K})$-C^*-algebra. In particular, the Fourier transform maps $C^*(G_{n,\mu})$ onto the subalgebra $D^*(G_{n,\mu})$ of $l^\infty(\Gamma)$.*

Proof. Propositions 4.8 and 4.13 show that the Fourier transform maps $C^*(G_{n,\mu})$ into $D^*(G_{n,\mu})$. The conditions on $D^*(G_{n,\mu})$ imply that $D^*(G_{n,\mu})$ is a closed involutive subspace of $l^\infty(\Gamma)$. It follows from [**ILL**] that $D^*(G_{n,\mu})$ is a C*-subalgebra of $l^\infty(\Gamma)$ and that $\mathcal{F}_{n,\mu}(C^*(G_{n,\mu})) = D^*(G_{n,\mu})$. □

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