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# Global existence of generalized rotational hypersurfaces with prescribed mean curvature in Euclidean spaces, I

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**Abstract.** We prove that for a given continuous function $H(s)$, $(-\infty < s < \infty)$, there exists a globally defined generating curve of a rotational hypersurface in a Euclidean space such that the mean curvature is $H(s)$. We also prove a similar theorem for generalized rotational hypersurfaces of $O(l+1) \times O(m+1)$-type. The key lemmas in this paper show the existence of solutions for singular initial value problems of ordinary differential equations satisfied using generating curves of those hypersurfaces.

## 1. Introduction.

Surfaces of revolution with constant mean curvature in the Euclidean three-space $\mathbb{R}^3$ can be uniquely extended infinitely using Delaunay's rolling construction method [**1**]. Delaunay's method was generalized to higher dimensions by Hsiang and Yu [**7**]. Other proofs of the Delaunay–Hsiang–Yu theorem are given in Kenmotsu [**8**] for $n = 3$ and by Dorfmeister and Kenmotsu [**2**] for $n \geq 3$. The extension property in the Delaunay–Hsiang–Yu theorem holds for the case of non-constant mean curvature. Indeed, Kenmotsu [**9**] showed that for a given continuous function $H(s)$ on $\mathbb{R}$, there exists a global surface of revolution such that the mean curvature is $H(s)$. In 2009, Dorfmeister and Kenmotsu [**3**] extended this result to higher dimensions under the condition that the mean curvature function is real-analytic. In this paper, we show that the result of [**3**] holds for the case of the given function $H(s)$ being continuous, which is Theorem 2.1. Then, in Section 3, we apply the theorem to generalize some results in [**4**] to the case of non-constant mean curvature.

In 1982, Hsiang extended the notion of rotational hypersurfaces and studied interesting properties of such generalized rotational hypersurfaces with constant mean curvature in Hsiang [**4**] and Hsiang and Huynh [**5**]. In this paper, we prove the extension theorem such as Theorem 2.1 for some classes of generalized rotational hypersurfaces. Let $M$ be a generalized rotational hypersurface in the $n$-dimensional Euclidean space $\mathbb{R}^n$. $M$ is invariant under an isometric transformation group $(G, \mathbb{R}^n)$ with codimension two principal orbit type, where $G$ denotes a compact Lie group acting on $\mathbb{R}^n$. Such transformation groups $(G, \mathbb{R}^n)$ have already been classified in [**6**]; for $n = 3$, we have only $G = O(2)$, and for $n \geq 4$, there are 14 Lie groups. When $G = O(n-1)$, the generalized rotational hypersurface of $O(n-1)$-type is the same as the usual rotational one.

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$M$ is generated using a plane curve $\gamma$ on the orbit space $\mathbb{R}^n/G$, called the generating curve of $M$. The generating curve of $M$ is the solution of an ordinary differential equation on the orbit space $\mathbb{R}^n/G$ and it is determined using the mean curvature of $M$. Conversely, for a given real-valued function $H(s)$ defined on $-\infty < s < \infty$, we obtain the generating curve $\gamma(s)$, $(s \in I)$, by applying the usual existence theorem of solutions of the first order ordinary differential equations, where $I$ is the maximal interval of existence of the solution curve. If $G$ is $O(n-1)$ and the given function $H(s)$ is continuous on $\mathbb{R}$, then Theorem 2.1 in this paper proves $I = \mathbb{R}$.

Next, we prove the same type of extension theorem for generalized rotational hypersurfaces of $O(l+1) \times O(m+1)$-type, $(l,\ m \geq 1,\ (l+1)+(m+1)=n)$, represented as Theorem 4.1.

In the proof of these results, a problem occurs at the point where the generating curve touches the boundary of the orbit space. At this point, we have to solve a singular initial value problem arising from the differential equation of the generating curve. The existence of solutions of singular initial value problems is proved by Banach's fixed-point theorem on an appropriate class of functions, and thus, the main objective of this paper is to estimate the terms that appear in differential equations of generating curves.

The analysis and geometry of generalized rotational hypersurfaces with prescribed mean curvature function of other types will be studied in subsequent papers.

## 2. Global existence of generalized rotational hypersurfaces of $O(n-1)$-type.

In this section, we prove the global existence of generalized rotational hypersurfaces of $O(n-1)$-type with prescribed mean curvature function. Let $(x(s), y(s))$, $y(s) > 0$ $(s \in \mathbb{R})$ be a plane curve parametrized by arc length in the Euclidean two-plane $\mathbb{R}^2$ defined by $x_3 = \cdots = x_n = 0$, where $x_i$, $i = 1, 2, \ldots, n$, are the standard coordinates of $\mathbb{R}^n$. A generalized rotational hypersurface $M$ of $O(n-1)$-type is defined by

$$M = \{(x(s), y(s)S^{n-2}) \in \mathbb{R}^n \mid s \in \mathbb{R}\}, \tag{1}$$

where $S^{n-2}$ is an $(n-2)$-dimensional unit sphere with center origin in the $(n-1)$-dimensional Euclidean space defined by $x_1 = 0$. The plane curve $(x(s), y(s))$ is called the generating curve of $M$.

The mean curvature of $M$ is a function of a variable $s$, denoted by $H = H(s)$, and it satisfies

$$(n-1)H(s) = (n-2)\frac{x'(s)}{y(s)} + x''(s)y'(s) - x'(s)y''(s), \quad \text{for} \quad s \in \mathbb{R}. \tag{2}$$

Component functions of the generating curve satisfy $y(s) > 0$ and

$$x'(s)^2 + y'(s)^2 = 1, \quad \text{for} \quad s \in \mathbb{R}, \tag{3}$$

because the parameter $s$ denotes arc length.

Conversely, given a continuous function $H(s)$ defined on the whole line $\mathbb{R}$, (2) and (3) are a system of ordinary differential equations for $x(s)$ and $y(s)$.

Let us fix an $s_0 \in \mathbb{R}$. Given any $c > 0$ and any real numbers $c'$, $d'$ satisfying ${c'}^2 + {d'}^2 = 1$, the usual existence theorem of ordinary differential equations implies that there exists a local solution curve $(x(s), y(s))$ on $\mathbb{R}$, of the system (2) and (3) with the initial conditions $x(s_0) = 0$, $y(s_0) = c$, $x'(s_0) = c'$, and $y'(s_0) = d'$.

When we extend the domain of definition of these component functions to $\mathbb{R}$, a problem occurs at the point where $y(s)$ passes through the $x$-axis at some finite $s$. From Dorfmeister and Kenmotsu [**3**, Proposition 3.2, p. 706], we have

Proposition 2.1. *Suppose that* $\lim_{s\to b} y(s) = 0$ *for some* $b \in \mathbb{R}$. *Then, there exists limit of* $x'(s)$ *as* $s \to b$ *and* $\lim_{s\to b} x'(s) = 0$.

Remark 2.1. The proof in [**3**] requires $y'(s) \neq 0$ near $b$, which was not proved. However, we can show this in a similar manner to the proof of Lemma 4.1 of this paper.

We can show that the solution can be extended beyond $b$. Without loss of generality, we may assume that $b = 0$, because the system of (2) and (3) is invariant under the translation parallel to $x$-axis. It follows from Proposition 2.1 that the mapping from $y$ to $s$ is one-to-one near 0, and therefore, the inverse function $s = s(y)$ exists. Now, we rewrite our equation considering $y$ as the independent variable. To do this, let us put $q = x'/y'$. Then, we have

$$\frac{dq}{dy} = \frac{1}{{y'}^3}\left\{(n-1)H(s) - (n-2)\frac{x'}{y}\right\}.$$

From (3) and $x'(0) = 0$, we have $y'(0) = \pm 1$. Hence, $y'(s)$ does not vanish, and $\operatorname{sgn} y'(s) = \operatorname{sgn} y'(0) \neq 0$ on a neighborhood of $s = 0$. From $(1+q^2){y'}^2 = 1$, it follows that

$$y'(s)^3 = (\operatorname{sgn} y'(s))\{(y'(s))^2\}^{3/2} = (\operatorname{sgn} y'(0))(1+q(s)^2)^{-3/2}.$$

Setting $\tilde{H}(y) = (\operatorname{sgn} y'(0))H(s(y))$, we obtain

$$\begin{aligned} \frac{dx}{dy} &= q, \\ y\frac{dq}{dy} &= -(n-2)q - (n-2)q^3 + (n-1)\tilde{H}(y)y(1+q^2)^{3/2}, \end{aligned}$$

where $x$ and $q$ are unknown functions of $y$.

Let us consider the singular initial value problem

$$\begin{cases} y\dfrac{dq}{dy} = -(n-2)q - (n-2)q^3 + (n-1)\tilde{H}(y)y(1+q^2)^{3/2}, & \text{for } y > 0, \\ q(0) = 0. \end{cases} \tag{4}$$

Multiplying $y^{n-3}$ to the first equation of (4), we see

$$\frac{d}{dy}(y^{n-2}q) = \{-(n-2)q^3 + (n-1)\tilde{H}(y)y(1+q^2)^{3/2}\}y^{n-3}.$$

Let us integrate the above equation on $[0, y]$. Then, by $y^{n-2}q(y)|_{y=0} = 0$, we have

$$y^{n-2}q(y) = \int_0^y \{-(n-2)q(\eta)^3 + (n-1)\tilde{H}(\eta)\eta(1+q(\eta)^2)^{3/2}\}\eta^{n-3}d\eta,$$

which leads to the integral equation

$$q(y) = \Phi(q)(y), \tag{5}$$

where we set

$$\Phi(q)(y) = y^{2-n}\int_0^y \{-(n-2)q(\eta)^3 + (n-1)\tilde{H}(\eta)\eta(1+q(\eta)^2)^{3/2}\}\eta^{n-3}d\eta.$$

We find a fixed point of mapping $\Phi$ in an appropriate class of functions. To do this, let us define a function space and its subclass by

$$X_Y = \{q \in C(0, Y] \mid \|q\|_X < \infty\}, \quad X_{Y,M} = \{q \in X_Y \mid \|q\|_X \leq M\},$$

where $Y$ and $M$ are positive constants and

$$\|q\|_X = \sup_{y \in (0,Y]} \left|\frac{q(y)}{y}\right|.$$

$X_Y$ is a Banach space with the norm $\|\cdot\|_X$. When $q \in X_{Y,M}$, we note that $|q(y)| \leq \|q\|_X|y| \to 0$ as $y \to +0$. Setting $q(0) = 0$, $q$ is an element of $C[0, Y]$.

Proposition 2.2. (i) *Suppose that $\tilde{H}$ is bounded on $[0, Y]$. For a sufficiently large $M$, and small $Y$, there exists a unique solution $q$ of the integral equation* (5) *on $X_{Y,M}$.*

(ii) *Suppose that $\tilde{H}$ is bounded and continuous on $[0, Y]$. Then, the function $q$ obtained in the above* (i) *is a solution of the initial value problem* (4).

Proof. Suppose that $\tilde{H}$ is bounded on $[0, Y]$. From now on, $C$ denotes a positive constant that depends on $n$ and $\sup_y |\tilde{H}(y)|$; however, it does not depend on $M$ and $Y$.

First, we show that $\Phi$ is a mapping from $X_{Y,M}$ to $X_{Y,M}$ for a certain large $M$ and small $Y$. Take $q \in X_{Y,M}$. Then, we have $\Phi(q) \in C(0, Y]$. We show $\|\Phi(q)\|_X \leq M$ as follows: By using $|q(\eta)| \leq M\eta$ for $\eta \in (0, Y]$, we see that

$$\begin{aligned}
&\left|\{-(n-2)q(\eta)^3 + (n-1)\tilde{H}(\eta)\eta(1+q(\eta)^2)^{3/2}\}\eta^{n-3}\right| \\
&\quad \leq C\{|q(\eta)|^3 + \eta(1+|q(\eta)|^3)\}\eta^{n-3} \leq C(M^3\eta^n + \eta^{n-2} + M^3\eta^{n+1}).
\end{aligned}$$

Hence,

$$\left|\frac{\Phi(q)(y)}{y}\right| \leq Cy^{1-n}\int_0^y (M^3\eta^n + \eta^{n-2} + M^3\eta^{n+1})d\eta \leq C(M^3y^2 + 1 + M^3y^3).$$

Take $M$ and $Y$ such that

$$C(M^3Y^2 + 1 + M^3Y^3) \leq M.$$

Then, $\|\Phi(q)\|_X \leq M$.

Next, if necessary, taking a much smaller $Y$, we show that $\Phi$ is a contraction mapping from $X_{Y,M}$ to itself. Take $q_1$, $q_2 \in X_{Y,M}$. Then,

$$\Phi(q_1)(y) - \Phi(q_2)(y) = y^{2-n}\int_0^y (\phi(q_1(\eta)) - \phi(q_2(\eta)))\eta^{n-3}d\eta,$$

where

$$\phi(q) = -(n-2)q^3 + (n-1)\tilde{H}(\eta)\eta(1+q^2)^{3/2}.$$

By the mean value theorem, there is a $q_*$ between $q_1$ and $q_2$, such that

$$\phi(q_1) - \phi(q_2) = \phi'(q_*)(q_1 - q_2).$$

We note that

$$|q_*(\eta)| \leq \max\{|q_1(\eta)|, |q_2(\eta)|\} \leq M\eta,$$

and since we have $\phi'(q) = -3(n-2)q^2 + 3(n-1)\tilde{H}(\eta)\eta(1+q^2)^{1/2}q$,

$$\begin{aligned}|\phi'(q_*(\eta))| &\leq C\big\{|q_*(\eta)|^2 + \eta(1+|q_*(\eta)|^2)^{1/2}|q_*(\eta)|\big\}\\ &\leq C(M^2\eta^2 + \eta(1+M\eta)M\eta) = C(M^2\eta^2 + M\eta^2 + M^2\eta^3).\end{aligned}$$

Combining these two estimates, we obtain

$$\begin{aligned}\left|\frac{\Phi(q_1)(y) - \Phi(q_2)(y)}{y}\right| &\leq Cy^{1-n}\int_0^y (M^2\eta^2 + M\eta^2 + M^2\eta^3)|q_1(\eta) - q_2(\eta)|\eta^{n-3}d\eta\\ &\leq Cy^{1-n}\|q_1 - q_2\|_X\int_0^y (M^2\eta^2 + M\eta^2 + M^2\eta^3)\eta^{n-2}d\eta\\ &\leq C(M^2y^2 + My^2 + M^2y^3)\|q_1 - q_2\|_X.\end{aligned}$$

Consequently, by choosing $Y$ such that $C(M^2Y^2 + MY^2 + M^2Y^3) < 1$, Banach's fixed point theorem implies that there exists a unique fixed point $q$ of $\Phi$ on $X_{Y,M}$, which satisfies (5).

To prove (ii), let $q$ be a solution of (5). Then, we have

$$y^{n-2}q(y) = \int_0^y \phi(q(\eta))\eta^{n-3}d\eta.$$

Since $q \in C[0,Y]$, if $\tilde{H}$ is continuous, then $\phi(q(\eta))$ is also continuous on $[0,Y]$. Thus, the right-hand side of the above equation is differentiable, which implies that $q$ is also differentiable. Differentiating the above formula, we obtain the first equation of (4). From $q \in X_{Y,M}$, we have $q(0) = 0$. □

By replacing [**3**, Proposition 3.3] to Proposition 2.2 of this paper, the proof of [**3**, Theorem 3.4] can be used to prove the following:

Theorem 2.1. *Let $H(s)$ be a continuous function on $\mathbb{R}$ and fix an $s_0 \in \mathbb{R}$. Then, for any $c > 0$ and any real numbers $c'$, $d'$ satisfying ${c'}^2 + {d'}^2 = 1$, there exists a global solution curve $(x(s), y(s))$, for $s \in \mathbb{R}$, of the system* (2) *and* (3) *with the initial conditions $x(s_0) = 0$, $y(s_0) = c$, $x'(s_0) = c'$, and $y'(s_0) = d'$.*

Remark 2.2. The global solution curve in Theorem 2.1 is extended smoothly in the $(x,y)$-plane with $y < 0$ when it touches the $x$-axis. The global generating curve is obtained using the reflection of the solution curve with respect to the $x$-axis.

## 3. Properties of generalized rotational hypersurfaces of $O(n-1)$-type.

In this section, we apply Theorem 2.1 to study some properties of generalized rotational hypersurfaces of $O(n-1)$-type with non-constant mean curvature function $H(s)$. We note that interesting properties of these hypersurfaces with constant mean curvature were studied by Hsiang and Yu [**7**] in 1981.

Given any continuous $H(s)$ on $\mathbb{R}$ and any $c > 0$, as per Theorem 2.1 there exists uniquely a global solution curve $(x_c(s), y_c(s))$, for $s \in \mathbb{R}$, of the system (2) and (3) with the initial conditions $x_c(0) = 0$, $y_c(0) = c > 0$, $x_c'(0) = 1$, and $y_c'(0) = 0$. Let $\Gamma_c = (x_c(s), |y_c(s)|)$. The curve $\Gamma_c$ is the generating curve of a generalized rotational hypersurface, say $M_c$, of $O(n-1)$-type. $\Gamma_c$ has possibly the singularity for the induced metric at $y_c(s) = 0$, because the first fundamental form of $M_c$ is the direct product of $ds^2$ and the conformal metric of $S^{n-2}$ with the conformal factor $y(s)^2$. We prove

Theorem 3.1. *Let $H(s)$ be an absolutely continuous function on $\mathbb{R}$ with $H(0) > 0$ such that $H'(s) \geq 0$ a.e. $s \in (0,\infty)$ and $H'(s) \leq 0$ a.e. $s \in (-\infty, 0)$. Then, for any $c$ satisfying $c > 1/H(0)$, $y_c(s)$ is positive on $\mathbb{R}$ and $M_c$ is an immersed hypersurface in $\mathbb{R}^n$.*

Proof. By contraries, suppose that there exists an $s_0 \in \mathbb{R}^+$ such that $y_c(s_0) = 0$ and $y_c(s) > 0$ on $s \in [0, s_0)$. By [**3**, (3.2)] and the initial conditions of $\Gamma_c$, we have

$$y_c^{n-2}(s)x_c'(s) = (n-1)\int_0^s H(t)y_c^{n-2}(t)y_c'(t)\,dt + c^{n-2}, \text{ for } s \in [0, s_0].$$

Since the left-hand side of the above formula is zero at $s = s_0$ by the assumption and (3), we have

$$\begin{aligned} c^{n-2} &= -(n-1)\int_0^{s_0} H(t)y_c^{n-2}(t)y_c'(t)\,dt \\ &= -\big[H(t)y_c^{n-1}(t)\big]_0^{s_0} + \int_0^{s_0} H'(t)y_c^{n-1}(t)\,dt \\ &\geq H(0)c^{n-1}. \end{aligned}$$

This contradicts the assumption of $c$. If $s_0 < 0$, then we have the same contradiction, proving Theorem 3.1. $\square$

REMARK 3.1. We also have the following: Let $H(s)$ be an absolutely continuous function on $\mathbb{R}$ with $H(0) > 0$ such that $H'(s) \leq 0$ a.e. $s \in (0,\infty)$ and $H'(s) \geq 0$ a.e. $s \in (-\infty, 0)$. Then, for any $c$ satisfying $0 < c < 1/H(0)$, $y_c(s)$ is positive on $\mathbb{R}$ and $M_c$ is an immersed hypersurface in $\mathbb{R}^n$.

Theorem 3.1 and Remark 3.1 extend the results for the constant mean curvature case [**8**], [**7**].

To continue the study of hypersurfaces with non-constant mean curvature $H(s)$, we make the asymptotic analysis of $\Gamma_c$ when $c \to \infty$. To do this, let $\Gamma_\infty$ be the plane curve parametrized using the arc length such that the curvature is $-(n-1)H(s)$. Indeed, a plane curve parametrized by arc length is determined using its curvature only up to a rigid motion. $\Gamma_\infty$ is defined on $\mathbb{R}$ by the fundamental theorem of curve theory. In fact, we have

$$\Gamma_\infty = \left(\int_0^s \cos\eta(u)du, -\int_0^s \sin\eta(u)du\right), \text{ for } s \in \mathbb{R}, \tag{6}$$

where we set $\eta(u) = (n-1)\int_0^u H(t)dt$. Set $F_c(s) = y_c(s)y_c'(s)/c$ and $G_c(s) = y_c(s)x_c'(s)/c$. Then, [**3**, Lemma 4.1] implies that

$$\begin{cases} F_c' = -(n-1)H(s)G_c + \dfrac{1}{c}\left\{1 + \dfrac{(n-3)G_c^2}{F_c^2+G_c^2}\right\}, & F_c(0) = 0, \\[2ex] G_c' = (n-1)H(s)F_c - \dfrac{1}{c}\left\{\dfrac{(n-3)F_cG_c}{F_c^2+G_c^2}\right\}, & G_c(0) = 1, \end{cases}$$

for $s \in \mathbb{R}$. We note that the functions $G_c^2/(F_c^2+G_c^2)$ and $F_cG_c/(F_c^2+G_c^2)$ are globally defined and differentiable on $\mathbb{R}$. Let $(F_\infty, G_\infty)$ be the unique solution to the system

$$\begin{cases} F_\infty' = -(n-1)H(s)G_\infty, & F_\infty(0) = 0, \\ G_\infty' = (n-1)H(s)F_\infty, & G_\infty(0) = 1. \end{cases}$$

This is integrated as $F_\infty(s) = -\sin\eta(s)$, and $G_\infty(s) = \cos\eta(s)$. With these conventions, we prove

Theorem 3.2. *It holds that*

$$\lim_{c\to\infty} F_c(s) = F_\infty(s), \quad \lim_{c\to\infty} G_c(s) = G_\infty(s),$$

*compactly uniformly with respect to* $s \in \mathbb{R}$.

Proof. Let us define $\tilde{F}_c$ and $\tilde{G}_c$ by $F_c = F_\infty + \tilde{F}_c$ and $G_c = G_\infty + \tilde{G}_c$, respectively. Then, we get

$$\begin{cases} \tilde{F}_c{}' = -(n-1)H(s)\tilde{G}_c + \dfrac{1}{c}\left\{1 + \dfrac{(n-3)G_c^2}{F_c^2+G_c^2}\right\}, & \tilde{F}_c(0) = 0, \\ \tilde{G}_c{}' = (n-1)H(s)\tilde{F}_c - \dfrac{1}{c}\left\{\dfrac{(n-3)F_cG_c}{F_c^2+G_c^2}\right\}, & \tilde{G}_c(0) = 0. \end{cases}$$

This implies that

$$\begin{aligned} \frac{d}{ds}\big(\tilde{F}_c{}^2 + \tilde{G}_c{}^2\big) &= \frac{2}{c}\left[\left\{1 + \frac{(n-3)G_c^2}{F_c^2+G_c^2}\right\}\tilde{F}_c - \frac{(n-3)F_cG_c}{F_c^2+G_c^2}\tilde{G}_c\right] \\ &\le \frac{2\sqrt{2}(n-2)}{c}\sqrt{\tilde{F}_c{}^2 + \tilde{G}_c{}^2}. \end{aligned}$$

Consequently, we have

$$\tilde{F}_c{}^2 + \tilde{G}_c{}^2 \le \frac{2(n-2)^2 s^2}{c^2} \quad \text{for } s \ge 0.$$

The same estimate for $s < 0$ follows from

$$\frac{d}{ds}\big(\tilde{F}_c{}^2 + \tilde{G}_c{}^2\big) \ge -\frac{2\sqrt{2}(n-2)}{c}\sqrt{\tilde{F}_c{}^2 + \tilde{G}_c{}^2},$$

proving Theorem 3.2. □

In view of $F_c(s)^2 + G_c(s)^2 = y_c(s)^2/c^2$, $\Gamma_c$ has the following expression: for $c > 0,\ s \in \mathbb{R}$,

$$\Gamma_c = \left(\int_0^s \frac{G_c(u)}{\sqrt{F_c(u)^2 + G_c(u)^2}}\,du,\ \int_0^s \frac{F_c(u)}{\sqrt{F_c(u)^2 + G_c(u)^2}}\,du + c\right).$$

As the geometric application of Theorem 3.2 and the above formula, we have

Corollary 3.1. *In the limit $c \to \infty$, curves $\Gamma_c - (0, c)$ tend to $\Gamma_\infty$.*

Proof. It follows from $\Gamma_c' \to \Gamma_\infty' = (\cos\eta(s), -\sin\eta(s))$ as $c \to \infty$, thus proving the corollary. □

Now, we derive the asymptotic expansion formulas of $F_c$ and $G_c$, which are applied to study periodic generating curves. Set

$$U(s) = \begin{pmatrix} \cos\eta(s) & -\sin\eta(s) \\ \sin\eta(s) & \cos\eta(s) \end{pmatrix}.$$

Since

$$U'(s)U(s)^{-1} = (n-1)H(s)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we have

$$\begin{aligned}
\left\{U(s)^{-1}\begin{pmatrix} F_c(s) \\ G_c(s) \end{pmatrix}\right\}' &= -U(s)^{-1}U(s)'U^{-1}(s)\begin{pmatrix} F_c(s) \\ G_c(s) \end{pmatrix} + U(s)^{-1}\begin{pmatrix} F_c'(s) \\ G_c'(s) \end{pmatrix} \\
&= U(s)^{-1}\left\{ -(n-1)H(s)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_c(s) \\ G_c(s) \end{pmatrix} + \begin{pmatrix} F_c'(s) \\ G_c'(s) \end{pmatrix}\right\} \\
&= \frac{1}{c}U(s)^{-1}\begin{pmatrix} 1+(n-3)P_c(s) \\ (n-3)Q_c(s) \end{pmatrix},
\end{aligned}$$

where

$$P_c(s) = \frac{G_c(s)^2}{F_c(s)^2+G_c(s)^2}, \quad Q_c(s) = -\frac{F_c(s)G_c(s)}{F_c(s)^2+G_c(s)^2}.$$

Integrating this, we obtain

$$\begin{pmatrix} F_c(s) \\ G_c(s) \end{pmatrix} = \begin{pmatrix} F_\infty(s) \\ G_\infty(s) \end{pmatrix} + \frac{1}{c}U(s)\int_0^s U^{-1}(t)\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{(n-3)G_c(t)}{F_c(t)^2+G_c(t)^2}\begin{pmatrix} G_c(t) \\ -F_c(t) \end{pmatrix}\right\}dt. \quad (7)$$

Setting $\epsilon = 1/c$, we have the following asymptotic expansions of $F_c(s)$ and $G_c(s)$.

Theorem 3.3. *For a continuous $H(s)$ on $\mathbb{R}$, it holds that*

$$F_c(s) = \sum_{k=0}^{2}\epsilon^k F_\infty^{(k)}(s) + \mathcal{O}(\epsilon^3), \quad G_c(s) = \sum_{k=0}^{2}\epsilon^k G_\infty^{(k)}(s) + \mathcal{O}(\epsilon^3),$$

*where*

(i) $\begin{pmatrix} F_\infty^{(0)}(s) \\ G_\infty^{(0)}(s) \end{pmatrix} = \begin{pmatrix} F_\infty(s) \\ G_\infty(s) \end{pmatrix}$,

(ii) $\begin{pmatrix} F_\infty^{(1)}(s) \\ G_\infty^{(1)}(s) \end{pmatrix} = U(s)\int_0^s \begin{pmatrix} (n-2)\cos\eta(t) \\ -\sin\eta(t) \end{pmatrix}dt$,

(iii) $\begin{pmatrix} F_\infty^{(2)}(s) \\ G_\infty^{(2)}(s) \end{pmatrix} = (n-2)(n-3)U(s)\displaystyle\int_0^s \left( \int_0^t \cos\eta(u)du \begin{pmatrix} \sin\eta(t) \\ -\cos\eta(t) \end{pmatrix} \right) dt.$

Proof. (i) follows from Theorem 3.2. To prove (ii), we compute formally, using (7),

$$\begin{aligned}
\begin{pmatrix} F_\infty^{(1)}(s) \\ G_\infty^{(1)}(s) \end{pmatrix} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{ \begin{pmatrix} F_c(s) \\ G_c(s) \end{pmatrix} - \begin{pmatrix} F_\infty^{(0)}(s) \\ G_\infty^{(0)}(s) \end{pmatrix} \right\} \\
&= U(s)\int_0^s U^{-1}(t)\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{(n-3)G_\infty(t)}{F_\infty(t)^2+G_\infty(t)^2}\begin{pmatrix} G_\infty(t) \\ -F_\infty(t) \end{pmatrix} \right\} dt \\
&= U(s)\int_0^s \begin{pmatrix} (n-2)\cos\eta(t) \\ -\sin\eta(t) \end{pmatrix} dt.
\end{aligned}$$

It holds for this that

$$\begin{aligned}
&\begin{pmatrix} F_c(s) \\ G_c(s) \end{pmatrix} - \begin{pmatrix} F_\infty^{(0)}(s) \\ G_\infty^{(0)}(s) \end{pmatrix} - \epsilon\begin{pmatrix} F_\infty^{(1)}(s) \\ G_\infty^{(1)}(s) \end{pmatrix} \\
&= \epsilon(n-3)U(s)\int_0^s U^{-1}(t)\left\{ \frac{G_c(t)}{F_c(t)^2+G_c(t)^2}\begin{pmatrix} G_c(t) \\ -F_c(t) \end{pmatrix} \right. \\
&\qquad\qquad \left. - \frac{G_\infty(t)}{F_\infty(t)^2+G_\infty(t)^2}\begin{pmatrix} G_\infty(t) \\ -F_\infty(t) \end{pmatrix} \right\} dt.
\end{aligned}$$

For any compact interval $I$, there exists $C_I$ such that

$$\begin{aligned}
&\sup_{t\in I}\left| \frac{G_c(t)}{F_c(t)^2+G_c(t)^2}\begin{pmatrix} G_c(t) \\ -F_c(t) \end{pmatrix} - \frac{G_\infty(t)}{F_\infty(t)^2+G_\infty(t)^2}\begin{pmatrix} G_\infty(t) \\ -F_\infty(t) \end{pmatrix} \right| \\
&\leq C_I\left( \sup_{t\in I}|G_c(t)-G_\infty(t)| + \sup_{t\in I}|F_c(t)-F_\infty(t)| \right) \leq C_I\epsilon.
\end{aligned}$$

The above estimate implies that

$$\begin{pmatrix} F_c(s) \\ G_c(s) \end{pmatrix} - \begin{pmatrix} F_\infty^{(0)}(s) \\ G_\infty^{(0)}(s) \end{pmatrix} - \epsilon U(s)\int_0^s \begin{pmatrix} (n-2)\cos\eta(t) \\ -\sin\eta(t) \end{pmatrix} dt = \mathcal{O}(\epsilon^2)$$

and that the convergence

$$\frac{1}{\epsilon}\left\{ \begin{pmatrix} F_c(s) \\ G_c(s) \end{pmatrix} - \begin{pmatrix} F_\infty^{(0)}(s) \\ G_\infty^{(0)}(s) \end{pmatrix} \right\} \to U(s)\int_0^s \begin{pmatrix} (n-2)\cos\eta(t) \\ -\sin\eta(t) \end{pmatrix} dt \quad \text{as} \quad \epsilon \to 0$$

is compactly uniform, which proves (ii).

For the proof of (iii), the above consideration yields

$$\begin{aligned}\begin{pmatrix}F_\infty^{(2)}(s)\\G_\infty^{(2)}(s)\end{pmatrix} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\left\{\begin{pmatrix}F_c(s)\\G_c(s)\end{pmatrix}-\begin{pmatrix}F_\infty^{(0)}(s)\\G_\infty^{(0)}(s)\end{pmatrix}-\epsilon\begin{pmatrix}F_\infty^{(1)}(s)\\G_\infty^{(1)}(s)\end{pmatrix}\right\}\\ &= \lim_{\epsilon\to 0}\frac{n-3}{\epsilon}U(s)\int_0^s U^{-1}(t)\left\{\frac{G_c(t)}{F_c(t)^2+G_c(t)^2}\begin{pmatrix}G_c(t)\\-F_c(t)\end{pmatrix}\right.\\ &\qquad\left.-\frac{G_\infty(t)}{F_\infty(t)^2+G_\infty(t)^2}\begin{pmatrix}G_\infty(t)\\-F_\infty(t)\end{pmatrix}\right\}dt.\end{aligned}$$

For any compact interval $I$, there exists $C_I$ such that

$$\begin{aligned}&\sup_{t\in I}\frac{1}{\epsilon}\left|\frac{G_c(t)}{F_c(t)^2+G_c(t)^2}\begin{pmatrix}G_c(t)\\-F_c(t)\end{pmatrix}-\frac{G_\infty(t)}{F_\infty(t)^2+G_\infty(t)^2}\begin{pmatrix}G_\infty(t)\\-F_\infty(t)\end{pmatrix}\right|\\ &\le\frac{C_I}{\epsilon}\left(\sup_{t\in I}|G_c(t)-G_\infty(t)|+\sup_{t\in I}|F_c(t)-F_\infty(t)|\right)\le C_I.\end{aligned}$$

Therefore, we can exchange the order of the limit as $\epsilon\to 0$ and the integration by the dominated convergence theorem, and we obtain

$$\begin{aligned}\begin{pmatrix}F_\infty^{(2)}(s)\\G_\infty^{(2)}(s)\end{pmatrix} &= (n-3)U(s)\int_0^s U^{-1}(t)\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{\frac{G_c(t)}{F_c(t)^2+G_c(t)^2}\begin{pmatrix}G_c(t)\\-F_c(t)\end{pmatrix}\right.\\ &\qquad\left.-\frac{G_\infty(t)}{F_\infty(t)^2+G_\infty(t)^2}\begin{pmatrix}G_\infty(t)\\-F_\infty(t)\end{pmatrix}\right\}dt\\ &= (n-3)U(s)\int_0^s U^{-1}(t)\frac{\partial}{\partial\epsilon}\left\{\frac{G_c(t)}{F_c(t)^2+G_c(t)^2}\begin{pmatrix}G_c(t)\\-F_c(t)\end{pmatrix}\right\}\Bigg|_{\epsilon=0}dt.\end{aligned}$$

Since

$$\begin{aligned}&\frac{\partial}{\partial\epsilon}\left\{\frac{G_c(t)}{F_c(t)^2+G_c(t)^2}\begin{pmatrix}G_c(t)\\-F_c(t)\end{pmatrix}\right\}\Bigg|_{\epsilon=0}\\ &=\left\{\frac{G_\infty^{(1)}(t)}{F_\infty^{(0)}(t)^2+G_\infty^{(0)}(t)^2}-\frac{2G_\infty^{(0)}(t)(F_\infty^{(0)}(t)F_\infty^{(1)}(t)+G_\infty^{(0)}(t)G_\infty^{(1)}(t))}{(F_\infty^{(0)}(t)^2+G_\infty^{(0)}(t)^2)^2}\right\}\begin{pmatrix}G_\infty^{(0)}(t)\\-F_\infty^{(0)}(t)\end{pmatrix}\\ &\quad+\frac{G_\infty^{(0)}(t)}{F_\infty^{(0)}(t)^2+G_\infty^{(0)}(t)^2}\begin{pmatrix}G_\infty^{(1)}(t)\\-F_\infty^{(1)}(t)\end{pmatrix}\\ &=\begin{pmatrix}-2G_\infty^{(0)}(t)^2F_\infty^{(0)}(t) & G_\infty^{(0)}(t)(1-2G_\infty^{(0)}(t)^2)\\ G_\infty^{(0)}(t)(2F_\infty^{(0)}(t)^2-1) & F_\infty^{(0)}(t)(2G_\infty^{(0)}(t)^2-1)\end{pmatrix}\begin{pmatrix}F_\infty^{(1)}(t)\\G_\infty^{(1)}(t)\end{pmatrix}\\ &=\begin{pmatrix}\cos\eta(t)\sin 2\eta(t) & \sin\eta(t)\sin 2\eta(t)\\ -\cos\eta(t)\cos 2\eta(t) & -\sin\eta(t)\cos 2\eta(t)\end{pmatrix}\begin{pmatrix}F_\infty^{(1)}(t)\\G_\infty^{(1)}(t)\end{pmatrix}\\ &=\begin{pmatrix}\sin 2\eta(t)\\-\cos 2\eta(t)\end{pmatrix}(\cos\eta(t)\quad\sin\eta(t))U(t)\int_0^t\begin{pmatrix}(n-2)\cos\eta(u)\\-\sin\eta(u)\end{pmatrix}du\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} \sin 2\eta(t) \\ -\cos 2\eta(t) \end{pmatrix} (1 \ \ 0) \int_0^t \begin{pmatrix} (n-2)\cos\eta(u) \\ -\sin\eta(u) \end{pmatrix} du \\
&= (n-2) \begin{pmatrix} \sin 2\eta(t) \\ -\cos 2\eta(t) \end{pmatrix} \int_0^t \cos\eta(u)\, du,
\end{aligned}
$$

we have

$$
\begin{aligned}
\begin{pmatrix} F_\infty^{(2)}(s) \\ G_\infty^{(2)}(s) \end{pmatrix} &= (n-2)(n-3)U(s) \int_0^s U^{-1}(t) \begin{pmatrix} \sin 2\eta(t) \\ -\cos 2\eta(t) \end{pmatrix} \int_0^t \cos\eta(u)\, dudt \\
&= (n-2)(n-3)U(s) \int_0^s \begin{pmatrix} \sin\eta(t) \\ -\cos\eta(t) \end{pmatrix} \int_0^t \cos\eta(u)\, dudt,
\end{aligned}
$$

proving Theorem 3.3. $\square$

Remark 3.2. Similarly, we can show, for any $K \in \mathbb{N}$,

$$
F_c(s) = \sum_{k=0}^{K} \epsilon^k F_\infty^{(k)}(s) + \mathcal{O}(\epsilon^{K+1}), \quad G_c(s) = \sum_{k=0}^{K} \epsilon^k G_\infty^{(k)}(s) + \mathcal{O}(\epsilon^{K+1}),
$$

where, $k \geq 2$,

$$
\begin{pmatrix} F_\infty^{(k)}(s) \\ G_\infty^{(k)}(s) \end{pmatrix} = \frac{n-3}{(k-1)!} U(s) \int_0^s U^{-1}(t) \frac{\partial^{k-1}}{\partial \epsilon^{k-1}} \left\{ \frac{G_c(t)}{F_c(t)^2 + G_c(t)^2} \begin{pmatrix} G_c(t) \\ -F_c(t) \end{pmatrix} \right\} \bigg|_{\epsilon=0} dt.
$$

Now, we study the periodicity of the family $\{\Gamma_c\}$. If $H(s)$ is a non-zero constant, then $\Gamma_c$ is periodic for any $c > 0$ [**1**], [**7**] and $\Gamma_\infty$ is a circle with curvature $(n-1)|H(s)|$. The period of $\Gamma_c$ depends on the initial condition of the generating curve when $n > 3$. If $H(s)$ is not constant and $n > 3$, then, the situation is different from the cases of the constant mean curvature case and 2-dimensional [**9**].

Lemma 3.1. *Suppose that for any $c > 0$, $\Gamma_c$ is periodic with period $L(c)$. If $H(s)$ and $L(c)$ are differentiable for $s$ and $c$, respectively, then one of them is constant.*

Proof. Since $x_c$ and $y_c$ are periodic for every $c > 0$, we have $H(s) = H(s + L(c))$, for $s \in \mathbb{R}$ and $c > 0$. Differentiating both sides with respect to $c$, we have $0 = H'(s + L(c))L'(c)$, which proves Lemma 3.1. $\square$

We have

Theorem 3.4. *Let $n > 3$ and $H(s)$ denote a non-constant differentiable function on $\mathbb{R}$. Then, there does not exist a family $\{\Gamma_c\}$ such that*

(i) $\Gamma_c$ *is periodic with period* $L(c)$,
(ii) $L(c)$ *is differentiable for* $c > 0$,
(iii) $\Gamma_\infty$ *is a simple closed curve.*

PROOF. Suppose that there exists a family $\{\Gamma_c\}$ satisfying the three conditions stated in Theorem 3.4. As per Lemma 3.1, $L(c)$ is constant, say $L(c) = L > 0$. Since $\Gamma_c$ is periodic with period $L$ for any $c > 0$, $F_c(s)$ and $G_c(s)$ are periodic with period $L$ for any $c > 0$. Then, $F_\infty^{(k)}(s)$ and $G_\infty^{(k)}(s)$ are also periodic with period $L$. From Theorem 3.3, we have

$$\int_0^L \cos\eta(u)du = 0, \quad \int_0^L \sin\eta(u)du = 0,$$
$$(n-2)(n-3)\int_0^L \left(\sin\eta(u)\int_0^u \cos\eta(t)dt\right)du = 0.$$

The first formula above with (6) implies that $\Gamma_\infty$ is a closed smooth curve and the second one means that the signed area $A(\Gamma_\infty)$ of the bounded domain surrounded by $\Gamma_\infty$ is zero if $n > 3$. Indeed, $A(\Gamma_\infty)$ is given by $A(\Gamma_\infty) = \int_0^L x_\infty(u)y_\infty'(u)du$ and it is non-zero if $\Gamma_\infty$ is a simple closed curve, which provides us the contradiction, proving Theorem 3.4. $\square$

## 4. Global existence of generalized rotational hypersurfaces of $O(l+1)\times O(m+1)$-type.

In this section, we study generalized rotational hypersurfaces in $\mathbb{R}^n$ of $O(l+1)\times O(m+1)$-type with the prescribed mean curvature. The main task is analyzing the behavior of the generating curve in the neighborhood of singular points. This is done by obtaining an integral equation instead of the system of differential equations.

Let $(x(s), y(s))$, $s \in \mathbb{R}$, be a plane curve satisfying $x(s) > 0$, $y(s) > 0$ and let $s$ be the arc length. For natural numbers $l$ and $m$ with $(l+1)+(m+1) = n$, we decompose $\mathbb{R}^n$ as $(x_1, \dots, x_{l+1}, y_1, \dots, y_{m+1}) \in \mathbb{R}^{l+1}\times\mathbb{R}^{m+1}$. A generalized rotational hypersurface $M$ of $O(l+1)\times O(m+1)$-type is defined by

$$M = \{(x(s)S^l, y(s)S^m) \in \mathbb{R}^n \mid s \in \mathbb{R}\}, \tag{8}$$

where for $k = l, m$ and $S^k$ denotes the $k$-dimensional unit sphere with center origin in $\mathbb{R}^{k+1}$. Note that this is $O(m+1)\times O(l+1)$-type in the terms of [**4**]. The mean curvature $H$ of $M$ is the function of variable $s$, say $H = H(s)$, and we have

$$\begin{cases} l\dfrac{y'(s)}{x(s)} - m\dfrac{x'(s)}{y(s)} - (x''(s)y'(s) - x'(s)y''(s)) + (n-1)H(s) = 0, \\ x'(s)^2 + y'(s)^2 = 1,\ x(s) > 0,\ y(s) > 0,\ s \in \mathbb{R}. \end{cases} \tag{9}$$

Conversely, given a continuous function $H = H(s)$, $s \in \mathbb{R}$, we have a system of ordinary differential equations (9). Let $c > 0$ and $d > 0$ be any positive numbers and any real numbers $c'$, $d'$ with ${c'}^2 + {d'}^2 = 1$, there exists a local solution curve $(x(s), y(s))$, $s \in I$, where $I$ denotes a subinterval of $\mathbb{R}$, of (9) with the initial conditions $x(s_0) = c$, $y(s_0) = d$, $x'(s_0) = c'$, and $y'(s_0) = d'$.

We extend this curve to the whole line $\mathbb{R}$. We study the behavior of the solution curve when it passes through the $x$-axis, $y$-axis, or origin $(0,0)$. When the curve passes through the $y$-axis, by changing $x$ and $y$, we can analyze the behavior of the solution curve in the same way as the curve passes through $x$-axis. Therefore, it is sufficient to study two cases: (a) the curve $(x(s), y(s))$ passes through $x$-axis at a finite $s$, and (b) the curve $(x(s), y(s))$ passes through the origin $(0,0)$. We study cases (a) and (b) in Subsections 4.1 and 4.2, respectively.

### 4.1. The case (a).

Let us multiply $y^m y'$ to the first equation of (9). From the second equation of (9) and the equation $x'x'' + y'y'' = 0$ that is obtained by differentiation of the second equation of (9), we have

$$(y^m x')' = (n-1)H(s)y^m y' + l\frac{y^m}{x}y'^2. \tag{10}$$

For simplicity, we assume that $s_0 = 0$. The integration of (10) leads to

$$y^m(s)x'(s) = (n-1)\int_0^s H(t)y^m(t)y'(t)\,dt + l\int_0^s \frac{y^m(t)}{x(t)}y'(t)^2 dt + y^m(0)x'(0). \tag{11}$$

We show that

Proposition 4.1. *Suppose that* $\lim_{s\to b} y(s) = 0$ *and* $\lim_{s\to b} x(s) > 0$ *for some* $b \in I$. *Then, there exists the limit of* $x'(s)$ *as* $s \to b$ *and* $\lim_{s\to b} x'(s) = 0$.

To prove Proposition 4.1, we need the following Lemma.

Lemma 4.1. $y'(s)$ *does not vanish on a neighborhood of* $s = b$.

Proof. By contraries, suppose that there exists a sequence $\{s_j\}$ such that $s_j \to b$ as $j \to \infty$, and $y'(s_j) = 0$. The second equation of (9) and its differentiation imply $x'(s_j) = \pm 1$, $x''(s_j) = 0$. By inserting these values in the first equation of (9), we have

$$(n-1)H(s_j) \mp \frac{m}{y(s_j)} \pm y''(s_j) = 0.$$

This with $y(s_j) \to +0$ as $j \to \infty$ yields

$$y''(s_j) = \mp(n-1)H(s_j) + \frac{m}{y(s_j)} \to \infty \text{ as } j \to \infty.$$

Hence, for large $j$, $y(s_j)$ is the minimum. Consequently, it does not hold that $y(s) \to +0$ as $s \to b$, providing contradiction. We proved Lemma 4.1. □

We proceed with the proof of Proposition 4.1.

Proof of Proposition 4.1. Since $y(s) \to 0$ as $s \to b$, (11) implies that

$$(n-1)\int_0^b H(t)y^m(t)y'(t)\,dt + l\int_0^b \frac{y^m(t)}{x(t)}y'(t)^2dt + y^m(0)x'(0) = 0.$$

From (11), Lemma 4.1, and l'Hôpital's rule, it follows that

$$\begin{aligned}\lim_{s\to b} x'(s) &= \lim_{s\to b}\frac{(n-1)\int_0^s H(t)y^m(t)y'(t)\,dt + l\int_0^s (y^m(t)/x(t))y'(t)^2dt + y^m(0)x'(0)}{y^m(s)}\\ &= \lim_{s\to b}\frac{(n-1)H(s)y^m(s)y'(s) + l(y^m(s)/x(s))y'(s)^2}{my^{m-1}(s)y'(s)}\\ &= \frac{1}{m}\lim_{s\to b}\left\{(n-1)H(s)y(s) + l\frac{y(s)}{x(s)}y'(s)\right\} = 0,\end{aligned}$$

proving Proposition 4.1. □

In order to prove the existence of solutions of the system (9) under the assumption of Proposition 4.1, we transform (9) using a change of variable. Let $s = s(y)$ be the inverse function of $y = y(s)$ on a neighborhood of $s = 0$. Let us put $q(s) = x'(s)/y'(s)$. We then obtain

$$\begin{aligned} xy\frac{dq}{dy} &= (n-1)\tilde{H}(y)(1+q^2)^{3/2}xy - mxq(1+q^2) + ly(1+q^2),\\ x &= \int_0^y q(\xi)\,d\xi + x(0),\end{aligned}$$

where $\tilde{H}(y)$ is defined using the same way as in (4).

Then, we have the singular initial value problem

$$\begin{cases} y\dfrac{dq}{dy} = -mq - mq^3 + (n-1)\tilde{H}(y)y(1+q^2)^{3/2} + l\dfrac{y(1+q^2)}{\int_0^y q(\xi)\,d\xi + x(0)},\\ q(0) = 0. \end{cases} \tag{12}$$

Remark 4.1. Although it is supposed to be $l \ge 1$ in this section, if $l = 0$ and $m = n-2$, then (12) is reduced to (4) of $O(n-1)$-type treated in Sections 2–3. When $l \ge 1$, the last term on the right-hand side of the first equation of (12) is new.

We study (12) as follows: It follows from (12) that

$$\frac{d}{dy}(y^m q) = -my^{m-1}q^3 + (n-1)\tilde{H}(y)y^m(1+q^2)^{3/2} + l\frac{y^m(1+q^2)}{\int_0^y q(\xi)\,d\xi + x(0)}.$$

Let us integrate both sides of the above equation. By virtue of $y^m q|_{y=0} = 0$, we have an integral equation

$$q(y) = \Psi(q)(y), \tag{13}$$

where we set

$$\Psi(q)(y) = y^{-m} \int_0^y \left\{ -mq(\eta)^3 + (n-1)\tilde{H}(\eta)\eta(1+q(\eta)^2)^{3/2} + l \frac{\eta(1+q(\eta)^2)}{\int_0^\eta q(\xi)\, d\xi + x(0)} \right\} \eta^{m-1} d\eta.$$

We note that when $l = 0$, $m = n-2$, (13) is reduced to (5). We use the same notations $X_Y$ and $X_{Y,M}$, which are defined in Section 2.

Proposition 4.2. *Suppose that $x(0) > 0$ and $y(0) = 0$.*

( i ) *If $\tilde{H}$ is bounded, then there exist constants $M$ and $Y$ such that the integral equation* (13) *has a unique solution $q$ on $X_M$.*
( ii ) *If $\tilde{H}$ is bounded and continuous, then the solution $q$ obtained in* (i) *is a unique solution of the initial value problem* (12).

Proof. For any $q \in X_{Y,M}$, we know $\Psi(q) \in C(0,Y]$. To prove (i), we show the boundedness of $\Psi(q)$ as follows: By noting that

$$\left| \int_0^\eta q(\xi)\, d\xi \right| \le M \int_0^\eta \xi\, d\xi = \frac{M}{2}\eta^2,$$

we choose $M$ and $Y$ such that

$$\left| \int_0^\eta q(\xi)\, d\xi + x(0) \right| \ge x(0) - \frac{M}{2}\eta^2 \ge x(0) - \frac{M}{2}Y^2 > 0.$$

Then,

$$\begin{aligned} \left| \frac{1}{y} y^{-m} \int_0^y \frac{l\eta^m (1+q(\eta)^2)}{\int_0^\eta q(\xi)\, d\xi + x(0)}\, d\eta \right| &\le \frac{l \int_0^y (\eta^m + M^2 \eta^{m+2}) d\eta}{y^{m+1}(x(0) - (M/2)Y^2)} \\ &= \frac{l[y^{m+1}/(m+1) + \{M^2/(m+3)\} y^{m+3}]}{y^{m+1}\{x(0) - (M/2)Y^2\}} \\ &\le \frac{l[1/(m+1) + \{M^2/(m+3)\}Y^2]}{x(0) - (M/2)Y^2}. \end{aligned}$$

Since the estimates of other parts on the right-hand side of (13) are calculated using the same method as that in the proof of Proposition 2.2 in Section 2, there is a constant $C$ such that

$$\left| \frac{\Psi(q)(y)}{y} \right| \le C(M^3 Y^2 + 1 + M^3 Y^3) + \frac{l\{1/(m+1) + M^2 Y^2/(m+3)\}}{x(0) - (M/2)Y^2},$$

where $C$ may depend on $l$, $m$, $n$, and $\sup_y |\tilde{H}(y)|$; however, it is independent of $M$ and $Y$. We can choose a large $M$ and a small $Y$ such that

$$C(M^3Y^2+1+M^3Y^3)+\frac{l\{1/(m+1)+M^2Y^2/(m+3)\}}{x(0)-(M/2)Y^2}\le M,$$

so that $\Psi(q)\in X_{Y,M}$.

Next, we show that $\Psi : X_M \longrightarrow X_M$ is a contraction mapping. Let us decompose $\Psi(q)$ as $\Psi(q)=\Psi_1(q)+\Psi_2(q)$, where

$$\Psi_1(q)(y)=y^{-m}\int_0^y\big\{-m\eta^{m-1}q(\eta)^3+(n-1)\tilde H(\eta)\eta^m(1+q(\eta)^2)^{3/2}\big\}d\eta,$$

$$\Psi_2(q)(y)=ly^{-m}\int_0^y\frac{\eta^m(1+q(\eta)^2)}{\int_0^\eta q(\xi)\,d\xi+x(0)}\,d\eta.$$

For $\Psi_1(q)$, using the above result, there exists a constant $C_1(\in(0,1))$ such that

$$\left|\frac{\Psi_1(q_1)(y)-\Psi_1(q_2)(y)}{y}\right|<C_1\|q_1-q_2\|_X.$$

For $\Psi_2(q)$, we compute

$$\begin{aligned}
&\Psi_2(q_1)(y)-\Psi_2(q_2)(y)\\
&\quad=ly^{-m}\int_0^y\left\{\frac{\eta^m(1+q_1(\eta)^2)}{\int_0^\eta q_1(\xi)\,d\xi+x(0)}-\frac{\eta^m(1+q_2(\eta)^2)}{\int_0^\eta q_2(\xi)\,d\xi+x(0)}\right\}d\eta\\
&\quad=ly^{-m}\left\{\int_0^y\frac{\eta^m(1+q_1(\eta)^2)(\int_0^\eta q_2(\xi)\,d\xi+x(0))}{(\int_0^\eta q_1(\xi)\,d\xi+x(0))(\int_0^\eta q_2(\xi)\,d\xi+x(0))}d\eta\right.\\
&\qquad\qquad\left.-\int_0^y\frac{\eta^m(1+q_2(\eta)^2)(\int_0^\eta q_1(\xi)\,d\xi+x(0))}{(\int_0^\eta q_1(\xi)\,d\xi+x(0))(\int_0^\eta q_2(\xi)\,d\xi+x(0))}d\eta\right\}\\
&\quad=ly^{-m}\int_0^y\frac{\eta^m\int_0^\eta(q_2(\xi)-q_1(\xi))\,d\xi}{(\int_0^\eta q_1(\xi)\,d\xi+x(0))(\int_0^\eta q_2(\xi)\,d\xi+x(0))}\,d\eta\\
&\qquad+ly^{-m}\int_0^y\frac{\eta^m\{q_1(\eta)^2(\int_0^\eta q_2(\xi)\,d\xi+x(0))-q_2(\eta)^2(\int_0^\eta q_1(\xi)\,d\xi+x(0))\}}{(\int_0^\eta q_1(\xi)\,d\xi+x(0))(\int_0^\eta q_2(\xi)\,d\xi+x(0))}\,d\eta.
\end{aligned}$$

Let $\Psi_{21}$ and $\Psi_{22}$ denote the first and second terms in the above formula, respectively. We estimate above terms separately as follows:

$$\begin{aligned}
\frac{|\Psi_{21}|}{y}&\le\frac{l|\int_0^y\eta^m\int_0^\eta(q_2(\xi)-q_1(\xi))\,d\xi d\eta|}{y^{m+1}(x(0)-(M/2)Y^2)^2}\\
&\le y^{-m-1}C_2\|q_2-q_1\|_X\int_0^y\eta^m\eta^2\,d\eta\\
&=C_2y^{-m-1}\|q_2-q_1\|_X\frac{y^{m+3}}{m+3}\le\tilde C_2\|q_2-q_1\|_Xy^2.
\end{aligned}$$

Since we have

$$
\begin{aligned}
&\text{the numerator of the integrand of } \Psi_{22} \\
&\quad = \eta^m \Big\{ q_1^2 \Big( \int_0^\eta q_2(\xi)\,d\xi + x(0) \Big) - q_2^2 \Big( \int_0^\eta q_1(\xi)\,d\xi + x(0) \Big) \Big\} \\
&\quad = \eta^m \Big\{ (q_1^2 - q_2^2) \Big( \int_0^\eta q_2(\xi)\,d\xi + x(0) \Big) + q_2^2 \Big( \int_0^\eta q_2(\xi)\,d\xi - \int_0^\eta q_1(\xi)\,d\xi \Big) \Big\},
\end{aligned}
$$

we see that

$$
\begin{aligned}
|\Psi_{22}| \le{}& \frac{C}{(x(0) - (M/2)Y^2)^2} \\
&\times \Big[ y^{-m} \int_0^y \eta^m \Big\{ \Big| (q_1(\eta) + q_2(\eta))(q_1(\eta) - q_2(\eta)) \Big( \int_0^\eta q_2(\xi)\,d\xi + x(0) \Big) \Big| \\
&\qquad + M^2 \eta^2 C \|q_2 - q_1\|_X \frac{\eta^2}{2} \Big\} d\eta \Big] \\
\le{}& C y^{-m} \Big\{ \int_0^y \|q_2 - q_1\|_X \eta^{m+2} \Big( \|q_2\|_X \frac{\eta^2}{2} + x(0) \Big) + \tilde{C} M^2 \|q_2 - q_1\|_X \eta^{m+4} \Big\} d\eta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\Big| \frac{\Psi_{22}}{y} \Big| &\le y^{-m-1} \Big( C_3 \frac{y^{m+5}}{m+5} + C_4 \frac{y^{m+3}}{m+3} + C_5 \frac{y^{m+5}}{m+5} \Big) \|q_2 - q_1\|_X \\
&= \big( \tilde{C}_3 y^4 + \tilde{C}_4 y^2 \big) \|q_2 - q_1\|_X.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\Big| \frac{\Psi_2(q_1)(y) - \Psi_2(q_2)(y)}{y} \Big| &\le \hat{C}_2 \|q_2 - q_1\|_X Y^2 + \|q_2 - q_1\|_X \big( \tilde{C}_3 Y^4 + \tilde{C}_4 Y^2 \big) \\
&= \|q_2 - q_1\|_X \big( \hat{C}_3 Y^4 + \hat{C}_4 Y^2 \big).
\end{aligned}
$$

We choose $Y$ such that $C_1 + \hat{C}_3 Y^4 + \hat{C}_4 Y^2 < 1$.

Summarizing these computations, for the case of $x(0) > 0$ and $y(0) = 0$, we prove that $\Psi : X_{Y,M} \longrightarrow X_{Y,M}$ is a contraction mapping, and hence, Banach's fixed point theorem verifies (i) of Proposition 4.2. Since (ii) is proved similar to the proof of (ii) in Proposition 2.2, we completed the proof of Proposition 4.2. □

### 4.2. The case (b).

In this subsection, we study case (b), that is, the curve $(x(s), y(s))$ satisfying (9) passes through the origin of $\mathbb{R}^2$.

Proposition 4.3. *Suppose that* $\lim_{s\to b} y(s) = 0$ *and* $\lim_{s\to b} x(s) = 0$ *for some* $b \in I$. *Then, there exists the limit of* $x'(s)^2$ *as* $s \to b$, *and*

$$\lim_{s \to b} x'(s)^2 = \frac{l}{l+m}.$$

*Hence, we have*

$$\textit{if } s < b, \textit{ then } \lim_{s \to b-0} x'(s) = -\sqrt{\frac{l}{l+m}},$$
$$\textit{if } s > b, \textit{ then } \lim_{s \to b+0} x'(s) = \sqrt{\frac{l}{l+m}}.$$

The proof of Proposition 4.3 is divided into several Lemmas.

LEMMA 4.2. *If there exists* $\lim_{s \to b} x'(s)^2$, *then the formulas in Proposition* 4.3 *hold.*

PROOF. Set $\lim_{s \to b} x'(s)^2 = X$. From the second equation of (9), there exists $\lim_{s \to b} y'(s)^2 = Y$. Similar to the proof of Lemma 4.1, it is shown that $x'(s)$ and $y'(s)$ are not zero on a neighborhood of $s = b$. Similar to the proof of Proposition 4.1, we have

$$\lim_{s \to b} x'(s) = \frac{1}{m} \lim_{s \to b} \left\{ (n-1)H(s)y(s) + l\frac{y(s)}{x(s)}y'(s) \right\}.$$

Squaring both sides, we obtain

$$X = \left(\frac{1}{m}\right)^2 \lim_{s \to b} \left\{ (n-1)H(s)y(s) + l\frac{y(s)}{x(s)}y'(s) \right\}^2 = \left(\frac{l}{m}\right)^2 \lim_{s \to b} \left(\frac{y(s)}{x(s)}y'(s)\right)^2.$$

Suppose $X \neq 0$. Since $x(b) = y(b) = 0$, l'Hôpital's rule leads to

$$\lim_{s \to b} \left(\frac{y(s)}{x(s)}\right)^2 = \lim_{s \to b} \left(\frac{y'(s)}{x'(s)}\right)^2 = \frac{Y}{X}.$$

Hence,

$$X = \left(\frac{l}{m}\right)^2 \frac{Y^2}{X}.$$

This, with the second equation of (9) yields

$$X = \frac{l}{l+m}, \quad Y = \frac{m}{l+m}.$$

Last, we need to show $X \neq 0$. By contraries, suppose $X = 0$. Since $Y = 1$, the same computation as above implies that

$$0 = X = \left(\frac{l}{m}\right)^2 \lim_{s \to b} \left(\frac{y(s)}{x(s)}y'(s)\right)^2 = \left(\frac{l}{m}\right)^2 \lim_{s \to b} \left(\frac{y'(s)}{x'(s)}\right)^2 Y = \infty,$$

providing a contradiction. □

For simplicity, we consider the case $s < b$.

Lemma 4.3. *It holds that*

$$\liminf_{s\to b-0}\frac{y'(s)}{x'(s)} = \frac{m}{l}\liminf_{s\to b-0}\frac{x'(s)}{y'(s)}.$$

Proof. If $\lim_{s\to b-0} x'(s)^2$ exists, then Lemma 4.3 follows from Lemma 4.2. In case $\lim_{s\to b-0} x'(s)^2$ does not exist, we will find a contradiction as follows: When $s$ is near $b$, we have $-1 \le x'(s) < 0$. Hence,

$$-1 \le \liminf_{s\to b-0} x'(s) < \limsup_{s\to b-0} x'(s) \le 0.$$

Then, there is a sequence $\{s_j\}$ satisfying

$$\lim_{j\to\infty} s_j = b-0, \quad \lim_{j\to\infty} x'(s_j) = \limsup_{s\to b-0} x'(s) > -1,$$
$$\lim_{j\to\infty} y'(s_j) = \liminf_{s\to b-0} y'(s) < 0, \quad x''(s_j) = y''(s_j) = 0.$$

From the first equation of (9), we see that at $s = s_j$,

$$\frac{y(s_j)}{x(s_j)} = \frac{m}{l}\frac{x'(s_j)}{y'(s_j)} - \frac{(n-1)H(s_j)y(s_j)}{ly'(s_j)}.$$

When $j\to\infty$, since $y'(s_j)$ does not converge to 0, the second term in the right-hand side of the above equation tends to 0. Hence, $\lim_{j\to\infty}(y(s_j)/x(s_j))$ exists and we have

$$\lim_{j\to\infty}\frac{y(s_j)}{x(s_j)} = \frac{m}{l}\lim_{j\to\infty}\frac{x'(s_j)}{y'(s_j)}.$$

By applying l'Hôpital's rule for the inferior limit, we have

$$\lim_{j\to\infty}\frac{y(s_j)}{x(s_j)} \ge \liminf_{s\to b-0}\frac{y(s)}{x(s)} \ge \liminf_{s\to b-0}\frac{y'(s)}{x'(s)}.$$

Since $-\xi/\sqrt{1-\xi^2}$ is monotone decreasing on the interval $(-1,0]$, it holds that

$$\lim_{j\to\infty}\frac{x'(s_j)}{y'(s_j)} = \lim_{j\to\infty}\left(-\frac{x'(s_j)}{\sqrt{1-x'(s_j)^2}}\right) = \liminf_{s\to b-0}\left(-\frac{x'(s)}{\sqrt{1-x'(s)^2}}\right) = \liminf_{s\to b-0}\frac{x'(s)}{y'(s)}.$$

Consequently, we have

$$\liminf_{s\to b-0}\frac{y'(s)}{x'(s)}\le\frac{m}{l}\liminf_{s\to b-0}\frac{x'(s)}{y'(s)}.$$

Next, we show the opposite inequality. There exists a sequence $\{\tilde{s}_j\}$ such that

$$\lim_{j\to\infty}\tilde{s}_j=b-0,\quad \lim_{j\to\infty}x'(\tilde{s}_j)=\liminf_{s\to b-0}x'(s)<0,$$
$$x''(\tilde{s}_j)=y''(\tilde{s}_j)=0.$$

From the second equation of (9),

$$\frac{y'(\tilde{s}_j)}{x'(\tilde{s}_j)}=\frac{m}{l}\frac{x(\tilde{s}_j)}{y(\tilde{s}_j)}-\frac{(n-1)H(\tilde{s}_j)x(\tilde{s}_j)}{lx'(\tilde{s}_j)}.$$

As $j\to\infty$, we see

$$\lim_{j\to\infty}\frac{y'(\tilde{s}_j)}{x'(\tilde{s}_j)}=\frac{m}{l}\lim_{j\to\infty}\frac{x(\tilde{s}_j)}{y(\tilde{s}_j)}.$$

Thus, we have

$$\lim_{j\to\infty}\frac{y'(\tilde{s}_j)}{x'(\tilde{s}_j)}=\lim_{j\to\infty}\left(-\frac{\sqrt{1-x'(\tilde{s}_j)^2}}{x'(s_j)}\right)=\liminf_{s\to b-0}\left(-\frac{\sqrt{1-x'(s)^2}}{x'(s)}\right)=\liminf_{s\to b-0}\frac{y'(s)}{x'(s)},$$
$$\lim_{j\to\infty}\frac{x(\tilde{s}_j)}{y(\tilde{s}_j)}\ge\liminf_{s\to b-0}\frac{x(s)}{y(s)}\ge\liminf_{s\to b-0}\frac{x'(s)}{y'(s)},$$

and therefore, we obtain

$$\liminf_{s\to b-0}\frac{y'(s)}{x'(s)}\ge\frac{m}{l}\liminf_{s\to b-0}\frac{x'(s)}{y'(s)},$$

proving Lemma 4.3. □

LEMMA 4.4. *It holds that*

$$\limsup_{s\to b-0}\frac{y'(s)}{x'(s)}=\frac{m}{l}\limsup_{s\to b-0}\frac{x'(s)}{y'(s)}.$$

PROOF. By changing $x$ and $y$ in the previous computation in Lemma 4.3, we see that

$$\liminf_{s\to b-0}\frac{x'(s)}{y'(s)}=\frac{l}{m}\liminf_{s\to b-0}\frac{y'(s)}{x'(s)}.$$

By taking the inverse, Lemma 4.4 is proved. □

Lemma 4.5. *It holds that*

$$\liminf_{s\to b-0}\frac{x'(s)}{y'(s)}=\liminf_{s\to b-0}\frac{x(s)}{y(s)},\quad \limsup_{s\to b-0}\frac{x'(s)}{y'(s)}=\limsup_{s\to b-0}\frac{x(s)}{y(s)}.$$

Proof. If $\lim_{s\to b-0}(x'(s)/y'(s))$ exists, then Lemma 4.5 follows from l'Hôpital's rule. By contraries, suppose that $\lim_{s\to b-0}(x'(s)/y'(s))$ does not exist. By using the sequence $\{s_j\}$ used in the proof of Lemma 4.3, we see that

$$\begin{aligned}\liminf_{s\to b-0}\frac{x'(s)}{y'(s)}&=\lim_{j\to\infty}\frac{x'(s_j)}{y'(s_j)}=\frac{l}{m}\lim_{j\to\infty}\frac{y(s_j)}{x(s_j)}\\&\geq\frac{l}{m}\liminf_{s\to b-0}\frac{y(s)}{x(s)}\geq\frac{l}{m}\liminf_{s\to b-0}\frac{y'(s)}{x'(s)}=\liminf_{s\to b-0}\frac{x'(s)}{y'(s)}.\end{aligned}$$

Hence, we have

$$\liminf_{s\to b-0}\frac{y(s)}{x(s)}=\frac{m}{l}\liminf_{s\to b-0}\frac{x'(s)}{y'(s)}=\liminf_{s\to b-0}\frac{y'(s)}{x'(s)}.$$

The formula about the superior limit in Lemma 4.3 is shown by changing $x$ and $y$ in the above formula, proving Lemma 4.5. □

Lemma 4.6. *There exist* $\lim_{s\to b-0}(y'(s)/x'(s))$ *and* $\lim_{s\to b-0}x'(s)^2$.

Proof. Set

$$A(s)=\frac{y(s)}{x(s)},\quad B(s)=\frac{y'(s)}{x'(s)},\quad \liminf_{s\to b-0}A(s)=\underline{L},\quad \limsup_{s\to b-0}A(s)=\bar{L}.$$

If $\underline{L}=\bar{L}$, then the assertion follows from Lemma 4.5.

Assuming $\underline{L}<\bar{L}$, we find a contradiction. From Lemma 4.5, we have $\liminf_{s\to b-0}B(s)=\underline{L}$, $\limsup_{s\to b-0}B(s)=\bar{L}$. From Lemma 4.4, we have $\bar{L}\underline{L}=m/l$. Hence, our assumption implies that $\underline{L}<\sqrt{m/l}<\bar{L}$. Taking into consideration the shape of the generating curve, we choose the sequence $\{s_j\}$ such that the generating curve is tangential to the line $y=L_jx$ at $s=s_j$, $\lim_{j\to\infty}A(s_j)=\underline{L}$ and $\lim_{j\to\infty}L_j=\underline{L}$ when $\lim_{j\to\infty}s_j=b-0$. Next, we choose the sequence $\{\tilde{s}_j\}$ such that

$$s_j<\tilde{s}_j<s_{j+1}<\tilde{s}_{j+1},\ \lim_{j\to\infty}A(\tilde{s}_j)=\bar{L},\ \text{and}\ \lim_{j\to\infty}\tilde{s}_j=b-0.$$

The last property stated above implies that

$$B(s_j)=L_j\to\underline{L}\quad\text{as}\quad j\to\infty.$$

Set

$$B_\epsilon=\{(A,B)\in\mathbb{R}^2\mid(A-\underline{L})^2+(B-\underline{L})^2<\epsilon^2\}.$$

If $\epsilon > 0$ is sufficiently small, then we may assume that

$$(A(s_j), B(s_j)) \in B_\epsilon, \quad (A(\tilde{s}_j), B(\tilde{s}_j)) \in B_\epsilon^c.$$

Hence, there exists $\{\hat{s}_j\}$ such that

$$s_j < \hat{s}_j < \tilde{s}_j, \quad (A(s), B(s)) \in \bar{B}_\epsilon \quad \text{for} \quad s \in [s_j, \hat{s}_j), \quad (A(\hat{s}_j), B(\hat{s}_j)) \in \partial B_\epsilon.$$

In order to consider the behavior of $(A(s), B(s))$ on the interval $I_j = [s_j, \hat{s}_j]$, we now compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\left(A(s) - \sqrt{\frac{m}{l}}\right)^2 &= \left(A(s) - \sqrt{\frac{m}{l}}\right)A'(s) \\
&= \left(A(s) - \sqrt{\frac{m}{l}}\right)\frac{(y'(s)x(x) - y(s)x'(s))}{x(s)^2} \\
&= \frac{x'(s)}{x(s)}\left(A(s) - \sqrt{\frac{m}{l}}\right)(B(s) - A(s)).
\end{aligned}$$

When $s \in I_j$,

$$\left|A(s) - \sqrt{\frac{m}{l}}\right| \leq C,$$

$$|B(s) - A(s)| = |B(s) - \underline{L} - (A(s) - \underline{L})| \leq 2\epsilon.$$

Therefore, it holds that

$$\left|\frac{x'(s)}{x(s)} - \frac{y'(s)}{y(s)}\right| = |A(s) - B(s)|\left|\frac{x'(s)}{y(s)}\right| \leq 2\epsilon\left|\frac{x'(s)}{y(s)}\right|,$$

which implies that

$$\frac{x'(s)}{x(s)} = \frac{y'(s) + O(\epsilon)x'(s)}{y(s)}.$$

Consequently, we have

$$\left|\frac{1}{2}\frac{d}{ds}\left(A(s) - \sqrt{\frac{m}{l}}\right)^2\right| = \left|\frac{y'(s) + O(\epsilon)x'(s)}{y(s)}\right|O(\epsilon) = \frac{O(\epsilon)}{y(s)},$$

where we used $|x'(s)| \leq 1$, $|y'(s)| \leq 1$. In contrast,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\left(B(s)-\sqrt{\frac{m}{l}}\right)^2\\
&\quad=\left(B(s)-\sqrt{\frac{m}{l}}\right)B'(s)\\
&\quad=\left(B(s)-\sqrt{\frac{m}{l}}\right)\frac{(y''(s)x'(s)-y'(s)x''(s))}{(x'(s))^2}\\
&\quad=-\frac{1}{(x'(s))^2}\left(B(s)-\sqrt{\frac{m}{l}}\right)\left\{l\frac{y'(s)}{x(s)}-m\frac{x'(s)}{y(s)}+(n-1)H(s)\right\}\\
&\quad=-\frac{l}{x'(s)y(s)}\left(B(s)-\sqrt{\frac{m}{l}}\right)\left(A(s)B(s)-\frac{m}{l}\right)-\frac{(n-1)H(s)}{(x'(s))^2}\left(B(s)-\sqrt{\frac{m}{l}}\right).
\end{aligned}$$

Set $\underline{L}=\lambda\sqrt{m/l}$, and then, $0\le\lambda<1$. We have on $I_j$,

$$0<A(s)<\frac{1+\lambda}{2}\sqrt{\frac{m}{l}},\quad 0<B(s)<\frac{1+\lambda}{2}\sqrt{\frac{m}{l}},\quad -1\le x'(s)<0,\quad y(s)>0$$

for large $j$. Hence, there exists $\delta>0$ independent of $\epsilon$ such that

$$-\frac{l}{x'(s)y(s)}\left(B(s)-\sqrt{\frac{m}{l}}\right)\left(A(s)B(s)-\frac{m}{l}\right)\ge\frac{\delta}{y(s)}.$$

Since it holds on the interval $I_j$ that

$$\frac{1-(x'(s))^2}{(x'(s))^2}=\left(\frac{y'(s)}{x'(s)}\right)^2=\underline{L}^2(1+O(1))=O(1),$$

we have

$$\inf\left\{(x'(s))^2\,\middle|\,s\in\bigcup_j I_j\right\}>0.$$

Hence,

$$\left|\frac{(n-1)H(s)}{(x'(s))^2}\left(B(s)-\sqrt{\frac{m}{l}}\right)\right|\le C.$$

Consequently, we have

$$\frac{1}{2}\frac{d}{ds}\left\{\left(A(s)-\sqrt{\frac{m}{l}}\right)^2+\left(B(s)-\sqrt{\frac{m}{l}}\right)^2\right\}\ge\frac{1}{y(s)}(\delta+O(\epsilon))-C$$

on $I_j$. If $j$ is sufficiently large, then $y(s)>0$ is sufficiently small. Taking $\epsilon$ small, we have

$$\frac{1}{2}\frac{d}{ds}\left\{\left(A(s)-\sqrt{\frac{m}{l}}\right)^2+\left(B(s)-\sqrt{\frac{m}{l}}\right)^2\right\}\geq\frac{\delta}{2y(s)}>0$$

on $I_j$ for a large $j$. Hence,

$$\begin{aligned}\left(A(\hat{s}_j)-\sqrt{\frac{m}{l}}\right)^2+\left(B(\hat{s}_j)-\sqrt{\frac{m}{l}}\right)^2&\geq\left(A(s_j)-\sqrt{\frac{m}{l}}\right)^2+\left(B(s_j)-\sqrt{\frac{m}{l}}\right)^2\\&=2\left(L_j-\sqrt{\frac{m}{l}}\right)^2,\end{aligned}$$

where the equality follows from the fact $A(s_j)=B(s_j)=L_j$. Taking a suitable subsequence, we have $(A(\hat{s}_j),B(\hat{s}_j))\to(\hat{A},\hat{B})$ as $j\to\infty$, where

$$(\hat{A},\hat{B})\in\partial B_\epsilon\cap\left\{(A,B)\in\mathbb{R}^2\;\middle|\;\left(A-\sqrt{\frac{m}{l}}\right)^2+\left(B-\sqrt{\frac{m}{l}}\right)^2\geq2\left(\underline{L}-\sqrt{\frac{m}{l}}\right)^2\right\}.$$

This shows that

$$\hat{A}<\underline{L}\quad\text{or}\quad\hat{B}<\underline{L}.$$

This is a contradiction. Indeed, if $\hat{A}<\underline{L}$, then

$$\liminf_{s\to b-0}A(s)=\underline{L}>\hat{A}=\lim_{j\to\infty}A(\hat{s}_j)\geq\liminf_{s\to b-0}A(s).$$

If $\hat{B}<\underline{L}$, then

$$\liminf_{s\to b-0}B(s)=\underline{L}>\hat{B}=\lim_{j\to\infty}B(\hat{s}_j)\geq\liminf_{s\to b-0}B(s).$$

Hence, we have $\underline{L}=\bar{L}$, proving Lemma 4.6. □

Proposition 4.3 is proved using Lemmas 4.2–4.6.

We are now in a position to study the system (12) with $x(0)=0$ and $y(0)=0$. Proposition 4.3 tells us $q(0)^2=l/m$. We may suppose $q(0)=\sqrt{l/m}$, because the generating curve is in the domain of $x>0$ and $y>0$. Thus, in this case, we have the singular initial value problem

$$\begin{cases}y\dfrac{dq}{dy}=(1+q^2)\left(-mq+\dfrac{ly}{\int_0^y q(\xi)\,d\xi}\right)+(n-1)\tilde{H}(y)(1+q^2)^{3/2}y,\\[2ex] q(0)=\sqrt{\dfrac{l}{m}}.\end{cases}\tag{14}$$

Furthermore, we transform this equation by setting

$$q(y) = \sqrt{\frac{l}{m}} + r(y).$$

The new singular initial value problem for $r(y)$ is obtained in the following Lemma.

Lemma 4.7. *$r = r(y)$ satisfies*

$$\begin{cases} y\dfrac{dr}{dy} = -(l+m)r(y) + F_1(r(y)) + F_2(r(\cdot), y) + F_3(r(y), y), \\ r(0) = 0, \end{cases} \tag{15}$$

*where we set*

$$\begin{aligned} F_1(r(y)) &= -r(y)^2\big(mr(y) + 2\sqrt{lm}\big), \\ F_2(r(\cdot), y) &= -\sqrt{lm}\left\{1 + \left(r(y) + \sqrt{\frac{l}{m}}\right)^2\right\} \frac{(\sqrt{m}/y)\int_0^y r(\xi)\,d\xi}{(\sqrt{m}/y)\int_0^y r(\xi)\,d\xi + \sqrt{l}}, \\ F_3(r(y), y) &= (n-1)\tilde{H}(y)\left\{1 + \left(r(y) + \sqrt{\frac{l}{m}}\right)^2\right\}^{3/2} y. \end{aligned}$$

Proof. For the right-hand side of the first equation of (14), we compute

$$\begin{aligned} &(1+q^2)\left(-mq + \frac{ly}{\int_0^y q(\xi)\,d\xi}\right) + (n-1)\tilde{H}(y)(1+q^2)^{3/2}y \\ &= \left\{1 + \left(\sqrt{\frac{l}{m}} + r\right)^2\right\}\left\{-m\left(\sqrt{\frac{l}{m}} + r\right) + \frac{ly}{\int_0^y(\sqrt{l/m} + r(\xi))d\xi}\right\} \\ &\quad + (n-1)\tilde{H}(y)\left\{1 + \left(\sqrt{\frac{l}{m}} + r\right)^2\right\}^{3/2} y \\ &= -\left\{1 + \left(\sqrt{\frac{l}{m}} + r\right)^2\right\}mr + \left\{1 + \left(\sqrt{\frac{l}{m}} + r\right)^2\right\}\left\{-m\sqrt{\frac{l}{m}} + \frac{ly}{\int_0^y(\sqrt{l/m} + r(\xi))d\xi}\right\} \\ &\quad + (n-1)\tilde{H}(y)\left\{1 + \left(\sqrt{\frac{l}{m}} + r\right)^2\right\}^{3/2} y. \end{aligned}$$

Since we have

$$\begin{aligned} -\left\{1 + \left(\sqrt{\frac{l}{m}} + r\right)^2\right\}mr &= -\left(1 + \frac{l}{m} + 2\sqrt{\frac{l}{m}}r + r^2\right)mr = -\big(m + l + 2\sqrt{lm}r + mr^2\big)r \\ &= -(l+m)r - r^2\big(mr + 2\sqrt{lm}\big), \end{aligned}$$

and

$$-m\sqrt{\frac{l}{m}}+\frac{ly}{\int_0^y\big(\sqrt{l/m}+r(\xi)\big)d\xi}=-\sqrt{lm}+\frac{l\sqrt{m}}{\sqrt{l}+(\sqrt{m}/y)\int_0^y r(\xi)\,d\xi}$$
$$=-\frac{(m\sqrt{l}/y)\int_0^y r(\xi)\,d\xi}{\sqrt{l}+(\sqrt{m}/y)\int_0^y r(\xi)\,d\xi},$$

Lemma 4.7 is proved. □

REMARK 4.2. There is a non-local part $\int_0^y r(\xi)\,d\xi$ in $F_2$. Therefore, we should write $F_2(r(\cdot),y)$, not $F(r(y),y)$, i.e., $F_2$ is defined on (a function space) $\times\,\mathbb{R}$, not on $\mathbb{R}^2$.

Multiplying $y^{l+m-1}$ on the first equation of (15) and integrating it with respect to $y$, we obtain an integral equation

$$r(y)=\Theta(r)(y), \tag{16}$$

where we set

$$\Theta(r)(y)=\frac{1}{y^{l+m}}\int_0^y F(\eta)\eta^{l+m-1}d\eta,$$
$$F(y)=F_1(r(y))+F_2(r(\cdot),y)+F_3(r(y),y).$$

Using the same notations $X_Y$ and $X_{Y,M}$ defined in Section 2, we have

PROPOSITION 4.4. (i) *If $\tilde{H}$ is bounded, then there exist constants $M$ and $Y$ such that the integral equation* (16) *has a unique solution $r$.*

(ii) *If $\tilde{H}$ is bounded and continuous, then the solution $r$ given in* (i) *is a solution of* (15).

PROOF. We find a fixed point of the mapping $\Theta: X_{Y,M}\longrightarrow X_{Y,M}$. The proof is accomplished by the following two steps.

First, we show that there exist $M$ and $Y$ such that $\Theta$ is a mapping from $X_{Y,M}$ into itself. Take any $r\in X_{Y,M}$. Then, we have

$$|F_1(r(y))|\le M^2y^2\big(mMy+2\sqrt{lm}\big)\le C(M^2y^2+M^3y^3).$$

By virtue of

$$\left|\frac{1}{y}\int_0^y r(\xi)\,d\xi\right|\le\frac{1}{y}\int_0^y\|r\|_X\xi\,d\xi\le\frac{My}{2},$$

if constants $M$ and $Y$ satisfy $MY<2\sqrt{l/m}$, then we have

$$\sqrt{\frac{m}{l}}\frac{1}{y}\int_0^y r(\xi)\,d\xi+1\ge 1-\sqrt{\frac{m}{l}}\frac{My}{2}>C^{-1}>0.$$

Then,

$$|F_2(r(\cdot),y)| \le \sqrt{lm}\left\{1+\left(My+\sqrt{\frac{l}{m}}\right)^2\right\}\frac{(\sqrt{m}/2)My}{\sqrt{l}-(\sqrt{m}/2)My}$$
$$\le \frac{\sqrt{l}}{\sqrt{l}-(\sqrt{m}/2)My}\frac{l+m}{2}My + C(M^2y^2+M^3y^3).$$

Moreover, by virtue of the inequality

$$\left\{1+\left(My+\sqrt{\frac{l}{m}}\right)^2\right\}^{1/2} \le 1+My+\sqrt{\frac{l}{m}} \quad \text{for} \quad y>0,$$

we have

$$|F_3(r(y),y)| \le C(1+M^3y^3)y = C(y+M^3y^4).$$

Suppose that $M$ and $Y$ satisfy $\sqrt{m}MY \le \sqrt{l}$, which implies that

$$\frac{\sqrt{l}}{\sqrt{l}-(\sqrt{m}/2)M\eta} \le 2 \quad \text{for} \quad \eta \in (0,Y].$$

Then,

$$\left|\frac{\Theta(r)(y)}{y}\right| \le \frac{1}{y^{l+m+1}}\int_0^y (|F_1(r(\eta))| + |F_2(r(\cdot),\eta)| + |F_3(r(\eta),\eta)|)\eta^{l+m-1}d\eta$$
$$\le \frac{1}{y^{l+m+1}}\int_0^y \left\{\frac{\sqrt{l}}{\sqrt{l}-(\sqrt{m}/2)M\eta}\frac{l+m}{2}M\eta \right.$$
$$\left. + C(\eta+M^2\eta^2+M^3\eta^3+M^3\eta^4)\right\}\eta^{l+m-1}d\eta$$
$$\le \frac{l+m}{l+m+1}M + C(1+M^2Y+M^3Y^2+M^3Y^3).$$

We take certain $M$ and $Y$ satisfying these two conditions:

$$MY < \sqrt{\frac{l}{m}}, \tag{17}$$

$$\frac{l+m}{l+m+1}M + C(1+M^2Y+M^3Y^2+M^3Y^3) \le M. \tag{18}$$

Then, $\Theta$ maps $X_{M,Y}$ into itself.

Suppose that $M$ and $Y$ satisfy (17) and (18). Then, we note that any $M$ and $Y'(<Y)$ also satisfy the same conditions (17) and (18). Next, we show that taking a smaller $Y$ if

necessary, $\Theta$ is a contraction mapping from $X_{Y,M}$ into itself. Take $r_1$, $r_2 \in X_{Y,M}$. Then, we have

$$\Theta(r_1)(y) - \Theta(r_2)(y) = \frac{1}{y^{l+m}} \int_0^y \sum_{j=1}^3 (F_{j1}(\eta) - F_{j2}(\eta))\eta^{l+m-1} d\eta,$$

where

$$F_{1k}(y) = F_1(r_k(y)), \quad F_{2k}(y) = F_2(r_k(\cdot), y), \quad F_{3k}(y) = F_3(r_k(y), y) \quad (k = 1, 2).$$

For $F_{1k}(y)$, we see that

$$\begin{aligned}
&|F_{11}(y) - F_{12}(y)| \\
&\quad = \bigl|\bigl\{m(r_1(y)^2 + r_1(y)r_2(y) + r_2(y)^2) + 2\sqrt{lm}(r_1(y) + r_2(y))\bigr\}(r_1(y) - r_2(y))\bigr| \\
&\quad \le C(My^2 + M^2y^3)\|r_1 - r_2\|_X.
\end{aligned}$$

For $F_{2k}(y)$, we set

$$F_{21}(y) - F_{22}(y) = G_1(y) + G_2(y) + G_3(y),$$

where

$$\begin{aligned}
G_1(y) &= -\sqrt{lm}\left\{\left(r_1(y) + \sqrt{\frac{l}{m}}\right)^2 - \left(r_2(y) + \sqrt{\frac{l}{m}}\right)^2\right\}\frac{(\sqrt{m}/y)\int_0^y r_1(\xi)\,d\xi}{(\sqrt{m}/y)\int_0^y r_1(\xi)\,d\xi + \sqrt{l}}, \\
G_2(y) &= -\sqrt{lm}\left\{1 + \left(r_2(y) + \sqrt{\frac{l}{m}}\right)^2\right\}\frac{(\sqrt{m}/y)\int_0^y (r_1(\xi) - r_2(\xi))\,d\xi}{(\sqrt{m}/y)\int_0^y r_1(\xi)\,d\xi + \sqrt{l}}, \\
G_3(y) &= \sqrt{lm}\left\{1 + \left(r_2(y) + \sqrt{\frac{l}{m}}\right)^2\right\}\frac{\sqrt{m}}{y}\int_0^y r_2(\xi)\,d\xi \\
&\qquad \times \frac{(\sqrt{m}/y)\int_0^y (r_1(\xi) - r_2(\xi))\,d\xi}{\bigl((\sqrt{m}/y)\int_0^y r_1(\xi)\,d\xi + \sqrt{l}\bigr)\bigl((\sqrt{m}/y)\int_0^y r_2(\xi)\,d\xi + \sqrt{l}\bigr)},
\end{aligned}$$

and then, we have

$$\begin{aligned}
|G_1(y)| &= \left|\sqrt{lm}\left(r_1(y) + r_2(y) + 2\sqrt{\frac{l}{m}}\right)(r_1(y) - r_2(y))\frac{(\sqrt{m}/y)\int_0^y r_1(\xi)\,d\xi}{(\sqrt{m}/y)\int_0^y r_1(\xi)\,d\xi + \sqrt{l}}\right| \\
&\le 2\sqrt{lm}\left(My + \sqrt{\frac{l}{m}}\right)\|r_1 - r_2\|_X y \frac{(\sqrt{m}/2)My}{\sqrt{l} - (\sqrt{m}/2)My} \\
&\le 2\bigl(mMy + \sqrt{lm}\bigr)My^2\|r_1 - r_2\|_X \\
&\le C(M^2y^3 + My^2)\|r_1 - r_2\|_X,
\end{aligned}$$

$$
\begin{aligned}
|G_2(y)| &\leq \sqrt{lm}\left\{1+\left(My+\sqrt{\frac{l}{m}}\right)^2\right\}\frac{(\sqrt{m}/y)\int_0^y\|r_1-r_2\|_X\xi\,d\xi}{\sqrt{l}-(\sqrt{m}/2)My} \\
&\leq \bigl(m+mM^2y^2+2\sqrt{lm}My+l\bigr)y\|r_1-r_2\|_X \\
&\leq \{C(M^2y^3+My^2)+(l+m)y\}\|r_1-r_2\|_X,
\end{aligned}
$$

and

$$
\begin{aligned}
|G_3(y)| &\leq \sqrt{lm}\left\{1+\left(My+\sqrt{\frac{l}{m}}\right)^2\right\}\frac{\sqrt{m}}{2}My\frac{(\sqrt{m}/2)y\|r_1-r_2\|_X}{(\sqrt{l}-(\sqrt{m}/2)My)^2} \\
&\leq \sqrt{\frac{m}{l}}\bigl(m+mM^2y^2+2\sqrt{lm}My+l\bigr)My^2\|r_1-r_2\|_X \\
&\leq C(My^2+M^3y^4+M^2y^3)\|r_1-r_2\|_X.
\end{aligned}
$$

Therefore, it holds that

$$
|F_{21}(y)-F_{22}(y)| \leq \{C(My^2+M^3y^4)+(l+m)y\}\|r_1-r_2\|_X.
$$

Last, we estimate $F_{3k}(y)$. From the mean value theorem, for each $y\in[0,Y]$, there exists $r_*(y)$ between $r_1(y)$ and $r_2(y)$, such that

$$
\begin{aligned}
&\left\{1+\left(r_1(y)+\sqrt{\frac{l}{m}}\right)^2\right\}^{3/2}-\left\{1+\left(r_2(y)+\sqrt{\frac{l}{m}}\right)^2\right\}^{3/2} \\
&\quad = 3\left(r_*(y)+\sqrt{\frac{l}{m}}\right)\left\{1+\left(r_*(y)+\sqrt{\frac{l}{m}}\right)^2\right\}^{1/2}(r_1(y)-r_2(y)).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&|F_{31}(y)-F_{32}(y)| \\
&\quad = \left|(n-1)\tilde{H}(y)y\left[\left\{1+\left(r_1(y)+\sqrt{\frac{l}{m}}\right)^2\right\}^{3/2}-\left\{1+\left(r_2(y)+\sqrt{\frac{l}{m}}\right)^2\right\}^{3/2}\right]\right| \\
&\quad \leq 3(n-1)y\sup_{\eta\in[0,Y]}|\tilde{H}(\eta)|\left(My+\sqrt{\frac{l}{m}}\right)\left\{1+\left(My+\sqrt{\frac{l}{m}}\right)^2\right\}^{1/2}y\|r_1-r_2\|_X \\
&\quad \leq C(1+M^2y^2)y^2\|r_1-r_2\|_X.
\end{aligned}
$$

Consequently, we see that

$$
\begin{aligned}
&\left|\frac{\Theta(r_1)(y)-\Theta(r_2)(y)}{y}\right| \\
&\quad \leq \frac{\|r_1-r_2\|_X}{y^{l+m+1}}\int_0^y\{C(M\eta^2+M^3\eta^4+\eta^2+M^2\eta^4)+(l+m)\eta\}\eta^{l+m-1}d\eta
\end{aligned}
$$

$$\leq \left\{ C(MY + M^3Y^3 + Y + M^2Y^3) + \frac{l+m}{l+m+1} \right\} \|r_1 - r_2\|_X.$$

Let us choose $Y$ satisfying

$$C(MY + M^3Y^3 + Y + M^2Y^3) + \frac{l+m}{l+m+1} < 1.$$

Then, $\Theta$ is a contraction mapping from $X_{Y,M}$ to itself.

As per Banach's fixed point theorem, there exists uniquely a fixed point $r$ of $\Theta$ on $X_{Y,M}$. This $r$ is a solution of the integral equation (16). If $\tilde{H}$ is continuous, then $r$ satisfies the first equation in (15). Since $r \in X_Y$, it also satisfies the second equation of (15), proving Proposition 4.4. □

We note that $q(y) = \sqrt{l/m} + r(y)$ is a solution of (14). From Propositions 4.3 and 4.4, we finished the proof for case (b).

By replacing [**3**, Proposition 3.3] to Propositions 4.3 and 4.4 of this paper, we prove the following theorem in the same way as the proof of [**3**, Theorem 3.4].

THEOREM 4.1. *Let $H(s)$ be a continuous function on $\mathbb{R}$ and fix an $s_0 \in \mathbb{R}$. Then, for any positive numbers $c > 0$, $d > 0$, and any real numbers $c'$, $d'$ with ${c'}^2 + {d'}^2 = 1$, there exists a global solution curve $(x(s), y(s))$, $s \in \mathbb{R}$, of* (9) *with the initial conditions $x(s_0) = c$, $y(s_0) = d$, $x'(s_0) = c'$, and $y'(s_0) = d'$.*

REMARK 4.3. In case of $s \to b+0$, $l$ principal curvatures $-y'(s)/x(s)$ tends to $-\infty$, and $m$ principal curvatures $x'(s)/y(s)$ to $\infty$; however, the sum of all principal curvatures

$$-l\frac{y'(s)}{x(s)} + m\frac{x'(s)}{y(s)} + x''(s)y'(s) - x'(s)y''(s)$$

remains bounded and tends to $(n-1)H(b)$. This suggests that for the generalized rotational hypersurface $M$ of $O(l+1) \times O(m+1)$-type, the asymptotic shape of the $x(s)S^l$-part as $s \to b+0$ is the negatively curved $l+1$ dimensional cone with the top at $(x(b), y(b)) = (0, 0)$, and that of the $y(s)S^m$-part is the positively curved $m+1$ dimensional cone. When $s \to b-0$, the asymptotic shape is a similar to that with reverse orientation.

## References

[1] C. Delaunay, Sur la surface de révolution dont la courbure moyenne est constante, J. Math. Pures Appl. Ser.1, **6** (1841), 309–320.

[2] J. Dorfmeister and K. Kenmotsu, On a theorem by Hsiang and Yu, Ann. Global Anal. Geom., **33** (2008), 245–252.

[3] J. Dorfmeister and K. Kenmotsu, Rotational hypersurfaces of periodic mean curvature, Differential Geom. Appl., **27** (2009), 702–712.

[4] W.-Y. Hsiang, Generalized rotational hypersurfaces of constant mean curvature in the Euclidean spaces, I, J. Differential Geom., **17** (1982), 337–356.

[5] W.-Y. Hsiang and H.-L. Huynh, Generalized rotational hypersurfaces of constant mean curvature

in the Euclidean spaces, II, Pacific J. Math., **130** (1987), 75–95.
[6] W.-Y. Hsiang and H. B. Lawson, Jr., Minimal submanifolds of low cohomogeneity, J. Differential Geom., **5** (1971), 1–38.
[7] W.-Y. Hsiang and W.-C. Yu, A generalization of a theorem of Delaunay, J. Differential Geom., **16** (1981), 161–177.
[8] K. Kenmotsu, Surfaces of revolution with prescribed mean curvature, Tôhoku Math. J. (2), **32** (1980), 147–153.
[9] K. Kenmotsu, Surfaces of revolution with periodic mean curvature, Osaka J. Math., **40** (2003), 687–696.

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