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Griess algebras generated by the Griess algebras of two $3A$-algebras with a common axis

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Abstract. In this article, we study Griess algebras generated by two pairs of Ising vectors (a_0, a_1) and (b_0, b_1) such that each pair generates a $3A$-algebra U_{3A} and their intersection contains the W_3-algebra $\mathcal{W}(4/5) \cong L(4/5, 0) \oplus L(4/5, 3)$. We show that there are only 3 possibilities, up to isomorphisms and they are isomorphic to the Griess algebras of the VOAs $V_{F(1A)}$, $V_{F(2A)}$ and $V_{F(3A)}$ constructed by Höhn–Lam–Yamauchi.

1. Introduction.

The study of vertex operator algebra (VOA) as a module of a simple Virasoro VOA was first initiated by Dong–Mason–Zhu [**DMZ**], in which they showed that the famous Moonshine VOA $V^\natural$ has a full sub-VOA isomorphic to a tensor product of 48 copies of the simple Virasoro VOA $L(1/2, 0)$. Partially motivated by [**DMZ**] and Conway's work [**Co**], Miyamoto [**Mi1**] introduced the notion of simple conformal vectors of central charge $1/2$, which we call *Ising vectors* in this article. In addition, he developed a method to construct involutions in the automorphism group of a VOA V from Ising vectors. These automorphisms are often called Miyamoto involutions. When V is the famous Moonshine VOA $V^\natural$, Miyamoto [**Mi2**] also showed that there is a $1-1$ correspondence between the $2A$-involutions of the Monster group and Ising vectors in $V^\natural$ (cf. [**Hö**]). This correspondence turns out to be very important in the study of the Monster group. In particular, many mysterious phenomena associated with the $2A$-involutions of the Monster can be interpreted using the theory of VOA. For instance, the McKay's observation on the affine E_8-diagram has been studied in [**LYY**], [**LYY2**] using Miyamoto involutions and several VOAs generated by two Ising vectors have been constructed explicitly and studied. These VOAs are usually denoted by U_{nX}, where $nX = 1A$, $2A$, $3A$, $4A$, $5A$, $6A$, $4B$, $2B$, or $3C$ and we call them the nX-algebra. In [**Sa**], the Griess algebras generated by two Ising vectors contained in a VOA with a positive definite invariant bilinear form over $\mathbb{R}$ are classified. The main result is that the Griess algebras $\mathcal{G}U_{nX}$ of the nine VOAs U_{nX}, $nX \in \{1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A\}$, constructed in [**LYY**] exhaust all the possibilities. He thus established another natural correspondence between the dihedral groups generated by two $2A$-involutions and Griess sub-algebras generated by two Ising vectors in $V^\natural$.

In [**HLY**], certain mysterious relations between the Fischer group Fi_{24} and the affine E_6-diagram are studied. In particular, some VOAs generated by a pair of $3A$-algebras

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are constructed. These VOAs are denoted by $V_{F(1A)}$, $V_{F(2A)}$, and $V_{F(3A)}$ in [**HLY**]. In this paper, we will study Griess sub-algebras generated by the Griess algebras of two $3A$-algebras U and U' such that their intersection contains a sub-VOA isomorphic to $\mathcal{W}(4/5) = L(4/5,0)\bigoplus L(4/5,3)$. We will show that there are only three possibilities, up to isomorphism and they are isomorphic to the Griess algebras of $V_{F(1A)}$, $V_{F(2A)}$, and $V_{F(3A)}$. Our technique is similar to [**LS**], in which Griess algebras generated two $2A$-algebras with a common Ising vector were studied.

The main idea is to analyze various Griess sub-algebras generated by two Ising vectors using Sakuma's Theorem. The organization of the paper is as follows. In Section 2, we review some basic definitions and results about VOAs over $\mathbb{R}$ and Griess algebras. In particular, we recall the definition of Miyamoto involutions and some of the consequences. A result of Sakuma is also reviewed. In Section 3, we recall some facts and list some basic properties of the $3A$ and $6A$-algebras. An automorphism associated to a W_3-algebra $\mathcal{W}_3(4/5) = L(4/5,0)\bigoplus L(4/5,3)$ and its real form $\mathcal{W}_{\mathbb{R}}^+$ will also be reviewed. In Section 4, we will state and prove our main theorem by case and case analysis.

2. Preliminary.

The following theorem is well-known (cf. [**FLM**, Theorem 8.9.5]).

THEOREM 2.1. *Let $(V,Y,\mathbf{1},\omega)$ be a VOA with $V=\bigoplus_{n\in\mathbb{Z}}V_n$, $V_n=0$ for $n<0$, $\dim V_0=1$, and $V_1=0$. Then the weight 2 space V_2 has a commutative (non-associative) algebra structure defined by the product,*

$$a\cdot b = a_{(1)}b\ (= b_{(1)}a).$$

Moreover, there is a symmetric bilinear form $\langle\cdot,\cdot\rangle$ defined by

$$\langle a,b\rangle\mathbf{1} = a_{(3)}b\ (= b_{(3)}a),\quad a,b\in V_2,$$

which is the restriction of the contragredient (cf. [**FHL**, Section 5.2]*) bilinear form of V on V_2. Note that the bilinear form on V_2 is invariant in the sense that*

$$\langle a\cdot b,c\rangle = \langle a,b\cdot c\rangle \quad \text{for all } a,b,c\in V_2. \tag{1}$$

DEFINITION 2.2. The algebra $\mathcal{G}=\mathcal{G}V=(V_2,\cdot,\langle\cdot,\cdot\rangle)$ in Theorem 2.1 is called the *Griess algebra*. An automorphism of $\mathcal{G}$ is a linear automorphism that preserves the product and the bilinear form. The group of all automorphisms of $\mathcal{G}$ is denoted by $\mathrm{Aut}(\mathcal{G})$. It is clear that $f\in\mathrm{Aut}(V)$ implies $f|_{\mathcal{G}}\in\mathrm{Aut}(\mathcal{G})$.

In this article, all VOAs are over the real field $\mathbb{R}$, unless otherwise stated. The following is our main assumption.

ASSUMPTION 1. *Let $(V,Y,\mathbf{1},\omega)$ be a VOA over $\mathbb{R}$ with $V=\bigoplus_{n\in\mathbb{Z}}V_n$, $V_n=0$ for $n<0$, $\dim V_0=1$ and $V_1=0$. We assume the contragredient bilinear form of V is positive definite.*

The next theorem is important to our discussion. The proof can be found in [**Mi1**, Theorem 6.3].

THEOREM 2.3 (Norton inequality). *Let V be a VOA satisfying Assumption 1. Then for all a, b in $\mathcal{G} = V_2$, we have*

$$\langle a \cdot a, b \cdot b \rangle \geq \langle a \cdot b, a \cdot b \rangle.$$

In particular, if a, b are idempotents in $\mathcal{G}$, then $\langle a, b \rangle = \langle a \cdot a, b \cdot b \rangle \geq \langle a \cdot b, a \cdot b \rangle \geq 0$.

Next we recall the basic notion of Ising vectors and Miyamoto involutions. We mainly follow the notations in [**Mi1**].

DEFINITION 2.4. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA such that $V_n = 0$ for $n < 0$, $\dim V_0 = 1$ and $V_1 = 0$. An element $e \in V_2$ is called an *Ising vector* if the sub-VOA $\mathrm{Vir}(e)$ generated by e is isomorphic to the simple Virasoro VOA $L(1/2, 0)$.

To define the automorphisms τ_e and σ_e, we need to know the decomposition of V as a $\mathrm{Vir}(e)$-module. If V is a VOA over $\mathbb{C}$, the decomposition is shown in [**Mi1**]; this decomposition also holds for a VOA over $\mathbb{R}$ with a positive definite contragredient form [**Mi4**, Theorem 2.4].

PROPOSITION 2.5 ([**Mi1**], [**Mi4**]). *Let $(V, Y, \mathbf{1}, \omega)$ be a VOA over $\mathbb{R}$ with a positive definite contragredient form. For an Ising vector $e \in V$, and a constant $h \in \mathbb{R}$, let $V_e(h)$ be the sum of all irreducible* $\mathrm{Vir}(e)$*-submodules of V isomorphic to $L(1/2, h)$. Then we have the submodule decomposition*

$$V = V_e(0) \bigoplus V_e\left(\frac{1}{2}\right) \bigoplus V_e\left(\frac{1}{16}\right). \tag{2}$$

DEFINITION 2.6. Define a linear map $\tau_e : V \to V$ by

$$\tau_e = \begin{cases} 1 & \text{on } V_e(0) \bigoplus V_e\left(\frac{1}{2}\right), \\ -1 & \text{on } V_e\left(\frac{1}{16}\right). \end{cases}$$

Let V^{τ_e} be the fixed point subspace of τ_e in V, i.e.,

$$V^{\tau_e} = \{v \in V \mid \tau_e(v) = v\} = V_e(0) \bigoplus V_e\left(\frac{1}{2}\right).$$

Define a linear map $\sigma_e : V^{\tau_e} \to V^{\tau_e}$ by

$$\sigma_e = \begin{cases} 1 & \text{on } V_e(0), \\ -1 & \text{on } V_e\left(\frac{1}{2}\right). \end{cases}$$

Theorem 2.7 (cf. [**Mi1**, Theorem 4.7 and Theorem 4.8]). *Let e be an Ising vector of a VOA V. Then the map τ_e defined in Definition 2.6 is an automorphism of V. Moreover, for any $\rho \in \mathrm{Aut}(V)$, we have $\rho\tau_e\rho^{-1} = \tau_{\rho(e)}$.*

On the fixed point sub-VOA V^{τ_e}, we have $\sigma_e \in \mathrm{Aut}(V^{\tau_e})$. In addition, for any $\varrho \in \mathrm{Aut}(V^{\tau_e})$, we have $\varrho\sigma_e\varrho^{-1} = \sigma_{\varrho(e)}$.

Remark 2.8. Note that for any Ising vector e and $x \in \mathcal{G}$, $x + \tau_e(x) \in V^{\tau_e}$ and thus $\sigma_e(x + \tau_e(x))$ is well-defined.

The following lemma can be found in [**Sa**, (2.2)].

Lemma 2.9. *Let e be an Ising vector of a VOA V. Let $\mathcal{G}_e(h) = \{x \in \mathcal{G} \mid e \cdot x = hx\}$ be the h-eigenspace of e for $h = 0, 2, 1/2, 1/16$. Then for any $x \in \mathcal{G} = V_2$, we have the decomposition $x = x_0 + x_2 + x_{1/2} + x_{1/16}$, where $x_h \in \mathcal{G}_e(h)$. Moreover,*

$$x_{1/16} = \frac{1}{2}(x - \tau_e(x)), \quad x_{1/2} = \frac{1}{2}\left(\frac{1}{2}(x + \tau_e(x)) - \sigma_e\left(\frac{1}{2}(x + \tau_e(x))\right)\right), \quad x_2 = 4\langle e, x\rangle e.$$

Hence

$$e \cdot x = 8\langle e, x\rangle e + \frac{1}{2^2}\left(\frac{1}{2}(x + \tau_e(x)) - \sigma_e\left(\frac{1}{2}(x + \tau_e(x))\right)\right) + \frac{1}{2^5}(x - \tau_e(x)).$$

If $\tau_e(x) = x$, then

$$e \cdot x = 8\langle e, x\rangle e + \frac{1}{2^2}(x - \sigma_e(x)).$$

In particular, $e \cdot e = 2e$ and $\langle e, e\rangle = 1/2^2$.

In [**Sa**], the Griess algebras generated by two Ising vectors in a VOA satisfying Assumption 1 has been classified.

Notation 2.10. For $x_1, \dots, x_n \in V_2$, we denote by $\mathcal{G}\{x_1, \dots, x_n\}$ the Griess subalgebra generated by $x_1, \dots, x_n$.

Theorem 2.11 (cf. [**Sa**] and [**IPSS**]). *Let V be a VOA satisfying Assumption 1. Let x_0, x_1 be Ising vectors in V_2. Then the Griess sub-algebra $\mathcal{G}\{x_0, x_1\}$ generated by x_0 and x_1 is isomorphic to one of the following 9 cases.*

$\mathcal{G}\{x_0, x_1\}$	$\mathcal{G}U_{1A}$	$\mathcal{G}U_{2A}$	$\mathcal{G}U_{2B}$	$\mathcal{G}U_{3A}$	$\mathcal{G}U_{3C}$	$\mathcal{G}U_{4A}$	$\mathcal{G}U_{4B}$	$\mathcal{G}U_{5A}$	$\mathcal{G}U_{6A}$
$\langle x_0, x_1\rangle$	$\frac{1}{2^2}$	$\frac{1}{2^5}$	0	$\frac{13}{2^{10}}$	$\frac{1}{2^8}$	$\frac{1}{2^7}$	$\frac{1}{2^8}$	$\frac{3}{2^9}$	$\frac{5}{2^{10}}$

We will refer to [**Sa**] *and* [**LYY2**] *for the exact structures of the Griess algebras $\mathcal{G}U_{nX}$ (cf.* [**IPSS**, Tabe 3]).

Remark 2.12. By Sakuma's Theorem (Theorem 2.11), it is easy to see that $a = b$

if and only if $\langle a, b\rangle = 1/2^2$ for any two Ising vectors a, b.

3. $3A$-algebra U_{3A} and $6A$-algebra U_{6A}.

In this section, we will review and list some properties of the $3A$-algebra U_{3A} and U_{6A} (cf. [**LYY2**], [**IPSS**]).

3.1. $3A$-algebra.

Let $\mathcal{G}U_{3A}$ be the Griess algebra of U_{3A}. Then $\dim \mathcal{G}U_{3A} = 4$ and it is spanned by three Ising vectors x_0, x_1, x_2 and a Virasoro vector μ of central charge $4/5$ (cf. [**IPSS**, Table 3]).

For $\{i, j, k\} = \{0, 1, 2\}$, the multiplication and the bilinear form are given by

$$x_i \cdot x_j = \frac{1}{2^4}(2x_i + 2x_j + x_k) - \frac{135}{2^{10}}\mu, \tag{3}$$

$$x_i \cdot \mu = \frac{2}{3^2}(2x_i - x_j - x_k) + \frac{5}{2^4}\mu, \tag{4}$$

$$\mu \cdot \mu = 2\mu, \tag{5}$$

and

$$\langle x_i, x_j\rangle = \frac{13}{2^{10}}, \quad \langle x_i, \mu\rangle = \frac{1}{2^4}, \quad \langle \mu, \mu\rangle = \frac{2}{5}. \tag{6}$$

Moreover, we have

$$\tau_{x_i}(x_j) = x_k \quad \text{and} \quad \tau_{x_i}(\mu) = \mu. \tag{7}$$

For $i \in \{0, 1, 2\}$, the fixed point sub-algebra $\mathcal{G}^{\tau_{x_i}}$ has dimension 3 and is spanned by x_i, $x_j + x_k$ and μ. Moreover we have

$$\sigma_{x_i}(x_j + x_k) = -\frac{3x_i}{2^4} + \frac{x_j + x_k}{2^2} + \frac{135\mu}{2^7}, \tag{8}$$

$$\sigma_{x_i}(\mu) = \frac{2x_i}{3^2} + \frac{8(x_j + x_k)}{3^2} - \frac{\mu}{2^2}. \tag{9}$$

We call the ordered set (x_0, x_1, x_2, μ) a *normal* $\mathcal{G}U_{3A}$ *basis.*

3.2. $6A$-algebra.

Let $\mathcal{G}U_{6A}$ be the Griess algebra of U_{6A}. Then $\dim \mathcal{G}U_{6A} = 8$ and there is a basis $(x_0, x_1, x_2, x_3, x_4, x_5, x, \mu)$ for $\mathcal{G}U_{6A}$ such that the multiplication and the bilinear form are given as follows (cf. [**LYY2**] and [**IPSS**, Table 3]).

- For $k \equiv i + 2 \pmod 6$, $m \equiv i - 2 \pmod 6$, the quadruple (x_i, x_k, x_m, μ) forms a normal $\mathcal{G}U_{3A}$ basis. Hence their structures are shown as in $\mathcal{G}U_{3A}$.
- For $l \equiv i+3 \pmod 6$, the triple (x_i, x_l, x) forms a normal $\mathcal{G}U_{2A}$ basis. In particular, we have $x_i \cdot x_l = (1/4)(x_i + x_l - x)$.

- For $j \equiv i+1 \,(\mathrm{mod}\, 6)$, $\{i,j,k,l,m,n\} = \{0,1,2,3,4,5\}$, we have

$$x_i \cdot x_j = \frac{1}{2^5}(x_i + x_j - x_k - x_l - x_m - x_n + x) + \frac{45}{2^{10}}\mu. \tag{10}$$

We also have

$$x \cdot \mu = 0, \qquad \langle x, \mu \rangle = 0 \tag{11}$$

and

$$\langle x_i, x_j \rangle = \frac{5}{2^{10}} \quad \text{for } j \equiv i+1 \,(\mathrm{mod}\, 6). \tag{12}$$

Moreover, for $i, j \in \mathbb{Z}_6$, we have

$$\tau_{x_i}(x_j) = x_{2i-j}. \tag{13}$$

The fixed point sub-algebra $\mathcal{G}^{\tau_{x_i}}$ has dimension 6 and is spanned by x_i, x_l, x, μ, $x_j + x_n$, $x_k + x_m$, where $l \equiv i+3 \,(\mathrm{mod}\, 6)$, $j \equiv i+1 \,(\mathrm{mod}\, 6)$, $n \equiv i-1 \,(\mathrm{mod}\, 6)$, $k \equiv i+2 \,(\mathrm{mod}\, 6)$, $m \equiv i-2 \,(\mathrm{mod}\, 6)$. Moreover we have

$$\sigma_{x_i}(x_j + x_n) = \frac{x_i}{2^4} + \frac{x_l}{2^2} + (x_j + x_n) + \frac{x_k + x_m}{2^2} - \frac{x}{2^2} - \frac{45\mu}{2^7}.$$

We call the ordered set $(x_0, x_1, x_2, x_3, x_4, x_5, x, \mu)$ a *normal* $\mathcal{G}U_{6A}$ *basis.*

3.3. The order 3 automorphism g induced by $\mathcal{W}_{\mathbb{R}}^+$.

Let $L_{\mathbb{C}}(4/5, 0)$ be the Virasoro VOA of central charge $4/5$ and $L_{\mathbb{C}}(4/5, 3)$ be the irreducible $L_{\mathbb{C}}(4/5, 0)$-module of highest weight 3 over the complex field $\mathbb{C}$.

In [**Mi2**], the real form $\mathcal{W}_{\mathbb{R}}^+$ of the W_3-algebra $\mathcal{W}_{\mathbb{C}}(4/5) = L_{\mathbb{C}}(4/5, 0) \oplus L_{\mathbb{C}}(4/5, 3)$ (cf. [**KMY**], [**LLY**]) has been studied.

PROPOSITION 3.1 ([**Mi2**, Theorem 6.1]). *There is a unique real sub-VOA $\mathcal{W}_{\mathbb{R}}^+$ of $\mathcal{W}_{\mathbb{C}}(4/5)$ which possesses a positive definite invariant bilinear form over $\mathbb{R}$ and $\mathcal{W}_{\mathbb{R}}^+ \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{W}_{\mathbb{C}}(4/5)$. This VOA $\mathcal{W}_{\mathbb{R}}^+$ is rational.*

THEOREM 3.2 ([**Mi2**, Theorem 6.2]). *Assume that a VOA V over $\mathbb{R}$ contains a sub-VOA $X \cong \mathcal{W}_{\mathbb{R}}^+$. Then there is an order 3 automorphism $g = g_X$ of V induced by X.*

Now suppose $U \cong U_{3A}$ is contained in a real VOA V satisfying Assumption 1. Let (a_0, a_1, a_2, μ) be a normal $\mathcal{G}U_{3A}$ basis of U. Then U contains a unique sub-VOA X isomorphic to $\mathcal{W}_{\mathbb{R}}^+$ (cf. [**LYY2**], [**SY**]). In this case, the Virasoro element of X is μ. By the theorem above, g defines an order 3 automorphism on V and U.

LEMMA 3.3 ([**LYY2**], [**SY**]). *Let (a_0, a_1, a_2, μ) be a normal $\mathcal{G}U_{3A}$ basis of U and let g be as in Theorem 3.2. Then $\tau_{a_0}\tau_{a_1} = g$ or g^{-1}.*

4. Main Result.

In [**HLY**], McKay's E_6-observation and the Fischer group Fi_{24} were studied. Along with other results, three VOAs $V_{F(1A)}$, $V_{F(2A)}$, and $V_{F(3A)}$ generated by two $3A$ algebras were constructed. We will denote their Griess algebras by $\mathcal{G}V_{F(1A)}$, $\mathcal{G}V_{F(2A)}$, and $\mathcal{G}V_{F(3A)}$ respectively.

The following is our main theorem.

THEOREM 4.1. *Let V be a VOA satisfying Assumption 1. Let $U \cong U_{3A}$ and $U' \cong U_{3A}$ be sub-VOAs of V such that $U \cap U'$ contains a sub-VOA isomorphic to $\mathcal{W}_{\mathbb{R}}^{+}$. Let (a_0, a_1, a_2, μ) and (b_0, b_1, b_2, μ) be normal $\mathcal{G}U_{3A}$ bases of $\mathcal{G}U$ and $\mathcal{G}U'$ respectively and let $\mathcal{G}$ be the sub-Griess algebra generated by $\mathcal{G}U$ and $\mathcal{G}U'$. Then one of the following three cases occur.*

1. $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{1A}$ *and* $\mathcal{G} \cong \mathcal{G}V_{F(1A)}$.
2. $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{2A}$ *or* $\mathcal{G}U_{6A}$ *and* $\mathcal{G} \cong \mathcal{G}V_{F(2A)}$.
3. $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{3A}$ *and* $\mathcal{G} \cong \mathcal{G}V_{F(3A)}$.

REMARK 4.2. In [**HLY**], it was shown that $V_{F(1A)} \cong U_{3A}$, $V_{F(2A)} \cong U_{6A}$ and $V_{F(3A)}$ is isomorphic to the ternary code VOA associated to the ternary tetra code (see [**KMY**]). Its Griess algebra is of dimension 12 and is spanned by nine Ising vectors $x_{i,j}$, $i, j \in \{0,1,2\}$, and four Virasoro vectors μ_1, μ_2, μ_3 and μ_4 of central charge $4/5$ subject to a relation

$$32 \sum_{i,j \in \{0,1,2\}} x_{i,j} - 45(\mu_1 + \mu_2 + \mu_3 + \mu_4) = 0.$$

By Theorem 2.11, there are nine possible structures for $\mathcal{G}\{a_0, b_0\}$. We will prove Theorem 4.1 by analyzing these nine cases in details.

First we recall the order 3 automorphism $g = g_X$ discussed in Section 3.3 for a sub-VOA $X \cong \mathcal{W}_{\mathcal{R}}^{+}$ of $U \cap U'$. By reindexing a_0 and a_1 or b_0 and b_1 if necessary, we may assume that (see Lemma 3.3)

$$\tau_{a_0}\tau_{a_1} = \tau_{b_0}\tau_{b_1} = g.$$

LEMMA 4.3. *We have $\tau_{a_i} g = g^{-1}\tau_{a_i}$ and g commutes with $\tau_{a_i}\tau_{b_j}$ for any $i, j \in \{0,1,2\}$.*

PROOF. Since $g = \tau_{a_0}\tau_{a_1} = \tau_{b_0}\tau_{b_1}$, both τ_{a_i} and τ_{b_j} invert g. Hence, we have

$$\tau_{a_i}\tau_{b_j} g = \tau_{a_i} g^{-1} \tau_{b_j} = g \tau_{a_i}\tau_{b_j}$$

as desired. □

4.1. Case: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{1A}$.

In this case, $a_0 = b_0$. Hence $\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$ and $\mathcal{G} \cong \mathcal{G}U_{3A}$ by the following proposition.

PROPOSITION 4.4. *Suppose $a_i = b_j$ for some $i, j \in \{0,1,2\}$. Then $\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$ and $\mathcal{G} \cong \mathcal{G}U_{3A}$. In particular, $\mathcal{G} \cong \mathcal{G}U_{3A}$ if $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{1A}$.*

PROOF. Without loss, we may assume $a_0 = b_0$. Then by (4) and (6),

$$\begin{aligned}\langle a_0 \cdot \mu, b_1\rangle &= \left\langle \frac{2}{3^2}(2a_0 - a_1 - a_2) + \frac{5}{2^4}\mu, b_1 \right\rangle \\ &= \frac{2}{3^2}\left(2 \cdot \frac{13}{2^{10}} - \langle a_1, b_1\rangle - \langle a_2, b_1\rangle\right) + \frac{5}{2^4} \cdot \frac{1}{2^4}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle a_0, \mu \cdot b_1\rangle &= \left\langle b_0, \frac{2}{3^2}(2b_1 - b_0 - b_2) + \frac{5}{2^4}\mu \right\rangle \\ &= \frac{2}{3^2}\left(2 \cdot \frac{13}{2^{10}} - \frac{1}{2^2} - \frac{13}{2^{10}}\right) + \frac{5}{2^4} \cdot \frac{1}{2^4}\end{aligned}$$

by (4) and (6). Since $\langle a_0 \cdot \mu, b_1\rangle = \langle a_0, \mu \cdot b_1\rangle$ by (1), we have $\langle a_1, b_1\rangle + \langle a_2, b_1\rangle = 267/2^{10}$, which implies $\max\{\langle a_1, b_1\rangle, \langle a_2, b_1\rangle\} \geq (1/2)\cdot(267/2^{10}) > 1/2^5$. Thus, we have $b_1 = a_1$ or $b_1 = a_2$ since by Theorem 2.11 and Remark 2.12, $\langle a_i, b_j\rangle \leq 1/2^5$ if $a_i \neq b_j$. In either case, we have $\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$ and $\mathcal{G}$ is isomorphic to $\mathcal{G}U_{3A}$. □

4.2. Case: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{2A}$.

In this case, set $c_0 = \sigma_{a_0}(b_0)$. Then by [**IPSS**, Table 3], we have $\mathcal{G}\{a_0, b_0\} = \mathrm{Span}\{a_0, b_0, c_0\}$,

$$a_0 \cdot b_0 = \frac{1}{2^2}(a_0 + b_0 - c_0) \quad \text{and} \quad \langle a_0, b_0\rangle = \frac{1}{2^5}. \tag{14}$$

PROPOSITION 4.5. *Suppose $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{2A}$. Then $\mathcal{G} = \mathcal{G}\{a_0, b_1\} = \mathcal{G}\{a_0, b_2\} \cong \mathcal{G}U_{6A}$.*

PROOF. We will first calculate the values of $\langle a_0, b_j\rangle$ for $j = 1, 2$. By (14) and (6), we have

$$\langle a_0 \cdot b_0, b_1\rangle = \left\langle \frac{1}{2^2}(a_0 + b_0 - c_0), b_1 \right\rangle = \frac{1}{2^2}\left(\langle a_0, b_1\rangle + \frac{13}{2^{10}} - \langle c_0, b_1\rangle\right),$$

and by (3)

$$\begin{aligned}\langle a_0, b_0 \cdot b_1\rangle &= \left\langle a_0, \frac{1}{2^4}(2b_0 + 2b_1 + b_2) - \frac{135}{2^{10}}\mu \right\rangle \\ &= \frac{2}{2^4}\cdot\frac{1}{2^5} + \frac{2}{2^4}\langle a_0, b_1\rangle + \frac{1}{2^4}\langle a_0, b_2\rangle - \frac{135}{2^{10}}\cdot\frac{1}{2^4}.\end{aligned}$$

Since $\langle a_0 \cdot b_0, b_1\rangle = \langle a_0, b_0 \cdot b_1\rangle$ by (1), we obtain

$$\langle c_0, b_1\rangle = \frac{123}{2^{12}} + \frac{1}{2}\langle a_0, b_1\rangle - \frac{1}{2^2}\langle a_0, b_2\rangle. \tag{15}$$

Since by Theorem 2.11,

$$\langle a_0, b_1\rangle, \langle a_0, b_2\rangle, \langle c_0, b_1\rangle \in \left\{\frac{1}{2^2}, \frac{1}{2^5}, \frac{13}{2^{10}}, \frac{1}{2^7}, \frac{3}{2^9}, \frac{5}{2^{10}}, \frac{1}{2^8}, 0\right\}, \tag{16}$$

we have

$$\langle c_0, b_1\rangle = \frac{123}{2^{12}} + \frac{1}{2}\langle a_0, b_1\rangle - \frac{1}{2^2}\langle a_0, b_2\rangle \le \frac{123}{2^{12}} + \frac{1}{2}\cdot\frac{1}{2^2} < \frac{1}{2^2}.$$

Hence $c_0 \neq b_1$ and $\langle c_0, b_1\rangle \le 1/2^5$.

We also note that $a_0 \neq b_1$ and $a_0 \neq b_2$; otherwise, $\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$ by Proposition 4.4 and $\mathcal{G}\{a_0, b_0\} \not\cong \mathcal{G}U_{2A}$. Therefore, $\langle a_0, b_1\rangle \le 1/2^5$ and $\langle a_0, b_2\rangle \le 1/2^5$.

Now by (15), we have

$$\langle c_0, b_1\rangle = \frac{123}{2^{12}} + \frac{1}{2}\langle a_0, b_1\rangle - \frac{1}{2^2}\langle a_0, b_2\rangle \ge \frac{123}{2^{12}} - \frac{1}{2^2}\cdot\frac{1}{2^5} = \frac{91}{2^{12}} > \frac{13}{2^{10}},$$

and hence

$$\langle c_0, b_1\rangle = \frac{1}{2^5}. \tag{17}$$

Therefore by (15), we have

$$2^{11}\langle a_0, b_1\rangle = 2^{10}\langle a_0, b_2\rangle + 5. \tag{18}$$

Note that $2^{11}\langle a_0, b_1\rangle$ is an even integer, so $2^{10}\langle a_0, b_2\rangle$ is an odd integer and hence $\langle a_0, b_2\rangle = 5/2^{10}$ or $13/2^{10}$ by (16). If $\langle a_0, b_2\rangle = 13/2^{10}$, then $\langle a_0, b_1\rangle = 9/2^{10}$ which is impossible. Hence, we have $\langle a_0, b_2\rangle = 5/2^{10}$ and $\langle a_0, b_1\rangle = 5/2^{10}$. That means $\mathcal{G}\{a_0, b_1\} \cong \mathcal{G}\{a_0, b_2\} \cong \mathcal{G}U_{6A}$ and $\mathcal{G}\{c_0, b_1\} \cong \mathcal{G}U_{2A}$.

Claim. $\mathcal{G} = \mathcal{G}\{a_0, b_1\}$.

Let $(a_0, b_1, x_2, x_3, x_4, x_5, e, \mu')$ be the normal $\mathcal{G}U_{6A}$ basis for $\mathcal{G}\{a_0, b_1\}$ (see Section 3.2 for the definition). We will show that $x_3 = b_0$, $x_5 = b_2$, $\{x_2, x_4\} = \{a_1, a_2\}$, $e = c_0$, $\mu' = \mu$ and $\mathcal{G} = \mathcal{G}\{a_0, b_1\}$.

Since $\mathcal{G}\{c_0, a_0\} \cong \mathcal{G}\{c_0, b_0\} \cong \mathcal{G}\{c_0, a_1\} \cong \mathcal{G}\{c_0, b_1\} \cong \mathcal{G}U_{2A}$ and $\mathcal{G}$ is generated by a_0, a_1, b_0, b_1, the map σ_{c_0} is well-defined on $\mathcal{G}$. Moreover,

$$\tau_{b_0}\sigma_{c_0}\tau_{b_0} = \sigma_{\tau_{b_0}(c_0)} = \sigma_{c_0}, \tag{19}$$

i.e., τ_{b_0} commutes with σ_{c_0}. Therefore,

$$\tau_{a_0} = \tau_{\sigma_{c_0}(b_0)} = \sigma_{c_0}\tau_{b_0}\sigma_{c_0} = \tau_{b_0}$$

and hence by (13),

$$x_5 = \tau_{a_0}(b_1) = \tau_{b_0}(b_1) = b_2.$$

Since (b_1, x_5, x_3, μ') is a normal $\mathcal{G}U_{3A}$ basis for $\mathcal{G}\{b_1, b_2\}$, we have

$$x_3 = \tau_{b_1}(x_5) = \tau_{b_1}(b_2) = b_0 \quad \text{and} \quad \mu' = \mu.$$

Note that μ and μ' are both determined by b_0 $(= x_3)$, b_1, b_2 $(= x_5)$ using (3).

Recall that $(a_0, b_1, x_2, x_3, x_4, x_5, e, \mu')$ is the normal $\mathcal{G}U_{6A}$ basis for $\mathcal{G}\{a_0, b_1\}$. Thus, we have

$$e = \sigma_{a_0}(x_3) = \sigma_{a_0}(b_0) = c_0.$$

Finally, we will show that $\{a_1, a_2\} = \{x_2, x_4\}$. By (8), we have

$$\begin{aligned}
\sigma_{a_0}(a_1 + a_2) &= -\frac{3}{2^4}a_0 + \frac{a_1 + a_2}{2^2} + \frac{135}{2^7}\mu, \\
\sigma_{a_0}(x_2 + x_4) &= -\frac{3}{2^4}a_0 + \frac{x_2 + x_4}{2^2} + \frac{135}{2^7}\mu'.
\end{aligned}$$

Note that $\mu = \mu'$ and hence

$$\begin{aligned}
&\langle a_1 + a_2, x_2 + x_4 \rangle \\
&\quad = \langle \sigma_{a_0}(a_1 + a_2), \sigma_{a_0}(x_2 + x_4) \rangle \\
&\quad = \left\langle -\frac{3}{2^4}a_0 + \frac{a_1 + a_2}{2^2} + \frac{135}{2^7}\mu, -\frac{3}{2^4}a_0 + \frac{x_2 + x_4}{2^2} + \frac{135}{2^7}\mu \right\rangle \\
&\quad = \frac{3}{2^4} \cdot \frac{3}{2^4} \cdot \frac{1}{2^2} + \frac{1}{2^4}\langle a_1 + a_2, x_2 + x_4 \rangle + \frac{135^2}{2^{14}} \cdot \frac{2}{5} - 2 \cdot \frac{3}{2^4} \cdot \frac{1}{2^2}\left(\frac{13}{2^{10}} + \frac{13}{2^{10}}\right) \\
&\qquad - 2 \cdot \frac{3}{2^4} \cdot \frac{135}{2^7} \cdot \frac{1}{2^4} + 2 \cdot \frac{1}{2^2} \cdot \frac{135}{2^7}\left(\frac{1}{2^4} + \frac{1}{2^4}\right) \\
&\quad = \frac{1}{2^4}\langle a_1 + a_2, x_2 + x_4 \rangle + \frac{8070}{2^{14}},
\end{aligned}$$

which implies

$$\langle a_1 + a_2, x_2 + x_4 \rangle = \frac{269}{2^9}.$$

On the other hand, we also have

$$\langle a_1 + a_2, a_1 + a_2 \rangle = \frac{1}{2^2} + \frac{1}{2^2} + 2 \cdot \frac{13}{2^{10}} = \frac{269}{2^9},$$

and similarly

$$\langle x_2 + x_4, x_2 + x_4 \rangle = \frac{269}{2^9}.$$

Thus, by the Schwartz inequality, we get $a_1 + a_2 = x_2 + x_4$.

Taking inner product with a_1, we get

$$\langle a_1, x_2 \rangle + \langle a_1, x_4 \rangle = \langle a_1, x_2 + x_4 \rangle = \langle a_1, a_1 + a_2 \rangle = \frac{1}{2^2} + \frac{13}{2^{10}} = \frac{77}{2^{10}},$$

which implies $\max\{\langle a_1, x_2 \rangle, \langle a_1, x_4 \rangle\} \geq (1/2) \cdot (77/2^{10}) > 1/2^5$. Then by Theorem 2.11, we have

$$(\langle a_1, x_2 \rangle, \langle a_1, x_4 \rangle) = \left(\frac{1}{2^2}, \frac{13}{2^{10}} \right) \text{ or } \left(\frac{13}{2^{10}}, \frac{1}{2^2} \right).$$

It implies $x_2 = a_1$ or $x_4 = a_1$. In either case, $\{x_2, x_4\} = \{a_1, a_2\}$. Therefore, $\mathcal{G} \subset \mathcal{G}\{a_0, b_1\}$ and thus $\mathcal{G} = \mathcal{G}\{a_0, b_1\}$. $\square$

4.3. Case: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{2B}$.

In this case, $a_0 \cdot b_0 = 0$ and $\langle a_0, b_0 \rangle = 0$ (cf. [**IPSS**, Table 3]). Then, we have

$$\begin{aligned} 0 = \langle a_0 \cdot b_0, \mu \rangle &= \langle a_0, b_0 \cdot \mu \rangle \\ &= \left\langle a_0, \frac{2}{3^2}(2b_0 - b_1 - b_2) + \frac{5}{2^4}\mu \right\rangle \\ &= \frac{-2}{3^2}(\langle a_0, b_1 \rangle + \langle a_0, b_2 \rangle) + \frac{5}{2^4} \cdot \frac{1}{2^4} \end{aligned}$$

by (1), (4) and (6). Therefore we have

$$\langle a_0, b_1 \rangle + \langle a_0, b_2 \rangle = \frac{45}{2^9},$$

which implies $\max\{\langle a_0, b_1 \rangle, \langle a_0, b_2 \rangle\} \geq (1/2) \cdot (45/2^9) > 1/2^5$. It means $a_0 = b_1$ or $a_0 = b_2$ since $\langle a_i, b_j \rangle \leq 1/2^5$ if $a_i \neq b_j$. It is impossible since $\langle b_0, b_1 \rangle = \langle b_0, b_2 \rangle = 13/2^{10}$ by our assumption.

4.4. Case: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{3C}$.

In this case,

$$\langle a_0, b_0 \rangle = \frac{1}{2^8} \tag{20}$$

and there is an Ising vector $c_0 \in \mathcal{G}$ such that

$$a_0 \cdot b_0 = \frac{1}{2^5}(a_0 + b_0 - c_0) \tag{21}$$

(cf. [**IPSS**, Table 3]). Therefore,

$$\begin{aligned}\langle a_0 \cdot b_0, b_1\rangle &= \left\langle \frac{1}{2^5}(a_0 + b_0 - c_0), b_1 \right\rangle \\ &= \frac{1}{2^5}\left(\langle a_0, b_1\rangle + \frac{13}{2^{10}} - \langle c_0, b_1\rangle \right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle a_0, b_0 \cdot b_1\rangle &= \left\langle a_0, \frac{1}{2^4}(2b_0 + 2b_1 + b_2) - \frac{135}{2^{10}}\mu \right\rangle \\ &= \frac{1}{2^4}(2\langle a_0, b_1\rangle + \langle a_0, b_2\rangle) - \frac{127}{2^{14}}\end{aligned}$$

by (3), (6) and (20). Hence (1) implies that

$$0 = (3\langle a_0, b_1\rangle + 2\langle a_0, b_2\rangle + \langle c_0, b_1\rangle) - \frac{267}{2^{10}}.$$

By Proposition 4.4, it is clear that $a_0 \neq b_1$, $a_0 \neq b_2$, $c_0 \neq b_1$. Thus, $\langle a_0, b_1\rangle, \langle a_0, b_2\rangle, \langle c_0, b_1\rangle \leq 1/2^5$ and hence $(3\langle a_0, b_1\rangle + 2\langle a_0, b_2\rangle + \langle c_0, b_1\rangle) - (267/2^{10}) \leq 6 \cdot (1/2^5) - (267/2^{10}) = -75/2^{10} < 0$, which contradicts the above equation. So this case is impossible.

4.5. Case: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{4A}$.

In this case, there exist Ising vectors c_0, d_0, and a Virasoro vector u of central charge 1 (cf. [**IPSS**, Table 3]) so that

$$\mathcal{G}\{a_0, b_0\} = \mathrm{Span}\{a_0, b_0, c_0, d_0, u\}.$$

In addition, $\tau_{a_0}(b_0) = d_0$ and $\mathcal{G}\{b_0, d_0\} \cong \mathcal{G}U_{2B}$. Applying τ_{a_0} to the normal $\mathcal{G}U_{3A}$ basis (b_0, b_1, b_2, μ), we get another normal $\mathcal{G}U_{3A}$ basis $(d_0, \tau_{a_0}(b_1), \tau_{a_0}(b_2), \mu)$. Since $\mathcal{G}\{b_0, d_0\} \cong \mathcal{G}U_{2B}$, this case is also impossible by the analysis of $\mathcal{G}U_{2B}$ (see Section 4.3).

4.6. Case: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{4B}$.

In this case, there exist Ising vectors c_0, d_0, $e \in \mathcal{G}$ such that

$$\mathcal{G}\{a_0, b_0\} = \mathrm{Span}\{a_0, b_0, c_0, d_0, e\},$$

$\mathcal{G}\{a_0, c_0\} = \mathrm{Span}\{a_0, c_0, e\} \cong \mathcal{G}U_{2A}$ and $\mathcal{G}\{b_0, d_0\} = \mathrm{Span}\{b_0, d_0, e\} \cong \mathcal{G}U_{2A}$ (cf. [**IPSS**, Table 3]). Moreover,

$$a_0 \cdot b_0 = \frac{1}{2^5}(a_0 + b_0 - c_0 - d_0 + e), \tag{22}$$

$$\langle a_0, b_0\rangle = \frac{1}{2^8}, \tag{23}$$

and

$$\tau_{b_0}(a_0) = c_0.$$

Applying τ_{b_0} to the normal $\mathcal{G}U_{3A}$ basis (a_0, a_1, a_2, μ), we get another normal $\mathcal{G}U_{3A}$ basis $(c_0, \tau_{b_0}(a_1), \tau_{b_0}(a_2), \mu)$. Then by Proposition 4.5, we have

$$\mathcal{G}\{a_0, a_1, a_2, c_0, \tau_{b_0}(a_1), \tau_{b_0}(a_2), \mu\} = \mathcal{G}\{c_0, a_1\} = \mathcal{G}\{a_0, \tau_{b_0}(a_1)\} \cong \mathcal{G}U_{6A}.$$

Set $x_0 = a_0$, $x_1 = \tau_{b_0}(a_1)$, $x_3 = c_0$, $x_5 = \tau_{b_0}(a_2)$. Then there exists $\{x_2, x_4\} = \{a_1, a_2\}$ such that $(x_0, x_1, x_2, x_3, x_4, x_5, e, \mu)$ forms a normal $\mathcal{G}U_{6A}$ basis for $\mathcal{G}\{c_0, a_1\}$.

Similarly, set $y_0 = b_0$, $y_1 = \tau_{a_0}(b_1)$, $y_3 = d_0$, $y_5 = \tau_{a_0}(b_2)$. There exists $\{y_2, y_4\} = \{b_1, b_2\}$, such that $(y_0, y_1, y_2, y_3, y_4, y_5, e, \mu)$ forms a normal $\mathcal{G}U_{6A}$ basis for $\mathcal{G}\{d_0, b_1\}$.

LEMMA 4.6. *For $i = 1, 2, 4, 5$, $\mathcal{G}\{x_0, y_i\} \cong \mathcal{G}\{x_3, y_i\} \cong \mathcal{G}U_{6A}$, and hence $\langle x_0, y_i\rangle = \langle x_3, y_i\rangle = 5/2^{10}$. Similarly, $\langle x_i, y_0\rangle = \langle x_i, y_3\rangle = 5/2^{10}$ for $i = 1, 2, 4, 5$.*

PROOF. Since (x_0, x_2, x_4, μ), (y_0, y_2, y_4, μ) are normal $\mathcal{G}U_{3A}$ bases, by Lemma 4.3, the order 3 element $\tau_{y_i}\tau_{y_0}$ commutes with $\tau_{y_0}\tau_{x_0}$ for $i = 2, 4$. Since $\mathcal{G}\{x_0, y_0\} \cong \mathcal{G}U_{4B}$, $\tau_{y_0}\tau_{x_0}$ has order 2 or 4. Hence $\tau_{y_i}\tau_{y_0}\cdot\tau_{y_0}\tau_{x_0}$ has order 6 or 12. Since $\tau_{y_i}\tau_{y_0}\cdot\tau_{y_0}\tau_{x_0} = \tau_{y_i}\tau_{x_0}$, by 6-transposition property (Theorem 2.11), $\tau_{y_i}\tau_{x_0}$ must have order ≤ 6 and hence has order 6 and $\mathcal{G}\{x_0, y_i\} \cong \mathcal{G}U_{6A}$ for $i = 2, 4$.

Since $(a_0, d_0) = (x_0, y_3)$, we have $\mathcal{G}\{x_0, y_3\} = \mathcal{G}\{a_0, d_0\} \cong \mathcal{G}U_{4B}$. Since (x_0, x_2, x_4, μ), (y_1, y_3, y_5, μ) form normal $\mathcal{G}U_{3A}$ bases, $\tau_{y_i}\tau_{y_3}$ commutes with $\tau_{y_3}\tau_{x_0}$ for $i = 1, 5$ and thus we also have $\mathcal{G}\{x_0, y_i\} \cong \mathcal{G}U_{6A}$ for $i = 1, 5$ by the same arguments as before. $\square$

PROPOSITION 4.7. *It is impossible that $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{4B}$.*

PROOF. By Lemma 4.6, (23) and (10), we have

$$\begin{aligned}\langle x_1 \cdot x_0, y_0\rangle &= \left\langle \frac{1}{2^5}(x_0 + x_1 - x_2 - x_3 - x_4 - x_5 + e) + \frac{45}{2^{10}}\mu, y_0\right\rangle \\ &= \frac{7}{2^{11}},\end{aligned}$$

and by (22), Lemma 4.6 and (12),

$$\begin{aligned}\langle x_1, x_0 \cdot y_0\rangle &= \left\langle x_1, \frac{1}{2^5}(x_0 + y_0 - x_3 - y_3 + e)\right\rangle \\ &= \frac{3}{2^{12}}.\end{aligned}$$

Hence by (1) we get a contradiction. So this case is impossible. $\square$

4.7. Case: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{5A}$.

In this case, $\tau_{a_0}\tau_{b_0}$ has order 5.

PROPOSITION 4.8. *It is impossible that* $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{5A}$.

PROOF. By Lemma 4.3, the order 3 element $\tau_{a_1}\tau_{a_0}$ commutes with $\tau_{a_0}\tau_{b_0}$ and hence $\tau_{a_1}\tau_{a_0} \cdot \tau_{a_0}\tau_{b_0}$ has order 15. But $\tau_{a_1}\tau_{a_0} \cdot \tau_{a_0}\tau_{b_0} = \tau_{a_1}\tau_{b_0}$, which has order ≤ 6 by the 6-transposition property (Theorem 2.11). It is a contradiction. □

4.8. Case: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{6A}$.

In this case, set $x_0 = a_0$, $x_1 = b_0$. Then there exist x_2, x_3, x_4, x_5, e, μ' such that the ordered set $(x_0, x_1, x_2, x_3, x_4, x_5, e, \mu')$ forms a normal $\mathcal{G}U_{6A}$ basis for $\mathcal{G}\{a_0, b_0\}$.

PROPOSITION 4.9. *Suppose* $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{6A}$. *Then* $\mathcal{G} = \mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{6A}$.

PROOF. Since $\tau_{x_i}(x_j) = x_{2i-j}$ by (13) and μ is fixed by $\tau_{x_0} = \tau_{a_0}$ and $\tau_{x_1} = \tau_{b_0}$, we have

$$\langle x_4, \mu\rangle = \langle \tau_{x_0} x_2, \mu\rangle = \langle x_2, \mu\rangle = \langle \tau_{x_1} x_0, \mu\rangle = \langle x_0, \mu\rangle = \frac{1}{2^4}.$$

Similarly, we also have

$$\langle x_3, \mu\rangle = \langle \tau_{x_1} x_5, \mu\rangle = \langle x_5, \mu\rangle = \langle \tau_{x_0} x_1, \mu\rangle = \langle x_1, \mu\rangle = \frac{1}{2^4}.$$

Now let $h = \tau_{b_0}\tau_{a_0} = \tau_{x_1}\tau_{x_0}$. Then $\mathcal{G}\{h(b_0), h(b_1)\} \cong \mathcal{G}\{b_0, b_1\} \cong \mathcal{G}U_{3A}$ and the set $(h(b_0), h(b_1), h(b_2), h(\mu)) = (x_3, h(b_1), h(b_2), \mu)$ will form a normal $\mathcal{G}U_{3A}$ basis for $\mathcal{G}\{h(b_0), h(b_1)\}$. Note that $h(b_0) = h(x_1) = x_3$ and $h(\mu) = \tau_{b_0}\tau_{a_0}(\mu) = \mu$.

Since $\mathcal{G}\{a_0, x_3\} \cong \mathcal{G}U_{2A}$ and $\{a_0, x_3, e\}$ forms a basis for $\mathcal{G}\{a_0, x_3\}$, by Proposition 4.5, we have $\mathcal{G}\{a_0, a_1, x_3, h(b_1)\} = \mathcal{G}\{a_0, h(b_1)\} \cong \mathcal{G}U_{6A}$. Hence $\langle a_i, e\rangle = 1/2^5$ for $i = 1, 2$ and $\langle e, \mu\rangle = 0$. Similarly we can also prove $\langle b_i, e\rangle = 1/2^5$ for $i = 1, 2$.

Finally, we will show that $\{a_1, a_2\} = \{x_2, x_4\}$ and $\{b_1, b_2\} = \{x_3, x_5\}$. By the structure of the $6A$-algebra, we have

$$\begin{aligned}\langle b_0 \cdot a_0, \mu\rangle &= \left\langle \frac{1}{2^5}(x_0 + x_1 - x_2 - x_3 - x_4 - x_5 + e) + \frac{45}{2^{10}}\mu', \mu \right\rangle \\ &= -\frac{1}{2^8} + \frac{45}{2^{10}}\langle \mu', \mu\rangle\end{aligned}$$

by (10) and (6), and

$$\begin{aligned}\langle b_0, a_0 \cdot \mu\rangle &= \left\langle b_0, \frac{2}{3^2}(2a_0 - a_1 - a_2) + \frac{5}{2^4}\mu \right\rangle \\ &= \frac{50}{2^8 \cdot 3^2} - \frac{2}{3^2}(\langle b_0, a_1\rangle + \langle b_0, a_2\rangle)\end{aligned}$$

by (4) and (12). By (1), it implies

$$\langle \mu', \mu \rangle = \frac{2^2}{3^4 \cdot 5}\big(59 - 2^9(\langle b_0, a_1\rangle + \langle b_0, a_2\rangle)\big). \tag{24}$$

Since $\mathcal{G}\{x_0, x_2\} \cong \mathcal{G}U_{3A}$, we have

$$\begin{aligned}\langle x_2 \cdot x_0, \mu \rangle &= \left\langle \frac{1}{2^4}(2x_0 + 2x_2 + x_4) - \frac{135}{2^{10}}\mu', \mu \right\rangle \\ &= \frac{5}{2^8} - \frac{135}{2^{10}}\langle \mu', \mu\rangle\end{aligned}$$

by (3) and

$$\begin{aligned}\langle x_2, x_0 \cdot \mu \rangle = \langle x_2, a_0 \cdot \mu \rangle &= \left\langle x_2, \frac{2}{3^2}(2a_0 - a_1 - a_2) + \frac{5}{2^4}\mu \right\rangle \\ &= \frac{58}{2^8 \cdot 3^2} - \frac{2}{3^2}(\langle x_2, a_1\rangle + \langle x_2, a_2\rangle)\end{aligned}$$

by (4), which implies

$$\langle \mu', \mu \rangle = \frac{2^2}{3^5 \cdot 5}\big(- 13 + 2^9(\langle x_2, a_1\rangle + \langle x_2, a_2\rangle)\big) \tag{25}$$

by (1). From (24) and (25), we get

$$3\langle b_0, a_1\rangle + 3\langle b_0, a_2\rangle + \langle x_2, a_1\rangle + \langle x_2, a_2\rangle = \frac{95}{2^8},$$

which implies

$$\max\{\langle b_0, a_1\rangle, \langle b_0, a_2\rangle, \langle x_2, a_1\rangle, \langle x_2, a_2\rangle\} \geq \frac{95}{2^8(3+3+1+1)} > \frac{1}{2^5}. \tag{26}$$

By Proposition 4.4, $a_i \neq b_j$ for any $i, j \in \{0, 1, 2\}$ and thus we must have $x_2 = a_1$ or $x_2 = a_2$. A similar argument also shows that $x_3 = b_1$ or b_2. Therefore, $\mathcal{G} = \mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{6A}$. □

4.9. Case: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{3A}$.

In this case, there exists c_0 and μ_0 such that (a_0, b_0, c_0, μ_0) forms a normal $\mathcal{G}U_{3A}$ basis.

Lemma 4.10. *Let (a_0, a_1, a_2, μ) and (b_0, b_1, b_2, μ) be normal $\mathcal{G}U_{3A}$ bases. Suppose $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{3A}$. Then either*

1. *$\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$ and $\mathcal{G} \cong \mathcal{G}U_{3A}$; or*
2. *$\mathcal{G}\{a_i, b_j\} \cong \mathcal{G}U_{3A}$ for $i, j \in \mathbb{Z}_3$.*

Proof. By Lemma 4.3, for $i = 1, 2$, the order 3 element $\tau_{a_i}\tau_{a_0}$ commutes with $\tau_{a_0}\tau_{b_0}$, which has order 3 by assumption. Hence $\tau_{a_i}\tau_{a_0} \cdot \tau_{a_0}\tau_{b_0} = \tau_{a_i}\tau_{b_0}$ has order 1 or 3 for $i = 1, 2$.

CASE 1. If $\tau_{a_i}\tau_{b_0}$ is of order 1, then $\tau_{a_i}\tau_{a_0} = (\tau_{a_0}\tau_{b_0})^{-1}$ and we have

$$a_j = \tau_{a_i}\tau_{a_0}a_0 = \tau_{b_0}\tau_{a_0}a_0 = c_0,$$

where $\{0, i, j\} = \{0, 1, 2\}$. Thus, by Proposition 4.4, we have $b_0 \in \{a_0, a_1, a_2\}$ and $\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$.

CASE 2. If $\tau_{a_i}\tau_{b_0}$ has order 3, then $\mathcal{G}\{a_i, b_0\} \cong \mathcal{G}U_{3A}$, $\mathcal{G}U_{3C}$ or $\mathcal{G}U_{6A}$.

By the discussion in Section 4.4, $\mathcal{G}\{a_i, b_0\} \cong \mathcal{G}U_{3C}$ is impossible.

If $\mathcal{G}\{a_i, b_0\} \cong \mathcal{G}U_{6A}$, then by Proposition 4.9, $\langle a_0, b_0\rangle = 1/32$ or $5/2^{10}$, which is again impossible since $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{3A}$. Therefore, $\mathcal{G}\{a_i, b_0\} \cong \mathcal{G}U_{3A}$ is the only possible case. Similarly, we also have $\mathcal{G}\{a_i, b_j\} \cong \mathcal{G}U_{3A}$ for any $i, j = 0, 1, 2$. □

From now on, we assume $\{a_0, a_1, a_2\} \neq \{b_0, b_1, b_2\}$, which implies $\mathcal{G}\{a_i, b_j\} \cong \mathcal{G}U_{3A}$ for all $i \neq j$.

Recall that $g = \tau_{a_0}\tau_{a_1} = \tau_{a_2}\tau_{a_0}$ is of order 3.

NOTATION 4.11. Let $h = \tau_{a_0}\tau_{b_0}$. Then h is of order 3 and it commutes with g by Lemma 4.3. Moreover, we have

$$\tau_{a_2}\tau_{b_0} = \tau_{a_2}\tau_{a_0} \cdot \tau_{a_0}\tau_{b_0} = gh,$$
$$\tau_{a_1}\tau_{b_0} = \tau_{a_1}\tau_{a_0} \cdot \tau_{a_0}\tau_{b_0} = g^2h.$$

For $i, j = 0, 1, 2$, denote

$$x_{i,j} = h^i g^j(a_0).$$

Note that $x_{0,0} = a_0$, $x_{0,1} = g(a_0) = a_1$, $x_{0,2} = g^2(a_0) = a_2$, and $x_{1,0} = h(a_0) = b_0$. By definition, it is also easy to see that

$$h^k g^\ell(x_{i,j}) = x_{i+k,\, j+\ell}, \quad \text{for } i, j, k, \ell \in \mathbb{Z}_3.$$

NOTATION 4.12. For any $(i, j) \neq (0, 0)$, denote

$$\mathcal{G}_{i,j,0} = \mathcal{G}\{x_{0,0}, x_{i,j}\} \cong \mathcal{G}U_{3A}.$$

Then there exists a Virasoro vector $\mu_{i,j,0}$ of central charge $4/5$ such that $(x_{0,0}, x_{i,j}, x_{2i,2j}, \mu_{i,j,0})$ forms a normal $3A$-basis of $\mathcal{G}_{i,j,0}$. For $k = 1, 2$, we denote

$$\mathcal{G}_{0,1,k} = h^k(\mathcal{G}_{0,1,0}) = h^k(\mathcal{G}_{0,2,0}).$$

Then $\mathcal{G}_{0,1,k} \cong \mathcal{G}U_{3A}$ and there is a Virasoro vector $\mu_{0,1,k}$ of central charge $4/5$ such that $(x_{k,0}, x_{k,1}, x_{k,2}, \mu_{0,1,k})$ forms a normal basis for $\mathcal{G}_{0,1,k}$.

REMARK 4.13. By our assumption, we have $\mu_{0,1,0} = \mu_{0,1,1} = \mu_{0,1,2} = \mu$. We use $\mu_{0,1}$ to denote $\mu_{0,1,0} = \mu_{0,1,1} = \mu_{0,1,2}$. Note that $\mu_{0,1}$ is fixed by $\tau_{x_{i,j}}$ for all i, j.

NOTATION 4.14. For $(i,j) \neq (0,0), (0,1)$ and $(0,2)$, we denote

$$\mathcal{G}_{i,j,k} = g^k(\mathcal{G}_{i,j,0}).$$

Then, $\mathcal{G}_{i,j,k} \cong \mathcal{G}U_{3A}$ for any $k = 0,1,2$. Let $\mu_{i,j,k}$ be the Virasoro vector of central charge $4/5$ such that $(x_{0,k}, x_{i,j+k}, x_{2i,2j+k}, \mu_{i,j,k})$ forms a normal $\mathcal{G}U_{3A}$ basis for $\mathcal{G}_{i,j,k}$. Note that $\mu_{i,j,k} = \mu_{2i,2j,k}$ and $g^\ell(\mu_{i,j,k}) = \mu_{i,j,k+\ell}$ for any $i \neq 0$.

We will show $\mu_{1,i,j} = \mu_{1,i,k}$ for all i, j, k (Lemma 4.23). This turns out to be the most complicated part of the proof.

LEMMA 4.15. *For any $n, i, k, \ell \in \mathbb{Z}_3$, we have*

$$\tau_{x_{n,n\ell+k}}(\mu_{1,\ell,i}) = \mu_{1,\ell,-k-i}. \tag{27}$$

PROOF. By Lemma 4.3, we have

$$\tau_{x_{i,j}}(x_{k,\ell}) = h^i g^j \tau_{a_0} g^{-j} h^{-i} h^k g^\ell(a_0) = h^{-k+2i} g^{-\ell+2j} \tau_{a_0}(a_0) = x_{-i-k,-j-\ell}.$$

Thus, $\tau_{x_{n,n\ell+k}}$ maps the normal $\mathcal{G}U_{3A}$ basis $(x_{0,i}, x_{1,i+\ell}, x_{2,i+2\ell}, \mu_{1,\ell,i})$ to

$$(x_{-n,-n\ell-k-i}, x_{-n-1,-n\ell-k-i-\ell}, x_{-n-2,-n\ell-k-i-2\ell}, \tau_{x_{n,n\ell+k}}(\mu_{1,\ell,i})).$$

Then we have

$$\begin{aligned}
&\{x_{-n,-n\ell-k-i}, x_{-n-1,-n\ell-k-i-\ell}, x_{-n-2,-n\ell-k-i-2\ell}\} \\
&\quad = \{x_{0,-k-i}, x_{-1,-k-i-\ell}, x_{-2,-k-i-2\ell}\} \\
&\quad = \{x_{0,-k-i}, x_{2,-k-i+2\ell}, x_{1,-k-i+\ell}\}.
\end{aligned}$$

Since $(x_{0,-k-i}, x_{1,-k-i+\ell}, x_{2,-k-i+2\ell}, \mu_{1,\ell,-k-i})$ forms a normal $\mathcal{G}U_{3A}$ basis, we have that $\tau_{x_{n,n\ell+k}}(\mu_{1,\ell,i}) = \mu_{1,\ell,-k-i}$. $\square$

LEMMA 4.16. *For any $i, j \in \mathbb{Z}_3$, $y \in \{\mu_{0,1}, \mu_{1,0,k}, \mu_{1,1,k}, \mu_{1,2,k} \mid k = 0,1,2\}$, we have*

$$\langle x_{i,j}, y\rangle = \frac{1}{2^4} \tag{28}$$

and

$$\langle \mu_{1,i,j}, \mu_{1,k,\ell}\rangle = 0, \quad \langle \mu_{0,1}, \mu_{1,i,j}\rangle = 0 \tag{29}$$

for all i, j, and for $k \neq i$.

PROOF. By (3),

$$\begin{aligned}\langle x_{0,0}, x_{0,1}\cdot x_{1,0}\rangle &= \left\langle x_{0,0}, \frac{1}{2^4}(2x_{0,1}+2x_{1,0}+x_{2,2}) - \frac{135}{2^{10}}\mu_{1,2,1}\right\rangle \\ &= \frac{65}{2^{14}} - \frac{135}{2^{10}}\langle x_{0,0}, \mu_{1,2,1}\rangle\end{aligned}$$

and

$$\begin{aligned}\langle x_{0,0}, x_{0,1}\cdot x_{1,0}\rangle &= \langle x_{0,0}\cdot x_{0,1}, x_{1,0}\rangle \\ &= \left\langle \frac{1}{2^4}(2x_{0,1}+2x_{1,0}+x_{2,2}) - \frac{135}{2^{10}}\mu_{0,1}, x_{1,0}\right\rangle \\ &= \frac{65}{2^{14}} - \frac{135}{2^{10}}\cdot\frac{1}{2^4}.\end{aligned}$$

Therefore, we have $\langle x_{0,0}, \mu_{1,2,1}\rangle = 1/2^4$. Similarly, we can get (28).

By (28),

$$\begin{aligned}\langle \mu_{1,0,1}, x_{0,1}\cdot x_{1,0}\rangle &= \left\langle \mu_{1,0,1}, \frac{1}{2^4}(2x_{0,1}+2x_{1,0}+x_{2,2}) - \frac{135}{2^{10}}\mu_{1,2,1}\right\rangle \\ &= \frac{5}{2^8} - \frac{135}{2^{10}}\langle \mu_{1,0,1}, \mu_{1,2,1}\rangle\end{aligned}$$

and

$$\begin{aligned}\langle \mu_{1,0,1}, x_{0,1}\cdot x_{1,0}\rangle &= \langle \mu_{1,0,1}\cdot x_{0,1}, x_{1,0}\rangle \\ &= \left\langle \frac{2}{3^2}(2x_{0,1}-x_{2,1}-x_{1,1}) + \frac{5}{2^4}\mu_{1,0,1}, x_{1,0}\right\rangle \\ &= \frac{5}{2^8}.\end{aligned}$$

Thus, we get $\langle \mu_{1,0,1}, \mu_{1,2,1}\rangle = 0$. Similar argument gives (29). □

LEMMA 4.17. *We have*

$$\mu_{1,i,j}\cdot\mu_{1,k,\ell} = 0 \tag{30}$$

for $i \neq k$, and

$$\mu_{0,1}\cdot\mu_{1,i,j} = 0 \tag{31}$$

for $i \in \mathbb{Z}_3$.

PROOF. By Theorem 2.3, (5) and (29), we have

$$\begin{aligned}\langle \mu_{1,i,j}\cdot\mu_{1,k,\ell},\mu_{1,i,j}\cdot\mu_{1,k,\ell}\rangle &\leq \langle \mu_{1,i,j}\cdot\mu_{1,i,j},\mu_{1,k,\ell}\cdot\mu_{1,k,\ell}\rangle\\ &= \langle 2\mu_{1,i,j},2\mu_{1,k,\ell}\rangle\\ &= 0.\end{aligned}$$

Since the inner product is positive definite by Assumption 1, we have (30). Similarly, we can get (31). □

LEMMA 4.18. *For $x \in \{x_{i,j} \mid i,j\}$, $\mu' \in \{\mu_{0,1},\mu_{1,i,j} \mid i,j\}$, we have*

$$x\cdot\mu' = \frac{1}{2}x + \frac{5}{2^5}\mu' + \frac{3}{2^5}\tau_x(\mu') - \frac{1}{2^3}\sigma_x(\mu'+\tau_x(\mu')). \tag{32}$$

PROOF. By Lemma 2.9 and (28), we have

$$\begin{aligned}x\cdot\mu' &= 8\langle x,\mu'\rangle x + \frac{1}{2^2}\left(\frac{1}{2}(\mu'+\tau_x(\mu')) - \sigma_x\left(\frac{1}{2}(\mu'+\tau_x(\mu'))\right)\right) + \frac{1}{2^5}(\mu'-\tau_x(\mu'))\\ &= \frac{1}{2}x + \frac{5}{2^5}\mu' + \frac{3}{2^5}\tau_x(\mu') - \frac{1}{2^3}\sigma_x(\mu'+\tau_x(\mu'))\end{aligned}$$

as desired. □

LEMMA 4.19. *For $i \in \{0,1,2\}$, we have*

$$\langle \mu_{1,i,0},\mu_{1,i,2}\rangle = \langle \mu_{1,i,1},\mu_{1,i,2}\rangle = \langle \mu_{1,i,0},\mu_{1,i,1}\rangle. \tag{33}$$

PROOF. Since $g \in \mathrm{Aut}(\mathcal{G})$ preserve the inner product, we have

$$\langle \mu_{1,i,0},\mu_{1,i,1}\rangle = \langle g^j(\mu_{1,i,0}),g^j(\mu_{1,i,1})\rangle = \langle \mu_{1,i,j},\mu_{1,i,1+j}\rangle$$

for any $j = 0,1,2$. □

LEMMA 4.20. *For $x = x_{k,\ell}$, $\mu' = \mu_{1,i,j}$, $\mu'' = \tau_x(\mu')$, we have*

$$\langle \sigma_x(\mu'+\mu''),\mu'\rangle = \frac{-1}{2^2} + \frac{3}{2^2}\langle \mu',\mu''\rangle. \tag{34}$$

PROOF. By (32), (28), and (29), we have

$$\begin{aligned}\langle x\cdot\mu',\mu'\rangle &= \left\langle \frac{1}{2}x + \frac{5}{2^5}\mu' + \frac{3}{2^5}\mu'' - \frac{1}{2^3}\sigma_x(\mu'+\mu''),\mu'\right\rangle\\ &= \frac{3}{2^5} + \frac{3}{2^5}\langle \mu',\mu''\rangle - \frac{1}{2^3}\langle \sigma_x(\mu'+\mu''),\mu'\rangle.\end{aligned}$$

By (5), we also have

$$\langle x \cdot \mu', \mu' \rangle = \langle x, \mu' \cdot \mu' \rangle = 2\langle x, \mu' \rangle = \frac{1}{2^3}.$$

Hence we get

$$\langle \sigma_x(\mu' + \mu''), \mu' \rangle = \frac{-1}{2^2} + \frac{3}{2^2}\langle \mu', \mu'' \rangle$$

as desired. □

LEMMA 4.21. *Let $\mu' = \mu_{i,j,k}$ and $\mu'' = \mu_{i',j',k'}$. If $(i,j) \neq (i',j')$ or $(2i', 2j')$, then we have*

$$\langle \sigma_x(\mu' + \tau_x(\mu')), \mu'' \rangle = \frac{1}{2^2} \tag{35}$$

for any $x = x_{k,\ell}$.

PROOF. By Lemma 4.17, we have $\mu' \cdot \mu'' = 0$ and $\langle \mu', \mu'' \rangle = \langle \tau_x(\mu'), \mu'' \rangle = 0$. Hence by (1) and (32),

$$\begin{aligned} 0 &= \langle x, \mu' \cdot \mu'' \rangle = \langle x \cdot \mu', \mu'' \rangle \\ &= \left\langle \frac{1}{2}x + \frac{5}{2^5}\mu' + \frac{3}{2^5}\tau_x(\mu') - \frac{1}{2^3}\sigma_x(\mu' + \tau_x(\mu')), \mu'' \right\rangle \\ &= \frac{1}{2^5} - \frac{1}{2^3}\langle \sigma_x(\mu' + \tau_x(\mu')), \mu'' \rangle, \end{aligned}$$

which implies (35). □

LEMMA 4.22. *We have*

$$\begin{aligned} &6075\mu_{0,1} \cdot \mu_{1,1,1} \\ &\quad = 64x_{0,1} - 656(x_{0,0} + x_{0,2}) - 576(x_{1,2} + x_{2,0}) + 384(x_{1,0} + x_{1,1} + x_{2,1} + x_{2,2}) \\ &\qquad + 810\mu_{0,1} + 1260\mu_{1,1,1} - 135(\mu_{1,1,0} + \mu_{1,1,2}) + 360(\mu_{1,0,1} + \mu_{1,2,1}) \\ &\qquad + 45(\mu_{1,0,0} + \mu_{1,0,2} + \mu_{1,2,0} + \mu_{1,2,2}) - 720(\sigma_{x_{0,1}}(\mu_{1,0,0} + \mu_{1,0,2} + \mu_{1,2,0} + \mu_{1,2,2}) \\ &\qquad + 180\big(\sigma_{x_{0,0}}(\mu_{1,1,1} + \mu_{1,1,2}) + \sigma_{x_{0,2}}(\mu_{1,1,0} + \mu_{1,1,1})\big) \\ &\quad = 0. \end{aligned}$$

PROOF. We will expand both sides of the equality

$$\sigma_{x_{0,1}}\big((x_{0,2} + x_{0,0}) \cdot (x_{1,2} + x_{2,0})\big) = \sigma_{x_{0,1}}(x_{0,2} + x_{0,0}) \cdot \sigma_{x_{0,1}}(x_{1,2} + x_{2,0}).$$

By (8), we have

$$
\begin{aligned}
&\sigma_{x_{0,1}}\big((x_{0,2}+x_{0,0})\cdot(x_{1,2}+x_{2,0})\big)\\
&\quad=\sigma_{x_{0,1}}\bigg(\frac{1}{2^4}(2x_{0,2}+2x_{1,2}+x_{2,2})+\frac{1}{2^4}(2x_{0,2}+2x_{2,0}+x_{1,1})+\frac{1}{2^4}(2x_{0,0}+2x_{1,2}+x_{2,1})\\
&\qquad\qquad+\frac{1}{2^4}(2x_{0,0}+2x_{2,0}+x_{1,0})-\frac{135}{2^{10}}(\mu_{1,0,2}+\mu_{1,2,2}+\mu_{1,2,0}+\mu_{1,0,0})\bigg)\\
&\quad=\frac{1}{2^4}x_{0,0}-\frac{15}{2^7}x_{0,1}+\frac{1}{2^4}x_{0,2}+\frac{1}{2^6}x_{1,0}+\frac{1}{2^6}x_{1,1}+\frac{1}{2^4}x_{1,2}+\frac{1}{2^4}x_{2,0}+\frac{1}{2^6}x_{2,1}\\
&\qquad+\frac{1}{2^6}x_{2,2}+\frac{135}{2^9}\mu_{0,1}+\frac{135}{2^9}\mu_{1,1,1}+\frac{135}{2^{11}}\mu_{1,0,1}+\frac{135}{2^{11}}\mu_{1,2,1}\\
&\qquad-\frac{135}{2^{10}}\sigma_{x_{0,1}}(\mu_{1,0,0}+\mu_{1,0,2})-\frac{135}{2^{10}}\sigma_{x_{0,1}}(\mu_{1,2,0}+\mu_{1,2,2}).
\end{aligned}
\tag{36}
$$

By (8), (32), and (27), we also have

$$
\begin{aligned}
&\sigma_{x_{0,1}}(x_{0,2}+x_{0,0})\cdot\sigma_{x_{0,1}}(x_{1,2}+x_{2,0})\\
&\quad=\bigg(\frac{-3}{2^4}x_{0,1}+\frac{1}{2^2}x_{0,2}+\frac{1}{2^2}x_{0,0}+\frac{135}{2^7}\mu_{0,1}\bigg)\cdot\bigg(\frac{-3}{2^4}x_{0,1}+\frac{1}{2^2}x_{1,2}+\frac{1}{2^2}x_{2,0}+\frac{135}{2^7}\mu_{1,1,1}\bigg)\\
&\quad=\frac{187}{2^{10}}x_{0,0}-\frac{33}{2^8}x_{0,1}+\frac{187}{2^{10}}x_{0,2}-\frac{7}{2^7}x_{1,0}-\frac{7}{2^7}x_{1,1}+\frac{43}{2^8}x_{1,2}+\frac{43}{2^8}x_{2,0}\\
&\qquad-\frac{7}{2^7}x_{2,1}-\frac{7}{2^7}x_{2,2}+\frac{945}{2^{13}}\mu_{0,1}-\frac{135}{2^{14}}(\mu_{1,0,0}+\mu_{1,0,2}+\mu_{1,2,0}+\mu_{1,2,2})+\frac{135}{2^{12}}\mu_{1,1,1}\\
&\qquad+\frac{405}{2^{14}}(\mu_{1,1,0}+\mu_{1,1,2})-\frac{135}{2^{12}}\big(\sigma_{x_{0,0}}(\mu_{1,1,1}+\mu_{1,1,2})+\sigma_{x_{0,2}}(\mu_{1,1,0}+\mu_{1,1,1})\big)\\
&\qquad+\frac{18225}{2^{14}}\mu_{0,1}\cdot\mu_{1,1,1}.
\end{aligned}
\tag{37}
$$

Hence we have by (31), (36), (37),

$$
\begin{aligned}
0&=6075\mu_{0,1}\cdot\mu_{1,1,1}\\
&=64x_{0,1}-656(x_{0,0}+x_{0,2})-576(x_{1,2}+x_{2,0})+384(x_{1,0}+x_{1,1}+x_{2,1}+x_{2,2})\\
&\quad+810\mu_{0,1}+1260\mu_{1,1,1}-135(\mu_{1,1,0}+\mu_{1,1,2})+360(\mu_{1,0,1}+\mu_{1,2,1})\\
&\quad+45(\mu_{1,0,0}+\mu_{1,0,2}+\mu_{1,2,0}+\mu_{1,2,2})-720(\sigma_{x_{0,1}}(\mu_{1,0,0}+\mu_{1,0,2}+\mu_{1,2,0}+\mu_{1,2,2})\\
&\quad+180\big(\sigma_{x_{0,0}}(\mu_{1,1,1}+\mu_{1,1,2})+\sigma_{x_{0,2}}(\mu_{1,1,0}+\mu_{1,1,1})\big),
\end{aligned}
$$

as desired. □

LEMMA 4.23. *For $i,k,\ell\in\{0,1,2\}$, we have*

$$
\langle\mu_{1,i,k},\mu_{1,i,\ell}\rangle=\frac{2}{5}.
\tag{38}
$$

Hence, $\mu_{1,i,k} = \mu_{1,i,\ell}$ *for any* i, k, ℓ.

Proof. By Lemma 4.22 and (33), (34), and (35), we have

$$\begin{aligned}
0 &= \langle 6075\mu_{0,1} \cdot \mu_{1,1,1}, \mu_{1,0,0}\rangle \\
&= 64 \cdot \frac{1}{2^4} - 656\left(\frac{1}{2^4} + \frac{1}{2^4}\right) - 576\left(\frac{1}{2^4} + \frac{1}{2^4}\right) + 384\left(\frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4}\right) \\
&\quad + 810 \cdot 0 + 1260 \cdot 0 - 135(0+0) + 360(\langle \mu_{1,0,0}, \mu_{1,0,1}\rangle + 0)r \\
&\quad + 45\left(\frac{2}{5} + \langle \mu_{1,0,0}, \mu_{1,0,1}\rangle + 0 + 0\right) - 720\left(-\frac{1}{2^2} + \frac{3}{2^2}\langle \mu_{1,0,0}, \mu_{1,0,1}\rangle + \frac{1}{2^2}\right) \\
&\quad + 180\left(\frac{1}{2^2} + \frac{1}{2^2}\right) \\
&= 54 - 135\langle \mu_{1,0,0}, \mu_{1,0,1}\rangle,
\end{aligned}$$

which implies $\langle \mu_{1,0,0}, \mu_{1,0,1}\rangle = 2/5$. Similarly, one can prove $\langle \mu_{1,i,k}, \mu_{1,i,\ell}\rangle = 2/5$, also. □

Notation 4.24. We denote $\mu_{1,i,0} = \mu_{1,i,1} = \mu_{1,i,2}$ by $\mu_{1,i}$ for $i \in \{0, 1, 2\}$.

Proposition 4.25. *For any* $(i, j) \neq (i', j')$, *we have*

$$\mu_{i,j} \cdot \mu_{i',j'} = 0. \tag{39}$$

Moreover,

$$\begin{aligned}
&\mu_{0,1} + \mu_{1,0} + \mu_{1,1} + \mu_{1,2} \\
&\quad = \frac{32}{45}(x_{0,0} + x_{0,1} + x_{0,2} + x_{1,0} + x_{1,1} + x_{1,2} + x_{2,0} + x_{2,1} + x_{2,2}).
\end{aligned} \tag{40}$$

Therefore, the dimension of $\mathcal{G}$ *is* 12.

Proof. The first assertion follows from (29) and Lemma 4.23.

To prove (40), let

$$\begin{aligned}
\tilde{\mu} &= \mu_{0,1} + \mu_{1,0} + \mu_{1,1} + \mu_{1,2}, \\
\tilde{x} &= \frac{32}{45}(x_{0,0} + x_{0,1} + x_{0,2} + x_{1,0} + x_{1,1} + x_{1,2} + x_{2,0} + x_{2,1} + x_{2,2}).
\end{aligned}$$

Then by Lemmas 4.16, 4.22, and $\langle \mu_{i,j}, \mu_{i',j'}\rangle = 0$ for $(i, j) \neq (i'j')$, we have

$$\langle \tilde{\mu} - \tilde{x}, \tilde{\mu} - \tilde{x}\rangle = 0$$

and thus $\tilde{\mu} = \tilde{x}$ as desired.

To check the dimension of $\mathcal{G}$, for $\{a_1, a_2, \ldots, a_{12}\} = \{x_{i,j}, \mu_{0,1}, \mu_{1,0}, \mu_{1,1} \mid i, j \in \mathbb{Z}_3\}$, we can get $\det(\langle a_i, a_j\rangle) = 3^{42}/2^{86} \cdot 5^2 \neq 0$ by computer. Hence the dimension of $\mathcal{G}$

is 12. □

REMARK 4.26. From our proof, we have shown that $(x_{i_0,j_0}, x_{i_1,j_1}, x_{i_2,j_2}, \mu_{i,j})$ forms a normal $\mathcal{G}U_{3A}$ basis of $\mathcal{G}\{x_{i_0,j_0}, x_{i_1,j_1}\}$ if and only if

$$\begin{cases} (i_0, j_0) + (i_1, j_1) + (i_2, j_2) \equiv (0, 0) \pmod 3, \\ (i_1, j_1) - (i_0, j_0) \equiv \pm(i, j) \pmod 3. \end{cases}$$

The Griess algebra $\mathcal{G}$ is isomorphic to $\mathcal{G}V_{F(3A)}$ and the structure is summarized as in Figure 1.

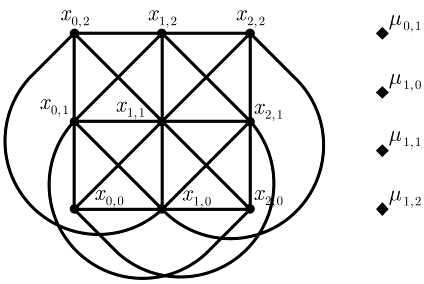

Figure 1. Configuration for $\mathcal{G}V_{F(3A)}$.

References

[Co] J. H. Conway, A simple construction for the Fischer-Griess monster group, Invent. Math., **79** (1985), 513–540.

[DMZ] C. Y. Dong, G. Mason and Y. C. Zhu, Discrete series of the Virasoro algebra and the moonshine module, In: Algebraic Groups and Their Generalizations: Quantum and Infinite-Dimensional Methods, Pennsylvania, 1991, (eds. William J. Haboush and Brian J. Parshall), Proc. Sympos. Pure Math., **56**, part 2, Amer. Math. Soc., Providence, RI, 1994, pp. 295–316.

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On Axiomatic Approaches to Vertex Operator Algebras and Modules, Mem. Amer. Math. Soc., **104**, no. 494, Amer. Math. Soc., Providence, RI, 1993.

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure Appl. Math., **134**, Academic Press, New York, 1988.

[Hö] G. Höhn, The group of symmetries of the shorter moonshine module, Abh. Math. Semin. Univ. Hambg., **80** (2010), 275–283.

[HLY] G. Höhn, C. H. Lam and H. Yamauchi, McKay's E_6-observation on the largest Fischer group, Comm. Math. Phys., **310** (2012), 329–365.

[IPSS] A. A. Ivanov, D. V. Pasechnik, Á. Seress and S. Shpectorov, Majorana representations of the symmetric group of degree 4, J. Algebra, **324** (2010), 2432–2463.

[KMY] M. Kitazume, M. Miyamoto and H. Yamada, Ternary codes and vertex operator algebras, J. Algebra, **223** (2000), 379–395.

[Li] H. S. Li, Symmetric invariant bilinear forms on vertex operator algebras, J. Pure Appl. Algebra, **96** (1994), 279–297.

[LLY] C. H. Lam, N. Lam and H. Yamauchi, Extension of unitary Virasoro vertex operator algebra by a simple module, Int. Math. Res. Not., **11** (2003), 577–611.

[LS] C. H. Lam and C. S. Su, Griess algebras generated by 3 Ising vectors of central 2A-type, J. Algebra, **374** (2013), 141–166.

[LYY] C. H. Lam, H. Yamada and H. Yamauchi, Vertex operator algebras, extended E_8 diagram, and McKay's observation on the Monster simple group, Trans. Amer. Math. Soc., **359** (2007), 4107–4123.

[LYY2] C. H. Lam, H. Yamada and H. Yamauchi, McKay's observation and vertex operator algebras generated by two conformal vectors of central charge 1/2, IMRP Int. Math. Res. Pap., **3** (2005), 117–181.

[Mi1] M. Miyamoto, Griess algebras and conformal vectors in vertex operator algebras, J. Algebra, **179** (1996), 523–548.

[Mi2] M. Miyamoto, 3-state Potts model and automorphisms of vertex operator algebras of order 3, J. Algebra, **239** (2001), 56–76.

[Mi3] M. Miyamoto, Vertex operator algebras generated by two conformal vectors whose τ-involutions generate S_3, J. Algebra, **268** (2003), 653–671.

[Mi4] M. Miyamoto, A new construction of the moonshine vertex operator algebra over the real number field, Ann. of Math. (2), **159** (2004), 535–596.

[Sa] S. Sakuma, 6-transposition property of τ-involutions of vertex operator algebras, Int. Math. Res. Not. IMRN, **2007**, Art. ID rnm 030.

[SY] S. Sakuma and H. Yamauchi, Vertex operator algebra with two Miyamoto involutions generating S_3, J. Algebra, **267** (2003), 272–297.

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