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Complete linear Weingarten hypersurfaces immersed in the hyperbolic space

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Abstract. In this paper, we apply the Hopf's strong maximum principle in order to obtain a suitable characterization of the complete linear Weingarten hypersurfaces immersed in the hyperbolic space $\mathbb{H}^{n+1}$. Under the assumption that the mean curvature attains its maximum and supposing an appropriated restriction on the norm of the second fundamental form, we prove that such a hypersurface must be either totally umbilical or isometric to a hyperbolic cylinder of $\mathbb{H}^{n+1}$.

1. Introduction.

In the seminal paper [**4**], Cheng and Yau introduced a new self-adjoint differential operator $\square$ acting on smooth functions defined on Riemannian manifolds. As a by-product of such approach they were able to classify closed hypersurfaces M^n with constant normalized scalar curvature R satisfying $R \geq c$ and nonnegative sectional curvature immersed in a real space form Q_c^{n+1} of constant sectional curvature c. Later on, Li [**5**] extended the results due to Cheng and Yau [**4**] in terms of the squared norm of the second fundamental form of the hypersurface M^n. In [**11**], Shu have used the so-called generalized maximum principle of Omori-Yau [**9**], [**13**] to prove that a complete hypersurface M^n in the hyperbolic space $\mathbb{H}^{n+1}$ with constant normalized scalar curvature $R \geq -1$ and nonnegative sectional curvature must be either totally umbilical or isometric to a hyperbolic cylinder $\mathbb{S}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, where $c_1 > 0$, $c_2 < 0$ and $(1/c_1) + (1/c_2) = -1$.

In [**6**], Li studied the rigidity of compact hypersurfaces with nonnegative sectional curvature immersed in a unit sphere with scalar curvature proportional to mean curvature. More recently, Li et al. [**7**] extended the result of [**4**] and [**6**] by considering *linear Weingarten* hypersurfaces immersed in a unit sphere, that is, hypersurfaces of $\mathbb{S}^{n+1}$ whose mean curvature H and normalized scalar curvature R satisfy $R = aH + b$, for some $a, b \in \mathbb{R}$. In this setting, Li showed that if M^n is a compact linear Weingarten hypersurface with nonnegative sectional curvature immersed in $\mathbb{S}^{n+1}$, such that $R = aH + b$ with $(n-1)a^2 + 4n(b-1) \geq 0$, then M^n is either totally umbilical or isometric to $\mathbb{S}^{n-k}(c_1) \times \mathbb{S}^k(c_2)$, where $1 \leq k \leq n-1$, $c_1, c_2 > 0$ and $(1/c_1) + (1/c_2) = 1$. Thereafter, Shu [**12**] obtained some rigidity theorems concerning to linear Weingarten hypersurfaces with two distinct principal curvatures immersed in a real space form.

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The aim of our work is to establish a characterization theorem concerning complete linear Weingarten hypersurfaces immersed in the hyperbolic space. Under the assumption that the mean curvature H attains its maximum along the hypersurface M^n and supposing an appropriated restriction on the norm of the second fundamental form B of M^n, the Hopf's strong maximum principle enable us to prove the following:

THEOREM 1.1. *Let M^n be a complete linear Weingarten hypersurface immersed in $\mathbb{H}^{n+1}$, such that $R = aH + b$ with $H^2 \geq 1$ and $b > -1$. If H attains its maximum on M^n and*

$$|B|^2 \leq nH^2 + \left(\mathcal{R}_H^+\right)^2,$$

where

$$\mathcal{R}_H^+ = \frac{1}{2}\sqrt{\frac{n}{n-1}}\left(\sqrt{n^2H^2 - 4(n-1)} - (n-2)H\right),$$

then M^n is either totally umbilical or isometric to a hyperbolic cylinder $\mathbb{S}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, if $R > 0$, or $\mathbb{S}^1(c_1) \times \mathbb{H}^{n-1}(c_2)$, if $R < 0$, where $c_1 > 0$, $c_2 < 0$ and $(1/c_1) + (1/c_2) = -1$.

The proof of Theorem 1.1 is given in Section 3.

2. Preliminaries.

In this section we will introduce some basic facts and notations that will appear on the paper. In what follows, we will suppose that all considered hypersurfaces are orientable and connected.

Let M^n be an n-dimensional hypersurface in $\mathbb{H}^{n+1}$. We choose a local field of orthonormal frame $\{e_A\}_{1 \leq A \leq n+1}$ in $\mathbb{H}^{n+1}$, with dual coframe $\{\omega_A\}_{1 \leq A \leq n+1}$, such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n and e_{n+1} is normal to M^n. We will use the following convention for the indices:

$$1 \leq A, B, C, \ldots \leq n+1,\ 1 \leq i, j, k, \ldots \leq n.$$

In this setting, denoting by $\{\omega_{AB}\}$ the connection forms of $\mathbb{H}^{n+1}$, we have that the structure equations of $\mathbb{H}^{n+1}$ are given by:

$$d\omega_A = \sum_i \omega_{Ai} \wedge \omega_i + \omega_{An+1} \wedge \omega_{n+1}, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = -(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

Next, we restrict all the tensors to M^n. First of all, $\omega_{n+1} = 0$ on M^n, so $\sum_i \omega_{n+1i} \wedge$

$\omega_i = d\omega_{n+1} = 0$ and by *Cartan's Lemma* [**3**] we can write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \ h_{ij} = h_{ji}. \tag{2.4}$$

This gives the second fundamental form of M^n, $B = \sum_{ij} h_{ij}\omega_i\omega_j e_{n+1}$. Furthermore, the mean curvature H of M^n is defined by $H = (1/n)\sum_i h_{ii}$.

The structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.5}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l. \tag{2.6}$$

Using the structure equations we obtain the Gauss equation

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.7}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

The Ricci curvature and the normalized scalar curvature of M^n are given, respectively, by

$$R_{ij} = -(n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj} \tag{2.8}$$

and

$$R = \frac{1}{n(n-1)}\sum_i R_{ii}. \tag{2.9}$$

From (2.8) and (2.9) we obtain

$$\begin{aligned} |B|^2 &= n^2H^2 - n(n-1)(R+1) \\ &= nH^2 + n(n-1)(H^2 - H_2), \end{aligned} \tag{2.10}$$

where $|B|^2 = \sum_{i,j} h_{ij}^2$ is the square of the length of the second fundamental form B of M^n, and $H_2 = (2/n(n-1))S_2$ denotes the mean value of the second elementary symmetric function S_2 on the eigenvalues of B. In particular, since (from the Cauchy-Schwarz inequality) $H^2 - H_2 \geq 0$, it follows from (2.10) that M^n is totally umbilical if, and only if, $|B|^2 = nH^2$.

The components h_{ijk} of the covariant derivative ∇B satisfy

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{ik}\omega_{kj} + \sum_k h_{jk}\omega_{ki}. \tag{2.11}$$

The *Codazzi equation* and the *Ricci identity* are, respectively, given by

$$h_{ijk} = h_{ikj} \tag{2.12}$$

and

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}, \tag{2.13}$$

where h_{ijk} and h_{ijk} denote the first and the second covariant derivatives of h_{ij}.

The Laplacian Δh_{ij} of h_{ij} is defined by $\Delta h_{ij} = \sum_k h_{ijkk}$. From equations (2.12) and (2.13), we obtain that

$$\Delta h_{ij} = \sum_k h_{kkij} + \sum_{k,l} h_{kl}R_{lijk} + \sum_{k,l} h_{li}R_{lkjk}. \tag{2.14}$$

Since $\Delta|B|^2 = 2\big(\sum_{i,j} h_{ij}\Delta h_{ij} + \sum_{i,j,k} h_{ijk}^2\big)$, from (2.14) we get

$$\frac{1}{2}\Delta|B|^2 = |\nabla B|^2 + \sum_{i,i,k} h_{ij}h_{kkij} + \sum_{i,j,k,l} h_{ij}h_{lk}R_{lijk} + \sum_{i,j,k,l} h_{ij}h_{il}R_{lkjk}. \tag{2.15}$$

Consequently, taking a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$, from equation (2.15) we obtain the following Simons-type formula

$$\frac{1}{2}\Delta|B|^2 = |\nabla B|^2 + \sum_i \lambda_i(nH)_{,ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{2.16}$$

Now, let $\phi = \sum_{i,j} \phi_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by

$$\phi_{ij} = nH\delta_{ij} - h_{ij}.$$

Following Cheng-Yau [**4**], we introduce a operator $\square$ associated to ϕ acting on any smooth function f by

$$\square f = \sum_{i,j} \phi_{ij}f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}. \tag{2.17}$$

Setting $f = nH$ in (2.17) and taking a local frame field $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$, from equation (2.10) we obtain the following:

$$\begin{aligned}\Box(nH) &= nH\Delta(nH) - \sum_i \lambda_i (nH)_{,ii} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_{,i}^2 - \sum_i \lambda_i (nH)_{,ii} \\ &= \frac{n(n-1)}{2}\Delta R + \frac{1}{2}\Delta|B|^2 - n^2|\nabla H|^2 - \sum_i \lambda_i (nH)_{,ii}.\end{aligned}$$

Consequently, taking into account equation (2.16), we get

$$\Box(nH) = \frac{n(n-1)}{2}\Delta R + |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{2.18}$$

3. Proof of Theorem 1.1.

In order to prove our result, it will be necessary some auxiliary lemmas. The first one is a classic algebraic lemma due to M. Okumura in [**8**], and completed with the equality case proved in [**2**] by H. Alencar and M. do Carmo.

LEMMA 3.1. *Let $\mu_1, \ldots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where β is constant and $\beta \geq 0$. Then*

$$-\frac{(n-2)}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{(n-2)}{\sqrt{n(n-1)}}\beta^3, \tag{3.1}$$

and equality holds if, and only if, either at least $(n-1)$ of the numbers μ_i are equal to $\beta/\sqrt{(n-1)/n}$ or at least $(n-1)$ of the numbers μ_i are equal to $-\beta/\sqrt{(n-1)/n}$.

To obtain the second lemma, we will reason as in the proof of Lemma 2.1 of [**7**].

LEMMA 3.2. *Let M^n be a linear Weingarten hypersurface in $\mathbb{H}^{n+1}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that*

$$(n-1)a^2 + 4n(b+1) \geq 0. \tag{3.2}$$

Then

$$|\nabla B|^2 \geq n^2|\nabla H|^2. \tag{3.3}$$

Moreover, if the inequality (3.2) *is strict and the equality holds in* (3.3) *on M^n, then H is constant on M^n.*

PROOF. Since we are supposing that $R = aH + b$, from equation (2.10) we get

$$2\sum_{i,j} h_{ij}h_{ijk} = (2n^2H - n(n-1)a)H_{,k}.$$

Thus,

$$4\sum_k\Big(\sum_{i,j}h_{ij}h_{ijk}\Big)^2=(2n^2H-n(n-1)a)^2|\nabla H|^2.$$

Consequently, using Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned}4|B|^2|\nabla B|^2&=4\Big(\sum_{i,j}h_{ij}^2\Big)\Big(\sum_{i,j,k}h_{ijk}^2\Big)\\&\geq 4\sum_k\Big(\sum_{i,j}h_{ij}h_{ijk}\Big)^2\\&=(2n^2H-n(n-1)a)^2|\nabla H|^2.\end{aligned}\tag{3.4}$$

On the other hand, since $R=aH+b$, from equation (2.10) we easily see that

$$(2n^2H-n(n-1)a)^2=n^2(n-1)((n-1)a^2+4n(b+1))+4n^2|B|^2.$$

Consequently, from (3.4) we have

$$|B|^2|\nabla B|^2\geq n^2|B|^2|\nabla H|^2.$$

Therefore, we obtain either $|B|=0$ and $|\nabla B|^2=n^2|\nabla H|^2$ or $|\nabla B|^2\geq n^2|\nabla H|^2$. Moreover, if $(n-1)a^2+4n(b+1)>0$, from the previous identity we get that $(2n^2H+n(n-1)a)^2>4n^2|B|^2$. Consequently, if $|\nabla B|^2=n^2|\nabla H|^2$ holds on M^n, from (3.4) we conclude that $\nabla H=0$ on M^n and, hence, H is constant on M^n. □

In what follows, we will consider the Cheng-Yau's modified operator

$$L=\square-\frac{n-1}{2}a\Delta.\tag{3.5}$$

Related to such operator, we have the following sufficient criteria of ellipticity.

Lemma 3.3. *Let M^n be a linear Weingarten hypersurface immersed in $\mathbb{H}^{n+1}$, such that $R=aH+b$ with $b>-1$. Then, L is elliptic.*

Proof. From equation (2.10), since $R=aH+b$ with $b>-1$, we easily see that H can not vanish on M^n and, by choosing the appropriate Gauss mapping, we may assume that $H>0$ on M^n.

Let us consider the case that $a=0$. Since $R=b>-1$, from equation (2.10) if we choose a (local) orthonormal frame $\{e_1,\ldots,e_n\}$ on M^n such that $h_{ij}=\lambda_i\delta_{ij}$, we have that $\sum_{i<j}\lambda_i\lambda_j>0$. Consequently,

$$n^2H^2 = \sum_i \lambda_i^2 + 2\sum_{i<j}\lambda_i\lambda_j > \lambda_i^2$$

for every $i = 1, \dots, n$ and, hence, we have that $nH - \lambda_i > 0$ for every i. Therefore, in this case, we conclude that L is elliptic.

Now, suppose that $a \neq 0$. From equation (2.9) we get that

$$a = -\frac{1}{n(n-1)H}(S - n^2H^2 + n(n-1)(b+1)).$$

Consequently, for every $i = 1, \dots, n$, with a straightforward algebraic computation we verify that

$$\begin{aligned} nH - \lambda_i - \frac{n-1}{2}a &= nH - \lambda_i + \frac{1}{2nH}(S - n^2H^2 + n(n-1)(b+1)) \\ &= \frac{1}{2nH}\Bigg(\sum_{j\neq i}\lambda_j^2 + \Big(\sum_{j\neq i}\lambda_j\Big)^2 + n(n-1)(b+1)\Bigg). \end{aligned}$$

Therefore, since $b > -1$, we also conclude in this case that L is elliptic. $\square$

Now, we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let us choose a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$. Since $R = aH + b$, from (2.18) and (3.5) we have that

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i - \lambda_j)^2. \tag{3.6}$$

Thus, since from (2.7) we have that $R_{ijij} = \lambda_i\lambda_j - 1$, from (3.6) we get

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + n^2H^2 - n|B|^2 - |B|^4 + nH\sum_i\lambda_i^3. \tag{3.7}$$

Now, set $\Phi_{ij} = h_{ij} - H\delta_{ij}$. We will consider the following symmetric tensor

$$\Phi = \sum_{i,j}\Phi_{ij}\omega_i\omega_j.$$

Let $|\Phi|^2 = \sum_{i,j}\Phi_{ij}^2$ be the square of the length of Φ. It is easy to check that Φ is traceless and

$$|\Phi|^2 = |B|^2 - nH^2. \tag{3.8}$$

With respect the frame field $\{e_1, \dots, e_n\}$ on M^n, we have that $\Phi_{ij} = \mu_i\delta_{ij}$ and, with a straightforward computation, we verify that

$$\sum_i \mu_i = 0, \ \sum_i \mu_i^2 = |\Phi|^2 \text{ and } \sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \tag{3.9}$$

Thus, using Gauss equation (2.7) jointly with (3.9) into (3.7), we get

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + nH\sum_i \mu_i^3 + |\Phi|^2(-|\Phi|^2 + nH^2 - n). \tag{3.10}$$

By applying Lemmas 3.1 and 3.2, from (3.10) we have

$$\begin{aligned} L(nH) &\geq |\Phi|^2\Big(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + nH^2 - n\Big) \\ &= |\Phi|^2 P_H(|\Phi|), \end{aligned} \tag{3.11}$$

where

$$P_H(|\Phi|) = -|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + nH^2 - n. \tag{3.12}$$

Since we are supposing that $H^2 \geq 1$, from (3.12) it is easy to verify that $P_H(|\Phi|)$ has two real roots $\mathcal{R}_H^-$ and $\mathcal{R}_H^+$ given by

$$\mathcal{R}_H^- = -\frac{1}{2}\sqrt{\frac{n}{n-1}}\big(\sqrt{n^2H^2 - 4(n-1)} + (n-2)H\big)$$

and

$$\mathcal{R}_H^+ = \frac{1}{2}\sqrt{\frac{n}{n-1}}\big(\sqrt{n^2H^2 - 4(n-1)} - (n-2)H\big).$$

Consequently, we have that

$$P_H(|\Phi|) = (|\Phi| - \mathcal{R}_H^-)(\mathcal{R}_H^+ - |\Phi|). \tag{3.13}$$

Thus, since our restriction on $|B|^2$ guarantees that $|\Phi| \leq \mathcal{R}_H^+$, from (3.13) we conclude that $P_H(|\Phi|) \geq 0$. Hence, from (3.11) we get

$$L(nH) \geq |\Phi|^2 P_H(|\Phi|) \geq 0. \tag{3.14}$$

Since Lemma 3.3 guarantees that L is elliptic and as we are supposing that H attains its maximum on M^n, from (3.14) we conclude that H is constant on M^n. Thus, taking into account equation (3.6), we get

$$|\nabla B|^2 = n^2|\nabla H|^2 = 0,$$

and it follows that λ_i is constant for every $i = 1, \dots, n$.

If $|\Phi| < \mathcal{R}_H^+$, then from (3.14) we have that $|\Phi| = 0$ and, hence, M^n is totally umbilical. If $|\Phi| = \mathcal{R}_H^+$, since the equality holds in (3.1) of Lemma 3.1, we conclude that M^n is either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple. Therefore, from the classification of the complete isoparametric hypersurfaces having at most two distinct principal curvatures due to Ryan [**10**] (see also [**1**, Theorem 5.1]), we conclude that M^n is either totally umbilical or isometric to a hyperbolic cylinder $\mathbb{S}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, if $R > 0$, or $\mathbb{S}^1(c_1) \times \mathbb{H}^{n-1}(c_2)$, if $R < 0$, where $c_1 > 0$, $c_2 < 0$ and $(1/c_1) + (1/c_2) = -1$. $\square$

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