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$\mathbb{C}^*$-equivariant degenerations of curves and normal surface singularities with $\mathbb{C}^*$-action

Dedicated to Professor Oswald Riemenschneider on his seventieth birthday

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Abstract. This paper presents a definition of $\mathbb{C}^*$-equivariant degeneration families of compact complex curves over $\mathbb{C}$. Those families are called $\mathbb{C}^*$-pencils of curves. We give the canonical method to construct them and prove some results on relations between them and normal surface singularities with $\mathbb{C}^*$-action. We also define $\mathbb{C}^*$-equivariant degeneration families of compact complex curves over $\mathbb{P}^1$. From this, it is possible to introduce a notion of dual $\mathbb{C}^*$-pencils of curves naturally. Associating it, we prove a duality for cyclic covers of normal surface singularities with $\mathbb{C}^*$-action.

1. Introduction.

In [**Ko**], K. Kodaira gave a definition of the local one-parameter degeneration families of compact complex curves (we call them *pencils of curves* in this paper).

Definition 1.1. Let S be a non-singular complex surface and let Δ ($\subset \mathbb{C}$) be a small open disc around the origin. If $\Phi\colon S \longrightarrow \Delta$ is a proper surjective holomorphic map whose generic fiber $S_t\colon = \Phi^{-1}(t)$ ($t \neq 0$) is a smooth curve (but not necessarily connected), then it is called a *quasi-pencil of curves*. Furthermore, if S_t is a smooth connected curve of genus g, it is called a *pencil of curves of genus* g. For the irreducible decomposition $\mathrm{supp}(S_0) = \bigcup_{i=1}^{r} E_i$, $m := \gcd\{\mathrm{Coeff}_{E_i}(S_0) \mid i = 1, \dots, r\}$ is called *the multiplicity* of Φ. If $m > 1$ (resp. $m = 1$), then Φ is a multiple (resp. non-multiple) pencil of curves.

In the case of $g = 1$, Kodaira [**Ko**] classified singular fibers and the homological monodromies and constructed pencils of curves with such numerical data. The corresponding work in the case of $g = 2$ was done by Y. Namikawa and K. Ueno ([**NU**]) according to the argument of [**Ko**]. Now, from local and global points of

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view in algebraic geometry and topology, there are many results related to pencils of curves (see [**AI**], [**AK**] and [**MM**]).

On the other hand, normal surface singularity theory has been developed for a long time by many mathematicians. Regarding the relation between surface singularities and pencils of curves, several works have been researched. In [**Ku**], V. Kulikov observed that all of Arnold's unimodal and bimodal singularities ([**Ar**]) relate elliptic pencils in [**Ko**]. Furthermore, M. Reid [**Re**] classified hypersurface minimally elliptic singularities and pointed out that they relate elliptic pencils in [**Ko**]. After their works, U. Karras ([**Kar**]) introduced the notion of Kodaira singularities in terms of pencils of curves and applied to research deformation theory of elliptic singularities (also see [**EW**], [**Stev**], [**To3**], [**To5**] and [**To6**]). If $\Phi\colon S \longrightarrow \Delta$ is a pencil of curves, then any connected one-dimensional analytic proper subset E in $\mathrm{supp}(S_o)$ has a negative definite intersection matrix from Zariski's lemma ([**BPV**, p. 90]). Therefore, E is contracted to a normal surface singularity by Grauert's result ([**G**, p. 367]). Conversely, the author proved the following.

THEOREM 1.2 ([**To6**, Theorem 2.4]). *Let (X, o) be a normal surface singularity and $h \in \mathfrak{m}_{X,o}$ (the maximal ideal of $\mathcal{O}_{X,o}$). Let $\pi\colon (\tilde{X}, E) \longrightarrow (X, o)$ be a good resolution such that* $\mathrm{red}((h \circ \pi)_{\tilde{X}})$ *is a simple normal crossing divisor on $\tilde{X}$. Then there exists a quasi-pencil of curves $\Phi\colon S \longrightarrow \Delta$ with $(\tilde{X}, E) \subset (S, \mathrm{supp}(S_o))$ such that $\Phi|_{\tilde{X}} = h \circ \pi$ and all connected components of* $\mathrm{supp}(S_o)\backslash E$ *are minimal $\mathbb{P}^1$-chains started from E. Furthermore, if h is not a perfect power element of $\mathcal{O}_{X,o}$ (see Notations and Terminologies), then $\Phi\colon S \longrightarrow \Delta$ above is a pencil of curves.*

Based on the above, the author defined the holomorphic invariants of (X, o) and (X, o, h) as follows:

$$p_e(X, o) = \min\{\text{the genus of a pencil of curves including a resolution of } (X, o)\},$$
$$\text{and } p_e(X, o, h) = \min\{\text{the genus of a pencil of curves with } \Phi|_{\tilde{X}} = h \circ \pi\},$$

where $h \in \mathfrak{m}_{X,o}$ is not a perfect power element. We have $p_e(X, o) = \min\{p_e(X, o, h) \mid h \in \mathfrak{m}_{X,o} \text{ is not a perfect power element}\}$ ([**To6**, Theorem 2.13]). These invariants are called the *pencil genus* of (X, o) (resp. a pair of (X, o) and h). We use them in this paper.

Normal singularities with $\mathbb{C}^*$-action have been studying from several points of view for a long time. The affine rings of these singularities are finitely generated graded rings [**OW1**]. Consequently, these singularities make very special class among singularities on complex spaces. However, this class contains many

important singularities in complex geometry. For example, it contains all quotient singularities and all normal isolated singularities obtained by blowing-down of the zero sections of negative line bundles on complex manifolds. Furthermore, the links (i.e., the intersection set of a small sphere around a singular point and the singular set) of these singularities provide many important examples of manifolds in geometry since the famous work by E. Brieskorn on exotic spheres, and many mathematicians have been studying them actively (see [**BG**], [**Sav**]). In 2-dimensional case, normal surface singularities with good $\mathbb{C}^*$-action have been investigated in depth and widely since P. Orlik and P. Wagreich's work ([**OW1**], [**OW2**] and [**OW3**]).

In this paper, we define pencils of curves with $\mathbb{C}^*$-action and study some relations between them and normal surface singularities with good $\mathbb{C}^*$-action.

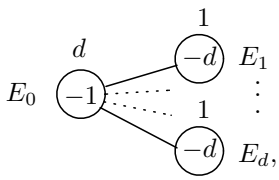
Definition 1.3.

(i) Let $\Phi : S \longrightarrow \mathbb{C}$ be a quasi-pencil, where we consider $\mathbb{C}$ as Δ with infinite radius. If there exists an effective holomorphic $\mathbb{C}^*$-action on S satisfying $\Phi(t \cdot p) = t^d \Phi(p)$ for any $t \in \mathbb{C}^*$ and $p \in S$ and for some $d \in \mathbb{N}$, then we call $\Phi : S \longrightarrow \mathbb{C}$ *a* $\mathbb{C}^*$*-quasi-pencil of curves of degree* d. Furthermore, if Φ is a pencil of curves of genus g and degree d and multiplicity m, then (d, g, m) is called *the type of* Φ.

(ii) Suppose that $\mathrm{red}(S_0)$ (the reduced divisor of S_0) is simple normal crossing. If there is no (-1)-curve (i.e., $\mathbb{P}^1$ whose self-intersection number is -1) in $\mathrm{supp}(S_0)$ which contains a one-dimensional $\mathbb{C}^*$-orbit, then Φ is called *a minimal good* $\mathbb{C}^*$*-pencil of curves.*

(iii) Let $\Phi_i : S_i \longrightarrow \mathbb{C}$ be a $\mathbb{C}^*$-pencil of curves of genus g $(i = 1, 2)$. *A holomorphic isomorphism* from Φ_1 to Φ_2 is defined to be a pair of $\mathbb{C}^*$-equivariant biholomorphic maps $\varphi_1 : S_1 \longrightarrow S_2$ and $\varphi_2 : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\varphi_2 \circ \Phi_1 = \Phi_2 \circ \varphi_1$.

Example 1.4. Let us consider a non-singular surface $S = \{([z_0 \colon z_1 \colon z_2], \zeta) \in \mathbb{P}^2 \times \mathbb{C} \mid z_0^d + z_1^d + \zeta z_2^d = 0\}$. Let $\Phi : S \longrightarrow \mathbb{C}$ be the restriction of the projection map $\mathbb{P}^2 \times \mathbb{C} \longrightarrow \mathbb{C}$. Then Φ gives a pencil of curves. Consider a $\mathbb{C}^*$-action on S defined by $t \cdot ([z_0 \colon z_1 \colon z_2], \zeta) = ([tz_0 \colon tz_1 \colon z_2], t^d \zeta)$. Therefore, $\Phi(t \cdot p) = t^d \Phi(p)$ for $t \in \mathbb{C}^*$ and $p \in S$. Consequently, $\Phi : S \longrightarrow \mathbb{C}$ is a $\mathbb{C}^*$-pencil of curves of type $((d-1)(d-2)/2, d, 1)$; also every fiber S_ζ with $\zeta \neq 0$ is isomorphic to the Fermat curve of degree d. Moreover, let $\sigma : S' \longrightarrow S$ be the blowing-up at a point $([0 \colon 0 \colon 1], 0)$. Then $\Phi' := \Phi \circ \sigma : S' \longrightarrow \mathbb{C}$ is also a $\mathbb{C}^*$-pencil of curves of same type as Φ. The weighted dual graph (=w.d.graph) of S'_0 is given by



where all components are $\mathbb{P}^1$; also $E_0^2 = -1$, $E_i^2 = -d$, $E_0 E_i = 1$ $(i = 1, \ldots, d)$, $E_i E_j = 0$ $(1 \leq i < j \leq d)$ and $\mathrm{Coeff}_{E_0} \Phi' = d$ means that the vanishing order of Φ' on E_0 is d, and so on.

In the general dimensional case, degenerate families with $\mathbb{C}^*$-action for polarized algebraic schemes were defined in [**Do**]. In 2-dimensional case, Orlik and Wagreich [**OW3**] studied $\mathbb{C}^*$-equivariant completions of $\mathbb{C}^*$-equivariant resolution spaces of normal surface singularities with good $\mathbb{C}^*$-action. Analogously, we define complete $\mathbb{C}^*$-pencils of curves. They play important roles for consideration of the relation between $\mathbb{C}^*$-pencils of curves and cyclic covers of normal surface singularities with good $\mathbb{C}^*$-action in Section 5.

DEFINITION 1.5.

(i) Let $\hat{S}$ be a compact complex surface with an effective holomorphic $\mathbb{C}^*$-action and $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ be a surjective holomorphic map. Let $S := \hat{S} \backslash \mathrm{supp}(\hat{\Phi}^{-1}(\infty))$ and $S^* := \hat{S} \backslash \mathrm{supp}(\hat{\Phi}^{-1}(\mathbf{0}))$, where $\infty = [1 : 0]$ and $\mathbf{0} = [0 : 1]$. If $\Phi := \hat{\Phi}|_S : S \longrightarrow \mathbb{C}$ and $\Phi^* := (1/\hat{\Phi})|_{S^*} : S^* \longrightarrow \mathbb{C}$ are $\mathbb{C}^*$-quasi-pencils of curves, then we call $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ a *complete* $\mathbb{C}^*$*-quasi-pencils of curves*. If Φ is a $\mathbb{C}^*$-pencil of curves of type (d, g, m), then $\hat{\Phi}$ is called *a* $\mathbb{C}^*$*-pencil of curves of type* (d, g, m).

(ii) If Φ and Φ^* are minimal good $\mathbb{C}^*$-pencil of curves, then $\hat{\Phi}$ is called *a minimal good complete* $\mathbb{C}^*$*-pencil of curves*.

(iii) Let $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ be a complete $\mathbb{C}^*$-quasi-pencil of curves. We also represent Φ (resp. Φ^*) as $\hat{\Phi}_L$ (resp. $\hat{\Phi}_R$) and call $\hat{\Phi}_L$ (resp. $\hat{\Phi}_R$) the *left* (resp. *right*) part of $\hat{\Phi}$. Furthermore, let put $\hat{S}_0 = \mathrm{supp}(\hat{\Phi}^{-1}(\mathbf{0}))$ and $\hat{S}_\infty = \mathrm{supp}(\hat{\Phi}^{-1}(\infty))$; hence $\hat{S}_0 = \hat{S}_{L,0}$ and $\hat{S}_\infty = \hat{S}_{L,\infty}$.

(iv) Let $\hat{\Phi}_i : \hat{S}_i \longrightarrow \mathbb{P}^1$ be a complete $\mathbb{C}^*$-quasi-pencil of curves $(i = 1, 2)$. *A holomorphically isomorphism* from $\hat{\Phi}_1$ to $\hat{\Phi}_2$ is defined to be a pair of $\mathbb{C}^*$-equivariant biholomorphic maps $\hat{\varphi}_1 : \hat{S}_1 \longrightarrow \hat{S}_2$ and $\varphi_2 : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ such that $\varphi_2 \circ \Phi_1 = \Phi_2 \circ \varphi_1$.

(v) Two $\mathbb{C}^*$-pencils of curves Φ and Φ^* are said to be *mutually dual*. Namely, Φ^* (resp. Φ) is the dual of Φ (resp. Φ^*).

We remark on Definition 1.5 (i). In Theorem 2.4 (i) and Corollary 2.11, we give a way to construct all complete $\mathbb{C}^*$-pencils of curves up to $\mathbb{C}^*$-equivariantly

biholomorphic equivalences, which is called the canonical construction. From the way, we can easily see that the types for Φ and Φ^* coincide.

From the definition above, we can see that the homological monodromy transformation associated to $\hat{\Phi}_L$ and $\hat{\Phi}_R$ are inverse to each other. Next we describe some fundamental facts for pencils of curves.

Remark 1.6. Let $\Phi\colon S \longrightarrow \Delta$ be a pencil of curves.

(i) For any non-zero constant c, the product $c\Phi$ also gives a pencil of curves that is isomorphic to Φ.

(ii) If Φ^m $(m \geq 2)$ is the power of the function Φ, then $\Phi^m\colon S \longrightarrow \Delta$ is a quasi-pencil of curves but not a pencil of curves, because the generic fiber of Φ^m has m connected components. Conversely, if $\Phi\colon S \longrightarrow \Delta$ is a quasi-pencil of curves but not a pencil of curves, then there is a pencil of curves $\Psi\colon S \longrightarrow \Delta$ such that $\Phi = \Psi^\ell$ for $\ell \geq 2$.

(iii) If Φ is a $\mathbb{C}^*$-pencil of curves, then similar statements as (ii) are also true. Namely, for $m \geq 2$, Φ^m is a $\mathbb{C}^*$-quasi-pencil of curves but not a $\mathbb{C}^*$-pencil of curves as in (ii). If $\Phi\colon S \longrightarrow \mathbb{C}$ is a $\mathbb{C}^*$-quasi-pencil of curves but not a $\mathbb{C}^*$-pencil of curves, then there is a $\mathbb{C}^*$-pencil of curves $\Psi\colon S \longrightarrow \mathbb{C}$ such that $\Phi = \Psi^\ell$ for $\ell \geq 2$.

Proof. Since (i) and the first statement of (ii) are obvious, we prove the second statement of (ii). Let ℓ be the number of connected components of the general fiber of Φ. From the Stein factorization of Φ, we have the following diagram:

$$\begin{tikzcd} S \arrow[rr, "\Phi"] \arrow[dr, "\Psi_1"'] & & \Delta \\ & \tilde{\Delta} \arrow[ur, "\eta_1"'] & \end{tikzcd},$$

where η_1 is a finite map and Ψ_1 is a connected map. Then Ψ_1 is a pencil of curves and $t = \eta_1(\zeta) = u(\zeta)\zeta^\ell$, where $u(0) \neq 0$. Let $u_1(\zeta)$ be a holomorphic function with $u_1^\ell = u$ on a small open disc $\tilde{\Delta}_\varepsilon$. Consider a coordinate change $\zeta_1 := \varphi(\zeta) := u_1(\zeta)\zeta$ on $\tilde{\Delta}_\varepsilon$. Let $\Psi := \varphi \circ \Psi_1$ and $\eta := \eta_1 \circ \varphi^{-1}$. Then, from $t = \eta(\zeta_1) = {\zeta_1}^\ell$, we have $\Phi = \eta_1 \circ \Psi_1 = \eta_1 \circ \varphi^{-1} \circ \varphi \circ \Psi_1 = \eta \circ \Psi = \Psi^\ell$ on an open set $\Psi^{-1}(\bar{\Delta}_\varepsilon)$. Hence, $\Phi = \Psi^\ell$ on S.

Since the first statement of (iii) is obvious, we prove the second statement. Similarly as (ii), we take the Stein factorization $\Phi = \eta \circ \Psi$, where $\Psi\colon S \longrightarrow \mathbb{C}$ is a pencil of curves and η is a finite map. Let Δ be a small disc around the origin of $\mathbb{C}$ and let consider a quasi-pencil of curves $\Phi_\Delta : S_\Delta = \Phi^{-1}(\Delta) \longrightarrow \Delta$. From (ii), there exists a pencil of curves Ψ_Δ satisfying $\Phi_\Delta = \Psi_\Delta^\ell$ on an open set $\Psi^{-1}(\Delta)$. Therefore, we have $\Phi = \Psi^\ell$ on S. Since $\Psi^\ell(tP) = \Phi(tP) = t^d\Phi(P) = t^d\Psi^\ell(P)$ for any $t \in \mathbb{C}^*$ and $P \in S$, we have $\Psi(tP) = \omega t^{d/\ell}\Psi(P)$, where $\omega^\ell = 1$. Since

$\Psi(P) = \omega\Psi(P)$ for $S \setminus \operatorname{supp}(S_0)$, we have $\omega = 1$. Hence Ψ is a $\mathbb{C}^*$-pencil of curves. $\square$

REMARK 1.7. Let $\Phi\colon S \longrightarrow \Delta$ be a quasi-pencil of curves.

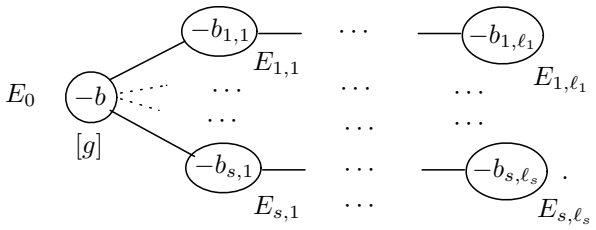
(i) If f is a non-constant holomorphic function on S, then a holomorphic function g on Δ exists such that $f = g \circ \Phi$. In fact, there is a well-defined function g on Δ by $g(t) = f(S_t)$ for any $t \in \Delta$. Considering a holomorphic local section s of Φ, g is holomorphic from $g = f \circ s$.

(ii) If $\Phi\colon S \longrightarrow \mathbb{C}$ is a $\mathbb{C}^*$-quasi-pencil of curves of degree d and f is a $\mathbb{C}^*$-equivariant holomorphic function on S of degree ℓ (i.e., $f(tp) = t^d f(p)$ for any $t \in \mathbb{C}^*$ and $p \in S$), then $d|\ell$ and f coincides to $c\Phi^{\ell/d}$ for a constant c. In fact, g of (i) with $f = g \circ \Phi$ is written as $ct^{\ell/d}$ because g is homogeneous. Therefore, any $\mathbb{C}^*$-equivariant holomorphic function f on S is given as a power of the fibering map $c_1\Phi$ for c_1 with $c = c_1^d$; namely $f = (c_1\Phi)^{\ell/d}$.

(iii) Let $\Phi : S \longrightarrow \mathbb{C}$ be a $\mathbb{C}^*$-quasi-pencil of curves. Let ζ_1 and ζ_2 be different non-zero elements of $\mathbb{C}$. Let ω be a fixed element of $\mathbb{C}^*$ satisfying $\omega^d = \zeta_2/\zeta_1$. Then we have a biholomorphic map from S_{ζ_1} to S_{ζ_2} by $P \mapsto \omega P$. Therefore, all general fibers are mutually biholomorphic.

Next we review some facts related to normal surface singularities with $\mathbb{C}^*$-action. Let (X, o) be an n-dimensional normal isolated singularity with $\mathbb{C}^*$-action. From [**OW1**], there is an embedding $(X, o) \subset (\mathbb{C}^{N+1}, o)$ such that the $\mathbb{C}^*$-action on (X, o) is induced from a diagonal action $t \cdot (z_0, \dots, z_N) = (t^{q_0} z_0, \dots, t^{q_N} z_N)$ on $\mathbb{C}^{N+1}$, where $q_i > 0$ for any i. If $\gcd(q_0, \dots, q_N) = 1$, then the action is called *a good* $\mathbb{C}^*$*-action*. In this paper, we usually consider singularities with good $\mathbb{C}^*$-action; abbreviate "good" in the following. If a polynomial $f(z_0, \dots, z_N)$ is expressed as a linear combination of monomials $z_0^{i_0} \dots z_N^{i_N}$ satisfying $\sum_{j=0}^N q_j i_j = d$, then f is called a quasi-homogeneous polynomial of type $(q_0, \dots, q_N; d)$ or of type $(q_0, \dots, q_N)$ and of degree d. The affine ring R_X of X is generated by quasi-homogeneous polynomials of type $(q_0, \dots, q_N)$ ([**OW1**]). It becomes a graded ring.

From now on, it is assumed usually that X is 2-dimensional. We often use star-shaped w.d.graphs given as follows:



(1.1)

The component E_0 is a compact smooth algebraic curve of genus g and $E_0^2 = -b$. It is called the *central curve*. Each $\mathbb{P}^1$-chain $\bigcup_{j=1}^{\ell_i} E_{i,j}$ is contracted to a cyclic quotient singularity of type C_{d_i,e_i}, where

$$\frac{d_i}{e_i} = [[b_{i,1}, \dots, b_{i,\ell_i}]] := b_{i,1} - \cfrac{1}{\ddots - \cfrac{1}{b_{i,\ell_i}}}.$$

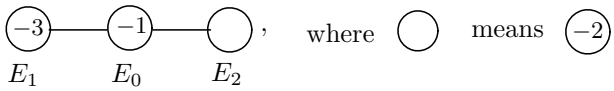
If $b - \sum_{i=1}^{s} e_i/d_i > 0$, then it is well-known that the configuration of (1.1) represents the exceptional set of a resolution space of a surface singularity ([**P**]). Furthermore, if $b = \sum_{i=1}^{s} e_i/d_i$, then we prove that there exists a $\mathbb{C}^*$-pencil of curves $\Phi\colon S \longrightarrow \mathbb{C}$ such that the configuration of (1.1) represents the support of S_0 (Theorem 2.4).

Let (X, o) be a normal surface singularity with $\mathbb{C}^*$-action.

THEOREM 1.8 ([**OW1**], [**P**] and [**To4**]). *There exists a $\mathbb{C}^*$-equivariant resolution $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ uniquely which satisfies the following*:

(i) *The w.d.graph of E is a star-shaped graph of* (1.1) *and* $b - \sum_{j=1}^{s} e_j/d_j > 0$.
(ii) *The $\mathbb{C}^*$-action on $\tilde{X}$ acts trivially on the central curve E_0; each irreducible component of E except for E_0 contains a one-dimensional orbit.*
(iii) *Each $\mathbb{P}^1$-chain $\bigcup_{j=1}^{\ell_i} E_{i,j}$ does not contain a (-1)-curve.*

In this paper, a resolution satisfying the three conditions above is called *a minimal $\mathbb{C}^*$-good resolution*. For cases aside from cyclic quotient singularities, the minimal $\mathbb{C}^*$-good resolution is the minimal good resolution. However, for cyclic quotient singularities, this is not always true. For such singularities, there are countably many natural $\mathbb{C}^*$-actions for each one. For example, let (X, o) be $(\mathbb{C}^2, o)$ with $\mathbb{C}^*$-action $t \cdot (x, y) = (t^2 x, t^3 y)$ for $t \in \mathbb{C}^*$. The minimal $\mathbb{C}^*$-good resolution of $(\mathbb{C}^2, o)$ with $\mathbb{C}^*$-action above is given as



(1.2)

and E_0 is the central curve. However, it is not the minimal good resolution. Please refer to [**To4**] for further details.

The structure of a normal surface singularity is determined by the analytic structures of the central curve E_0 and the normal bundle of E_0 in the minimal good resolution and intersection points of E_0 and $\mathbb{P}^1$-chains. This fact was described explicitly by A. Fujiki [**Fu1**] and H. Pinkham [**P**]. Assuming that the w.d.graph of E is given as (1.1), then let H be the restriction of the conormal bundle $N^*_{E_0/\tilde{X}}$

onto E_0 and $P_i := E_0 \cap E_{i,1}$ for any i. For affine graded ring R_X of (X, o), Pinkham [**P**] proved an isomorphism of graded rings as

$$R_X \cong \bigoplus_{k=0}^{\infty} H^0\big(E_o, \mathcal{O}_{E_o}(D^{(k)})\big)t^k, \tag{1.3}$$

where $D^{(k)} = kH - \sum_{j=1}^{s} \lceil e_j k/d_j \rceil P_j$ and $\lceil a \rceil$ is the round up of $a \in \mathbb{R}$. In [**De**], M. Demazure generalized this formula in higher dimensional case, and K-i. Watanabe applied it to finitely generated graded ring theory ([**Wke**]). We call a $\mathbb{Q}$-coefficient divisor $D = H - \sum_{j=1}^{s}(e_j/d_j)P_j$ *the Pinkham-Demazure divisor* of (X, o). Also the representation of R_X of (1.3) is called *the Pinkham-Demazure construction.*

In Section 2, we present a canonical method to construct (resp. complete) $\mathbb{C}^*$-pencils of curves (Section 2) and call such objects (resp. complete) $\mathbb{C}^*$-pencils of curves by the canonical construction. We prove that any complete $\mathbb{C}^*$-pencil of curves is $\mathbb{C}^*$-equivariantly and holomorphically isomorphic to one by the canonical construction. Furthermore, considering complete $\mathbb{C}^*$-pencils of curves, we introduce the notion of dual $\mathbb{C}^*$-pencils of curves.

In Section 3, some results on cyclic quotient singularities and its cyclic coverings are proven as the preparation of Section 4 and Section 5. Our main result is Theorem 3.4, which is proven according to the argument by Fujiki ([**Fu2**]).

In Section 4, some results on $\mathbb{C}^*$-pencils of curves are shown. We prove a $\mathbb{C}^*$-equivariant version of Theorem 1.2. (Theorem 4.1). In [**Fu1**] and [**P**], they prove that any normal surface singularity with $\mathbb{C}^*$-action is obtained as a finite group quotient of a holomorphic line bundle on a compact smooth complex curve. As its analogy, we prove Theorem 4.6.

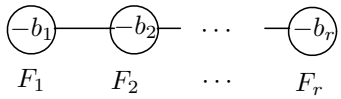
In Section 5, we prove a relation between complete $\mathbb{C}^*$-pencils of curves and cyclic coverings of normal surface singularities with $\mathbb{C}^*$-action (Theorem 5.4). Let (X, o) be a normal surface singularity with $\mathbb{C}^*$-action and h a homogeneous element of R_X of degree d. Let (Y_i, o) be the normalization of the cyclic covering over (X, o) defined by $w_i^{m_i} = h$ $(i = 1, 2)$. Suppose that $m_1 + m_2 \equiv 0$ (d) and Y_1, Y_2 are irreducible. Then we can see some duality phenomenon between (Y_1, o) and (Y_2, o) (Theorem 5.4, Remark 5.7). In [**Ko**], Kodaira already recognized "a dual" of elliptic pencils from the point of view of homological monodromy theory. Recently, for general pencils of curves, Lu and Tan [**LT**] studied the notion of dual pencils from the point of view of n-th root fibrations. In Section 5, we explain their definition for dual pencils. As an application of Theorem 5.4, we prove that the dual as $\mathbb{C}^*$-pencils of curves coincides with the notion of dual as cyclic coverings in [**LT**] up to the $\mathbb{C}^*$-equivariant birational map.

NOTATIONS AND TERMINOLOGIES. Let R be a ring and h a non-zero element

of R. Then h is called a *perfect power element* if there is an element $g \in R$ satisfying $h = g^k$ for some positive integer $k \geq 2$.

In this paper, we use weighted dual graphs to represent the configurations of singular fibers of pencils of curves and the exceptional sets of resolutions of surface singularities. Please refer [**To6**] for them.

Let $A = \bigcup_{j=1}^{r} A_j \subset M$ be the irreducible decomposition of a complex curve A in a complex surface M. Let $D = \sum_{i=1}^{r} d_i A_i$ be a divisor on M. In this paper, we put $\mathrm{supp}(D) = \bigcup_{j=1}^{r} A_j$ (the support of D), $\mathrm{red}(D) = \sum_{j=1}^{r} A_j$ (the reduced divisor of D) and $\mathrm{Coeff}_{A_j} D = d_j$. Furthermore, if A is a reduced divisor with $\mathrm{supp}(A) \subset \mathrm{supp}(D)$, then we put $\mathrm{supp}(D) \backslash A := \mathrm{supp}(\mathrm{red}(D) - A)$. Suppose that $A_j^2 \leq 0$ for any j and that $A = \sum_{j=1}^{r} A_j$ is a simple normal crossing divisor on M. Let $F = \bigcup_{i=1}^{r} F_i$ ($\subset A$). If $\bigcup_{i=2}^{r-1} F_i$ does not intersect other components except for F_1 and F_r and the w.d.graph of F is given by



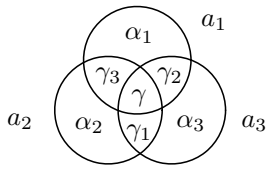
,

then F is called *a* $\mathbb{P}^1$*-chain of type* $(b_1, \ldots, b_r)$ or *type* $\langle d, e \rangle$, where $d/e = [[b_1, \ldots, b_r]]$ and $\gcd(d, e) = 1$. If $b_i \geq 2$ for any i, then F is called a *minimal* $\mathbb{P}^1$*-chain.*

For non-negative integers $a_1, \ldots, a_s$ with $a_1 + \cdots + a_s > 0$ and $0 < r < s$, we define an integer as

$$[a_1, \ldots, a_r \mid a_{r+1}, \ldots, a_s] = \frac{\gcd(a_1, \ldots, a_r)}{\gcd(a_1, \ldots, a_r, \mathrm{lcm}(a_{r+1}, \ldots, a_s))}. \tag{1.4}$$

In this paper, we often use it in the case of $(r, s) = (1, 2)$ or $(2, 1)$. It is readily apparent that $[a_1 \mid a_2, a_3] = [a_1 \mid a_3, a_2]$ and $[a_1, a_2 \mid a_3] = [a_2, a_1 \mid a_3]$. For integers a_1, a_2, a_3, the following figure is convenient to represent integers above.



$\gamma := \gcd(a_1, a_2, a_3)$,

$\alpha_1 := [a_1 | a_2, a_3]$,

$\gamma_1 := [a_2, a_3 | a_1]$, and so on.

The following can be checked readily:

(i) α_1, α_2 and α_3 are relatively prime.
(ii) γ_1, γ_2 and γ_3 are relatively prime.
(iii) α_i and γ_i are relatively prime for $i = 1, 2, 3$.

2. Canonical construction of complete $\mathbb{C}^*$-pencils of curves.

In this section, we present a method to construct complete $\mathbb{C}^*$-pencils of curves from $\mathbb{P}^1$-bundles on smooth compact complex curves (Theorem 2.4). In addition, we obtain $\mathbb{C}^*$-pencils of curves as subsets of complete $\mathbb{C}^*$-pencils of curves constructed by the method. Such method is called *the canonical construction* for (complete) $\mathbb{C}^*$-pencils of curves. Moreover, we prove that every (complete) $\mathbb{C}^*$-pencil of curves is isomorphic to one by the canonical construction (Theorem 2.10 and Corollary 2.11).

LEMMA 2.1.

(i) *Let* $[[b_1, \dots, b_\ell]] = n/q$ *and let* q', q'' *be numbers satisfying* $qq' \equiv 1\ (n)$ $(0 < q' < n)$ *and* $q''n = qq' - 1$. *Then, for any real number* a, *we have*

$$[[b_1, \dots, b_{\ell-1}, b_\ell + a]] = \frac{n + q'a}{q + q''a}.$$

(ii) *Let* n, q_1 *and* q_2 *be positive integers such that* n *and* q_i *are relatively prime and* $0 < q_i < n$ *for* $i = 1, 2$. *If we put* $n/q_1 = [[b_{1,1}, \dots, b_{1,\ell_1}]]$ *and* $n/q_2 = [[b_{2,1}, \dots, b_{2,\ell_2}]]$, *then*

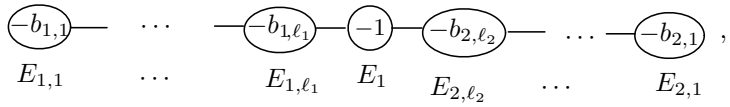
$$[[b_{1,1}, \dots, b_{1,\ell_1}, 1, b_{2,\ell_2}, \dots, b_{2,1}]] = 0 \quad \textit{if and only if} \quad q_1 + q_2 = n. \tag{2.1}$$

PROOF. Since (i) was proven in [**To4**], we prove (ii). From (i),

$$\begin{aligned}[[b_{1,1}, \dots, b_{1,\ell_1}, 1, b_{2,\ell_2}, \dots, b_{2,1}]] &= \left[\left[b_{1,1}, \dots, b_{1,\ell_1-1}, b_{1,\ell_1} - \frac{n}{n - q_2'}\right]\right] \\ &= \frac{n + q_1'(-n/(n - q_2'))}{q_1 + q_1''(-n/(n - q_2'))} = \frac{n(n - (q_1' + q_2'))}{nq_1 - q_1q_2' - nq_1''},\end{aligned}$$

where q_i' and q_i'' are defined for q_i and n as in (i) $(i = 1, 2)$. From this, if $q_1 + q_2 = n$, then $q_1' + q_2' = n$ and so the value above is zero. Conversely, if the left hand side of (2.1) is zero, then we have $q_1' + q_2' = n$. Consequently, $q_1 + q_2 = n$. □

Let n and q be relatively prime integers with $1 \leq q < n$. Let put $n/q = [[b_{1,1}, \dots, b_{1,\ell_1}]]$ and $n/(n - q) = [[b_{2,1}, \dots, b_{2,\ell_2}]]$. Let Δ be a small open disc around the origin in $\mathbb{C}$. From (2.1), there exists uniquely a successive $\mathbb{C}^*$-blowing-up $\sigma_{n,q} : V_{n,q} \longrightarrow \Delta \times \mathbb{P}^1$ started from $(0, 0)$ such that the w.d.graph of ${\sigma_{n,q}}^{-1}(\{0\} \times \mathbb{P}^1)$ is given as



(2.2)

where $E_{2,1} = (\sigma_{n,q})_*^{-1}(\{0\} \times \mathbb{P}^1)$ (= strict transform of $\{0\} \times \mathbb{P}^1$ through $\sigma_{n,q}$). Let $E(0) = \bigcup_{k=1}^{\ell_1} E_{1,k}$ and $E(\infty) = \bigcup_{k=1}^{\ell_2} E_{2,k}$. Also, let $\tau : V_{n,q} \longrightarrow \bar{V}_{n,q}$ be the contraction of $E(0)$ and $E(\infty)$. Therefore, the complex surface $\bar{V}_{n,q}$ has two cyclic quotient singularities $P_0 := \tau(E(0))$ and $P_\infty := \tau(E(\infty))$ such that $(\bar{V}_{n,q}, P_0) \cong C_{n,q}$ and $(\bar{V}_{n,q}, P_\infty) \cong C_{n,n-q}$. Furthermore, let $U_{n,q} := V_{n,q} \backslash E(\infty)$ and let $\bar{U}_{n,q}$ be the complex surface obtained by the contraction of $E(0)$ in $U_{n,q}$. Thus, $\bar{U}_{n,q}$ has only one cyclic quotient singularity of type $C_{n,q}$.

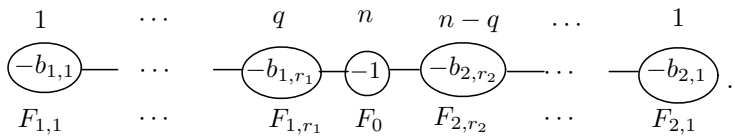
Let put $g = \left(\begin{smallmatrix} e_n & 0 \\ 0 & e_n^q \end{smallmatrix}\right)$ for $e_n := \exp(2\pi\sqrt{-1}/n)$ and consider the natural action of $G_{n,q} := \langle g \rangle$ on $\Delta \times \mathbb{C}$. Then the action is extended naturally onto $\Delta \times \mathbb{P}^1$. Hence we have the following:

$$\begin{tikzcd} \Delta \times \mathbb{C} \arrow[r, hook] \arrow[d, "p"'] & \Delta \times \mathbb{P}^1 \arrow[d, "\bar{p}"] \\ (\Delta \times \mathbb{C})/G_{n,q} \arrow[r, hook] & (\Delta \times \mathbb{P}^1)/G_{n,q}. \end{tikzcd}$$

The complex surface $(\Delta \times \mathbb{C})/G_{n,q}$ has a cyclic quotient singularity of type $C_{n,q}$ at $\bar{p}(0,0)$ and $(\Delta \times \mathbb{P}^1)/G_{n,q}$ has another cyclic quotient singularity of type $C_{n,n-q}$ at $\bar{p}(0,\infty)$. Let $\sigma : Y_{n,q} \longrightarrow (\Delta \times \mathbb{P}^1)/G_{n,q}$ be the minimal resolution of $\bar{p}(0,0)$ and $\bar{p}(0,\infty)$; also let $X_{n,q} \longrightarrow (\Delta \times \mathbb{C})/G_{n,q}$ be the minimal resolution of $\bar{p}(0,0)$. Hence, $X_{n,q} \subset Y_{n,q}$. The following was already described explicitly in [**Fu1**], but it is written in Japanese. Then we give the proof in a slightly different way.

Lemma 2.2. *The complex surface $(\Delta \times \mathbb{P}^1)/G_{n,q}$ (resp. $(\Delta \times \mathbb{C})/G_{n,q}$) is $\mathbb{C}^*$-equivariantly biholomorphic to $\bar{V}_{n,q}$ (resp. $\bar{U}_{n,q}$). Therefore, $Y_{n,q}$ (resp. $X_{n,q}$) is $\mathbb{C}^*$-equivariantly biholomorphic to $V_{n,q}$ (resp. $U_{n,q}$).*

Proof. Let z be the coordinate of Δ. Because z^n is $G_{n,q}$-invariant, we have a holomorphic function h induced from it on $(\Delta \times \mathbb{P}^1)/G_{n,q}$. Let A be a divisor on $(\Delta \times \mathbb{P}^1)/G_{n,q}$ defined by h. Let F_0 be the strict transform of A by σ; then $v_{F_0}(h \circ \sigma) = n$. Therefore, the divisor by $h \circ \sigma$ on $Y_{n,q}$ is given as



Let $\varphi : Y_{n,q} \longrightarrow Z$ be the contraction of $\bigcup_{j=1}^{r_1} F_{1,j} \cup F_0 \cup \bigcup_{j=2}^{r_2} F_{2,j}$. Then, Z

is a non-singular surface and $\varphi(F_{2,1})$ is a non-singular projective line in Z whose intersection number equal to zero from Lemma 2.1. Because the projection map $p_1 : \Delta \times \mathbb{P}^1 \longrightarrow \Delta$ is $G_{n,q}$-equivariant, we can get a proper surjective holomorphic map $\bar{p} : Z \longrightarrow \Delta$ such that any fiber of $\bar{p}$ is $\mathbb{P}^1$. This is isomorphic to the trivial $\mathbb{P}^1$-bundle on Δ. Consequently, from the construction of $V_{n,q}$, $Y_{n,q}$ is $\mathbb{C}^*$-equivariantly biholomorphic to $V_{n,q}$. $\square$

Let $\varphi : L \longrightarrow E_0$ be a holomorphic $\mathbb{P}^1$-bundle on a compact smooth complex curve E_0. For any $P_0 \in E_0$, a small open neighborhood U_{P_0} of P_0 is chosen such that $U_{P_0} \cong \Delta$ (= a small open disc around the origin in $\mathbb{C}$) and $\varphi^{-1}(U_{P_0}) \cong U_{P_0} \times \mathbb{P}^1$. Corresponding to $\sigma_{n,q} : V_{n,q} \longrightarrow U_{P_0} \times \mathbb{P}^1$, there is a successive $\mathbb{C}^*$-blowing-up $\sigma_{n,q}(P_0) : w_{P_0}(n,q)L \longrightarrow L$ satisfying the following diagram:

$$\begin{array}{ccc} w_{P_0}(n,q)L & \xrightarrow{\sigma_{n,q}(P_0)} & L \\ \cup & & \cup \\ V_{n,q} & \xrightarrow{\sigma_{n,q}} & U_{P_0} \times \mathbb{P}^1 \simeq \varphi^{-1}(U_{P_0}). \end{array}$$

DEFINITION 2.3. Let $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ be a complete $\mathbb{C}^*$-pencil of curves and $m_0 := \gcd\{\mathrm{Coeff}_{E_i} S_0 \mid E_i \text{ is an irreducible component of } \mathrm{supp}(S_0)\}$. If $m_0 > 1$, then we call $\hat{\Phi}$ is *a multiple complete $\mathbb{C}^*$-pencil of curves of multiplicity m_0*. This is equivalent to the condition that $\hat{\Phi}_L : \hat{S}|_L \longrightarrow \mathbb{C}$ is a multiple pencil of curves of multiplicity m_0.

THEOREM 2.4. *Let E_0 be a compact smooth complex curve of genus g_0 and N a non-positive holomorphic line bundle on E_0. Assume that $dN \sim -\sum_{j=1}^{s} d_j P_j$ (linearly equivalent) for positive integers $d, d_1, \ldots, d_s$ with $0 < d_j < d$ $(s \geq 0)$ and mutually distinct s points $P_1, \ldots, P_s \in E_0$. Let $m_0 := \gcd(d, d_1, \ldots, d_s)$ and $g := 1 + (1/2)\{d(2g_0 - 2 + s) - \sum_{j=1}^{s} \gcd(d, d_j)\}$.*

(i) *There exists a minimal good complete $\mathbb{C}^*$-quasi-pencil of curves $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ of degree d satisfying the following properties*:
  - (i-1) *the w.d.graphs of the singular fibers S_0 and S_∞ are star-shaped*;
  - (i-2) *the central curves of S_0 and S_∞ are holomorphically isomorphic to E_0*;
  - (i-3) *if E_0 is considered as the central curve of S_0, then $N_{E_0/\hat{S}}|_{E_0} \simeq N$ as holomorphic line bundles on E_0.*

(ii) *Let $\hat{\Phi}$ be a complete $\mathbb{C}^*$-quasi-pencil of curves constructed in* (i).
  - (ii-1) *If $m_0 = 1$, then $\hat{\Phi}$ is a non-multiple complete $\mathbb{C}^*$-pencil of curves of type $(d, g, 1)$.*
  - (ii-2) *Supposing that $m_0 > 1$, then $\hat{\Phi}$ is a multiple complete $\mathbb{C}^*$-pencil of*

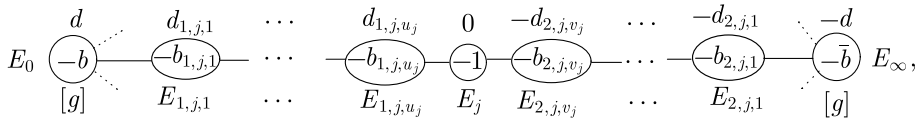
curves of type (g, d, m_0) *if and only if* $(d/m_0)N + \sum_{j=1}^{s}(d_j/m_0)P_j$ *is a torsion bundle of order* m_0.

Proof. (i) Let $-b$ be the degree of N ($b \geq 0$). Then $bd = \sum_{j=1}^{s} d_j$. When $s = 1$, we have $db = b_1 > 0$. This contradicts the hypothesis $0 < d_1 < d$; hence we have $s = 0$ or $s \geq 2$. Let H be a holomorphic line bundle $N + [P_1 + \cdots + P_s]$ on E_0 and $\varphi : H \longrightarrow E_0$ the projection map. Furthermore, let $\bar{\varphi} : \bar{H} \longrightarrow E_0$ be a $\mathbb{P}^1$-bundle on E_0 associated to H. We choose an open coordinate covering $\bigcup_\alpha U_\alpha$ of E_0 such that $\bar{\varphi}^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{P}^1$ (so $\varphi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}$). Let ζ_α be a fiber coordinate function on $\varphi^{-1}(U_\alpha)$. We choose a meromorphic function h_α on U_α such that $(h_\alpha) = \sum_{j=1}^{s}(d - d_j)P_j$ on U_α. Because $\{h_\alpha\}$ is a meromorphic section of dH satisfying $\zeta_\alpha^d/h_\alpha = \zeta_\beta^d/h_\beta$ on $\varphi^{-1}(U_\alpha) \cap \varphi^{-1}(U_\beta)$, there is a meromorphic function Φ' on H such that $\Phi'|_{\varphi^{-1}(U_\alpha)} = \zeta_\alpha^d/h_\alpha$ for any α. The natural $\mathbb{C}^*$-action on H is given by $t \cdot (\zeta_\alpha, P) = (t\zeta_\alpha, P)$ on $\varphi^{-1}(U_\alpha)$. Hence we have $\Phi'(t \cdot (\zeta_\alpha, P)) = (t\zeta_\alpha)^d/h_\alpha(P) = t^d\Phi'(P)$. Then Φ' is extended to a meromorphic function on $\bar{H}$. Let $\bar{\Phi} := [\bar{\Phi}_0 \colon \bar{\Phi}_1] : \bar{H} \longrightarrow \mathbb{P}^1$ be a holomorphic map which is given by Φ', where $\bar{\Phi}_0$ and $\bar{\Phi}_1$ are holomorphic functions with $\bar{\Phi}_0 h_\alpha = \zeta_\alpha^d \bar{\Phi}_1$. Then $\bar{\Phi}$ is a $\mathbb{C}^*$-equivariant holomorphic map with respect to a $\mathbb{C}^*$-action defined by $t \cdot [\xi_0 \colon \xi_1] = [t^d\xi_0 \colon \xi_1]$ on $\mathbb{P}^1$ for any $t \in \mathbb{C}^*$.

When $s = 0$, let assume that $\hat{S} := \bar{S}$ and $\hat{\Phi} := \bar{\Phi}$. In this case, H is a torsion or the trivial line bundle on E_0 from the definition. Next, let consider the case of $s \geq 2$; hence we have $bd = \sum_{j=1}^{s} d_j > 0$. Let $d/d_j = [[b_{1,j,1}, \ldots, b_{1,j,u_j}]]$ and $d/(d - d_j) = [[b_{2,j,1}, \ldots, b_{2,j,v_j}]]$. If we put $E_{2,j,1} = \varphi^{-1}(P_j)$ (so $P_j = E_0 \cap E_{2,j,1}$), then $v_{E_{2,j,1}}(\bar{\Phi})$ (= the vanishing order of $\bar{\Phi}$ on $E_{2,j,1}$) is equal to $-d_j$ and $v_{E_0}(\bar{\Phi}) = d$. For a small open neighborhood U_j of P_j, we consider a successive $\mathbb{C}^*$-equivariant blowing-up $\sigma_j := \sigma_{d,d_j}(P_j) : w_{P_j}(d, d_j)(\bar{\varphi}^{-1}(U_j)) \longrightarrow \bar{\varphi}^{-1}(U_j)$ as in (2.2). Let $\hat{S} := w_{P_s}(d, d_s) \cdots w_{P_1}(d, d_1)\bar{H}$ and $\sigma := \sigma_s \circ \cdots \circ \sigma_1 : \hat{S} \longrightarrow \bar{H}$. We consider the following holomorphic map

$$\hat{\Phi} = \bar{\Phi} \circ \sigma : \hat{S} \longrightarrow \mathbb{P}^1.$$

Then the figure of the divisor for $\hat{\Phi}$ on $\hat{S}$ is represented as



where $E_{2,j,1}$ (resp. E_∞) is the strict transform of $\varphi^{-1}(P_j)$ (resp. the infinity section of $\bar{H}$); also $d_{i,j,k} > 0$ for $i = 1, 2$ and $d_{1,j,1} = d - d_j$ and $d_{2,j,1} = d_j$. Since σ_j is a $\mathbb{C}^*$-equivariant map, the $\mathbb{C}^*$-action on $\bar{H}$ is lifted onto $\hat{S}$ and $\hat{\Phi}$ is

a $\mathbb{C}^*$-equivariant map. In the construction of $\hat{S}$, we took only one blowing-up at P_j for each j. Therefore, the restriction of the normal bundle $N_{E_0/\hat{S}}$ onto E_0 is linearly equivalent to N.

Now let $S := \hat{S}\backslash \operatorname{supp}(\hat{\Phi}^{-1}(0)) = \hat{S}\backslash(E_\infty \cup (\bigcup_{j=1}^{s}\bigcup_{k=1}^{v_j} E_{2,j,k}))$ and consider a holomorphic map $\Phi := \hat{\Phi}|_S : S \longrightarrow \mathbb{C}$. We prove that Φ is a $\mathbb{C}^*$-quasi-pencil of curves. Because $E_\infty \cup (\bigcup_{j=1}^{s}\bigcup_{k=1}^{v_j} E_{2,j,k})$ is a $\mathbb{C}^*$-invariant set, S is also a $\mathbb{C}^*$-invariant set and Φ is a $\mathbb{C}^*$-equivariant holomorphic map. Because $\operatorname{supp}(\Phi^{-1}(0)) = E_0 \cup (\bigcup_{j=1}^{s}\bigcup_{k=1}^{u_j} E_{1,j,k})$ and this is a compact complex curve in S, the restriction map $\Phi : \Phi^{-1}(\Delta_\varepsilon) \longrightarrow \Delta_\varepsilon$ is a proper map for sufficiently small ε with $0 < \varepsilon \ll 1$ (see [**Stei**] and [**Fi**, p. 56]). Since $\operatorname{red}(\Phi^{-1}(0))$ is a simple normal crossing divisor, $\Phi^{-1}(t)$ is non-singular for any $t \in \Delta_\varepsilon - \{0\}$. Because, if we write $\Phi(x,y) = x^a y^b$ locally, then any point P satisfying $\partial\Phi/\partial x(P) = \partial\Phi/\partial y(P) = 0$ is included in $\{xy = 0\}$ ($\subset \operatorname{supp}(\Phi^{-1}(0))$). Then $\Phi^{-1}(t)$ is a compact smooth complex curve for any $t \in \Delta_\varepsilon - \{0\}$. Then $\Phi : \Phi^{-1}(\Delta_\varepsilon) \longrightarrow \Delta_\varepsilon$ is a quasi-pencil of curves.

It is easy to confirm that $\Phi(tP) = t^d\Phi(P)$ for any $t \in \mathbb{C}^*$. Consequently, we need only to show that $\Phi : S \longrightarrow \mathbb{C}$ is a quasi-pencil of curves. Namely, for any $\zeta \in \mathbb{C}^*$, we must prove that $\Phi^{-1}(\zeta)$ is a smooth compact complex curve. For any $\zeta \in \mathbb{C}$ and any $t \in \mathbb{C}^*$, we have

$$t\Phi^{-1}(\zeta) = \Phi^{-1}(t^d\zeta). \tag{2.3}$$

In fact, for any $tP \in t\Phi^{-1}(\zeta)$, we have $\Phi(tP) = t^d\Phi(P) = t^d\zeta$. Therefore, $tP \in \Phi^{-1}(t^d\zeta)$ and so $t\Phi^{-1}(\zeta) \subset \Phi^{-1}(t^d\zeta)$. Conversely, for any $P \in \Phi^{-1}(t^d\zeta)$, we have $\Phi(P) = t^d\zeta$ and so $\zeta = (1/t^d)\Phi(P) = \Phi((1/t)P)$. Then $(1/t)P \in \Phi^{-1}(\zeta)$ and so $P \in t\Phi^{-1}(\zeta)$. Therefore, we have (2.3). For any $\zeta \in \mathbb{C}$, we take $t_0 \in \mathbb{C}^*$ such that $t_0^d\zeta \in \Delta_\varepsilon$. From (2.3), $t_0\Phi^{-1}(\zeta) = \Phi^{-1}(t_0^d\zeta)$ is a smooth compact complex curve. Because $t_0 : \Phi^{-1}(\zeta) \longrightarrow t_0\Phi^{-1}(\zeta)$ gives a biholomorphic map, $\Phi^{-1}(\zeta)$ is also smooth and compact. Then Φ is a $\mathbb{C}^*$-quasi-pencil of curves. Let $S^* := \hat{S}\backslash \operatorname{supp}(\hat{\Phi}^{-1}(\infty)) = \hat{S}\backslash(E_0 \cup (\bigcup_{j=1}^{s}\bigcup_{k=1}^{u_j} E_{1,j,k}))$. Similarly, we can show that $\Phi^* := 1/(\hat{\Phi})|_{S^*} : S^* \longrightarrow \mathbb{C}$ is a $\mathbb{C}^*$-quasi-pencil of curves; hence $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ is a complete $\mathbb{C}^*$-quasi-pencil of curves.

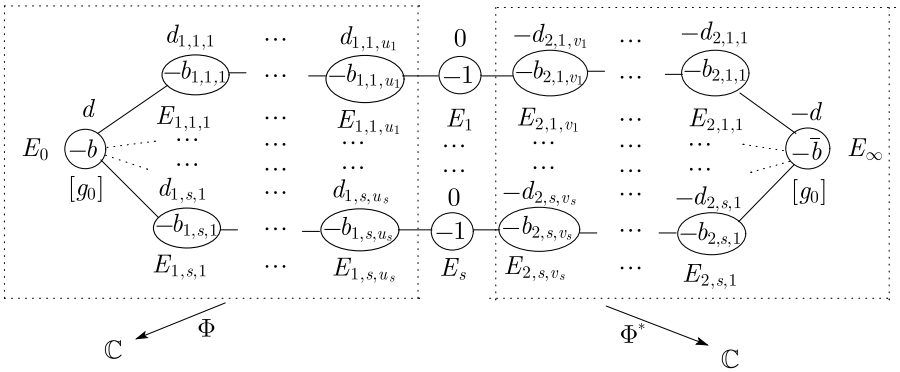
(ii) Considering the restriction map $\Phi_\varepsilon : \Phi^{-1}(\Delta_\varepsilon) \longrightarrow \Delta_\varepsilon$ constructed in (i), first we prove (ii-1). Consider the Stein factorization of Φ_ε as

$$\begin{tikzcd} \Phi^{-1}(\Delta_\varepsilon) \arrow[rr, "\Phi_\varepsilon"] \arrow[dr, "\Phi'_\varepsilon"'] & & \Delta \\ & \bar{\Delta} \arrow[ur, "\eta"'] & \end{tikzcd},$$

where η is a finite map and Φ'_ε is a proper map whose fiber is connected. Then η is

given by $t = \eta(v) = v^n \eta_1(v)$ $(n \geq 1)$, where v is a coordinate on $\bar{\Delta}$ and $\eta_1(0) \neq 0$. Therefore, we have $S_0 = n\bar{S}_0$ for an effective divisor $\bar{S}_0$ determined by $v \circ \Phi'_\varepsilon$. Then $n \mid d$ and $n \mid d_i$ for any i. Consequently, $(d/n)N \sim -\sum_{j=1}^{s}(d_j/n)P_j$ and $n \mid m_0$. Therefore, we have $n = 1$ from $m_0 = 1$. Then, any fiber of Φ_ε is connected. Hence $\hat{\Phi}$ gives a $\mathbb{C}^*$-pencil of curves because all general fiber of $\hat{\Phi}$ are isomorphic from Remark 1.7 (iii). Therefore, $\hat{\Phi}$ is a non-multiple complete $\mathbb{C}^*$-pencil of curves from $m_0 = 1$.

Next we consider the genus of $\hat{\Phi}$. Assume that the following figure is associated to $\hat{\Phi}$:



(2.4)

where we have the following:

$$\begin{aligned} &\text{(i)} \quad \big[\big[b_{1,j,1}, \ldots, b_{1,j,u_j}, 1, b_{2,j,v_j}, \ldots, b_{2,j,1}\big]\big] = 0, \\ &\text{(ii)} \quad b + \bar{b} = s \ \text{ and } \ d_{1,j,1} + d_{2,j,1} = d. \end{aligned} \tag{2.5}$$

By the adjunction formula (see [**BPV**, p. 68]), $g = 1 + (1/2)\{d(b + 2g_0 - 2) + \sum_{j=1}^{s}\sum_{k=1}^{u_j} d_{1,j,k}(b_{1,j,k} - 2)\}$. For any j, we obtain a simultaneous linear equation $d_{1,j,k-1} - b_{1,j,k}d_{1,j,k} + d_{1,j,k+1} = 0$ for $(k = 1, \ldots, u_j)$, where $d_{1,j,0} = d$, $d_{1,j,1} = d_j$ and $d_{1,j,u_j+1} = 0$. Adding up them, we obtain an equation $d + \sum_{k=1}^{u_j-1} d_{1,j,k} - \sum_{k=1}^{u_j} b_{1,j,k}d_{1,j,k} + \sum_{k=2}^{u_j} d_{1,j,k} = 0$ for any j. Then $\sum_{j=1}^{s}\sum_{k=1}^{u_j} d_{1,j,k}(b_{1,j,k} - 2) = sd - \sum_{j=1}^{s} d_{1,j,u_j} - \sum_{j=1}^{s} d_{1,j,1}$. Because $d_{1,j,u_j} = \gcd(d, d_{1,j,1})$ and $\sum_{j=1}^{s} d_{1,j,1} = db$, we obtain the formula of g.

(ii-2) Let put $F =: (d/m_0)N + \sum_{j=1}^{s}(d_j/m_0)P_j$. First, consider the "only if" part. Let $k_0 = \min\{k \mid 1 \leq k \leq m_0, kF \sim 0\}$. Assume that $k_0 < m_0$. Since $m_0F \sim 0$ and $k_0F \sim 0$, there exists a positive integer k_1 $(k_1 < m_0)$ such that $k_1 \mid m_0$ and $k_1F \sim 0$. Using (i), there exists a quasi-complete $\mathbb{C}^*$-pencil of curves $\hat{\Phi}_0 : \hat{S} \longrightarrow \mathbb{P}^1$ associated to k_1F. From Remark 1.7 (ii), we have $\hat{\Phi} = c\hat{\Phi}_0^{m_0/k_1}$

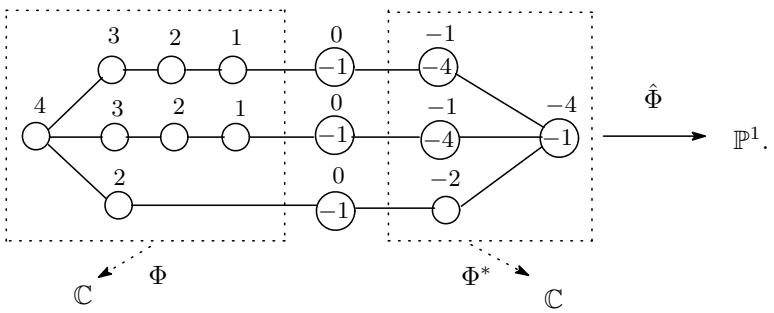
for a constant c. Therefore, $\hat{\Phi}$ is not a pencil of curves. This is a contradiction. Consequently, $k_0 = m_0$, and F is a torsion line bundle of order m_0.

Second, consider the "if" part. If $\hat{\Phi}$ is not a multiple pencil of curves, then it is not a pencil of curves. Namely, it is not a connected map. Using Stein factorization as in (ii-1), we can show that $S_0 = n\bar{S}_0$ for a quasi-pencil of curves $\bar{\Phi} : \bar{S} \longrightarrow \mathbb{C}$, where n (> 1) is the number of connected components of the general fiber. Since $\bar{S}_0 E_0 \sim 0$, we have $n \mid m_0$ and $(d/n)N \sim -\sum_{j=1}^{s}(d_j/n)P_j$. Therefore, $(m_0/n)F \sim 0$ and this contradicts the hypothesis that the order of F is m_0. $\square$

Let $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ be a complete $\mathbb{C}^*$-pencil of curves constructed in 2.4 (i). It is denoted by $\hat{\Phi}(E_0, N, \sum_{j=1}^{s} d_j P_j)$ or $\hat{\Phi}(A)$, where $A := (E_0, N, \sum_{j=1}^{s} d_j P_j)$. The degree of $\hat{\Phi}$ is given as $-(\sum_{j=1}^{s} d_j)/\deg(N)$.

REMARK 2.5. (i) In the construction of complete $\mathbb{C}^*$-pencils of curves of order d, the generic fiber of $\hat{\Phi}_L$ is also the generic fiber of $\hat{\Phi}_R$. Therefore, $\hat{\Phi}_L$ is the inverse of the homological monodromy transformation associated to $\hat{\Phi}_R$ because $\hat{\Phi}_L = 1/\hat{\Phi}_R$ on the generic fiber. Moreover, the order of the monodromy group is equal to d. Generally, the monodromy group of any pencil of curves whose singular fiber is star-shaped is a finite group, and the order is equal to the coefficient of the central curves of the singular fiber S_0 (see Section 4 in [**MM**]).

(ii) In Kodaira's list of elliptic pencils in [**Ko**], we consider pencils of curves except for type I_m and I_m^* $(m \geq 1)$. Then they have star-shaped singular fibers. It is easy to see that they are realized as $\mathbb{C}^*$-pencils of curves. For example, III and III* in [**Ko**] are embedded into a complete $\mathbb{C}^*$-pencil of curves as follows:

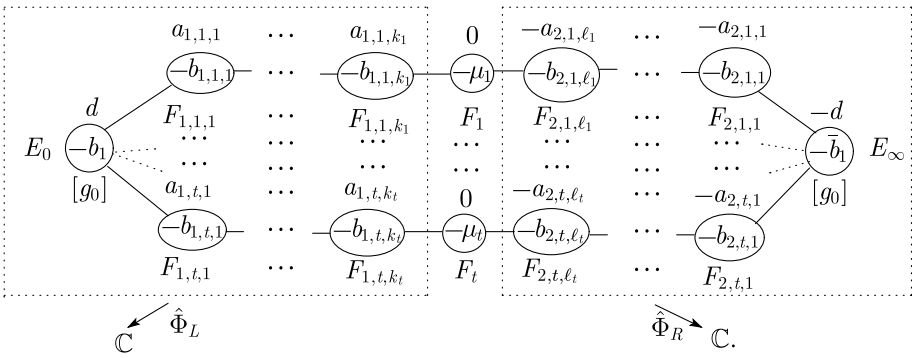


Therefore, they are dual mutually in our sense. Using Theorem 4.6, we can see that the generic fiber of $\hat{\Phi}$ is the elliptic curve with complex multiplicative group of order 4.

(iii) Let Y be a compact complex algebraic surface. Let $\Psi : Y \longrightarrow \mathbb{P}^1$ be a pencil of curves. Let s be the number of its singular fibers. It is known that $s \geq 2$. Furthermore, if $s = 2$, then the geometric genus $p_g(Y)$ and the irregularity

$q(Y)$ are equal to zero. Please refer [**N**] for more detail. Any non-trivial complete $\mathbb{C}^*$-pencil of curves gives a pencil of curves with $s = 2$.

Definition 2.6. Let $\hat{\Phi}(0) : \hat{S}(0) \longrightarrow \mathbb{P}^1$ be a complete $\mathbb{C}^*$-pencil of curves obtained by the construction of Theorem 2.4 (i). Let $\tau : \hat{S} \longrightarrow \hat{S}(0)$ be a sequel of blowing-ups at $\mathbb{C}^*$-fixed points. Then $\hat{\Phi} := \hat{\Phi}(0) \circ \tau : \hat{S} \longrightarrow \mathbb{P}^1$ is also a complete $\mathbb{C}^*$-pencil of curves. We call such one *a complete* $\mathbb{C}^*$*-pencil of curves by the canonical construction.* It is figured as follows:



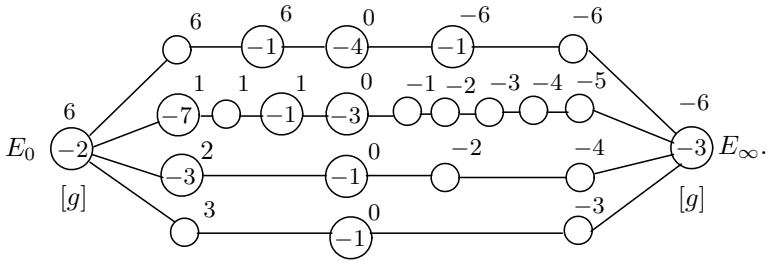
(2.6)

In this paper, we call the figure above *the configuration associated to* $\hat{\Phi}$. Let put $F_L = E_0 \cup (\bigcup_{j=1}^{t} \bigcup_{k=1}^{k_j} F_{1,j,k})$, $F_R = E_\infty \cup (\bigcup_{j=1}^{t} \bigcup_{k=1}^{\ell_j} F_{2,j,k})$ and $F_M = \bigcup_{j=1}^{t} F_j$. Therefore, $\mathbb{C}^*$ acts trivially on E_0 and E_∞. All intersection points of irreducible components in $F_L \cup F_R \cup F_M$ are fixed points of the $\mathbb{C}^*$-action. Furthermore, each component of $F_L \cup F_R \cup F_M$ except for E_0 and E_∞ is constructed by a one-dimensional orbit and two fixed points. As in (2.5), we obtain the following:

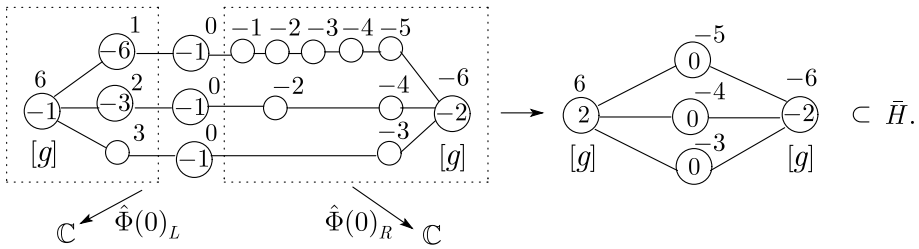
$$\begin{aligned} &\text{(i)} \quad \big[\big[b_{1,j,1}, \ldots, b_{1,j,k_j}, \mu_j, b_{2,j,\ell_j}, \ldots, b_{2,j,1}\big]\big] = 0, \\ &\text{(ii)} \quad a_{1,j,1} + a_{2,j,1} \equiv 0 \ (d), \end{aligned} \tag{2.7}$$

but it is not always true that $b_1 + \bar{b}_1 = t$. If F_L and F_R does not contain a (-1)-curve, then $\hat{\Phi}$ is called *a minimal good complete* $\mathbb{C}^*$*-pencil of curves.* For a given complete $\mathbb{C}^*$-pencil of curves, we can obtain a minimal good complete $\mathbb{C}^*$-pencil of curves after suitable contractions of (-1)-curves. The latter is called *the minimal good model* of the former.

Example 2.7. Let us consider a complete $\mathbb{C}^*$-pencil of curves $\hat{\Phi}\colon \hat{S} \longrightarrow \mathbb{P}^1$ whose configuration associated to $\hat{\Phi}$ is given as follows:



By a suitable contraction of (-1)-curves in $\hat{S}$, we obtain the minimal good model $\hat{\Phi}(0)\colon \hat{S}(0) \longrightarrow \mathbb{P}^1$ and an associated $\mathbb{P}^1$-bundle $\bar{H}$ on E_0 as follows:



In the following, using the slice theorem due to H. Holmann ([**H2**]), we prove that every $\mathbb{C}^*$-pencil of curves is $\mathbb{C}^*$-holomorphically isomorphic to one obtained by the canonical construction. We prepare some definitions and a lemma in the following.

DEFINITION 2.8 ([**H2**]). Let G be a topological transformation group acting on a topological space X.

(i) For any point $x \in X$, suppose that there exists a neighborhood U_x such that $G \times \bar{U}_x \longrightarrow \bar{U}_x$ $((h,p) \mapsto hp)$ is proper on the closure $\bar{U}_x$. Then the action of G on X is said to be *locally proper*.

(ii) The action of G on X is said to be *proper* if $G \times X \longrightarrow X \times X$ $((h,p) \mapsto (hp,p))$ is proper.

(iii) If the isotropy group $G_x := \{h \in G \mid hx = x\}$ is a finite group of G for any $x \in X$, then the action is said to be *isotropy finite*.

In the above definition, if the action of G is proper, then we can easily check that it is locally proper.

LEMMA 2.9. *Let $\Phi : S \longrightarrow \mathbb{C}$ be a $\mathbb{C}^*$-pencil of curves of degree d. The $\mathbb{C}^*$-action on $S' := S \setminus \operatorname{supp}(S_0)$ is proper and isotropy finite. Also, any isotropy group is a cyclic group whose order is a divisor of d.*

PROOF. Let $\varphi : \mathbb{C}^* \times S' \longrightarrow S' \times S'$ be a holomorphic map defined by $(t, x) \mapsto (tx, x)$. Let K be a compact subset in $S' \times S'$. Then there are suitable positive constants m_i and M_i $(i = 1, 2)$ such that we have $m_1 \leq |\Phi(x)| \leq M_1$ and $m_2 \leq |\Phi(y)| \leq M_2$ for any $(x, y) \in K$. Let p_i be the projection map to the i-th factor from $\mathbb{C}^* \times S'$ for $i = 1, 2$. For any $(t, x) \in \varphi^{-1}(K)$, we have $(tx, x) \in K$. Then we have $m_1 \leq |t|^d |\Phi(x)| \leq M_1$ and $m_2 \leq |\Phi(x)| \leq M_2$. Then we have $m_1/M_2 \leq |t|^d \leq M_1/m_2$ for any $t \in p_1^{-1}(K)$. Hence $p_1(\varphi^{-1}(K)) = \{t \in \mathbb{C}^* | (tx, x) \in K$ for an element $x \in S'\}$ is a bounded and closed set in $\mathbb{C}^*$; then it is a compact set. On the other hand, $p_2(\varphi^{-1}(K)) = q_2(K)$ can be checked readily, where q_2 is the projection map to the second factor from $S' \times S'$. Thus $p_2(\varphi^{-1}(K))$ is compact and so $p_1(\varphi^{-1}(K)) \times p_2(\varphi^{-1}(K))$ is compact in $S' \times S'$. Since $\varphi^{-1}(K)$ is a closed subset in $p_1(\varphi^{-1}(K)) \times p_2(\varphi^{-1}(K))$, $\varphi^{-1}(K)$ is compact. Then φ is a proper map.

Any finite subgroup of $\mathbb{C}^*$ is a cyclic group. For any $p \in S'$, let G_p be the isotropy group. For any $t \in G_p$, we have $\Phi(p) = \Phi(tp) = t^d \Phi(p) \neq 0$. Consequently, $t^d = 1$ and G_p is a cyclic group whose order is a divisor of d. $\square$

THEOREM 2.10. *Let $\Phi : S \longrightarrow \mathbb{C}$ be a $\mathbb{C}^*$-pencil of curves. Then it is $\mathbb{C}^*$-holomorphically isomorphic to a $\mathbb{C}^*$-pencil of curves obtained by the canonical construction.*

PROOF. Let g be the genus of Φ. If $g = 0$ and relatively minimal (i.e., no fiber contains (-1)-curve), then Φ is a trivial pencil of curves. Therefore, we can assume that $g \geq 1$. Let E_0 be an irreducible component of $E := \mathrm{supp}(S_0)$ such that there are infinitely many orbits whose closures in S intersect to it; hence $\mathbb{C}^*$ acts trivially on E_0. Let $\sigma : S \longrightarrow \bar{S}$ be the contraction map of all connected components of $E - E_0$. Then $\bar{S}$ is a normal surface. Since any connected component of $E - E_0$ is $\mathbb{C}^*$-invariant, the $\mathbb{C}^*$-action is preserved onto $\bar{S}$. Because $S' := S \backslash \mathrm{supp}(S_0)$ is identified with $\bar{S}' := \bar{S} \backslash \sigma(E_0)$, the $\mathbb{C}^*$-action on $\bar{S}'$ is proper from Lemma 2.9.

Let x_0 be any point in $\bar{S}'$. From Holmann's slice theorem ([**H2**, Satz 4, 8]), we have a $\mathbb{C}^*$-saturated neighborhood U_{x_0} in $\bar{S}'$ (i.e., $h \cdot U_{x_0} = U_{x_0}$ for any $h \in \mathbb{C}^*$) and a G_{x_0}-invariant smooth complex curve D_{x_0} in U_{x_0} such that there is a $\mathbb{C}^*$-equivariant biholomorphic map $\psi_{x_0} : U_{x_0} \longrightarrow (\mathbb{C}^* \times D_{x_0})/\tau(G_{x_0})$ with respect to the canonical $\mathbb{C}^*$-action on the right hand side (i.e., $t(\zeta, x) = (t\zeta, x)$ for any $t \in \mathbb{C}^*$), where $\tau : G_{x_0} \longrightarrow \mathrm{Aut}(\mathbb{C}^* \times D_{x_0})$ is defined by $\tau(h)(\zeta, x) = (h\zeta, h^{-1} \cdot x)$ for any $h \in G_{x_0}$. Here, $h\zeta$ means the usual product in $\mathbb{C}^*$ and $h^{-1} \cdot x$ means the $\mathbb{C}^*$-action. For any $x \in \bar{S}'$, let O_x be the $\mathbb{C}^*$-orbit of x; also $\bar{O}_x$ be the closure of O_x in $\bar{S}$. If we set $V_{x_0} := \bigcup_{x \in U_{x_0}} \bar{O}_x$, then it is an open set in $\bar{S}$ satisfying $V_{x_0} \cap \bar{S}' = U_{x_0}$. If we set $F := V_{x_0} \cap \bar{E}_0$, then $\mathbb{C}^*$ acts trivially on F. For any point $y \in F$, let x be a point of $\bar{S}'$ with $y \in \bar{O}_x$. Let $\tilde{\psi}_{x_0}(y) :=$

$\lim_{t\to 0} t\cdot\psi_{x_0}(x) \in (\{0\}\times D_{x_0})/\tau(G_{x_0})$. This is independent of the choice of x. In fact, if $x_1, x_2 \in S'$ with $y \in O_{x_1} = O_{x_2}$, then there exists $t_0 \in \mathbb{C}^*$ with $x_1 = t_0 x_2$. Since ψ_{x_0} is $\mathbb{C}^*$-equivariant, $\psi_{x_0}(x_1) = \psi_{x_0}(t_0 x_2) = t_0\cdot\psi_{x_0}(x_2)$. Hence $\lim_{t\to 0} t\cdot\psi_{x_0}(x_1) = \lim_{t\to 0} tt_0\cdot\psi_{x_0}(x_2) = \lim_{t\to 0} t\cdot\psi_{x_0}(x_2)$. Consequently, ψ_{x_0} is extended to a $\mathbb{C}^*$-equivariant homeomorphism $\tilde{\psi}_{x_0} : V_{x_0} \simeq (\mathbb{C}\times D_{x_0})/\tau(G_{x_0})$. Therefore, from the Riemann's removable singularity theorem ([**KK**, p. 308]), $\tilde{\psi}_{x_0}$ become a $\mathbb{C}^*$-equivariant biholomorphic map. On the other hand, $\tau(G_{x_0})$ is finite subgroup of $\mathbb{C}^*$; then it is a finite cyclic group. Therefore, V_{x_0} has at most a cyclic quotient singularity. Since σ is a resolution map of those cyclic quotient singularities, any connected component of $E\backslash E_0$ is a $\mathbb{P}^1$-chain and the w.d.graph of E is star-shaped. Hence, $\sigma^{-1}(V_{x_0})$ is $\mathbb{C}^*$-equivariantly biholomorphic to $V_{n,q}$ by Lemma 2.2. Consequently, we complete the proof. □

COROLLARY 2.11. *Let $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ be a complete $\mathbb{C}^*$-pencil of curves. Then it is $\mathbb{C}^*$-equivariantly biholomorphic to one obtained by the canonical construction.*

PROOF. Assume that $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ is minimal good and so the w.d.graph is given by (2.4). From 2.10, there exists a minimal good complete $\mathbb{C}^*$-pencil of curves $\hat{\Psi} : \hat{S}' \longrightarrow \mathbb{P}^1$ by the canonical construction such that $\hat{\Phi}_L : \hat{S}_L \longrightarrow \mathbb{C}$ and $\hat{\Psi}_L : \hat{S}'_L \longrightarrow \mathbb{C}$ are holomorphically isomorphic. Since $\hat{\Phi}$ and $\hat{\Psi}$ are minimal good, the w.d.graphs of $\hat{S}_{R,0}$ and $\hat{S}'_{R,0}$ coincide. Let $\varphi = (\varphi_1, \varphi_2)$ be a holomorphic isomorphism from $\hat{\Phi}_L$ to $\hat{\Psi}_L$; namely it satisfies the following commutative diagram:

$$\begin{array}{ccc} \hat{S}_L & \xrightarrow{\varphi_1} & \hat{S}'_L \\ {\scriptstyle \hat{\Phi}_L}\downarrow & & \downarrow{\scriptstyle \hat{\Psi}_L} \\ \mathbb{C} & \xrightarrow{\varphi_2} & \mathbb{C}. \end{array}$$

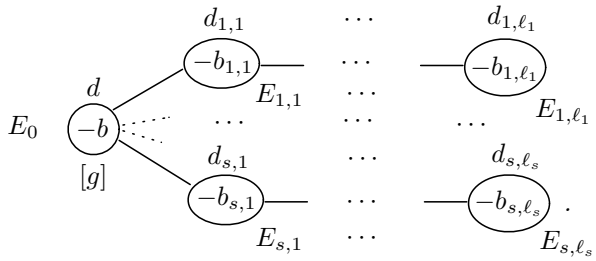
Let E_∞ (resp. F_∞) be the central curve in $\mathrm{supp}(\hat{\Phi}^{-1}(\infty))$ (resp. $\mathrm{supp}(\hat{\Psi}^{-1}(\infty))$. Let $\sigma_1 : \hat{S} \longrightarrow \bar{S}$ (resp. $\sigma_2 : \hat{S}' \longrightarrow \bar{S}'$) be the contraction map of $\mathrm{supp}(\hat{\Phi}^{-1}(\infty))\backslash E_\infty$ (resp. $\mathrm{supp}(\hat{\Psi}^{-1}(\infty))\backslash F_\infty$). Therefore, φ_1 gives a $\mathbb{C}^*$-equivariant biholomorphic map $\bar{\varphi}_1 : \bar{S}\backslash\sigma_1(E_\infty)$ to $\bar{S}'\backslash\sigma_2(F_\infty)$. Obviously, $\bar{\varphi}_1$ is extended to a homeomorphism from $\bar{S}$ to $\bar{S}'$. Then it becomes a biholomorphic map between normal complex spaces $\bar{S}$ and $\bar{S}'$ by the Riemann's removable singularity theorem; also it is $\mathbb{C}^*$-equivariant. Hence we have a $\mathbb{C}^*$-equivariant biholomorphic map $\hat{\varphi}_1$ from $\hat{S}$ to $\hat{S}'$. Defining as $\hat{\varphi}_2(\infty) = \infty$, φ_2 is extended to a $\mathbb{C}^*$-equivariant biholomorphic map $\hat{\varphi}_2$ of $\mathbb{P}^1$. Therefore, $\hat{\varphi} := (\hat{\varphi}_1, \hat{\varphi}_2)$ gives a $\mathbb{C}^*$-equivariantly holomorphic isomorphism between $\hat{\Phi}$ and $\hat{\Psi}$. □

Consequently, from Theorems 2.4 and 2.10, we have proved the following, which is an analogous result to that of Fujiki [**Fu1**] and Pinkham [**P**] for sur-

face singularities with $\mathbb{C}^*$-action. Especially, (iii) is an analogous formula of Pinkham-Demazure construction for normal surface singularities with $\mathbb{C}^*$-action.

COROLLARY 2.12. *Let $\Phi : S \longrightarrow \mathbb{C}$ be a $\mathbb{C}^*$-pencil of curves of degree d.*

(i) *After a suitable $\mathbb{C}^*$-birational transformation, the w.d.graph of the singular fiber S_0 is a star-shaped graph as*



(2.8)

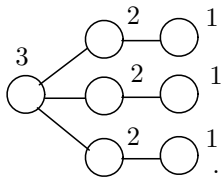
(ii) *The analytic type of a $\mathbb{C}^*$-pencil of curves is determined by the following data*:
  (ii-1) *The analytic types of the central curve E_0.*
  (ii-2) *The analytic types of the normal bundle of E_0 in S.*
  (ii-3) *The intersection points $P_j = E_0 \cap E_{j,1}$ $(j = 1, \ldots, s)$.*

(iii) *For the fibering map Φ, there is a following natural identification*:

$$H^0(E_0, \mathcal{O}_{E_0}(D^{(d)}))t^d \cong \mathbb{C}\Phi,$$

where $D^{(d)} = d(N^*_{E_0/S}|_{E_0}) - \sum_{j=1}^{s} \lceil e_j d/d_j \rceil P_j = d(N^*_{E_0/S}|_{E_0}) - \sum_{j=1}^{s} d_{j,1} P_j$ and Φ is a holomorphic function on S, which is constructed in Theorem 2.4 (i) *from an element of $H^0(E_0, \mathcal{O}_{E_0}(D^{(d)}))$ and the fiber coordinate ζ.*

EXAMPLE 2.13. Let $\Phi : S \longrightarrow \mathbb{C}$ be an elliptic pencil whose singular fiber is given as follows:



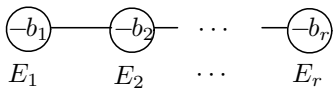
Then $D^{(k)} = 2kP_0 - \sum_{j=1}^{3} \lceil 2k/3 \rceil P_j$ and $\deg(D^{(k)}) \leq 0$ for any k. Moreover, $\deg(D^{(k)}) = 0$ if and only if $k \equiv 0 \pmod 3$. Therefore, $H^0(E_0, \mathcal{O}_{E_0}(D^{(3\ell)})) \cong \mathbb{C}\Phi^\ell$ and $H^0(E_0, \mathcal{O}_{E_0}(D^{(k)})) = 0$ if $k \not\equiv 0 \pmod 3$.

3. Cyclic covers of cyclic quotient singularities.

In this section, we prove some results on cyclic coverings of cyclic quotient singularities as the preparation of Sections 4 and 5. Let consider the natural $\mathbb{C}^*$-action $t \cdot (x, y) = (x, t^\ell y)$ on $D_\varepsilon \times \mathbb{C}$, where $D_\varepsilon = \{z \in \mathbb{C} \mid |z| < \varepsilon\} \subset \mathbb{C}$ for $0 < \varepsilon \leq \infty$ and ℓ is a positive integer. Let G be a finite cyclic group generated by $\left(\begin{smallmatrix} e_n & 0 \\ 0 & e_n^q \end{smallmatrix}\right)$ and consider the natural G-action on $D_\varepsilon \times \mathbb{C}$ and the following quotient complex space

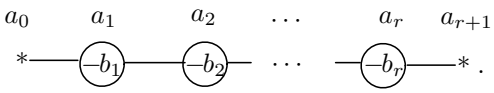
$$X := (D_\varepsilon \times \mathbb{C})/G, \tag{3.1}$$

and let $p\colon D_\varepsilon \times \mathbb{C} \longrightarrow X$ be the quotient map. Then the natural $\mathbb{C}^*$-action on $D_\varepsilon \times \mathbb{C}$ induces a $\mathbb{C}^*$-action on X and (X, o) is a cyclic quotient singularity $C_{n,q}$, where $o = p(0)$ for the origin 0 of $D_\varepsilon \times \mathbb{C}$. Let us consider a resolution $\pi\colon (\tilde{X}, E) \longrightarrow (X, o)$ whose w.d.graph of E is given as



where $b_i \geq 1$ for any i. We call such resolution *a Hirzebruch-Jung resolution* of type $\langle b_1, \ldots, b_r \rangle$ of $C_{n,q}$. If it is the minimal resolution (i.e., $b_i \geq 2$ for any i), then $n/q = [[b_1, \ldots, b_r]]$ and $n/q' = [[b_r, \ldots, b_1]]$ as $qq' \equiv 1\ (n)$ $(0 < q' < n)$ (see [**Ri**]). The $\mathbb{C}^*$-action on X induces a $\mathbb{C}^*$-action on $\tilde{X}$ such that any point in $p(D_\varepsilon \times \{0\})$ is a fixed point and $E = \bigcup_{i=1}^r E_i$ is invariant under the $\mathbb{C}^*$-action. Let E_0 and E_{r+1} be the strict transforms of non-compact curves $p(D_\varepsilon \times \{0\})$ and $p(\{0\} \times \mathbb{C})$ by π respectively. Thus, for $1 \leq i \leq r$, E_i is the sum of a one dimensional $\mathbb{C}^*$-orbit and two fixed points given by $E_i \cap E_{i-1}$ and $E_i \cap E_{i+1}$. In this paper, the $\mathbb{C}^*$-action defined on X and $\tilde{X}$ as above are called *standard $\mathbb{C}^*$-action.*

DEFINITION 3.1. Let (X, o) be a cyclic quotient singularity of type $C_{n,q}$. Let h be an element of $\mathfrak{m}_{X,o}$ such that the divisor $(h \circ \pi)_{\tilde{X}}$ is given by



Let us call it *a Hirzebruch-Jung divisor* on a Hirzebruch-Jung resolution $\tilde{X}$ and represent it as follows:

$$\langle\!\langle a_0 \mid a_1, \ldots, a_r \mid a_{r+1} \rangle\!\rangle \quad (r \geq 0). \tag{3.2}$$

In particular, if (X, o) is a non-singular point $C_{1,0}$, then the divisor is represented

by $\langle\langle a_0 \mid |a_1\rangle\rangle$. If $a_{r+1} = 0$, then we simplify $\langle\langle a_0 \mid a_1, \dots, a_r \mid 0\rangle\rangle$ to $\langle\langle a_0 \mid a_1, \dots, a_r\rangle\rangle$; also it is called a *right complete Hirzebruch-Jung divisor*. If $a_{r+1} > 0$, then $\langle\langle a_0 \mid a_1, \dots, a_r \mid a_{r+1}\rangle\rangle$ is included in $\langle\langle a_0 \mid a_1, \dots, a_t\rangle\rangle$ (i.e., $r+1 \le t$). The latter is called *the right completion* of the former.

From $a_i - a_{i+1}b_{i+1} + a_{i+2} = 0$ $(i = 0, 1, \dots, r-1)$, we obtain

$$a_1 = \frac{a_0 q + a_{r+1}}{n} \quad \text{and} \quad a_r = \frac{a_{r+1}q' + a_0}{n}. \tag{3.3}$$

Using (3.3) successively, we can easily compute that $a_1, \dots, a_r$ from n, q, a_0 and a_{r+1}.

Definition 3.2. Let $\pi\colon (\tilde{X}, E) \longrightarrow (X, o)$ be a Hirzebruch-Jung resolution of $C_{n,q}$.

(i) If $D = \langle\langle a_0 \mid a_1, \dots, a_r \mid a_{r+1}\rangle\rangle$ is a Hirzebruch-Jung divisor on the minimal resolution, then D is said to be *minimal* or a *minimal Hirzebruch-Jung divisor* (i.e., $a_{i-1} + a_{i+1} \ge 2a_i$ for $i = 1, \dots, r$).

(ii) Let D be a Hirzebruch-Jung divisor on $\tilde{X}$. Let $\bar{\pi}\colon (\bar{X}, \bar{E}) \longrightarrow (X, o)$ be the minimal resolution and $\sigma\colon (\tilde{X}, E) \longrightarrow (\bar{X}, \bar{E})$ the holomorphic map with $\pi = \bar{\pi} \circ \sigma$. Then a minimal Hirzebruch-Jung divisor $\sigma_*(D)$ is called the *minimalization* of D.

Let X be a quotient space $(D_\varepsilon \times \mathbb{C})/G$ of (3.1). Hence (X, o) is a cyclic quotient singularity of type $C_{n,q}$ which has the standard $\mathbb{C}^*$-action. Let h be an element of $\mathfrak{m}_{X,o}$ whose divisor $(h \circ \pi)_{\tilde{X}}$ is represented as $\langle\langle a_0 \mid a_1, \dots, a_r \mid a_{r+1}\rangle\rangle$, where $\pi\colon (\tilde{X}, E) \longrightarrow (X, o)$ is a Hirzebruch-Jung resolution. Let Y be a connected component of the normalization of the m-fold cyclic covering of X defined by $z^m = h$. Since the normalization of the singularity defined by $z^m = u^{a_i} v^{a_{i+1}}$ in $D_\varepsilon \times \mathbb{C}^2$ is a disjoint union of cyclic quotient singularities, the exceptional set of the minimal resolution of the normalization is a disjoint union of $\mathbb{P}^1$-chains. Therefore, the exceptional set of the minimal resolution of Y is also a $\mathbb{P}^1$-chain, and Y is holomorphically isomorphic to a neighborhood of a cyclic quotient singularity. In the following, we prove Theorem 3.4, which gives the type of Y as a cyclic quotient singularity from n, q, a_0, a_{r+1} and m.

Definition 3.3. Under the condition above, let λ_0, λ_1 be integers defined by

$$\lambda_0 = \gcd(a_0, \dots, a_{r+1}, m), \quad \lambda_1 = \gcd(a_0, a_{r+1}, m) \quad \text{and} \quad n_1 = \frac{n\lambda_0}{\lambda_1}.$$

Moreover, let ξ, ζ be integers defined by

$$m\xi \equiv \gcd(m, a_0)\ (a_0), \quad 0 < \xi \leq \frac{a_0}{\gcd(m, a_0)} \quad \text{and} \quad \zeta = \frac{m\xi - \gcd(m, a_0)}{a_0}.$$

From (3.3), we can easily check the following:

$$\lambda_0 = \gcd\left(m, a_0, \frac{a_0 q + a_{r+1}}{n}\right). \tag{3.4}$$

THEOREM 3.4. *Under the situation above, Y is a disjoint union of λ_o connected components and each connected component has one cyclic quotient singularity of type $C_{m_1 n_1, \delta_0}$, where $m_1 := [m \mid a_0, a_{r+1}]$ with respect to the notation of* (1.4) *and $\delta_0 \equiv (mq\xi + a_{r+1}\zeta)/\gcd(m, a_{r+1})\ (m_1 n_1)$ $(0 < \delta_0 < m_1 n_1)$.*

Furthermore, any connected component Y_i of Y has the standard the $\mathbb{C}^*$-action for $1 \leq i \leq \lambda_o$.

PROOF. From the definition, we can easily see that the number of connected components of Y is equal to λ_0. Therefore, let us determine the type of Y_i. Put $a := a_0$ and $b := a_{r+1}$; also put $\alpha_1 := [a \mid m, b]$, $\beta_1 := [b \mid m, a]$, $d_0 := [a, b \mid m]$, $d_1 := [m, b \mid a]$ and $d_2 := [m, a \mid b]$. We may assume that $h = \bar{x}^b \bar{y}^a$, where $\bar{x}, \bar{y}$ are elements of $\mathfrak{m}_{X,o}$ induced from coordinate functions x, y on $\mathbb{C}^2$. If we put $\bar{Z} = \{(z, x, y) \in D_\varepsilon \times \mathbb{C}^2 \mid z^m = x^b y^a\}$, then G acts on $\bar{Z}$ by $g(z, x, y) = (z, e_n x, e_n^q y)$. Let $\varphi_1 \colon Z \longrightarrow \bar{Z}$ be the normalization of $\bar{Z}$. The action of G on $\bar{Z}$ is lifted onto Z by the universality of the normalization ([**Or**, p. 44]). Let $Y := Z/G$ and let $\pi_1 \colon \tilde{Y} \longrightarrow Y$ be the minimal resolution. Then we have the following diagram:

$$\begin{tikzcd} & Z \arrow[r, "\varphi_1"] \arrow[d, "\psi_2"'] & \bar{Z} \arrow[r, "p"] & D_\varepsilon \times \mathbb{C} \arrow[d, "\psi_1"'] \\ \tilde{Y} \arrow[r, "\pi_1"] & Y = Z/G \arrow[rr, "\varphi_2"] & & X = (D_\varepsilon \times \mathbb{C})/G, \end{tikzcd} \tag{3.5}$$

where $p(z, x, y) = (x, y)$ and ψ_k $(k = 1, 2)$ is the quotient map by G. Let $\bar{Z} = \bigcup_{j=0}^{\lambda_1 - 1} \bar{Z}_j$ be the irreducible decomposition, where $\bar{Z}_j = \{(z, x, y) \in D_{\bar{\varepsilon}} \times \mathbb{C}^2 \mid z^{m/\lambda_1} = e_{\lambda_1}^j x^{b/\lambda_1} y^{a/\lambda_1}\}$ for $e_{\lambda_1} = \exp(2\pi\sqrt{-1}/\lambda_1)$ and $\bar{\varepsilon}$ with $\bar{\varepsilon}^{m_1 d_1} = \varepsilon$. Therefore, Z is a disjoint union of cyclic quotient singularities of same type; also any connected component Z_j of Z is the normalization of $\bar{Z}_j$. Let μ be a positive integer satisfying $\alpha_1 \mu + \beta_1 \equiv 0\ (m_1)$ $(0 < \mu < m_1)$. From Lemma 2.5 in [**To3**], Z_j is a cyclic quotient singularity of type $C_{m_1, \mu}$ and all Z_j are isomorphic to each other.

Hereafter, we prove that any connected component Y_j of Y is isomorphic to

$C_{m_1n_1,\mu}$. It is sufficient to check the case of $j=0$. We have the following diagram (see [**To2**, Lemma 2.5]):

$$\begin{tikzcd} & D_{\bar{\varepsilon}} \times \mathbb{C} \arrow[d, "\psi_3"', "/g_1"] \arrow[dr, "\varphi"] & \\ C_{m_1,\mu} \cong & Z_0 \arrow[r, "\varphi_1"] & \bar{Z}_0 \\ & Z \arrow[u, phantom, "\cap"] & \bar{Z}, \arrow[u, phantom, "\cap"] \end{tikzcd} \tag{3.6}$$

where $\varphi(u,v) := (u^{\beta_1 d_0} v^{\alpha_1 d_0}, u^{m_1 d_2}, v^{m_1 d_1})$ and g_1 is an action on $D_{\bar{\varepsilon}} \times \mathbb{C}$ defined by $g_1(u,v) = (e_{m_1} u, e_{m_1}^{\mu} v)$. Since $\varphi(g_1(u,v)) = \varphi(u,v)$ for any (u,v), it induces the normalization map φ_1 by Lemma 2.5 in [**To3**]. Let $g_0(z,x,y) := (z, e_{n_1} x, e_{n_1}^q y)$ for any $(z,x,y) \in \mathbb{C}^3$ and $G_0 := \langle g_0 \rangle$. Then G_0 acts on $\bar{Z}_0$ and G/G_0 gives an effective permutation among $\{\bar{Z}_0, \dots, \bar{Z}_{\lambda_1 - 1}\}$. Since $|G/G_0| = \lambda_1/\lambda_0$, the number of connected components of $Y = Z/G$ is equal to λ_0.

Because the action of G_0 is lifted onto Z_0 through φ_1, Z_0/G_0 is a cyclic quotient singularity of order $m_1 n_1$ and it is isomorphic to Y_i. Let consider an action $g_2 = \left(\begin{smallmatrix} e_{m_1 n_1} & 0 \\ 0 & e_{m_1 n_1}^{\delta} \end{smallmatrix}\right)$ on $D_{\bar{\varepsilon}} \times \mathbb{C}$ for a positive integer δ satisfying $\langle g_2^{n_1} \rangle = \langle g_1 \rangle$ and $\langle g_2^{m_1} \rangle = \langle \tilde{g}_0 \rangle$, where $\tilde{g}_0$ is the lifting of the action of g_0 through φ. We determine δ by m, b, q, ξ and ζ. Since $\varphi({g_2}^{\gamma}(u,v)) = g_0 \varphi(u,v)$ for any $(u,v) \in D_{\bar{\varepsilon}} \times \mathbb{C}$ and a suitable $\gamma \in \mathbb{N}$, we have the following:

$$\begin{aligned} &\left(e_{m_1 n_1}^{d_0 \gamma(\alpha_1 \delta + \beta_1)} u^{\beta_1 d_0} v^{\alpha_1 d_0}, e_{n_1}^{d_2 \gamma} u^{m_1 d_2}, e_{n_1}^{d_1 \gamma \delta} v^{m_1 d_1}\right) \\ = &\left(u^{\beta_1 d_0} v^{\alpha_1 d_0}, e_{n_1} u^{m_1 d_2}, e_{n_1}^{q} v^{m_1 d_1}\right). \end{aligned}$$

Therefore,

$$d_2 \gamma \equiv 1 \ (n_1) \quad \text{and} \quad d_1 \gamma \delta \equiv q \ (n_1). \tag{3.7}$$

Since $\varphi(g_2(u,v)) = g_0^{\gamma_1} \varphi(u,v)$ for a suitable $\gamma_1 \in \mathbb{N}$, we obtain $e_{m_1 n_1}^{d_0(\alpha_1 \delta + \beta_1)} = 1$. Hence

$$d_0(\alpha_1 \delta + \beta_1) \equiv 0 \ (m_1 n_1). \tag{3.8}$$

Since $\gcd(d_0, m_1) = 1$, we have $\alpha_1 \delta + \beta_1 \equiv 0 \ (m_1)$. Thus, $\delta \equiv \mu \ (m_1)$ from the definition of μ, and $\langle g_2^{n_1} \rangle = \langle g_1 \rangle$.

Since $d_1 \delta \equiv d_1 d_2 \delta \gamma \equiv d_2 q \ (n_1)$ from (3.7), we have $m_1 d_1 \delta \equiv m_1 d_2 q \ (m_1 n_1)$. From the definition of ξ and ζ, we have $m_1 d_1 \xi - \alpha_1 d_0 \zeta = 1$. Since $\alpha_1 d_0 \delta \zeta + \beta_1 d_0 \zeta \equiv$

$0\ (m_1n_1)$ from (3.8), we have $\alpha_1 d_0\delta\zeta \equiv -\beta_1 d_0\zeta\ (m_1n_1)$. Therefore,

$$\delta = \delta(m_1d_1\xi - \alpha_1 d_0\zeta) \equiv m_1d_2q\xi + \beta_1 d_0\zeta \equiv \frac{mq\xi + b\zeta}{\gcd(m,b)} \equiv \delta_0\ (m_1n_1).$$

Consequently, any connected component of Y is isomorphic to $(D_{\bar{\varepsilon}}\times\mathbb{C})/\langle g_2\rangle$ whose any connected component is holomorphically isomorphic to the cyclic quotient singularity of type $C_{m_1n_1,\delta_0}$.

Consider the standard $\mathbb{C}^*$-action on X, which is induced from the $\mathbb{C}^*$-action $t\cdot(x,y) = (x, t^{m_1d_1}y)$ on $D_\varepsilon \times \mathbb{C}$. The $\mathbb{C}^*$-action is lifted onto $\bar{Z}_j$ as $t\cdot(z,x,y) = (t^{\alpha_1 d_0}z, x, t^{m_1d_1}y)$. Therefore, the m/λ_1-fold cyclic covering map p is $\mathbb{C}^*$-equivariant. The $\mathbb{C}^*$-action on $\bar{Z}_j$ is also lifted onto $D_{\bar{\varepsilon}}\times\mathbb{C}$ as $t\cdot(u,v) = (u,tv)$. Then φ is $\mathbb{C}^*$-equivariant, and also the $\mathbb{C}^*$-action and the g_2-action are commutative on $(D_{\bar{\varepsilon}}\times\mathbb{C})$. Since Y_i is the quotient space $(D_{\bar{\varepsilon}}\times\mathbb{C})/\langle g_2\rangle$, it has the standard $\mathbb{C}^*$-action. $\square$

When we compute cyclic coverings of surface singularities, Theorem 3.4 is convenient. In [**KN**], using Theorem 3.4, Konno-Nagashima computed maximal ideal cycles for hypersurface singularities of Brieskorn type and compared with the fundamental cycles. They proved a necessary and sufficient condition that those two cycles coincide.

EXAMPLE 3.5. Let (X,o) be a cyclic quotient singularity $C_{30,7}$. Let h be an element of $\mathfrak{m}_{X,o}$ such that the divisor $(h\circ\pi)_{\tilde{X}}$ on the minimal resolution $\pi : \tilde{X} \longrightarrow X$ is given by $\langle\langle 30 \mid 9, 15, 21, 27 \mid 60\rangle\rangle$. Consider the normalization (Y,o) of the 45-fold cyclic cover of (X,o) defined by $z^{45} = h$. Since $m = 45$, $a = 30$ and $b = 60$, we have $\lambda_1 = 15$, $\lambda_0 = 3$ and n_1=6 and $m_1 = 3$; therefore $\xi = \zeta = 1$ and $\delta_0 = 7$. Hence, (Y,o) is a disjoint union of three cyclic quotient singularities of type $C_{18,7}$. If $\sigma : \tilde{Y}_i \longrightarrow (Y_i,o)$ is the minimal resolution of a connected component of (Y,o), then $(z\circ\sigma)_{\tilde{Y}_i}$ is given by $\langle\langle 2 \mid 1,1,2,3 \mid 4\rangle\rangle$.

COROLLARY 3.6. *In the situation of Definitions* 3.1–3.3 *and Theorem* 3.4, *we assume that* $a_{r+1} = 0$ (*i.e.,* $(h\circ\pi)_{\tilde{X}} = \langle\langle a_0 \mid a_1,\dots,a_r\rangle\rangle$). *Let* $\bar{a}_0 := [a_0 \mid m, a_1]$, $\bar{a}_1 := [a_1 \mid m, a_0]$ *and* $\bar{m} := [m \mid a_0, a_1]$. *If* $\bar{a}_0 > 1$ *and* δ_0 *is a positive integer defined by* $\bar{m}\delta_0 \equiv \bar{a}_1\ (\bar{a}_0)\ (0 < \delta_0 < \bar{a}_0)$, *then the normalization* (Y,o) *is the disjoint union of* $\gcd(m,a_0,a_1)$ *cyclic quotient singularities of type* $C_{\bar{a}_0,\delta_0}$. *Furthermore, if* $\bar{a}_0 = 1$, *then the normalization* (Y,o) *is the disjoint union of* $\gcd(m,a_0,a_1)$ *non-singular points.*

PROOF. We have $m_1 = 1$. From $n = a_0/\gcd(a_0,a_1)$, $q = a_1/\gcd(a_0,a_1)$ and Definition 3.3, we obtain $\lambda_0 = \gcd(m,a_0,a_1)$ and $\lambda_1 = \gcd(m,a_0)$. Then

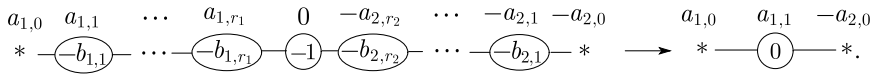
$$n_1 = \frac{n\lambda_0}{\lambda_1} = \frac{n \cdot \gcd(m, a_0, a_1)}{\gcd(m, a_0)} = \frac{a_0 \cdot \gcd(m, a_0, a_1)}{\gcd(a_0, m)\gcd(a_0, a_1)} = [a_0 \mid m, a_1] = \bar{a}_0.$$

Because it is easy to show the case of $\bar{a}_0 = 1$, we assume $\bar{a}_0 > 1$. Since $\bar{m}$, $\bar{a}_0$ and $\bar{a}_1$ are relatively prime, δ_0 is determined uniquely. Therefore, the normalization (Y, o) is the disjoint union of λ_0 cyclic quotient singularities of type $C_{\bar{a}_0,\delta_0}$, where $m\xi \equiv \gcd(m, a_0)\ (a_0)$ $(0 < \xi < a_0/\lambda_1)$ and $\delta_0 \equiv q\xi\ (\bar{a}_0)$ $(0 < \delta_0 < \bar{a}_0)$ from the definition of ξ and δ_0 and Theorem 3.4. If we put $\varepsilon := \gcd(m, a_1)/\lambda_0$, then we have $\bar{m}\varepsilon\xi \equiv 1\ (\bar{a}_0)$ and $\bar{m}\varepsilon\xi\bar{a}_1 \equiv \bar{a}_1\ (\bar{a}_0)$. Since $\delta_0 \equiv q\xi\ (\bar{a}_0)$ and $q = [a_1 \mid a_0] = \bar{a}_1\varepsilon$, we have $\bar{m}\delta_0 \equiv \bar{a}_1\ (\bar{a}_0)$. □

In the following, we prepare some terminologies and facts to prove Theorem 5.4.

Definition 3.7. Let D_1, D_2 be two right complete Hirzebruch-Jung divisors $\langle\!\langle a_{i,0} \mid a_{i,1}, \ldots, a_{i,r}\rangle\!\rangle$ for $i = 1, 2$.

(i) Assume that D_1 and D_2 are minimal. If $a_{1,1} + a_{2,1} = a_{1,0} = a_{2,0}$, then it is said that D_1 and D_2 are *located on the opposite side*, which means that the left divisor in the following figure is contracted to the right one as follows:



(ii) Assume that D_1 or D_2 is not minimal. If their minimalizations are located on the opposite side, then it is said that D_1 and D_2 are *located on the opposite side*. This condition is equivalent to be $a_{1,0} = a_{2,0}$ and $a_{1,1} + a_{2,1} \equiv 0\ (a_{1,0})$.

(iii) Consider two Hirzebruch-Jung divisors D_1, D_2. Let $\bar{D}_i$ be the minimalization of a right completion of D_i. Here D_1 and D_2 are said to be *located on the opposite side* if $\bar{D}_1$ and $\bar{D}_2$ are so.

Example 3.8. Let $D_1 = \langle\!\langle 14 \mid 18, 22, 4 \mid 2\rangle\!\rangle$ and $D_2 = \langle\!\langle 14 \mid 24, 10, 6 \mid 2\rangle\!\rangle$. Then the minimalizations of their right completions are given by $\langle\!\langle 14 \mid 4, 2\rangle\!\rangle$ and $\langle\!\langle 14 \mid 10, 6, 2\rangle\!\rangle$. Hence D_1 and D_2 are located on the opposite side.

Definition 3.9. Let (X, o) be a cyclic quotient singularity of type $C_{n,q}$ and $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ a Hirzebruch-Jung resolution and $h \in \mathfrak{m}_{X,o}$. Let $D = \langle\!\langle a_0 \mid a_1, \ldots, a_r \mid a_{r+1}\rangle\!\rangle$ $(r \geq 0)$ be a Hirzebruch-Jung divisor defined by $(h \circ \pi)_{\tilde{X}}$. Let (Y, o) be a cyclic quotient singularity, which is given as the normalization of the m-fold cyclic cover defined by $z^n = h$ over (X, o). Let $\pi : (\tilde{Y}, F) \longrightarrow (Y, o)$ be a Hirzebruch-Jung resolution. A Hirzebruch-Jung divisor $(z \circ \sigma)_{\tilde{Y}}$ is called *an m-fold cyclic lifting* of D, which is written by $D(m)$.

For example, let (X, o) be a cyclic quotient singularity of type $C_{7,4}$ and h an element of $\mathfrak{m}_{X,o}$ corresponding to $x^9y^{10} \in \mathbb{C}[x, y]$. Thus the Hirzebruch-Jung divisor D defined by h on the minimal resolution of (X, o) is given by $\langle\langle 10 \mid 7, 4 \mid 9\rangle\rangle$. From Theorem 3.4, the 21-fold cyclic cover defined by $z^{21} = h$ over (X, o) coincides with one cyclic quotient singularity of type $C_{49,34}$. From $49/34 = [[2, 2, 5, 4]]$, $D(21)$ is represented as $\langle\langle 10 \mid a_1, a_2, a_3, a_4 \mid 3\rangle\rangle$. Using (3.3) successively, we have $a_1 = 7$, $a_2 = 4$ and $a_3 = a_4 = 1$; then $D(21) = \langle\langle 10 \mid 7, 4, 1, 1 \mid 3\rangle\rangle$.

PROPOSITION 3.10. *Under the situation of Definition* 3.9, *let* m_1, m_2 *be positive integers satisfying* $m_1 + m_2 \equiv 0\ (a_0)$. *Then two cyclic liftings* $D(m_1)$, $D(m_2)$ *are located on the opposite side.*

PROOF. Let $\hat{D} = \langle\langle a_0 \mid a_1, \ldots, a_s\rangle\rangle$ be the right completion of D. Using Corollary 3.6, we can compute the m_i-fold cyclic lifting of $\hat{D}$ $(i = 1, 2)$. From $m_1 + m_2 \equiv 0\ (a_0)$, we have $[a_0 \mid m_1, a_1] = [a_0 \mid m_2, a_1]$ with respect to the notation of (1.4). Let $\bar{a}_0 := [a_0 \mid m_1, a_1] = [a_0 \mid m_2, a_1]$, $\bar{m}_i := [m_i \mid a_0, a_1]$, $\alpha_i := [a_1 \mid m_i, a_0]$ and $\gamma_i := [m_i, a_1 \mid a_0]$ for $i = 1, 2$. Therefore, $\alpha_1\gamma_1 = \alpha_2\gamma_2$ can be checked readily. From Corollary 3.6, we can check that the normalization of the m_i-fold cyclic cover associated to $\hat{D}$ is constructed by cyclic quotient singularities of type $C_{\bar{a}_0,\delta_i}$, where $\bar{m}_i\delta_i \equiv \alpha_i\ (\bar{a}_0)$ and $0 < \delta_i < \bar{a}_0$. Thus we need only to show that $\delta_1 + \delta_2 = \bar{a}_0$. Dividing $m_1 + m_2 \equiv 0\ (a_0)$ by $\gcd(m_i, a_0)$, we have $\bar{m}_1\gamma_1 + \bar{m}_2\gamma_2 \equiv 0$ $(\bar{a}_0)$. From $\gcd(\bar{a}_0, \alpha_1) = 1$, there exists an integer β with $\alpha_1\beta \equiv 1\ (\bar{a}_0)$; hence $\gamma_1 \equiv \alpha_2\gamma_2\beta\ (\bar{a}_0)$ from $\alpha_1\gamma_1 = \alpha_2\gamma_2$. Therefore, $(\bar{m}_1\alpha_2\beta + \bar{m}_2)\gamma_2 \equiv 0\ (\bar{a}_0)$; hence $\bar{m}_1\alpha_2\beta + \bar{m}_2 \equiv 0\ (\bar{a}_0)$ from $\gcd(\bar{a}_0, \gamma_2) = 1$. Consequently, $\bar{m}_1\alpha_1\alpha_2\beta + \bar{m}_2\alpha_1 \equiv 0$ $(\bar{a}_0)$ and also $\bar{m}_1\alpha_2 + \bar{m}_2\alpha_1 \equiv 0\ (\bar{a}_0)$ from $\alpha_1\beta \equiv 1\ (\bar{a}_0)$. By the definition of δ_i, we have $\bar{m}_1\bar{m}_2\delta_1 + \bar{m}_1\bar{m}_2\delta_2 \equiv \bar{m}_2\alpha_1 + \bar{m}_1\alpha_2 \equiv 0\ (\bar{a}_0)$. Hence, $\delta_1 + \delta_2 \equiv 0\ (\bar{a}_0)$ from $\gcd(\bar{a}_0, \bar{m}_i) = 1$ for $i = 1, 2$. Since $0 < \delta_i < \bar{a}_0$, we have $\delta_1 + \delta_2 = \bar{a}_0$. $\square$

EXAMPLE 3.11. Let D be a Hirzebruch-Jung divisor $\langle\langle 16 \mid 7, 5, 3 \mid 4\rangle\rangle$ on the minimal resolution of $C_{12,5}$. Since $16 \mid 22 + 26$, we consider $D(22)$ and $D(26)$. From Corollary 3.6, 22-fold (resp. 26-fold) cyclic cover of $C_{12,5}$ with the branch divisor D is $C_{66,41}$ (resp. $C_{78,29}$). Hence, $D(22) = \langle\langle 8 \mid b_1, \ldots, b_6 \mid 2\rangle\rangle$ on the minimal resolution of $C_{66,41}$; hence $D(22) = \langle\langle 8 \mid 5, 2, 1, 1, 1, 1 \mid 2\rangle\rangle$ from (3.3). Similarly we have $D(26) = \langle\langle 8 \mid 3, 1, 1, 1, 1, 1 \mid 2\rangle\rangle$. Therefore, their minimalizations are given by $\langle\langle 8 \mid 5, 2, 1\rangle\rangle$ and $\langle\langle 8 \mid 3, 1\rangle\rangle$ respectively. Hence $D(22)$ and $D(26)$ are located on the opposite side.

4. Some results on $\mathbb{C}^*$-pencils of curves and normal surface singularities with $\mathbb{C}^*$-action.

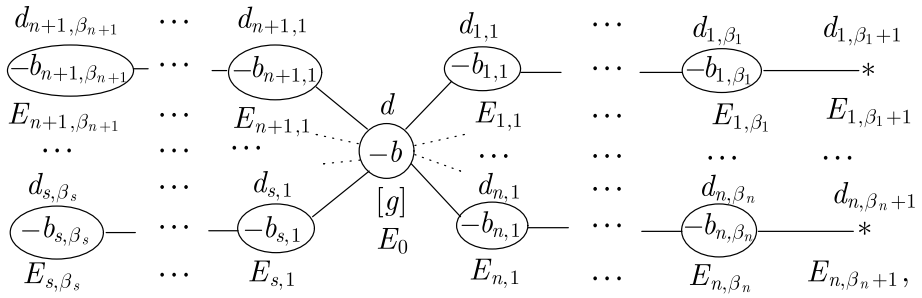
In this section, we prove a $\mathbb{C}^*$-equivariant version of Theorem 1.2 (Theorem 4.1), and also prove that any $\mathbb{C}^*$-pencil of curves is constructed as a resolution of a cyclic quotient of the trivial bundle on a curve (Theorem 4.6).

THEOREM 4.1. *Let (X, o) be a normal surface singularity with good $\mathbb{C}^*$-action. Let h be a homogeneous element of degree d in the affine graded ring R_X of (X, o). Let $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ be a $\mathbb{C}^*$-equivariant resolution such that* $\mathrm{red}(h\circ\pi)_{\tilde{X}}$ *is a simple normal crossing divisor. Then there exists a $\mathbb{C}^*$-quasi-pencil of curves $\Phi : S \longrightarrow \mathbb{C}$ which induces the following commutative diagram*:

$$\begin{tikzcd} (\tilde{X}, E) \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & (S, \mathrm{supp}(S_0)) \arrow[d, "\Phi"] \\ (X, o) \arrow[r, "h"] & \mathbb{C}, \end{tikzcd}$$

where ι is a $\mathbb{C}^*$-equivariant embedding with $\iota(\tilde{X}) \subset S$ and $\iota(E) \subset \mathrm{supp}(S_0)$. Furthermore, if h is not a perfect power element of R_X, then Φ is a $\mathbb{C}^*$-pencil of curves of degree d and genus $p_e(X, o, h)$.

PROOF. From Theorem 1.8, there exists a $\mathbb{C}^*$-equivariant resolution $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ such that the w.d.graph associated to E is star-shaped. Since h defines a $\mathbb{C}^*$-invariant divisor, the configuration of $(h \circ \pi)_{\tilde{X}}$ is given as follows:



(4.1)

where $0 < n \leq s$, $0 \leq \beta_j$, $2 \leq b_{j,k}$ and $d_{j,k} = v_{E_{j,k}}(h\circ\pi)$ for any j, k; also "$*$" means a non-exceptional irreducible component. In fact, if a non-exceptional irreducible component $*$ intersects E_{j_0,k_0} with $0 < k_0 < \beta_{j_0}$, then intersection points of E_{j_0,k_0} and $\mathrm{supp}(h \circ \pi)_{\tilde{X}} \backslash E_{j_0,k_0}$ are fixed points of the $\mathbb{C}^*$-action on $\tilde{X}$; also the number of those fixed points is greater than or equal to three. Then $\mathbb{C}^*$ acts trivially on

E_{j_0,k_0}, because any automorphism on $\mathbb{P}^1$ which fixes three points is the identity. This is a contradiction because each $E_{j,k}$ contains a one-dimensional orbit from (ii) of Theorem 1.8. Therefore we obtain the configuration above and $v_{E_0}(h\circ\pi)_{\tilde{X}} = d$ (see [**P**]). Since $E_{j,\beta_j+1}\backslash(E_{j,\beta_j+1}\cap E_{j,\beta_j})$ is a one-dimensional $\mathbb{C}^*$-orbit, $\mathbb{C}^*$ acts freely on the orbit. It follows that E_{j,β_j+1} $(=*)$ is a non-singular curve that is isomorphic to $\mathbb{C}$.

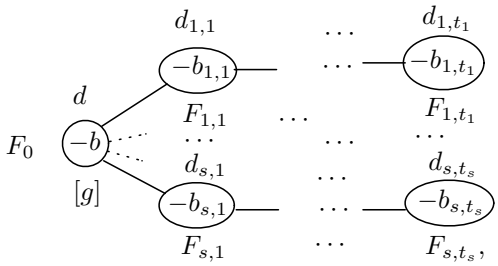
Now, let N be the holomorphic line bundle $N_{E_0/\tilde{X}}|_{E_0}$ on E_0 and $P_j = E_0\cap E_{j,1}$ for $j = 1,\dots,s$. Then we have $0 \sim (h\circ\pi)_{\tilde{X}}E_0 = dN + \sum_{j=1}^{s} d_{j,1}P_j$. Let $\hat{\Phi}(0)\colon \hat{S}(0) \longrightarrow \mathbb{P}^1$ be a minimal good complete $\mathbb{C}^*$-quasi-pencil of curves constructed in Theorem 2.4 (i.e., $\hat{\Phi}(E_0, N, \sum_{j=1}^{s} d_{j,1}P_{j,1})$). If we put $d_{j,\beta_j}/d_{j,\beta_j+1} = [[b_{j,\beta_j+1},\dots,b_{j,t_j}]]$ for $j = 1,\dots,n$, then

$$\big[\big[b_{j,1},\dots,b_{j,\beta_j},b_{j,\beta_j+1},\dots,b_{j,t_j}\big]\big] = d/\bar{d}_{j,1},$$

where $b_{j,\beta_j+1} \geq 1$, $\bar{d}_{j,1} \equiv d_{j,1}$ (d) and $0 < \bar{d}_{j,1} < d$. Consider a sequence of successive $\mathbb{C}^*$-blowing-ups

$$\hat{S}(0) \xleftarrow{\tau_1} \hat{S}(1) \xleftarrow{\tau_2} \cdots \xleftarrow{\tau_n} \hat{S} := \hat{S}(n),$$

where $\hat{S}(j-1) \xleftarrow{\tau_j} \hat{S}(j)$ is a successive $\mathbb{C}^*$-blowing-up which products a $\mathbb{P}^1$-chain of type $(b_{j,1},\dots,b_{j,t_j})$ from the minimal $\mathbb{P}^1$-chain of type $\langle d, \bar{d}_{j,1}\rangle$ intersecting E_0 at P_j for $j = 1,\dots,n$. If we put $\hat{\Psi} = \hat{\Phi}(0)\circ\tau_1\cdots\circ\tau_n$, then $\hat{\Psi} : \hat{S} \longrightarrow \mathbb{P}^1$ is a complete $\mathbb{C}^*$-quasi-pencil of curves, which is not necessarily to be minimal good. Furthermore, $\hat{\Psi}_L : S := \hat{S}_L \longrightarrow \mathbb{C}$ is a $\mathbb{C}^*$-quasi-pencil of curves whose singular fiber S_0 has the following configuration:



where $t_j = \beta_j$ for $j = n+1,\dots,s$. Let $K := \bigcup_{j=1}^{n}\bigcup_{k=\beta_j+2}^{t_j} F_{j,k}$, $\tilde{X}_1 := S\backslash K$ and $F := \bigcup_{j=1}^{s}\bigcup_{k=1}^{\beta_j} F_{j,k}$. Then $S_0|_F$ coincides to $(h\circ\pi)_{\tilde{X}}$ numerically. From the construction of $\hat{\Phi}$, there exists a biholomorphic map $\varphi : F_0 \xrightarrow{\sim} E_0$ which satisfies $\varphi^*(N_{E_0/\tilde{X}}|_{E_0}) \cong N_{F_0/\tilde{X}_1}|_{F_0}$ and $\varphi(Q_j) = P_j$, where $Q_j := F_{j,1}\cap F_0$ $(j = 1,\dots,s)$.

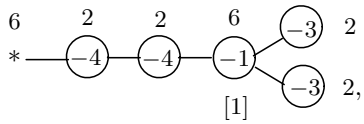
Thereby, from the result by Fujiki-Pinkham ([**Fu1**], [**P**]), there is a $\mathbb{C}^*$-equivariant biholomorphic map $\psi : (\tilde{X}_1, F) \longrightarrow (\tilde{X}, E)$ with $\psi|_{F_0} = \varphi$ and exists the following diagram:

$$\begin{tikzcd}
(\tilde{X}_1, F) \arrow[rr, "\psi"] \arrow[d, phantom, "\cap"] & & (\tilde{X}, E) \arrow[d, "\pi"'] \\
(S, \mathrm{supp}(S_0)) \arrow[dr, "\hat{\Psi}_L"'] & & (X, o) \arrow[dl, "h"] \\
& \mathbb{C}. &
\end{tikzcd}$$

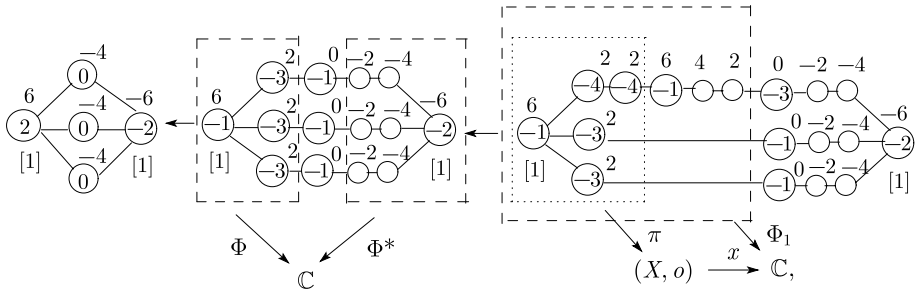
Let ι_0 be the embedding map $(\tilde{X}_1, F) \hookrightarrow (S, \mathrm{supp}(S_0))$. It follows that $h \circ \pi$ and $\hat{\Psi}_L \circ \iota_0 \circ \psi^{-1}$ correspond to non-zero elements of one-dimensional subspace $H^0(E_0, \mathcal{O}_{E_0}(dN^* - \sum_{i=1}^{s} d_{i,1}P_i))t^d$, where N^* is the dual line bundle of N. From Corollary 2.12 (iii), there exists a non-zero constant c with $h \circ \pi = (c\hat{\Psi}_L) \circ \iota_0 \circ \psi^{-1}$. Consequently, if we put $\hat{\Phi} = c\hat{\Psi}$ and $\iota = \iota_0 \circ \psi^{-1}$, then we have $h \circ \pi = \hat{\Phi}_L \circ \iota$.

Assume that h is not a perfect power element of R_X. If Φ is not a $\mathbb{C}^*$-pencil of curves, then there exists a $\mathbb{C}^*$-pencil of curves Ψ satisfying $\Phi = \Psi^\ell$ for an integer $\ell \geq 2$ from Remark 1.6 (ii). Then, from the above, there is a holomorphic function g with $\Psi \circ \iota = g \circ \pi$. Hence $g^\ell = h$ and this is a contradiction. $\square$

Example 4.2. Let (X, o) be a hypersurface singularity defined by a quasi-homogeneous polynomial $z^6 + x(y^4 + x^{10})$ of type $(6, 15, 11; 66)$. The minimal resolution $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ is given as a $\mathbb{C}^*$-equivariant resolution such that the divisor $(x \circ \pi)_{\tilde{X}}$ is given as follows:



where $*$ means a non-exceptional curve defined by $x \circ \pi$. Thus, the Pinkham-Demazure divisor is given by $D := Q - (1/3)P_1 - (1/3)P_2 - (4/15)P_3$, where P_1, P_2 and P_3 are intersection points of the central curve and three $\mathbb{P}^1$-chains. Since there are no elements of degree 3 and $\deg(x) = 6$, we have $3Q \not\sim P_1 + P_2 + P_3$ and $6Q \sim 2P_1 + 2P_2 + 2P_3$. We can construct a $\mathbb{C}^*$-pencil of curves satisfying the property of Theorem 4.1 as follows:



where Φ and Φ^* are minimal good $\mathbb{C}^*$-pencils of curves. Since $3Q - P_1 - P_2 - P_3$ is a torsion bundle of order 2, they are multiple pencils from 2.4 (ii). Furthermore, we remark that the genus of Φ is equal to 7 ($= p_e(X, o, x)$).

From now on, we consider cyclic coverings for normal surface singularities with $\mathbb{C}^*$-action and $\mathbb{C}^*$-pencils of curves. Let (X, o) be a normal surface singularity (not necessarily with $\mathbb{C}^*$-action) and h an element of $\mathfrak{m}_{X,o}$. Let $\pi\colon (\tilde{X}, E) \longrightarrow (X, o)$ be a resolution such that $\mathrm{red}(h\circ\pi)_{\tilde{X}}$ is a simple normal crossing divisor. Let $\bar{V}_m$ be the cyclic cover of (X, o) defined by $w^m = h$ (i.e., $\bar{V}_m = \{(p, w) \in X\times\mathbb{C} \mid w^m = h(p)\}$). Then, $\bar{V}_m$ is normal if and only if h is an reduced element ([**TW**, Theorem 3.2]). Let (V_m, o) be an irreducible component of the normalization of $\bar{V}_m$. Then a resolution $(\tilde{V}_m, E(m)) \longrightarrow (V_m, o)$ is constructed as follows (see (4.1) in [**To6**]):

$$\begin{tikzcd}
(V_m, o) \arrow[d, "\psi_m"'] & V'_m \arrow[l, "\phi_1"'] \arrow[d, "\psi'_m"'] & V''_m \arrow[l, "\phi_2"'] & (\tilde{V}_m, E(m)) \arrow[l, "\phi_3"'] \arrow[dll, "\tilde{\psi}_m"] \\
(X, o) & (\tilde{X}, E), \arrow[l, "\pi"'] & &
\end{tikzcd} \tag{4.2}$$

where V'_m is the fiber product $V_m \times_X \tilde{X}$ and ϕ_2 is the normalization map and ϕ_3 is a resolution map of V''_m; hence $\phi_1 \circ \phi_2 \circ \phi_3\colon (\tilde{V}_m, E(m)) \longrightarrow (V_m, o)$ is a resolution. Contracting (-1)-curves successively from $(\tilde{V}_m, E(m))$, we have the minimal good resolution (V_m, o).

Let $g \in \mathfrak{m}_{X,o}$ and put $(g \circ \pi)_{\tilde{X}} = \sum_{i=1}^r v_{E_i}(g \circ \pi)E_i + \sum_{j=1}^s v_{C_j}(g \circ \pi)C_j$. Let A be the set of singular points of $(\bigcup_{i=1}^r E_i) \cup (\bigcup_{j=1}^s C_j)$ (i.e., it is the set of all intersection points of all irreducible components of $\bigcup_{i=1}^r E_i$ and $\bigcup_{j=1}^s C_j$). Then $\tilde{\psi}_m$ is a finite map on $(\tilde{\psi}_m)^{-1}(A^c)$, where $A^c := \tilde{X}\backslash A$. Let $\tilde{E}_i$ (resp. $\tilde{C}_j$) be the closure of $(\tilde{\psi}_m)^{-1}(E_i \cap A^c)$ (resp. $(\tilde{\psi}_m)^{-1}(C_j \cap A^c)$) in $\tilde{V}_m$. They are not necessarily irreducible curves. With respect to the vanishing orders of z and $g \circ \pi$ on $\tilde{E}_i$ and $\tilde{C}_j$, we have the following.

LEMMA 4.3 ([**To3**, Lemma 3.1]). *Let F be an irreducible component E_i or*

C_j of $\operatorname{supp}((g\circ\pi)_{\tilde{X}})$ *and $\tilde{F}$ the strict transform of F by $\tilde{\psi}_m$. If we put $\phi = \phi_1\circ\phi_2 \circ\phi_3$, then $v_{\tilde{F}}(z\circ\phi) = v_F(h\circ\pi)/\gcd(m, v_F(h\circ\pi))$ and $v_{\tilde{F}}(g\circ\pi\circ\tilde{\psi}_m) = mv_F(g\circ \pi)/\gcd(m, v_F(h\circ\pi))$.*

The following seems to be already known among experts. However, we prove it because the author has not seen it.

LEMMA 4.4. *In the situation of* (4.2), *assume that (X, o) is a normal surface singularity with $\mathbb{C}^*$-action and h is a homogeneous element of R_X; also assume that π is a $\mathbb{C}^*$-equivariant resolution such that* $\operatorname{red}(h\circ\pi)_{\tilde{X}}$ *is a simple normal crossing divisor. Then V_m has a good $\mathbb{C}^*$-action and $\phi_1\circ\phi_2\circ\phi_3$ is a $\mathbb{C}^*$-equivariant resolution of V_m.*

PROOF. Let define a $\mathbb{C}^*$-action on $\bar{V}_m$ by $t\cdot(p, w) = (t^{\bar{m}}p, t^{\bar{d}}w)$, where d is the degree of h and $\bar{m} = [m \mid d]$, $\bar{d} = [d \mid m]$. This $\mathbb{C}^*$-action is lifted onto V_m by the universality of the normalization ([**Or**, p. 44]). Then, a natural $\mathbb{C}^*$-action can be defined on the fiber product $V'_m = V_m \times_X \tilde{X}$ such that ϕ_1 and ψ'_m are $\mathbb{C}^*$-equivariant maps. As in above, this action is lifted onto the normalization V''_m such that ϕ_2 is a $\mathbb{C}^*$-equivariant map. Also the action on V''_m is lifted onto $\tilde{V}_m$. Any one-dimensional $\mathbb{C}^*$-orbit on $\tilde{X}$ is lifted onto a one-dimensional $\mathbb{C}^*$-orbit on $\tilde{V}_m$ through $\tilde{\psi}_m$. Therefore, it is easy to see that $\tilde{V}_m$ has a $\mathbb{C}^*$-action which acts freely on $\tilde{V}_m\backslash E(m)$. Thus, $\mathbb{C}^*$ acts on $V_m - \{o\}$ freely. It follows that V_m has a good $\mathbb{C}^*$-action from Proposition 3 (iii) in ([**Or**, p. 47]). Also, $\phi = \phi_1\circ\phi_2\circ\phi_3\colon (\tilde{V}_m, E(m)) \longrightarrow (V_m, o)$ is a $\mathbb{C}^*$-equivariant resolution map. $\square$

Let consider the n-th root fibration for $\mathbb{C}^*$-pencils of curves (see [**BPV**, p. 92–93]).

LEMMA 4.5. *Let $\Phi\colon S \longrightarrow \mathbb{C}$ be a $\mathbb{C}^*$-pencil of curves of degree d. Then the n-th root fibration $\Phi^{(n)}$ is a $\mathbb{C}^*$-pencil of curves of degree $\bar{d}$, where $\bar{d} := [d \mid n]$.*

PROOF. We have the following diagram:

$$\begin{tikzcd} S \arrow[d, "\Phi"'] & S' \arrow[l, "\varphi_1"'] \arrow[d, "\Phi'"'] & S'' \arrow[l, "\varphi_2"'] & S^{(n)}, \arrow[l, "\varphi_3"'] \arrow[dll, "\Phi^{(n)}"] \\ \mathbb{C} & \mathbb{C} \arrow[l, "\eta"'] & & \end{tikzcd} \tag{4.3}$$

where $\xi := \eta(\zeta) = \zeta^n$ and $S' := S\times_{\mathbb{C}}\mathbb{C} = \{(p,\zeta) \mid \Phi(p) = \zeta^n\}$; also S'' is the normalization of S' and $S^{(n)}$ is the minimal $\mathbb{C}^*$-good resolution of S''. Consider a $\mathbb{C}^*$-action on S' by $t\cdot(p,\zeta) := (t^{\bar{n}}\cdot p, t^{\bar{d}}\zeta)$ for $t\in\mathbb{C}^*$, where $\bar{n} := [n|d]$. Then we can easily see that $\Phi(\varphi_1(t\cdot(p,\zeta))) = \Phi(\varphi_1(t^{\bar{n}}\cdot p, t^{\bar{d}}\zeta)) = \Phi(t^{\bar{n}}\cdot p) = t^{\bar{n}d}\Phi(p) = t^{\bar{n}d}\xi$

and $\eta \circ \Phi'(t \cdot (p, \zeta)) = \eta(\Phi'(t^{\bar{n}} p, t^{\bar{d}} \zeta)) = \eta(t^{\bar{d}} \zeta) = t^{\bar{n} d} \xi$. Hence the $\mathbb{C}^*$-action on S' is considered as a lifting of the $\mathbb{C}^*$-action on S. As in Lemma 4.4, the $\mathbb{C}^*$-action on S' is lifted onto $S^{(n)}$. Since $\bar{n} d = \bar{d} n$, the degree of $\Phi^{(n)}$ is $\bar{d}$. $\square$

In [**Fu1**] and [**P**], Fujiki and Pinkham proved that every normal surface singularity with $\mathbb{C}^*$-action is obtained as the quotient of a cone singularity (i.e., the blowing-down of the zero section of a negative line bundle on a curve) by a finite subgroup (not necessarily to be cyclic) of the automorphism group of the line bundle. From now on, we prove an analogous result for $\mathbb{C}^*$-pencils of curves.

Let $\Phi : S \longrightarrow \mathbb{C}$ be a $\mathbb{C}^*$-pencil of curves such that $\mathrm{supp}(S_0)$ is irreducible (i.e., $\mathrm{supp}(S_0) = E_0$ is the central curve). From the construction in Theorem 2.4, S is the total space of a holomorphic line bundle over E_0. We call such Φ *a simple $\mathbb{C}^*$-pencil of curves*. Moreover, if $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ is a complete and $\hat{\Phi}_L$ and $\hat{\Phi}_R$ are simple, then $\hat{\Phi}$ is said to be *simple*. From Theorem 2.4 (ii-2), it is true that if a simple $\mathbb{C}^*$-pencil of curves is given, it is considered as a holomorphic torsion line bundle and vice versa.

Theorem 4.6. *Let $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ be a complete $\mathbb{C}^*$-pencil of curves of type (d, g, m) and $d_1 := \min\{\ell \in \mathbb{N} \mid d | \operatorname{lcm}(\ell, m)\}$ and $d_0 := d/d_1$. Let $\tilde{\Phi} : \tilde{S} \longrightarrow \mathbb{K}$ be $\hat{\Phi}$ above or $\hat{\Phi}_L$, where $\mathbb{K}$ is $\mathbb{P}^1$ or $\mathbb{C}$. Let E_0 be the central curve of $\mathrm{supp}(\tilde{\Phi}^{-1}(0))$ and let E_g be the general fiber S_t of $\tilde{\Phi}$. Then there is a finite cyclic subgroup G of $\mathrm{Aut}(E_g)$ (= the holomorphic automorphism group of E_g) $\mathbb{C}^*$-equivariantly acting on $E_g \times \mathbb{K}$ such that there exists the following diagram*:

$$\begin{tikzcd}
E_g \times \mathbb{K} \arrow[rr, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "/G_0"] & & \mathbb{K} & \\
E_g \arrow[dr, "/G_0"'] & \tilde{L} \arrow[ur, "\tilde{\Psi}"] \arrow[d, "\tilde{\psi}"'] \arrow[dr, "/G_1"] & & \tilde{S} \arrow[ul, "\tilde{\Phi}"'] \arrow[dl, "\sigma"] \\
& E_1 \arrow[dr, "/G_1"'] & \tilde{L}/G_1 \arrow[uu, "\tilde{\Psi}_0"] \arrow[d, "\tilde{\psi}_0"'] & \\
& & E_0, &
\end{tikzcd}$$

where $G_0 := \{h \mid$ a fixed point free element of $G\}$ and it has $|G_0| = d_0$; also $G_1 := G/G_0$, $E_1 := E_g/G_0$, $\tilde{L} := (E_g \times \mathbb{K})/G_0$ and σ is a resolution of all cyclic quotient singularities on $\tilde{L}/G_1$ $(= (E_g \times \mathbb{K})/G)$. If $d_0 > 1$, then $\tilde{\psi} : \tilde{L} \longrightarrow E_1$ is a torsion $\mathbb{K}$-bundle of order d_0.

Proof. First, let us consider the case of $\tilde{\Phi} := \hat{\Phi}_L : \tilde{S} := \hat{S}_L \longrightarrow \mathbb{K} = \mathbb{C}$. Assume that the w.d.graph of $\tilde{S}_0 = \tilde{\Phi}^{-1}(0)$ is given by (2.8). It can be assumed

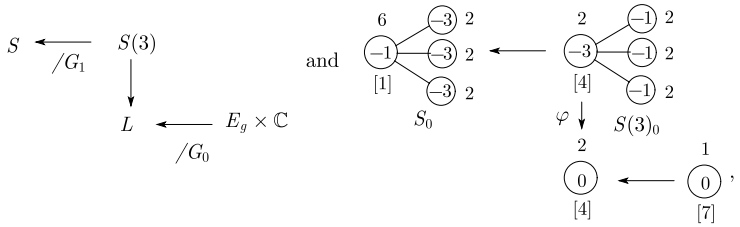
that $\tilde{\Phi}$ is minimal good (i.e., any $E_{i,j}$ is not a (-1)-curve for any i, j). Let $\sigma : \tilde{S} \longrightarrow \bar{S}$ be the contraction map of $\mathbb{P}^1$-chains $\bigcup_{j=1}^{\ell_i} E_{i,j}$ for $i = 1, \ldots, s$. Then $\bar{S}$ is a normal complex surface which has cyclic quotient singularities; each type of them is given by $C_{\bar{d}_i, \bar{d}_{i,1}}$ $(i = 1, \ldots, s)$, where $\bar{d}_i := [d \mid d_{i,1}]$ and $\bar{d}_{i,1} = [d_{i,1} \mid d]$. Let $\bar{\Phi}$ be a holomorphic function on $\bar{S}$ induced from $\tilde{\Phi}$. Let $\bar{S}^{(d_1)}$ be the d_1-fold cyclic covering of $\bar{S}$ defined by $z^{d_1} = \bar{\Phi}$. For computing $\bar{S}^{(d_1)}$, we consider the d_1-th root fibration $\tilde{\Phi}^{(d_1)} : \tilde{S}^{(d_1)} \longrightarrow \mathbb{C}$ of $\tilde{\Phi}$. Let F_i be the d_1-fold cyclic lifting of a Hirzebruch-Jung divisor $\sum_{j=1}^{\ell_i} d_{i,j} E_{i,j}$ for any i. From $d \mid \mathrm{lcm}(d_1, m)$, we have $[d \mid d_1, m] = 1$. Since $m | d_{i,1}$, we can easily check that $[d \mid d_1, d_{i,1}] = 1$ for any i. In $\tilde{S}^{(d_1)}$, $\mathrm{supp}(F_i)$ is contracted to a non-singular point from Corollary 3.6. Let $\tilde{\sigma} : \tilde{S}^{(d_1)} \longrightarrow \bar{S}^{(d_1)}$ be the contraction map of $\bigcup_{i=1}^{s} \mathrm{supp}(F_i)$. Then we obtain a simple $\mathbb{C}^*$-pencil of curves $\tilde{\Psi} : \tilde{L} := \bar{S}^{(d_1)} \longrightarrow \mathbb{C}$. From Theorem 2.4 (ii-2), $\tilde{L}$ is the total space of a line bundle on E_1, where E_1 is the d_1-fold cyclic covering of E_0 by the covering map $\pi^{(d_1)} : \tilde{S}^{(d_1)} \longrightarrow \tilde{S}$. Then we have the following diagram:

$$\begin{tikzcd}
E_1 \arrow[d] & \subset & \tilde{L} \arrow[d, "/G_1"] & = & \bar{S}^{(d_1)} \arrow[d] & & \tilde{S}^{(d_1)} \arrow[ll, "\tilde{\sigma}"'] \arrow[d, "\pi^{(d_1)}"] \\
E_0 & \subset & \tilde{L}/G_1 & = & \bar{S} & & \tilde{S}. \arrow[ll, "\sigma"']
\end{tikzcd}$$

From Remark 1.7 (iii), the analytic type of general fiber of any $\mathbb{C}^*$-pencil of curves is constant. Moreover, from the construction of $\pi^{(d_1)}$, the general fibers of $\tilde{S}^{(d_1)}$ and $\tilde{S}$ are biholomorphic. For any $t \in \mathbb{C}^*$, $(\pi^{(d_1)})^{-1}(\tilde{S}_t)$ is a disjoint union of d_1 connected components and each connected component is isomorphic to the general fiber E_g $(= \tilde{S}_t)$ of $\tilde{\Phi}$. Therefore, the general fiber of $\tilde{\Psi} : \tilde{L} \longrightarrow \mathbb{C}$ is also isomorphic to E_g. Then it becomes a simple $\mathbb{C}^*$-pencil of curves whose singular fiber is $d_0 E_1$ and the generic fiber is E_g. If $d_0 = 1$, then $E_1 = E_g$ and $\tilde{L}$ is the trivial line bundle on E_g. If $d_0 > 1$, $\tilde{L}$ is a torsion line bundle of order d_0 on E_1. The d_0-th root fibration of $\tilde{\Psi}$ is the trivial $\mathbb{C}^*$-pencil of curves on E_g and $\tilde{L} = (E_g \times \mathbb{C})/G_0$. From the construction above, we have $E_1 = E_g/G_0$ and $E_0 = E_1/G_1$; hence we complete our proof for the case of $\tilde{\Phi} = \hat{\Phi}_L$.

Second, let us consider the case of $\tilde{\Phi} = \hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$. Suppose $d_0 > 0$. From the case above, the d_1-th root fibrations of $\hat{\Phi}_L$ and $\hat{\Phi}_R$ are torsion line bundle of order d_0 which are mutually dual. Consequently, we complete the proof. $\square$

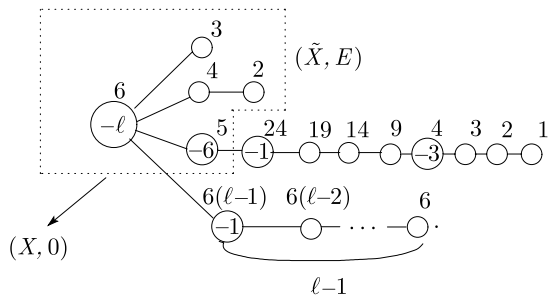
Example 4.7. Let $\Phi \colon S \longrightarrow \mathbb{C}$ be a multiple $\mathbb{C}^*$-pencil of curves of type $(6, 7, 2)$ in Example 4.2. Then we have the following diagrams:



where φ is the contraction map of three (-1)-curves. Let $\bar{Q} = \varphi^{-1}(Q)$ and $\bar{P}_i = \varphi^{-1}(P_i)$ for the points Q, P_1, P_2, P_3 defined in Example 4.2. Then L is a torsion line bundle represented by $3\bar{Q} - \sum_{i=1}^{3} \bar{P}_i$ on the central curve of genus 4.

REMARK 4.8. Log-canonical surface singularities have been studied from various points of view since Kawamata's classification [**Kaw**] (see [**I**], [**Oku1**], [**To1**], [**Ts**] and [**Wki**]). Let (X, o) be a log-canonical surface singularity with $\mathbb{C}^*$-action such that the central curve is $\mathbb{P}^1$ and the index is m (see [**Mat**]). It is known that (X, o) is obtained as a cyclic quotient of a simple elliptic singularity, which is the blowing-down of the zero-section of a negative line bundle on an elliptic curve with complex multiplicative group of order m. We explain a relation between such singularities and $\mathbb{C}^*$-pencils of elliptic curves. Since those pencil of curves are non-multiple, the degree is equal to the coefficient of the singular fiber on the central curve. Using the canonical construction, any pencil of elliptic curves with w.d.graph is realized as a $\mathbb{C}^*$-pencil of curves. For the minimal $\mathbb{C}^*$-good resolution $\pi : (\tilde{X}, E) \longrightarrow (X, o)$, there exists a $\mathbb{C}^*$-pencil of curves $\Phi : \tilde{S} \longrightarrow \mathbb{C}$ of degree m and a $\mathbb{C}^*$-equivariantly holomorphic embedding $(\tilde{X}, E) \subset (S, \mathrm{supp}(S_0))$. Then we obtain a homogeneous element $h \in R_X$ with $\Phi|_{\tilde{X}} = h \circ \pi$. Let (Y, o) be the m-fold cyclic covering defined by $z^m = h$. Then we can easily see that (Y, o) is a simple elliptic singularity resolved by an elliptic curve with complex multiplicative group of order m.

For example, let $[\ell; 1/2, 2/3, 1/6]$ be the type of the w.d.graph of (X, o) $(\ell \geq 2)$. The minimal good resolution $(\tilde{X}, E)$ is included into a $\mathbb{C}^*$-pencil of elliptic curves Φ of degree 6 as follows:



The 6-fold cyclic covering $(\tilde{Y}, F)$ of $(\tilde{X}, E)$ defined by $z^6 = h \circ \pi$ is included into $S^{(6)}$. Since $\Phi^{(6)}$ is a $\mathbb{C}^*$-blowing-up from the trivial bundle, F is isomorphic to the general fiber $S_t^{(6)}$ of $\Phi^{(6)}$. Since $S_t^{(6)}$ is an elliptic curve with complex multiple group of order 6 from Theorem 4.6, (Y, o) is a simple elliptic singularity, which is obtained as the blowing-down of the zero-section of a holomorphic line bundle of degree $-6\ell + 2$ on the general fiber $S_t^{(6)}$.

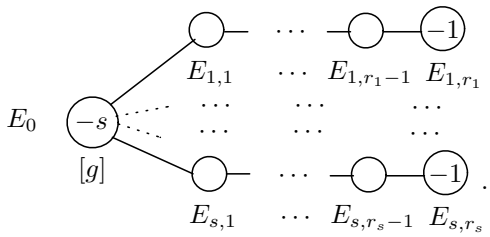
Definition 4.9. Let E_0 be a smooth compact complex curve of genus g. Let $P_1, \dots, P_s$ be points of E_0 and $H_i := \{P_i\} \times \mathbb{C}$ for $i = 1, \dots, s$. Let $\sigma_{i,1} : L_{i,1} \longrightarrow E_0 \times \mathbb{C}$ be the blowing-up at a $\mathbb{C}^*$-fixed point $(P_i, 0)$. Hence it is a $\mathbb{C}^*$-equivariant map. Let $\sigma_{i,2} : L_{i,2} \longrightarrow L_{i,1}$ be the blowing-up at a point $P_{i,1} := \sigma_{i,1}^{-1}(P_i) \cap (\sigma_{i,1})_*^{-1}(H_i)$. Next, let $\sigma_{i,3} : L_{i,3} \longrightarrow L_{i,2}$ be the blowing-up at a point $P_{i,2} := \sigma_{i,2}^{-1}(P_{i,1}) \cap (\sigma_{i,1} \circ \sigma_{i,2})_*^{-1}(H_i)$. Continuing this process r_i times ($r_i \geq 1$), we have the following sequence of $\mathbb{C}^*$-equivariant maps

$$L_{i,r_i} \xrightarrow{\sigma_{i,r_i}} L_{i,r_i-1} \xrightarrow{\sigma_{i,r_i-1}} \cdots \xrightarrow{\sigma_{i,2}} L_{i,1} \xrightarrow{\sigma_{i,1}} E_0 \times \mathbb{C}.$$

Let $\sigma_i := \sigma_{i,1} \circ \cdots \circ \sigma_{i,r_i} : L_{i,r_i} \longrightarrow E_0 \times \mathbb{C}$. Taking these processes s times for $E_0 \times \mathbb{C}$, we obtain

$$\tilde{L} = \tilde{L}_s \xrightarrow{\sigma_s} L_{s-1} \xrightarrow{\sigma_{s-1}} \dots \xrightarrow{\sigma_2} \tilde{L}_1 \xrightarrow{\sigma_1} E_0 \times \mathbb{C}.$$

Let $\sigma := \sigma_1 \circ \cdots \circ \sigma_s$. Therefore, the w.d.graph of $\sigma^{-1}(E_0 \times \{0\})$ is given as follows:



(4.4)

Let put $\tilde{X} := \tilde{L} \setminus \bigcup_{i=1}^{s} (\sigma_i)_*^{-1} H_i$ and $E := E_0 \cup (\bigcup_{i=1}^{s} \bigcup_{j=1}^{r_i-1} E_{j,j})$ and let $\pi \colon (\tilde{X}, E) \longrightarrow (X, o)$ be the contraction map. Then (X, o) is a normal surface singularity with $\mathbb{C}^*$-action. In this paper, we call such singularities *quasi-cone singularities.*

Every quasi-cone singularity (X, o) is a Kodaira singularity such that the maximal ideal cycle on the minimal resolution is reduced. The minimal cycle in the sense of Definition 1.2 in [**To2**] coincides to the central curve. In the situ-

ation of Definition 4.9, since the normal bundle of E_0 in $\tilde{X}$ is given by $[\sum_{i=1}^{s} P_i]$, the Pinkham-Demazure construction of R_X is written as $\bigoplus_{k=0}^{\infty} H^0(E, \mathcal{O}_{E_0}(\sum_{i=1}^{s} [k/r_i]P_i))t^k$, where $[a]$ means the Gaussian symbol.

Let (X, o) be a normal surface singularity with $\mathbb{C}^*$-action and h a homogeneous element of R_X. Let $\pi\colon (\tilde{X}, E) \longrightarrow (X, o)$ be a $\mathbb{C}^*$-equivariant good resolution such that the configuration of $(h \circ \pi)_{\tilde{X}}$ is given by (4.1). Let Y_ℓ be the normalization of the cyclic covering defined by $w^\ell = h$. Let $\ell_0 := \gcd\{\ell, d, d_{i,j} \mid i = 1, \dots, s; j = 1, \dots, \beta_i + 1\}$, where $d_{i,\beta_i+1} := 0$ for $i = n+1, \dots, s$. Then Y_ℓ is decomposed into ℓ_0 connected components and all connected components are isomorphic to each other. Let (Y_ℓ, o) be a connected component of Y_ℓ.

THEOREM 4.10. *Under the notation above, let* $m = \operatorname{lcm}(d, d_{1,\beta_1+1}, \dots, d_{n,\beta_n+1})$. *Then* (Y_m, o) *is a quasi-cone singularity. Therefore, every normal surface singularity with* $\mathbb{C}^*$*-action is realized as the quotient of a quasi-cone singularity by a finite cyclic group* G*, where* G *is a subgroup of the holomorphic automorphism group of the central curve for the quasi-cone singularity.*

PROOF. Consider a resolution space $(\tilde{Y}_m, F)$ of (Y_m, o) which is constructed as in (4.2). Assume that the w.d.graph is given by (4.1). We need only prove that the m-fold cyclic covering associated to the Hirzebruch-Jung divisor on the i-th $\mathbb{P}^1$-chain of (4.1) becomes a Hirzebruch-Jung divisor of type $\langle\!\langle 1 \mid 1, \dots, 1 \mid 1 \rangle\!\rangle$ (i.e., $\mathbb{P}^1$-chain of A_k-type) for $1 \le i \le n$. To apply Theorem 3.4, we put $a_0 := d$ and $a_{r+1} := d_{i,\beta_i+1}$. Furthermore, assume that $\bar{n}/q = [[b_{i,1}, \dots, b_{i,\beta_i}]]$ for relatively prime integers $\bar{n}, q$ with $1 < q < \bar{n}$. For a_0, a_{r+1} and m, let define the following positive integers:

$$d_0 := \gcd(m, a_0, a_{r+1}), \quad d_1 := [m, a_{r+1} \mid a_0],$$
$$d_2 := [m, a_0 \mid a_{r+1}] \ \text{ and } \ m_1 := [m \mid a_0, a_{r+1}].$$

Since $a_0 \mid m$ and $a_{r+1} \mid m$, we have $[a_0 \mid m, a_{r+1}] = [a_{r+1} \mid m, a_0] = [a_0, a_{r+1} \mid m] = 1$. Then we have $m = m_1 d_0 d_1 d_2$, $a_0 = d_0 d_2$ and $a_{r+1} = d_0 d_1$. Let $\lambda_0, \lambda_1, \xi$ and ζ be positive integers defined as in Definition 3.3. Then we have $\lambda_1 = d_0$, $\bar{n}_1 = \bar{n}\lambda_0/\lambda_1$, $\xi = 1$, and

$$\zeta = \frac{m_1 d_0 d_1 d_2 - d_0 d_2}{d_0 d_2} = m_1 d_1 - 1.$$

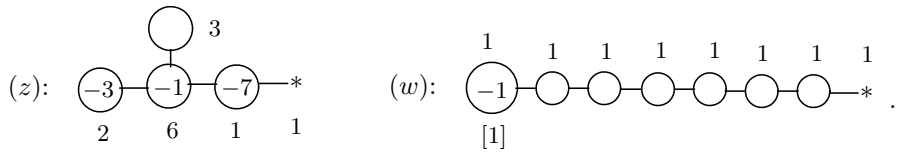
Hence, if δ_0 is the integer defined in Theorem 3.4, then $\delta_0 \equiv (m_1 d_0 d_1 d_2 q + d_0 d_1 (m_1 d_1 - 1))/d_0 d_1 \equiv m_1 d_2 q + m_1 d_1 - 1 \ (m_1 \bar{n}_1)$. If we prove

$$m_1 d_2 q + m_1 d_1 - 1 \equiv m_1 \bar{n}_1 - 1 \ (m_1 \bar{n}_1), \tag{4.5}$$

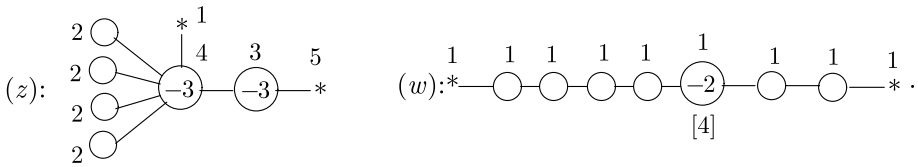
then the m-fold cyclic covering is a $\mathbb{P}^1$-chain of A_k-type. Hence we complete the proof in this case. Obviously, (4.5) is equivalent to $d_2 q + d_1 \equiv 0 \ (\bar{n}_1)$. Multiplying d_0, we have $d_0 d_2 q + d_0 d_1 \equiv 0 \ (\bar{n}_1 d_0)$. Since $a_0 = d_0 d_2$ and $a_{r+1} = d_0 d_1$ and $\bar{n}_1 d_0 = \lambda_0 \bar{n}$, (4.5) is equivalent to $a_0 q + a_{r+1} \equiv 0 \ (\lambda_0 \bar{n})$. This is obviously correct from (3.3) since $a_1 \ (= d_{i,1})$ is divided by λ_0.

Next, consider the case of $n+1 \leq i \leq s$. As in above, we can easily check that every cyclic covering is contracted to a non-singular point. The divisor of w on the minimal $\mathbb{C}^*$-good resolution of (Y_m, o) is reduced. Then we can easily see that the normal bundle of the central curve is linearly equivalent to $Q_1 + \cdots + Q_n$, where $Q_1, \ldots, Q_n$ are intersection points of the central curve and the above n $\mathbb{P}^1$-chains of type $\langle\langle 1 \mid 1, \ldots, 1 \mid 1 \rangle\rangle$. Then the resolution is contracted to the total space of the trivial line bundle. It completes the proof. $\square$

Example 4.11. (i) Let (X, o) be a hypersurface singularity defined by $x^2 + y^3 + z^7 = 0$. The divisor defined by z on the minimal good resolution is given by the left one in the following figure. Also the divisor defined by w on the minimal resolution of $x^2 + y^3 + w^{42} = 0$ (= a cyclic covering of (X, o) defined $w^7 = z$) is given by the right one.



(ii) Let (X, o) be a non-normal hypersurface singularity defined by $y^5(x^2 + y^3) + z^4 = 0$. As in (i), the divisor defined by z and the w.d.graph associated to the minimal resolution of $y^5(x^2 + y^3) + w^{80} = 0$ (= a cyclic covering of (X, o) defined $w^{20} = z$) are given as follows:

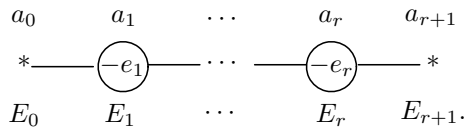


On the pencil genus $p_e(X, o, h)$ (see Section 1), we can see that $p_e(X, o, z) = 1$ for (i) and $p_e(X, o, z) = 4$ for (ii).

5. A duality for cyclic covers of normal surface singularities with $\mathbb{C}^*$-action.

In this section, we study some duality relations for cyclic covers of normal surface singularities with $\mathbb{C}^*$-action. The main result is Theorem 5.4. In addition, for hypersurface case, we can observe some dualities for two invariants (Remark 5.7). Furthermore, as an application of 5.4, some dualities are proven for n-th root fibrations of $\mathbb{C}^*$-pencils of curves (Theorem 5.8).

Here we prepare a few facts for Hirzebruch-Jung divisors on Hirzebruch-Jung resolution spaces of cyclic quotient singularities. Let (X, o) be a cyclic quotient singularity of type $C_{n,q}$ and consider a Hirzebruch-Jung resolution $(\tilde{X}, E)$ of type $\langle e_1, \dots, e_r \rangle$, where $e_i \geq 1$ for any i. Consider a Hirzebruch-Jung divisor D=$\langle\langle a_0 \mid a_1, \dots, a_r \mid a_{r+1} \rangle\rangle$ on the Hirzebruch-Jung resolution as follows:



The resolution is not necessarily to be minimal. Let E_i be an irreducible component of $\mathrm{supp}(D)$ with $\mathrm{Coeff}_{E_i} D = a_i$ for $i = 0, 1, \dots, r+1$. Hence E_0 and E_{r+1} are not compact curve. If $[[e_1, \dots, e_r]] = 0$, then $n = 1$ (i.e., (X, o) is a non-singular point). When $0 < q < n$ and $(\tilde{X}, E)$ above is the minimal resolution, it is well-known that $n/q' = [[e_r, \dots, e_1]]$ if $qq' \equiv 1\ (n)$ and $0 < q' < n$ (see [**Ri**]). Since it is possible to check the following easily, we omit the proof.

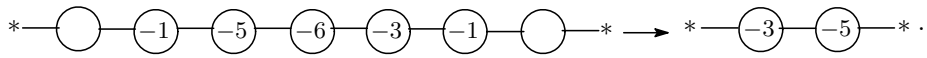
LEMMA 5.1. *In the situation above, suppose* $[[e_1, \dots, e_r]] \neq 0$. *Let* $\tilde{q}$, $\tilde{q}'$ *be positive integers such that* $n/\tilde{q} = [[e_1, \dots, e_r]]$ *and* $n/\tilde{q}' = [[e_r, \dots, e_1]]$ *and* $\gcd(n, \tilde{q}) = \gcd(n, \tilde{q}') = 1$. *Then we have the following.*

(i) $\tilde{q}\tilde{q}' \equiv 1\ (n)$.
(ii) $\tilde{q} \equiv q\ (n)$ *and* $\tilde{q}' \equiv q'\ (n)$; *also* $a_1 = (a_0\tilde{q} + a_{r+1})/n$ *and* $a_r = (a_{r+1}\tilde{q}' + a_0)/n$.
(iii) *If* σ *is a contraction map of* (-1)*-curves from the Hirzebruch-Jung resolution above to the minimal resolution, then we have the following*:

$$\left[\frac{\tilde{q}}{n}\right] = \textit{the number of blowing-ups at } P_o = E_o \cap E_1 \textit{ in } \sigma,$$

$$\left[\frac{\tilde{q}'}{n}\right] = \textit{the number of blowing-ups at } P_r = E_r \cap E_{r+1} \textit{ in } \sigma.$$

For example, we consider two Hirzebruch-Jung resolutions of $C_{14,5}$ whose w.d.graphs are given as



From $[[2,1,5,6,3,1,2]] = 14/19$ and $[[2,1,3,6,5,1,2]] = 14/31$, we have $\tilde{q} = 19$ and $\tilde{q}' = 31$.

LEMMA 5.2. *Let $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ be a complete $\mathbb{C}^*$-pencil of curves whose w.d.graph of the total fiber is given by* (2.6). *Under the natural identification of E_0 and E_∞, we have the following*:

$$N^*_{E_o/\hat{S}}|_{E_o} + N^*_{E_\infty/\hat{S}}|_{E_\infty} \sim \sum_{j=1}^{t} \frac{a_{1,j,1} + a_{2,j,1}}{d} P_j \ (\textit{linearly equivalent}),$$

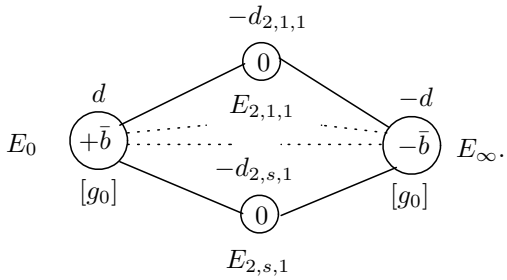
where $P_j = E_0 \cap F_{1,j,1} = E_\infty \cap F_{2,j,1}$ for any j.

PROOF. From Corollary 2.11, it can be assumed that $\hat{\Phi}$ is a complete $\mathbb{C}^*$-pencil of curves obtained by the canonical construction. Let $\tilde{\Phi} : \tilde{S} \longrightarrow \mathbb{P}^1$ be the minimal good $\mathbb{C}^*$-pencil of curves obtained from $\hat{\Phi}$. We have the following:

$$N_{E_o/\tilde{S}}|_{E_o} = N_{E_o/\hat{S}}|_{E_o} + \sum_{j=1}^{t} \left[\frac{a_{1,j,1}}{d}\right] P_j \quad \text{and}$$

$$N_{E_\infty/\hat{S}}|_{E_\infty} = N_{E_\infty/\tilde{S}}|_{E_\infty} - \sum_{j=1}^{t} \left[\frac{a_{2,j,1}}{d}\right] P_j.$$

Assume that the configuration associated to $\tilde{\Phi}$ is given by (2.4). Let $\sigma : \tilde{S} \longrightarrow \bar{L}$ be the contraction of $(\bigcup_{j=1}^{s} \bigcup_{k=1}^{u_j} E_{1,j,k}) \cup (\bigcup_{j=1}^{s} E_j) \cup (\bigcup_{j=1}^{s} \bigcup_{k=2}^{v_j} E_{2,j,k})$ in $\tilde{S}$. We have a holomorphic $\mathbb{P}^1$-bundle $\bar{L}$ on E_0 and a holomorphic map $\bar{\Phi} : \bar{L} \longrightarrow \mathbb{P}^1$ whose divisor is given as follows:



Therefore, $N_{E_o/\tilde{S}}|_{E_o} = N_{E_0/\bar{L}}|_{E_0} - \sum_{j=1}^{s} P_j$ and $N_{E_\infty/\tilde{S}}|_{E_\infty} = N_{E_\infty/\bar{L}}|_{E_\infty}$. Set

$d_{i,j,1} = a_{i,j,1} - [a_{i,j,1}/d]d$ for $i = 1, 2; j = 1, \dots, t$. Since $d \mid a_{1,j,1}$ for $d \mid a_{2,j,1}$ for $j = s+1, \dots, t$ and $d_{1,j,1} + d_{2,j,1} = d$ and $N_{E_o/\bar{L}}|_{E_o} + N_{E_\infty/\bar{L}}|_{E_\infty} \sim 0$, we have

$$\begin{aligned}
N^*_{E_o/\hat{S}}|_{E_o} + N^*_{E_\infty/\hat{S}}|_{E_\infty} &\sim \sum_{j=1}^{t} \left(\left[\frac{a_{1,j,1}}{d} \right] + \left[\frac{a_{2,j,1}}{d} \right] \right) P_j + \sum_{j=1}^{s} P_j \\
&= \sum_{j=1}^{s} \left(\left[\frac{a_{1,j,1}}{d} \right] + \left[\frac{a_{2,j,1}}{d} \right] + 1 \right) P_j + \sum_{j=s+1}^{t} \left(\left[\frac{a_{1,j,1}}{d} \right] + \left[\frac{a_{2,j,1}}{d} \right] \right) P_j \\
&= \sum_{j=1}^{s} \left(\frac{a_{1,j,1} - d_{1,j,1} + a_{2,j,1} - d_{2,j,1}}{d} + 1 \right) P_j + \sum_{j=s+1}^{t} \frac{a_{1,j,1} + a_{2,j,1}}{d} P_j \\
&= \sum_{j=1}^{t} \frac{a_{1,j,1} + a_{2,j,1}}{d} P_j. \qquad \square
\end{aligned}$$

Lemma 5.3. *Let (X, o) be a normal surface singularity and consider the situation of* (4.2). *Let E_α be an irreducible component of E with $v_{E_\alpha}(h \circ \pi) = d$ and $E(m)_\alpha$ an irreducible component of $E(m)$ with $\tilde{\psi}_m(E(m)_\alpha) = E_\alpha$.*

(i) *If $\gcd(m, d) = 1$, then $\tilde{\psi}_m$ gives a biholomorphic mapping from $E(m)_\alpha$ to E_α.*

(ii) *If m_1, m_2 are positive integers with $m_1 \equiv \pm m_2$ (d), then $E(m_1)_\alpha$ is biholomorphic to $E(m_2)_\alpha$.*

Proof. Since (i) is obvious, we prove (ii). Since $\gcd(m_1, d) = \gcd(m_2, d)$, we put $m_0 = \gcd(m_i, d)$ and $\bar{m}_i = [m_i \mid d]$ for $i = 1, 2$. Then the following diagram exists:

$$\begin{tikzcd}
V_{m_1} \arrow[r, "\varphi_1"] \arrow[dr, "\psi_{m_1}"'] & V_{m_0} \arrow[d, "\psi_{m_0}"] & V_{m_2} \arrow[l, "\varphi_2"'] \arrow[dl, "\psi_{m_2}"] \\
& X, &
\end{tikzcd}$$

where, for $j = 0, 1, 2$, ψ_{m_j} is a cyclic covering map defined by $w_j^{m_j} = h$ as in (4.2); φ_i is a cyclic covering map defined by $w_j^{\bar{m}_i} = w_0$ for $i = 1, 2$. Thus $\psi_{m_i} = \psi_{m_0} \circ \varphi_i$ and we have the following diagram:

$$\begin{tikzcd}
\tilde{V}_{m_1} \arrow[r, "\tilde{\varphi}_1"] \arrow[dr, "\tilde{\psi}_{m_1}"'] & \tilde{V}_{m_0} \arrow[d, "\tilde{\psi}_{m_0}"] & \tilde{V}_{m_2} \arrow[l, "\tilde{\varphi}_2"'] \arrow[dl, "\tilde{\psi}_{m_2}"] \\
& \tilde{X}. &
\end{tikzcd}$$

Let $E(m_0)_\alpha$ be an irreducible component of $E(m_0)$ with $\tilde{\psi}_{m_0}(E(m_0)_\alpha) = E_\alpha$; therefore $v_{E(m_0)_\alpha}(w_i) = d/m_0$. Since $\gcd(d/m_0, \bar{m}_i) = 1$, $\tilde{\varphi}_i|_{E(m_i)_\alpha} \colon E(m_i)_\alpha \longrightarrow E(m_0)_\alpha$ is a bijective holomorphic map from (i). Consequently, $\tilde{\varphi}_1^{-1} \circ \tilde{\varphi}_2$ gives a biholomorphic map from $E(m_2)_\alpha$ and $E(m_1)_\alpha$. $\square$

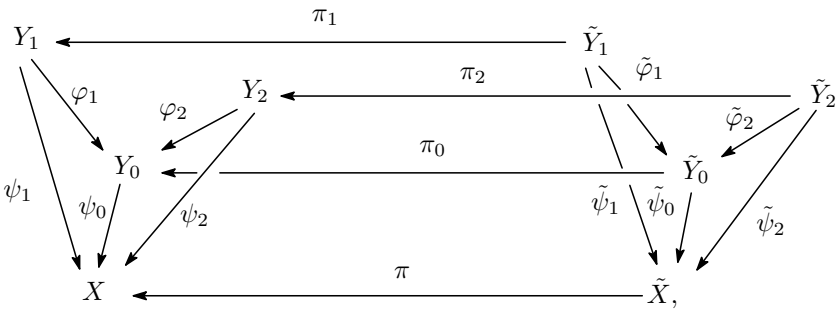
Now let (X, o) be a normal surface singularity with $\mathbb{C}^*$-action, and let h be a homogeneous element of degree d in R_X. Let (Y_i, o) be the normalization of a cyclic cover defined by $w_i^{m_i} = h$ over (X, o) $(i = 1, 2)$. Let $\pi_i \colon (\tilde{Y}_i, E(i)) \longrightarrow (Y_i, o)$ be the minimal $\mathbb{C}^*$-good resolution. The following is the main result of this section.

THEOREM 5.4. *Under the situation above, suppose that $m_1 + m_2 \equiv 0$ (d) and Y_i is connected for $i = 1, 2$. Then there exists a complete $\mathbb{C}^*$-pencil of curves $\hat{\Phi} \colon \hat{S} \longrightarrow \mathbb{P}^1$ satisfying the following $\mathbb{C}^*$-equivariant commutative diagram*:

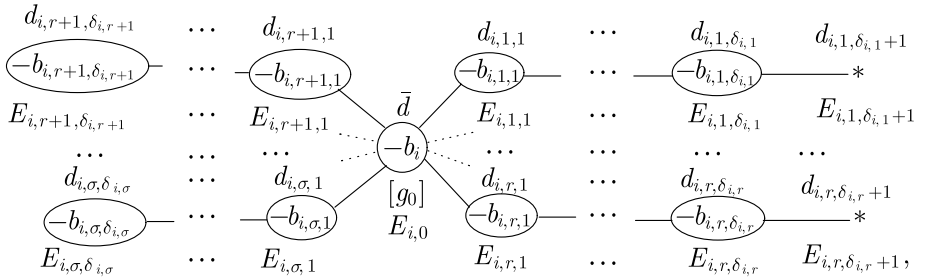
$$\begin{tikzcd}
(\tilde{Y}_i, E(i)) \arrow[r, hook, "\iota_i"] \arrow[d, "\pi_i"'] & (S_i, \mathrm{supp}(S_{i,o})) \arrow[d, "\Phi_i"'] \\
(Y_i, o) \arrow[r, "w_i"] & \mathbb{C}
\end{tikzcd} \qquad (i = 1, 2), \tag{5.1}$$

where $S_1 := \hat{S}_L$, $S_2 := \hat{S}_R$, $\Phi_1 := \hat{\Phi}_L$ and $\Phi_2 := \hat{\Phi}_R$.

PROOF. Assume that $(X, o) \subset (\mathbb{C}^N, o)$ and the $\mathbb{C}^*$-action on (X, o) is induced from $t \cdot (z_1, \ldots, z_N) = (t^{c_1} z_1, \ldots, t^{c_N} z_N)$ on $\mathbb{C}^N$, where $\gcd(c_1, \ldots, c_N) = 1$ and $c_\ell > 0$ for $\ell = 1, \ldots, N$. Let $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ be a $\mathbb{C}^*$-equivariant good resolution such that $\mathrm{red}(h \circ \pi)_{\tilde{X}}$ is simple normal crossing. Since h is homogeneous, we assume that the figure of $(h \circ \pi)_{\tilde{X}}$ is given by (4.1). Therefore, we have $c_\ell = v_{E_0}(z_\ell \circ \pi)$ for any ℓ. From the condition $d \mid m_1 + m_2$, we have $\gcd(d, m_1) = \gcd(d, m_2)$ and $[m_1, d \mid m_2] = [m_2, d \mid m_2] = 1$. Furthermore, let $m_0 := \gcd(d, m_i)$, $\bar{d} := d/m_0$ and $\bar{m}_i := m_i/m_0$ $(i = 1, 2)$ (i.e., $\bar{m}_1 = [m_1 \mid m_2, d]$ and $\bar{m}_2 = [m_2 \mid m_1, d]$). For $i = 0, 1, 2$, let $\pi_0 : (\tilde{Y}_0, E(0)) \longrightarrow (Y_0, o)$ be the minimal $\mathbb{C}^*$-good resolution of the normalization Y_0 of the m_0-fold cyclic cover over $\tilde{X}$ defined by $w_0^{m_0} = h$. Then there exists a generically finite m_i-fold cyclic covering map $\tilde{\psi}_i \colon \tilde{Y}_i \longrightarrow \tilde{X}$ $(i = 0, 1, 2)$. Also, there exists an $\bar{m}_i$-fold cyclic covering map $\varphi_i \colon Y_i \longrightarrow Y_0$ induced from $w_i^{\bar{m}_i} = w_0$ for $i = 1, 2$. Thus we have the following diagram:



where $\tilde{\varphi}_i$ is a generically finite $\bar{m}_i$-fold cyclic covering map corresponding to φ_i $(i = 1, 2)$. Let σ and r $(\sigma \geq r)$ be the numbers of Hirzebruch-Jung divisors and incomplete Hirzebruch-Jung divisors respectively that they are supported on $\mathrm{supp}(h \circ \pi \circ \tilde{\psi}_0)_{\tilde{Y}_0} \backslash E_{0,0}$, where $E_{0,0}$ is the central curve in $E(0)$. Since $v_{E_{0,0}}(h \circ \pi \circ \tilde{\psi}_0)_{\tilde{Y}_0} = \bar{d}$ and $\gcd(\bar{d}, \bar{m}_i) = 1$ for $i = 1, 2$, the numbers of Hirzebruch-Jung divisors and incomplete Hirzebruch-Jung divisors included in $\mathrm{supp}(h \circ \pi \circ \tilde{\psi}_i)_{\tilde{Y}_i} \backslash E_{i,0}$ coincide with σ and r respectively. The central curve $E_{0,0}$ is an m_0-fold cyclic covering of the central curve E_0 of E. Furthermore, from Lemma 5.3 (ii), $E_{1,0}$ and $E_{2,0}$ are biholomorphic to $E_{0,0}$ by $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ respectively. Let g_0 be the genus of $E_{i,0}$ $(i = 0, 1, 2)$. Thus, for $i = 1, 2$, we can give the configuration of $(w_i \circ \pi_i)_{\tilde{Y}_i}$ as follows:



(5.2)

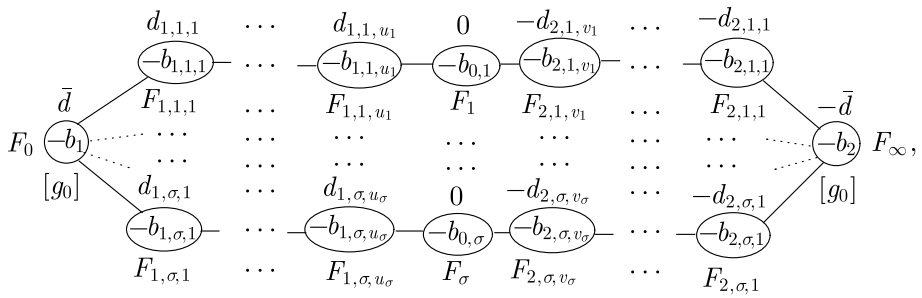
where $E(i) = E_{i,0} \cup \bigcup_{j=1}^{\sigma} \bigcup_{k=1}^{\delta_{i,j}} E_{i,j,k}$ and $b_{i,j,k} \geq 2$ for any i, j, k.

From Theorem 4.1, there exists a complete $\mathbb{C}^*$-pencil of curves $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ which satisfies the following $\mathbb{C}^*$-equivariant commutative diagram:

$$\begin{array}{ccc} (\tilde{Y}_1, E(1)) & \xhookrightarrow{\iota_1} & (\hat{S}_L, \mathrm{supp}(\hat{S}_{L,o})) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \hat{\Phi}_L} \\ (Y_1, o) & \xrightarrow{w_1} & \mathbb{C}. \end{array}$$

For $i = 1, 2$ and a fixed j with $1 \le j \le r$, consider a $\mathbb{P}^1$-chain $\bigcup_{k=1}^{\delta_{i,j}+1} E_{i,j,k}$ started from the central curve $E_{i,0}$ of $E(i)$. For each i, the $\mathbb{P}^1$-chain is mapped onto a $\mathbb{P}^1$-chain on E by $\tilde{\psi}_i$. We put it $\bigcup_{k=1}^{\beta_{j_0}+1} E_{j_0,k}$ if $j_0 \le n$ and $\bigcup_{k=1}^{\beta_{j_0}} E_{j_0,k}$ if $j_0 > n$, where n is the number of cyclic branches for incomplete Hirzebruch-Jung divisors in (4.1). On the $\mathbb{P}^1$-chain, if we put $D := \sum_{k=1}^{\beta_{j_0}+1} d_{j_0,k} E_{j_0,k}$ ($d_{j_0,\beta_{j_0}+1} := 0$ if $j_0 > n$), then $\sum_{k=1}^{\delta_{i,j}+1} d_{i,j,k} E_{i,j,k}$ coincides with $D(m_i)$ on E_i for $i = 1, 2$. From the assumption $m_1 + m_2 \equiv 0$ (d) and Proposition 3.10, $D(m_1)$ and $D(m_2)$ are located on the opposite side. By taking a suitable successive $\mathbb{C}^*$-blowing-up on $\hat{S}$, we can assume that the divisor $(w_2 \circ \pi_2)_{\tilde{Y}_2}$ is contained into the singular fiber $\hat{S}_{R,0}$ numerically. Namely, the w.d.graph of $E(2)$ is contained in the w.d.graph of $\mathrm{supp}(\hat{S}_{R,0})$ such that the coefficient of each component of $(w_2 \circ \pi_2)_{\tilde{Y}_2}$ coincides with the coefficient of the corresponding component for $\hat{S}_{R,0}$.

Let us draw the configuration associated to $\hat{\Phi}$ as follows:



(5.3)

where $\iota_1(E_{1,0}) = F_0$ and $\iota_1(E_{1,j,k}) = F_{1,j,k}$ for any $j = 1, \ldots, \sigma$ and $k = 1, \ldots, \delta_{1,j}$; also $b_{i,j,k} \ge 1$ for any i, j, k and $u_j = \delta_{1,j}$ and $v_j = \delta_{2,j}$ for $j \ge r+1$. Since $\sum_{k=1}^{u_j} d_{1,j,k} F_{1,j,k}$ and $\sum_{k=1}^{v_j} d_{2,j,k} F_{2,j,k}$ are complete minimal Hirzebruch-Jung divisors located on the opposite side for $j = r+1, \ldots, \sigma$, we have $b_{0,j} = 1$ in (5.3) for such j. Let $\tilde{\varphi}$ be the restriction of $\tilde{\varphi}_1^{-1} \circ \tilde{\varphi}_2$ onto $E_{2,0}$. From Lemma 5.3 (ii), $\tilde{\varphi}$ gives a biholomorphic mapping from $E_{2,0}$ to $E_{1,0}$ such that $\tilde{\varphi}(P_{2,j}) = P_{1,j}$, where $P_{i,j} := E_{i,0} \cap E_{i,j,1}$ for $i = 1, 2$ and $j = 1, \ldots, \sigma$. Furthermore, there are biholomorphic mappings $\iota_1 \colon E_{1,0} \xrightarrow{\sim} F_0$ and $\varphi \colon F_0 \xrightarrow{\sim} F_\infty$ such that $\iota_1(P_{1,j}) = F_0 \cap F_{1,j,1}$ and $\varphi(F_0 \cap F_{1,j,1}) = F_\infty \cap F_{2,j,1}$. Through biholomorphic mappings $\tilde{\varphi}$, ι_1 and φ, four points $P_{2,j}$, $P_{1,j}$, $F_0 \cap F_{1,j,1}$ and $F_\infty \cap F_{2,j,1}$ correspond to each other for any j. Therefore, in the following, we identify these four points and represent them by P_j for each j, and identify those four curves (i.e., $E_{1,0}$, $E_{2,0}$, F_0 and F_∞) through the biholomorphic mappings above. If there exists an isomorphism as between two normal bundles as follows:

$$N_{E_{2,0}/\tilde{Y}_2}|_{E_{2,0}} \simeq N_{F_\infty/\hat{S}}|_{F_\infty}, \tag{5.4}$$

then there exists a holomorphic embedding ι_2 of $\tilde{Y}_2$ into $\hat{S}_R$ with $\hat{\Phi}_R \circ \iota_2 = w_2 \circ \pi_2$ from the result due to Fujiki [**Fu1**] and Pinkham [**P**]. This completes our proof for the theorem.

From now on, we prove (5.4). It is done by representing two normal bundles above with two divisors that are linearly equivalent to them respectively. If we put $\hat{c}_{i,\ell} := v_{E_{i,0}}(z_\ell \circ \pi \circ \tilde{\psi}_i)$ for $i = 1, 2$ and $\ell = 1, \dots, N$, then

$$\hat{c}_{i,\ell} = \bar{m}_i c_\ell \tag{5.5}$$

from Lemma 4.3. If we assume $\tilde{\psi}_i(E_{i,j,\delta_{i,j}+1}) = E_{\hat{j},\beta_{\hat{j}}+1}$ and put $\hat{c}_{i,j,\ell} := v_{E_{i,j,\delta_{i,j}+1}}(z_\ell \circ \pi \circ \tilde{\psi}_i)$ and $c_{\hat{j},\ell} := v_{E_{\hat{j},\beta_{\hat{j}}+1}}(z_\ell \circ \pi)$, then we have

$$\hat{c}_{i,j,\ell} = \frac{m_i c_{\hat{j},\ell}}{\gcd(m_i, d_{\hat{j},\beta_{\hat{j}}+1})} \tag{5.6}$$

from Lemma 4.3 ($j = 1, \dots, r$). If $\delta_{i,j} > 1$, then $\bigcup_{k=1}^{\delta_{i,j}} E_{i,j,k}$ is contracted to a cyclic quotient singularity of type $C_{n_{i,j},q_{i,j}}$, where $n_{i,j}$ and $q_{i,j}$ are relatively prime positive integers with $1 \le q_{i,j} < n_{i,j}$. When $\delta_{i,j} > 1$, we have

$$v_{E_{i,j,1}}(z_\ell \circ \pi \circ \tilde{\psi}_i) = \frac{\hat{c}_{i,\ell} q_{i,j} + \hat{c}_{i,j,\ell}}{n_{i,j}} \tag{5.7}$$

from Lemma 5.1 (ii). From the assumption $m_1 + m_2 \equiv 0 \ (d)$, let K be an integer defined by $K\bar{d} = \bar{m}_1 + \bar{m}_2$. Let δ_1 be an integer defined by $\bar{m}_1\delta_1 \equiv 1 \ (\bar{d})$ and $0 < \delta_1 < \bar{d}$. If we put $\delta_2 := -\delta_1$, then $\bar{m}_i\delta_i \equiv 1 \ (\bar{d})$ for $i = 1, 2$. Let ε_i be an integer defined by $\bar{m}_i\delta_i + \bar{d}\varepsilon_i = 1$ for $i = 1, 2$. Then, we have $\varepsilon_2 = K\delta_1 + \varepsilon_1$ because of $\bar{d}(\varepsilon_2 - \varepsilon_1) = \bar{m}_1\delta_1 - \bar{m}_2\delta_2 = \delta_1(\bar{m}_1 + \bar{m}_2) = K\bar{d}\delta_1$. From $\gcd(c_1, \dots, c_N) = 1$, there exists $(\gamma_1, \dots, \gamma_N) \in \mathbb{Z}^N$ with $\sum_{\ell=1}^N c_\ell\gamma_\ell = \delta_1$. It is easy to check that $(-1)^{i+1}\sum_{\ell=1}^N \hat{c}_{i,\ell}\gamma_\ell + \bar{d}\varepsilon_i = 1$ from (5.5) for $i = 1, 2$. Consider a meromorphic function on $\tilde{X}$ given as $f := (z_1^{\gamma_1} \cdots z_N^{\gamma_N}) \circ \pi$. It follows that $v_{E_0}(f) = \sum_{\ell=1}^N c_\ell\gamma_\ell = \delta_1$ and $v_{E_{i,0}}(f \circ \tilde{\psi}_i) = \bar{m}_i\delta_1$ from Lemma 4.3. If we put $\tilde{f}_i := (w_i \circ \pi_i)^{\varepsilon_i} \cdot (f \circ \tilde{\psi}_i)^{(-1)^{i+1}}$, then $v_{E_{i,0}}(\tilde{f}_i) = \bar{d}\varepsilon_i + (-1)^{i+1}\bar{m}_i\delta_1 = \bar{m}_i\delta_i + \bar{d}\varepsilon_i = 1$ ($i = 1, 2$). Let $\xi_{i,j} := v_{E_{i,j,1}}(\tilde{f}_i)$. From $v_{E_{i,j,1}}(w_i \circ \pi_i) = d_{i,j,1}$ and (5.7), we have

$$\begin{aligned}\xi_{i,j} &= d_{i,j,1}\varepsilon_i + (-1)^{i+1}\sum_{\ell=1}^{N} v_{E_{i,j,1}}(z_\ell \circ \pi \circ \tilde{\psi}_i)\gamma_\ell \\ &= (-1)^{i+1}\sum_{\ell=1}^{N} \frac{\hat{c}_{i,\ell}q_{i,j} + \hat{c}_{i,j,\ell}}{n_{i,j}}\gamma_\ell + d_{i,j,1}\varepsilon_i. \end{aligned} \tag{5.8}$$

Since D_1 and D_2 are located on the opposite side, we have $\bar{d} \mid d_{1,j,1} + d_{2,j,1}$ from Definition 3.7 (ii). The following equality is proven for any j:

$$\xi_{1,j} + \xi_{2,j} = \frac{d_{1,j,1} + d_{2,j,1}}{\bar{d}}. \tag{5.9}$$

We complete the proof of (5.4) before to prove (5.9). There exists the following equivalences:

$$0 \sim -(\tilde{f}_i)_{\tilde{Y}_i} E_{i,0} = N^*_{E_{i,0}/\tilde{Y}_i}|_{E_{i,0}} - \sum_{j=1}^{\sigma} \xi_{i,j} P_j \quad (i = 1, 2).$$

Under the identification $E_{1,0} \simeq F_0 \simeq F_\infty$ by ι_1 and φ, we have the following:

$$\begin{aligned} N^*_{E_{1,0}/\tilde{Y}_1}|_{E_{1,0}} + N^*_{E_{2,0}/\tilde{Y}_2}|_{E_{2,0}} &\sim \sum_{j=1}^{\sigma}(\xi_{1,j} + \xi_{2,j})P_j \\ &\sim \sum_{j=1}^{\sigma} \frac{d_{1,j,1} + d_{2,j,1}}{\bar{d}} P_j \sim N^*_{F_0/\hat{S}}|_{F_0} + N^*_{F_\infty/\hat{S}}|_{F_\infty} \end{aligned}$$

from (5.9) and Lemma 5.2. Since $N^*_{E_{1,0}/\tilde{Y}_1}|_{E_{1,0}} \sim N^*_{F_0/\hat{S}}|_{F_0}$, we have (5.4).

In the following, for a fixed j, we prove (5.9). Consider a $\mathbb{P}^1$-chain $\bigcup_{k=1}^{\delta_{i,j}+1} E_{i,j,k}$ on $E(i)$ which corresponds to a $\mathbb{P}^1$-chain $\bigcup_{k=1}^{\beta_{\hat{j}}+1} E_{\hat{j},k}$ on E by $\tilde{\psi}_i$, where $\bigcup_{k=1}^{\beta_{\hat{j}}} E_{\hat{j},k}$ is contracted to a cyclic quotient singularity of type $C_{n_{\hat{j}},q_{\hat{j}}}$. Since $\gcd(m_1, d) = \gcd(m_2, d)$, we can define the following integers:

$$\hat{d} := d_{\hat{j},\beta_{\hat{j}}+1}, \quad \mu_i := [m_i \mid d, \hat{d}], \quad \hat{d}_i := [\hat{d} \mid m_i, d], \quad \rho_i := [m_i, \hat{d} \mid d] \ (i = 1, 2),$$

$$\lambda_1 := \gcd(m_i, d, \hat{d}), \quad \rho_3 := [d, \hat{d} \mid m_i],$$

$$\rho_4 := [m_i, d \mid \hat{d}] \text{ and } \lambda_0 := \gcd\left(m_i, d, \hat{d}, \frac{dq_{\hat{j}} + \hat{d}}{n_{\hat{j}}}\right).$$

From (5.8), we have

$$\xi_{1,j}+\xi_{2,j}=\sum_{\ell=1}^{N}\left(\frac{\hat{c}_{1,\ell}q_{1,j}+\hat{c}_{1,j,\ell}}{n_{1,j}}-\frac{\hat{c}_{2,\ell}q_{2,j}+\hat{c}_{2,j,\ell}}{n_{2,j}}\right)\gamma_\ell+d_{1,j,1}\varepsilon_1+d_{2,j,1}\varepsilon_2. \quad (5.10)$$

From (3.4) and Theorem 3.4, we have $n_{i,j}=\mu_i\lambda_0 n_{\hat{j}}/\lambda_1$. Also, $\hat{c}_{i,j,\ell}=\mu_i\rho_4 c_{\hat{j},\ell}$ from (5.6). Therefore, $\hat{c}_{i,j,\ell}/n_{i,j}=\mu_i\rho_4 c_{\hat{j},\ell}\lambda_1/\mu_i n_{\hat{j}}\lambda_0=\rho_4 c_{\hat{j},\ell}\lambda_1/n_{\hat{j}}\lambda_0$ and so $\hat{c}_{1,j,\ell}/n_{1,j}-\hat{c}_{2,j,\ell}/n_{2,j}=0$ for any j and ℓ. From (5.5) and $\bar{m}_i=\mu_i\rho_i$, we have $\hat{c}_{i,\ell}/n_{i,j}=\bar{m}_i c_\ell/n_{i,j}=\rho_i c_\ell\lambda_1/n_{\hat{j}}\lambda_0$. We put $\Delta:=((\rho_1 q_{1,j}-\rho_2 q_{2,j})\lambda_1)/n_{\hat{j}}\lambda_0$. From (5.10),

$$\begin{aligned}\xi_{1,j}+\xi_{2,j}&=\sum_{\ell=1}^{N}\left(\frac{\hat{c}_{1,\ell}q_{1,j}}{n_{1,j}}-\frac{\hat{c}_{2,\ell}q_{2,j}}{n_{2,j}}\right)\gamma_\ell+d_{1,j,1}\varepsilon_1+d_{2,j,1}\varepsilon_2\\&=\Delta\sum_{\ell=1}^{N}c_\ell\gamma_\ell+d_{1,j,1}\varepsilon_1+d_{2,j,1}\varepsilon_2=\Delta\delta_1+d_{1,j,1}\varepsilon_1+d_{2,j,1}\varepsilon_2.\end{aligned}$$

We put $B_j:=(d_{1,j,1}+d_{2,j,1})/\bar{d}$. From $\varepsilon_2=K\delta_1+\varepsilon_1$ and $\bar{m}\delta_1+\bar{d}\varepsilon_1=1$, we have

$$\begin{aligned}d_{1,j,1}\varepsilon_1+d_{2,j,1}\varepsilon_2&=d_{1,j,1}\varepsilon_1+d_{2,j,1}(K\delta_1+\varepsilon_1)=(d_{1,j,1}+d_{2,j,1})\varepsilon_1+K\delta_1 d_{2,j,1}\\&=\bar{d}\varepsilon_1 B_j+K\delta_1 d_{2,j,1}=(1-\bar{m}_1\delta_1)B_j+K\delta_1 d_{2,j,1}.\end{aligned}$$

Therefore, (5.9) is equivalent to the following equality:

$$\Delta=\bar{m}_1 B_j-Kd_{2,j,1}. \quad (5.11)$$

Hereafter, we prove (5.11). Put $d_{i,j}:=d_{i,j,\delta_{i,j}+1}$ for $i=1,2$ and $j=1,\dots,r$. Since $d_{i,j,1}=(\bar{d}q_{i,j}+d_{i,j})/n_{i,j}=((\bar{d}q_{i,j}+d_{i,j})\lambda_1)/\mu_i\lambda_0 n_{\hat{j}}$ from Lemma 5.1 (ii) and $\bar{m}_i=\mu_i\rho_i$, we have

$$\begin{aligned}\bar{d}(\bar{m}_1 B_j-Kd_{2,j,1})&=\bar{m}_1(d_{1,j,1}+d_{2,j,1})-(\bar{m}_1+\bar{m}_2)d_{2,j,1}=\bar{m}_1 d_{1,j,1}-\bar{m}_2 d_{2,j,1}\\&=\rho_1\frac{(\bar{d}q_{1,j}+d_{1,j})\lambda_1}{n_{\hat{j}}\lambda_0}-\rho_2\frac{(\bar{d}q_{2,j}+d_{2,j})\lambda_1}{n_{\hat{j}}\lambda_0}\\&=\frac{\bar{d}(\rho_1 q_{1,j}-\rho_2 q_{2,j})\lambda_1}{n_{\hat{j}}\lambda_0}+\frac{(\rho_1 d_{1,j}-\rho_2 d_{2,j})\lambda_1}{n_{\hat{j}}\lambda_0}.\end{aligned}$$

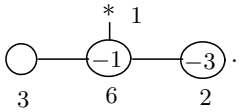
From the definition, we have that $\tilde{\psi}_i(E_{i,j,\delta_{i,j}+1}) = E_{\hat{j},k_{\hat{j}}+1}$, $\hat{d} = d_{\hat{j},\beta_{\hat{j}}+1} = \mathrm{Coeff}_{E_{\hat{j},k_{\hat{j}}+1}}(h \circ \pi)_{\tilde{X}}$ and $d_{i,j} = d_{i,j,\delta_{i,j}+1} = \mathrm{Coeff}_{E_{i,j,\delta_{i,j}+1}}(w_i \circ \pi_i)_{\tilde{Y}_i}$. Hence, we have $d_{i,j} = [\hat{d} \mid m_i] = \rho_3 \hat{d}_i$ from Lemma 4.3. Therefore,

$$\rho_1 d_{1,j} - \rho_2 d_{2,j} = \rho_1\rho_3\hat{d}_1 - \rho_2\rho_3\hat{d}_2 = \rho_3(\rho_1\hat{d}_1 - \rho_2\hat{d}_2) = 0$$

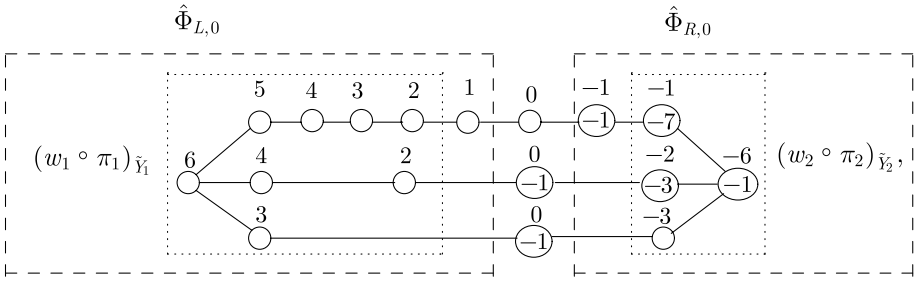
from $\rho_1\hat{d}_1 = \rho_2\hat{d}_2 = [\hat{d} \mid d, \hat{d}]$. Therefore, $\bar{m}_1 B_j - K d_{2,j,1} = (\lambda_1/n_{\hat{j}}\lambda_0)(\rho_1 q_{1,j} - \rho_2 q_{2,j}) = \Delta$ and it completes the proof of (5.11). $\square$

Example 5.5. Let (X, o) be a non-singular point $(\mathbb{C}^2, o)$ with $\mathbb{C}^*$-action defined by $t \cdot (x, y) = (t^3 x, t^2 y)$ for $t \in \mathbb{C}^*$. Let $\pi : (\tilde{X}, E) \longrightarrow (\mathbb{C}^2, 0)$ be the minimal $\mathbb{C}^*$-good resolution for the $\mathbb{C}^*$-action.

(i) Let $h = x^2 + y^3$. It is a weighted homogeneous polynomial of type $(3, 2; 6)$. Then the divisor $(h \circ \pi)_{\tilde{X}}$ on the minimal $\mathbb{C}^*$-good resolution (see (1.2)) of (X, o) is given as follows:

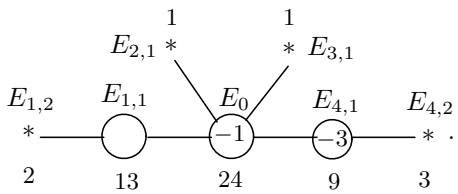


Let $m_1 = 5$ and $m_2 = 7$ and so $d = 6 \mid m_1 + m_2 = 12$. Consider two normal surface singularities $(Y_1, o) = \{w_1^5 = x^2 + y^3\}$ (rational double point of type E_8) and $(Y_2, o) = \{w_2^7 = x^2 + y^3\}$ (uni-modal singularity of type E_{12} in [**Ar**]). Thus we can easily check that there exists a complete $\mathbb{C}^*$-pencil of curves $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ as follows:

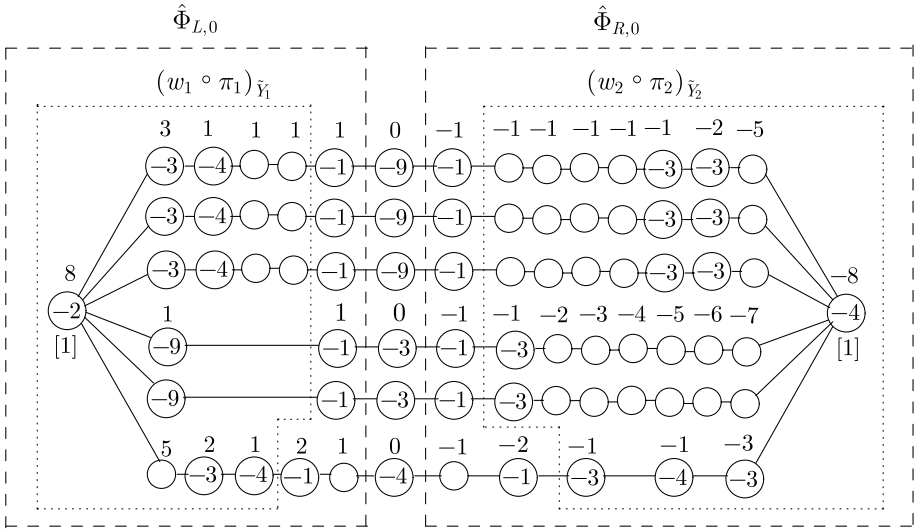


where π_i is a resolution of (Y_i, o) for $i = 1, 2$.

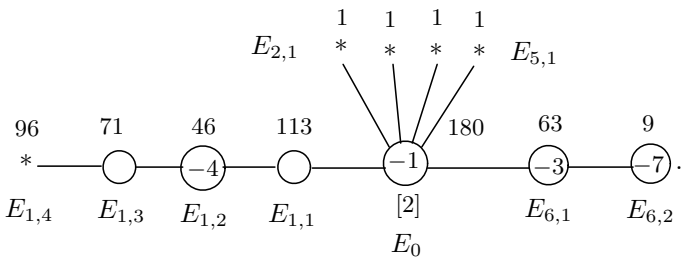
(ii) Let $h = x^2 y^3 (x^4 + y^6)$ and so it is a weighted homogeneous polynomial of type $(3, 2; 24)$. The w.d.graph of $(h \circ \pi)_{\tilde{X}}$ is given as follows:



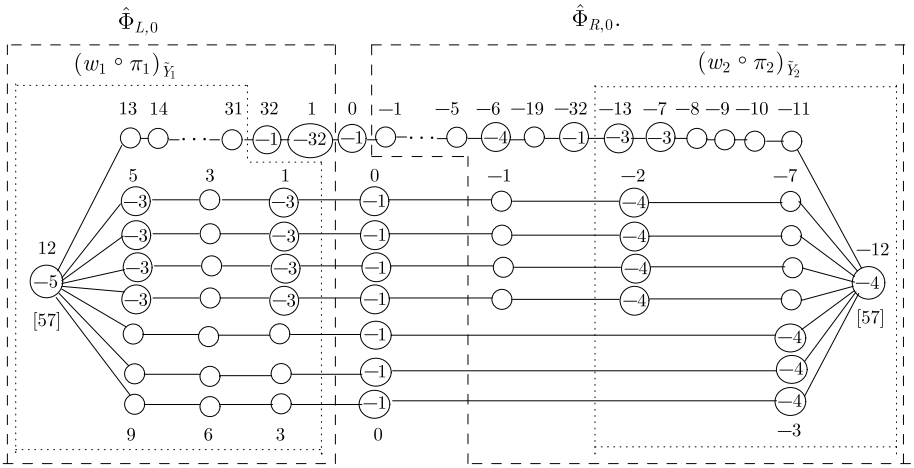
Let $m_1 = 27$ and $m_2 = 45$ and so $d = 24 \mid m_1 + m_2 = 72$. Let (Y_i, o) be the normalization of a non-normal hypersurface singularity $\{z^{m_i} = x^2y^3(x^2+y^3)\}$ $(i = 1, 2)$. Using Theorem 3.4, we can easily compute the minimal $\mathbb{C}^*$-good resolution $\pi_i : (\tilde{Y}_i, o) \longrightarrow (Y_i, o)$ and the divisor $(w_i \circ \pi_i)_{\tilde{Y}_i}$. In the case of $i = 1$, consider a Hirzebruch-Jung divisor D of type $\langle\langle 24 \mid 13 \mid 2 \rangle\rangle$ on $E_0 \cup E_{1,1} \cup E_{1,2}$. Let compute the 27-fold cyclic lifting $D(27)$ (see Definition 3.9). In the notation of Theorem 3.4, we have $\lambda_1 = \lambda_2 = \lambda_0 = 3$ and so $n_1 = 3$; also $\zeta = \xi = 1$ and $\hat{m}_1 := [27|24, 3] = 9$. Since $\lambda_0 = 3$, $D(27)$ has three connected components. Also, each of them coincides with a Hirzebruch-Jung divisor on a resolution of $C_{27,10}$ from $\hat{m}_1 n_1 = 27$ and $\delta_1 = 10$. Computing other cyclic coverings similarly, we can obtain $(w_1 \circ \pi_1)_{\tilde{Y}_1}$ and $(w_2 \circ \pi_2)_{\tilde{Y}_2}$. Thus there exists a complete $\mathbb{C}^*$-pencil of curves $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ as follows:



EXAMPLE 5.6. Let (X, o) be a normal surface singularity defined by an ideal $I = \langle x^2 + y^3 + z^5, y(x^2 + y^3) + w^{32} \rangle$ $(= \langle x^2 + y^3 + z^5, yz^5 + w^{32} \rangle)$; hence this is a weighted homogeneous complete intersection singularity of weight $(60, 40, 24, 5)$. Let h be a homogeneous polynomial $x(x^2 + z^5)$ of degree 180. The configuration of $(h \circ \pi)_{\tilde{Y}}$ for a minimal $\mathbb{C}^*$-good resolution $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ is given as follows:



Let $m_1 = 75$ and $m_2 = 105$ and so $d = m_1 + m_2 = 180$. Let (Y_i, o) be the m_i-fold cyclic covering of (X, o) defined by $w_i^{m_i} = h$. Compute the minimal $\mathbb{C}^*$-good resolutions $\pi : (\tilde{Y}_i, E(i)) \longrightarrow (Y_i, o)$ and the divisor $(w_i \circ \pi_i)_{\tilde{Y}_i}$. Consider a Hirzebruch-Jung divisor D of type $\langle\langle 180 \mid 113, 46, 71 \mid 96 \rangle\rangle$ on $E_0 \cup (\bigcup_{j=1}^{4} E_\ell)$. It is associated to a cyclic quotient singularity of type $C_{12,7}$. From Definition 3.3, we have $\lambda_0 = 1$, $\lambda_1 = 3$ and $n_1 = 4$. From Theorem 3.4, we have $\xi = 5$, $\zeta = 2$, $\hat{m}_1 = 5$ and $\delta_0 = 19$. Since $\hat{m}_1 n_1 = 20$, the 75-fold cyclic lifting $D(75)$ is a Hirzebruch-Jung divisor on the minimal resolution of $C_{20,19}$. Computing other cases similarly, we can obtain $(w_i \circ \pi_i)_{\tilde{Y}_i}$ $(i = 1, 2)$. Hence there exists a complete $\mathbb{C}^*$-pencil of curves $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$ as follows:



In this example, the number of Hirzebruch-Jung divisors started from F_0 is equal to 8, but $-(F_0^2 + F_\infty^2) = 9$. It is obvious that $[[2, 2, \ldots, 2, 1]] = 0$ and $[[1, 2, \ldots, 2, 4, 2, 1, 3, 3, 2, 2, 2, 2]] = 0$ for the Hirzebruch-Jung divisor of the first branch. In general, if there are such Hirzebruch-Jung divisors, then the sum of the numbers of Hirzebruch-Jung divisors started from F_0 and F_∞ does not coincide with $-(F_0^2 + F_\infty^2)$.

REMARK 5.7. In the situation of Theorem 5.4, the resolutions of (Y_1, o) and (Y_2, o) are embedded naturally into $(\hat{S}_L, \mathrm{supp}(\hat{S}_{L,0}))$ and $(\hat{S}_R, \mathrm{supp}(\hat{S}_{R,0}))$ for a complete $\mathbb{C}^*$-pencil of curves $\hat{\Phi} : \hat{S} \longrightarrow \mathbb{P}^1$. We consider the case of $(X, o) = (\mathbb{C}^2, o)$. We can observe some dualities between invariants of (Y_1, o) and (Y_2, o).

(i) Let μ_0 be the Milnor number of $(\{h = 0\}, 0)$ (cf. [**Mi**]), where $h \in \mathbb{C}[x, y]$ is a weighted homogeneous polynomial of type $[q_1, q_2; d]$. From Milnor and Orlik's formula ([**MO**]), we have $\mu_0 = (d/q_1 - 1)(d/q_2 - 1)$ and $\mu(Y_i, o) = \mu_0(m_i - 1)$ for $i = 1, 2$. Therefore, we can see the following:

(i-1) If $m_1 + m_2 \equiv 0 \ (d)$, then $\mu(Y_1, o) + \mu(Y_2, o) + 2\mu_0 \equiv 0 \ (d)$.

(i-2) If $m_1 \equiv m_2 \ (d)$, then $\mu(Y_1, o) \equiv \mu(Y_2, o) \ (d)$.

(ii) In [**GW**], Goto and Watanabe defined an invariant $a(R)$ for every finitely generated normal graded ring R and call it *a-invariant* of R. They showed that if R is a weighted homogeneous hypersurface singularity of type $(q_1, \ldots, q_n; d)$, then $a(R) = d - \sum_{i=1}^{n} q_i$. In the case of $n = 2$, $-a(R)$ coincides with the minimal exponent in the sense of Saito [**Sai**]. Here we consider a-invariants of two weighted homogeneous hypersurface singularities $(Y_i, o) = \{z^{m_i} = h(x, y)\}$ for $i = 1, 2$, where h is a weighted homogeneous polynomial of type $(q_1, q_2; d)$. Let $\bar{m}_i := m_i/\gcd(d, m_i)$ $(i = 1, 2)$ and $d_0 := d/\gcd(d, m_i)$. Since $z^{m_i} = h(x, y)$ is a weighted homogeneous polynomial of type $(\bar{m}_i q_1, \bar{m}_i q_2, d_0; \bar{m}_i d)$, we can easily see the following:

(ii-1) If $m_1 + m_2 \equiv 0 \ (d)$, then $a(Y_1, o) + a(Y_2, o) \equiv 0 \ (d_0)$.

(ii-2) If $m_1 \equiv m_2 \ (d)$, then $a(Y_1, o) \equiv a(Y_2, o) \ (d_0)$.

For example, let $(Y_1, o) = \{z^{42} = xy(x^3 + y^5)\}$ and $(Y_2, o) = \{z^4 = xy(x^3 + y^5)\}$. Their defining polynomials are weighted homogeneous polynomials of types $(210, 126, 23; 966)$ and $(20, 12, 23; 92)$. Hence we have $\mu_0 = 24$, $\mu_1 = 984$ and $\mu_2 = 72$ and $\mu_1 + \mu_2 + 2\mu_0 = 1104 \equiv 0 \ (23)$. Furthermore, we have $d_0 = 23$, $a(Y_1, o) = 607$ and $a(Y_2, o) = 37$. Then we have $a(X_1, o) + a(X_2, o) = 644 \equiv 0 \ (23)$.

Hereafter, we consider n-th root fibrations (see p. 92–93 in [**BPV**]) of $\mathbb{C}^*$-pencils of curves. Similarly as Lemma 5.4, we can easily see that n-th root fibrations of $\mathbb{C}^*$-pencils of curves are $\mathbb{C}^*$-pencils of curves.

In the classification of elliptic degenerations in [**Ko**], Kodaira implicitly described the notion of "dual" of pencil of curves from the point of view of homological monodromy theory. We can see it from the notation in [**Ko**]. The homological monodromy transformations of pencils of curves of type K and K^* are mutually

dual for K = I, II, III and IV in [**Ko**]. We can easily check that K^* in [**Ko**] is birational to the n-th root fibrations of K for a suitable n. Recently, for a pencil of curves $\Phi : S \longrightarrow \Delta$, J. Lu and S. L. Tan [**LT**] called the $(N-1)$-th root fibration $\Phi^{(N-1)}$ the "dual" of Φ, where $N := \mathrm{lcm}\{\mathrm{Coeff}_{E_i} S_o \mid$ any irreducible component E_i of $\mathrm{supp}(S_0)\}$. The reason they call it "dual" is that the homological monodromy transformation of $\Phi^{(N-1)}$ is the inverse of one of Φ.

For $\mathbb{C}^*$-pencils of curves, we could introduce the notion of "dual" holomorphically. In the following, as an application of Theorem 5.4, we prove that our "dual" as $\mathbb{C}^*$-pencils of curves coincides with "dual" as n-th root fibrations up to birational equivalences (see Theorem 5.8 (iii)).

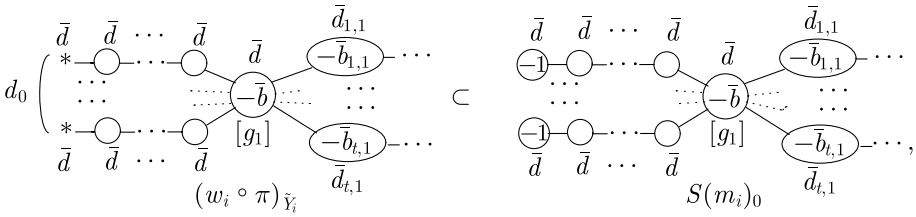
THEOREM 5.8. *Let $\Phi : S \longrightarrow \mathbb{C}$ be a $\mathbb{C}^*$-pencil of curves of degree d. Let $\Phi^{(m)}$ be the m-th root fibration of Φ.*

(i) *If $m_1 \equiv m_2 \ (d)$, then the minimal good $\mathbb{C}^*$-models of $\Phi^{(m_1)}$ and $\Phi^{(m_2)}$ are isomorphic.*
(ii) *If $m_1+m_2 \equiv 0 \ (d)$, then the minimal good $\mathbb{C}^*$-models of $\Phi^{(m_1)}$ and $(\Phi^{(m_2)})^*$ are isomorphic. Namely, there exists a complete $\mathbb{C}^*$-pencil of curves $\hat{\Phi}$ such that $\hat{\Phi}_L = \Phi^{(m_1)}$ and $\hat{\Phi}_R = \Phi^{(m_2)}$.*
(iii) *The minimal good $\mathbb{C}^*$-models of $\Phi^{(d-1)}$ and $(\Phi)^*$ are $\mathbb{C}^*$-equivariantly and holomorphically isomorphic.*

PROOF. Because (i) and (iii) are induced from (ii), we prove (ii). From $m_1 + m_2 \equiv 0 \ (d)$, we have $\gcd(m_1, d) = \gcd(m_2, d)$. We can assume that Φ is a minimal good $\mathbb{C}^*$-pencil of curves and the configuration of S_0 is given by (2.4). Let P_0 be a point of $E_0 \backslash \{P_1, \ldots, P_s\}$ and $O(P_0)$ the $\mathbb{C}^*$-orbit whose closure $\bar{O}(P_0)$ in S contains P_0. Let $\sigma : \tilde{S} \longrightarrow S$ be the blowing-up at P_0. Let $C_1 := \sigma_*^{-1}(\bar{O}(P_0))$, $\tilde{X} := \tilde{S} \backslash C_1$, $C_0 := \sigma^{-1}(P_0) \backslash C_1$ and $E := \sigma_*^{-1}(S_0)$, where $\sigma_*^{-1}(S_0)$ is the strict transform of S_0 by σ, and so on. Then, by the result of Grauert in [**G**], there is a $\mathbb{C}^*$-equivariant contraction $\pi : (\tilde{X}, E) \longrightarrow (X, o)$, where (X, o) is a normal surface singularity with $\mathbb{C}^*$-action. There exists a homogeneous element h of R_X such that $h \circ \pi = \Phi \circ \sigma|_{\tilde{X}}$. Then h is not a perfect power element. In fact, if $h = g^\ell$ for $g \in R_X$ $(\ell \geq 2)$, then $(g \circ \pi)^\ell = \Phi \circ \sigma|_{\tilde{X}}$. Since $\Phi \circ \sigma$ is a holomorphic function on $\tilde{S}$, there exists a holomorphic function $\tilde{g}$ on $\tilde{S}$ by the Riemann's removable singularity theorem such that $\tilde{g} = g \circ \pi$ on $\tilde{X}$. Hence, $\tilde{g}^\ell = \Phi \circ \sigma$ on $\tilde{S}$. This is a contradiction.

For $i = 1, 2$, let (Y_i, o) be the normalization of the m_i-fold cyclic covering of (X, o) defined by $w_i^{m_i} = h$. Then they are irreducible since h is not a perfect power element. Let $\pi_i : (\tilde{Y}_i, E(i)) \longrightarrow (Y_i, o)$ be the minimal $\mathbb{C}^*$-good resolution. From Theorem 5.4, there exists a complete $\mathbb{C}^*$-pencil of curves $\hat{\Psi} : \hat{W} \longrightarrow \mathbb{P}^1$ which satisfies a similar diagram as (5.1). Namely, $\iota_i : (\tilde{Y}_i, E(i)) \hookrightarrow (W_i, \mathrm{supp}(W_{i,0}))$ and

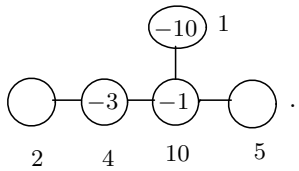
$w_i \circ \pi = \Psi_i \circ \iota_i|_{\tilde{Y}_i}$ $(i = 1, 2)$, where $W_1 := \hat{W}_L$, $W_2 := \hat{W}_R$, $\Psi_1 = \hat{\Psi}_L$ and $\Psi_2 = \hat{\Psi}_R$. The divisor $(w_i \circ \pi_i)_{\tilde{Y}_i}$ is situated on the m_i-th root fibration $(\tilde{S}^{(m_i)}, \mathrm{supp}(\tilde{S}_o^{(m_i)}))$ of Φ as follows:



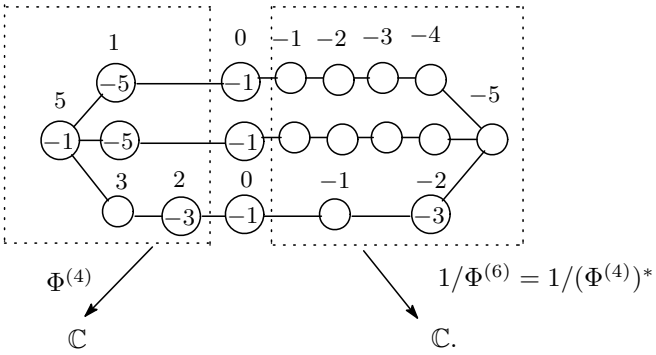
where $d_0 = \gcd(d, m_i)$, $\bar{d} = d/d_0$ and d_0 Hirzebruch-Jung divisors of type $\langle\langle \bar{d} \mid \bar{d} \ldots \bar{d} \rangle\rangle$ on $\tilde{Y}_i$ corresponds to the Hirzebruch-Jung divisor $dE_0 + dC_0$ determined by $h \circ \pi$ on $\tilde{X}$.

On the other hand, it is obvious that the w.d.graphs of $W_{i,0}$ and $\tilde{S}_0^{(m_i)}$ coincide. Since $(\tilde{Y}_i, E(i)) \subset (W_i, \mathrm{supp}(W_{i,0}))$ and $(\tilde{Y}_i, E(i)) \subset (\tilde{S}^{(m_i)}, \mathrm{supp}(\tilde{S}_0^{(m_i)}))$, the conditions (ii-1)–(ii-3) of Corollary 2.12 for $\tilde{\Psi}_i$ and $\tilde{\Phi}^{(m_i)}$ coincide. Thus, $\tilde{\Psi}_i$ and $\tilde{\Phi}^{(m_i)}$ are isomorphic for $i = 1, 2$. Then $\tilde{\Phi}^{(m_1)} : \tilde{S}^{(m_1)} \longrightarrow \mathbb{C}$ and $\tilde{\Phi}^{(m_2)} : \tilde{S}^{(m_2)} \longrightarrow \mathbb{C}$ are mutually dual. Let $E(i)_0$ (resp. F_i) be the central curve of $E(i)$ (resp. $\tilde{S}_0^{(m_i)}$). Then ι_i gives a biholomorphic map $E(i)_0 \cong F_i$ for $i = 1, 2$. Let $Q_1, \ldots, Q_{d_0}$ be the intersection points of $E(i)_0$ and d_0 $\mathbb{P}^1$-chains. Also, assume that their points represent intersection points of F_i and d_o $\mathbb{P}^1$-chains in $\tilde{S}_0^{(m_i)}$. Hence they correspond to P_0 by the covering map from Y_i to X. The difference between normal bundles $N_{F_i/\tilde{S}^{(m_i)}}$ and $N_{F_i/S^{(m_i)}}$ is linearly equivalent to the line bundle associated to $Q_1 + \cdots + Q_{d_0}$ for $i = 1, 2$. Therefore, $\Phi^{(m_1)} : S^{(m_1)} \longrightarrow \mathbb{C}$ and $\Phi^{(m_2)} : S^{(m_2)} \longrightarrow \mathbb{C}$ are mutually dual. $\square$

EXAMPLE 5.9. If h is a weighted homogeneous polynomial $x^2 + y^5$, then the pencil genus $p_e(\mathbb{C}^2, 0, h) = \mu(h)/2 = 2$ because of $\mu(h) = 4$ and Corollary 2.12 (iii) in [**To6**]. From Theorem 4.1, we can construct a $\mathbb{C}^*$-pencil of curves $\Phi : S \longrightarrow \mathbb{C}$ of type $(10, 2, 1)$ whose singular fiber is given as follows:



Let m_1 and m_2 be positive integers satisfying $m_1 \equiv 4$ (10) and $m_2 \equiv 6$ (10). Then $\Phi^{(4)}$ and $\Phi^{(6)}$ are mutually dual as follows:



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