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# Poisson integrals for standard weighted Laplacians in the unit disc

Dedicated to the memory of professor Mikael Passare

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**Abstract.** In this paper a counterpart of the classical Poisson integral formula is found for a class of standard weighted Laplace differential operators in the unit disc. In the process the corresponding Dirichlet boundary value problem is solved for arbitrary distributional boundary data. Boundary limits and representations of the associated solutions are studied within a framework of homogeneous Banach spaces. Special emphasis is put on the so-called relative completion of a homogeneous Banach space.

## 0. Introduction.

Let $\Omega$ be a domain in the complex plane $\mathbb{C}$ equipped with a weight function $w : \Omega \to (0, \infty)$ which provides us with a way to calculate weighted area using the weighted area element

$$dA_w(z) = w(z)dA(z), \quad z = x + iy \in \Omega, \tag{0.1}$$

where $dA(z) = dxdy$ is the usual planar Lebesgue area measure. By geometric considerations the area element (0.1) is the area element induced by the (Riemannian) metric

$$ds_w(z)^2 = w(z)|dz|^2, \quad z \in \Omega, \tag{0.2}$$

where $|dz|$ is the usual arclength element of the complex plane. We mention here Kobayashi and Nomizu [**24**], [**25**] as an extensive background on differential

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geometry.

A fundamental object of study in complex analysis is the Cauchy-Riemann differential operator

$$\bar{\partial}_z = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right), \quad z = x + iy \in \mathbb{C},$$

which we can think of as a vector field $\mathbb{C} \ni z \mapsto \bar{\partial}_z$ over the complex plane. The metric (0.2) suggests us to consider the weighted Cauchy-Riemann differential operator

$$\bar{\partial}_{w,z} = w(z)^{-1/2}\bar{\partial}_z, \quad z \in \Omega.$$

A straightforward calculation shows that the formal adjoint $(\bar{\partial}_w)^*$ of $\bar{\partial}_w$ with respect to the hermitian pairing

$$(u, v) = \iint_{\mathbb{C}} u(z)\overline{v(z)}dA(z)$$

is the differential operator

$$(\bar{\partial}_w)^*_z = -\partial_z w(z)^{-1/2}, \quad z \in \Omega.$$

At this point it is natural to consider the second order operator $(\bar{\partial}_w)^*\bar{\partial}_w$ given by $(\bar{\partial}_w)^*\bar{\partial}_w = -\Delta_w$, where $\Delta_w$ is the weighted Laplacian

$$\Delta_{w,z} = \partial_z w(z)^{-1}\bar{\partial}_z, \quad z \in \Omega. \tag{0.3}$$

We mention that weighted Laplacians of the form (0.3) seem first to have been systematically studied by Paul Garabedian [**12**].

In the study of Bergman spaces of the unit disc $\mathbb{D}$ one often considers so-called standard weights which are weight functions of the form

$$w_\alpha(z) = (1 - |z|^2)^\alpha, \quad z \in \mathbb{D},$$

where $\alpha > -1$ is a real parameter. For an account of recent developments in Bergman space theory we mention the monograph [**17**] by Hedenmalm, Korenblum and Zhu. The case $\alpha = 0$ is commonly referred to as the unweighted case, whereas the case $\alpha = 1$ has attracted special attention recently with contributions by Hedenmalm, Shimorin and others (see for instance [**18**], [**19**], [**20**], [**32**]).

To simplify notation let us denote by $\Delta_\alpha$ the weighted Laplace operator corresponding to the weight $w_\alpha$:

$$\Delta_{\alpha,z} = \partial_z (1 - |z|^2)^{-\alpha} \bar{\partial}_z, \quad z \in \mathbb{D},$$

where $\alpha > -1$. Note that for $\alpha \neq 0$ the differential operator $\Delta_\alpha$ has a certain singular or degenerate behavior on the boundary $\mathbb{T} = \partial\mathbb{D}$. Of particular interest is the $\alpha$-harmonic equation

$$\Delta_\alpha u = 0 \quad \text{in } \mathbb{D},$$

and its associated Dirichlet boundary value problem

$$\begin{cases} \Delta_\alpha u = 0 & \text{in } \mathbb{D}, \\ u = f & \text{on } \mathbb{T}. \end{cases} \tag{0.4}$$

Here the boundary data $f \in \mathcal{D}'(\mathbb{T})$ is a distribution on $\mathbb{T}$, and the boundary condition in (0.4) is to be understood as $u_r \to f$ in $\mathcal{D}'(\mathbb{T})$ as $r \to 1$, where we for a function $u$ in $\mathbb{D}$ employ the notation

$$u_r(e^{i\theta}) = u(re^{i\theta}), \quad e^{i\theta} \in \mathbb{T}, \tag{0.5}$$

for $0 \leq r < 1$.

We show that the Poisson kernel for the $\alpha$-harmonic Dirichlet problem (0.4) is the function given by the formula

$$P_\alpha(z) = \frac{(1 - |z|^2)^{\alpha+1}}{(1 - z)(1 - \bar{z})^{\alpha+1}}, \quad z \in \mathbb{D},$$

where as above $\alpha > -1$ is a real parameter (see Section 2). By this we mean that for every $f \in \mathcal{D}'(\mathbb{T})$ the distributional Dirichlet problem (0.4) is uniquely solvable by the Poisson integral

$$u(z) = P_\alpha[f](z) = P_{\alpha,r} * f(e^{i\theta}), \quad z = re^{i\theta} \in \mathbb{D},$$

where $*$ denotes convolution of distributions on $\mathbb{T}$ and $P_{\alpha,r}(e^{i\theta}) = P_\alpha(re^{i\theta})$ for $0 \leq r < 1$ and $e^{i\theta} \in \mathbb{T}$ as in (0.5). We also calculate an associated homogeneous power series expansion of the $\alpha$-harmonic Poisson kernel $P_\alpha$. The individual terms $e_{\alpha,k}$ for $k < 0$ in this homogeneous expansion are closely related to certain incomplete beta function ratios appearing in statistics (see Section 1).

To discuss boundary limits and representations of $\alpha$-harmonic functions we consider so-called *homogeneous Banach spaces*. Let us fix some terminology. By a Banach space of distributions on $\mathbb{T}$ we mean a Banach space $X$ continuously embedded in $\mathcal{D}'(\mathbb{T})$. We say that the space $X$ is translation invariant if it has the property that $f_{e^{i\theta}} \in X$ and $\|f_{e^{i\theta}}\|_X = \|f\|_X$ whenever $f \in X$ and $e^{i\theta} \in \mathbb{T}$; here the translation $f_{e^{i\theta}}$ of $f \in \mathcal{D}'(\mathbb{T})$ is defined in the usual sense of distribution theory. By a homogeneous Banach space on $\mathbb{T}$ we mean a Banach space $B$ of distributions on $\mathbb{T}$ which is translation invariant and has the property that for every element $f \in B$ the translation $\mathbb{T} \ni e^{i\theta} \mapsto f_{e^{i\theta}}$ is a continuous $B$-valued map on $\mathbb{T}$. Examples of homogeneous Banach spaces are plentiful (see Section 3). An influential treatment of homogeneous Banach spaces is Katznelson [**23**].

The kernel $P_\alpha$ has a property of bounded $L^1$-means

$$\sup_{0\leq r<1} \frac{1}{2\pi}\int_{\mathbb{T}} |P_\alpha(re^{i\theta})|d\theta = \lim_{r\to 1}\frac{1}{2\pi}\int_{\mathbb{T}} |P_\alpha(re^{i\theta})|d\theta = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha/2+1)^2}, \tag{0.6}$$

where $\Gamma$ is the Gamma function and $\alpha > -1$ (see Section 2). If $B$ is a homogeneous Banach space on $\mathbb{T}$ and $f \in B$, then standard arguments show that the above $L^1$-means bound (0.6) leads to the regularizing property $\lim_{r\to 1} u_r = f$ in $B$ when $u = P_\alpha[f]$ and $u_r$ is as in (0.5) (see Sections 3 and 5).

In order to discuss Poisson integral representations of $\alpha$-harmonic functions we use the so-called relative completion $\tilde{B}$ of a homogeneous Banach space $B$ on $\mathbb{T}$. We show that an $\alpha$-harmonic function $u$ in $\mathbb{D}$ has the property that $u_r \in B$ for $0 \leq r < 1$ and

$$\sup_{0\leq r<1} \|u_r\|_B < \infty$$

if and only if it has the form of a Poisson integral $u = P_\alpha[f]$ of some element $f \in \tilde{B}$ (see Section 5). For $\alpha = 0$ this result unifies classical results on Poisson integral representation of harmonic functions in the unit disc dating back to Herglotz and F. Riesz.

Let $B$ be a homogeneous Banach space on $\mathbb{T}$. The *relative completion* $\tilde{B}$ of $B$ is a translation invariant Banach space of distributions on $\mathbb{T}$ containing $B$ isometrically having the same Fourier spectra as $B$ and an additional property of relative completeness:

- If $f_n \in \tilde{B}$, $\|f_n\|_{\tilde{B}} \leq C$ for $n \geq 1$ and $f_n \to f$ in $\mathcal{D}'(\mathbb{T})$, then $f \in \tilde{B}$ and $\|f\|_{\tilde{B}} \leq C$.

We show that every homogeneous Banach space $B$ on $\mathbb{T}$ has a relative completion and that this relative completion is uniquely determined by its defining properties (see Section 4). We also discuss a canonical predual of the relative completion $\tilde{B}$ which in general is not a space of distributions on $\mathbb{T}$.

The relative completion seems first to have been introduced by Gagliardo [**11**] and then further studied by Braun and Feichtinger [**8**], and others. We mention that Braun and Feichtinger [**8**] consider function spaces living on more general topological groups and use an assumption of double module structures. It is evident that such an assumption of double module structures is not fulfilled in our setup of a general homogeneous Banach space $B$ on $\mathbb{T}$ where the Fourier spectrum of $B$ has to be taken into account in the analysis. As a matter of convenience we consider in this paper homogeneous Banach spaces continuously embedded in $\mathcal{D}'(\mathbb{T})$. Of course, other choices of embeddings are possible.

Here we also wish to mention the general study of existence and uniqueness of preduals initiated by Dixmier [**10**] and Grothendieck [**15**] and further pursued by Godefroy and others (see [**14**] for a survey); see also Kaijser [**22**]. We mention also that preduals of $Q_p$ spaces have recently been investigated by Aleman, Carlsson and Persson [**3**], [**4**] using Cauchy duality.

The term Fatou theorems is commonly used for results that guarantee existence of non-tangential boundary values almost everywhere. We provide some basic results of this type for $\alpha$-harmonic functions (see Section 6). The proofs of these results depend on a good control of kernels of the form

$$K_\alpha(z) = |P_\alpha(z)| = \frac{(1-|z|^2)^{\alpha+1}}{|1-z|^{\alpha+2}}, \quad z \in \mathbb{D},$$

where $\alpha > -1$, which allows us to apply known techniques from the study of Fatou theorems for harmonic functions.

## 1. Power series expansion of $\alpha$-harmonic functions.

In this section we prove power series expansion formulas for $\alpha$-harmonic functions. We begin with a lemma.

LEMMA 1.1. *Let $\alpha \in \mathbb{R}$. For a positive integer $k \in \mathbb{Z}^+$, denote by $\tilde{e}_{\alpha,-k}$ the function*

$$\tilde{e}_{\alpha,-k}(z) = \left(\int_0^1 t^{k-1}(1-t|z|^2)^\alpha dt\right)\bar{z}^k, \quad z \in \mathbb{D}.$$

*Then $\tilde{e}_{\alpha,-k}$ solves the $\bar{\partial}$-problem*

$$\bar\partial \tilde e_{\alpha,-k}(z) = w_\alpha(z)\bar z^{k-1}, \quad z\in\mathbb{D}.$$

Proof. Introduce the function $p_{\alpha,k}$ given by

$$p_{\alpha,k}(x) = \int_0^1 t^{k-1}(1-tx)^\alpha dt, \quad -1<x<1, \tag{1.1}$$

for $k\in\mathbb{Z}^+$. Now $\tilde e_{\alpha,-k}(z) = p_{\alpha,k}(|z|^2)\bar z^k$ for $z\in\mathbb{D}$, and a differentiation gives

$$\begin{aligned}\bar\partial \tilde e_{\alpha,-k}(z) &= p'_{\alpha,k}(|z|^2)z\bar z^k + p_{\alpha,k}(|z|^2)k\bar z^{k-1}\\ &= \big(|z|^2p'_{\alpha,k}(|z|^2) + kp_{\alpha,k}(|z|^2)\big)\bar z^{k-1}, \quad z\in\mathbb{D}.\end{aligned}$$

By the last equality we see that the conclusion of the lemma will follow if $p_{\alpha,k}$ satisfies the ordinary differential equation

$$xp'_{\alpha,k}(x) + kp_{\alpha,k}(x) = (1-x)^\alpha, \quad -1<x<1.$$

By differentiation under the integral we have

$$p'_{\alpha,k}(x) = -\int_0^1 t^k\alpha(1-tx)^{\alpha-1}dt, \tag{1.2}$$

and an integration by parts gives

$$xp'_{\alpha,k}(x) = (1-x)^\alpha - k\int_0^1 t^{k-1}(1-tx)^\alpha dt.$$

This completes the proof of the lemma. □

To ensure convergence of the power series expansion in Theorem 1.2 below we shall need some estimates of the functions $p_{\alpha,k}$ defined by (1.1) for $\alpha\in\mathbb{R}$ and $k\in\mathbb{Z}^+$. Recall by (1.2) that

$$p'_{\alpha,k}(x) = -\alpha p_{\alpha-1,k+1}(x). \tag{1.3}$$

When $\alpha\ge 0$ and $0\le r<1$ we have the estimate

$$|p_{\alpha,k}(x)| \le (1+r)^\alpha\int_0^1 t^{k-1}dt = \frac{(1+r)^\alpha}{k}, \quad |x|\le r,$$

and when $\alpha < 0$ we have $|(1-tx)^\alpha| \le (1-r)^\alpha$ if $|x| \le r$ and $t \in [0,1]$. Thus, for any $\alpha \in \mathbb{R}$ and $0 \le r < 1$ we have

$$|p_{\alpha,k}(x)| \le C_\alpha / k \quad \text{for } |x| \le r, \tag{1.4}$$

where $C_\alpha = \max\{(1+r)^\alpha, (1-r)^\alpha\}$.

THEOREM 1.2. *A function $u$ in $\mathbb{D}$ is $\alpha$-harmonic if and only if it is given by a convergent power series expansion of the form*

$$u(z) = \sum_{k=1}^{\infty} c_{-k}\bigg(\int_0^1 t^{k-1}(1-t|z|^2)^\alpha dt\bigg)\bar{z}^k + \sum_{k=0}^{\infty} c_k z^k, \quad z \in \mathbb{D}, \tag{1.5}$$

*for some sequence $\{c_k\}_{k=-\infty}^{\infty}$ of complex numbers such that*

$$\limsup_{|k|\to\infty} |c_k|^{1/|k|} \le 1.$$

PROOF. It is straightforward to check using Lemma 1.1 that every function $u$ of the form (1.5) is $\alpha$-harmonic.

Assume next that $u$ is $\alpha$-harmonic. Then the function $w_\alpha^{-1}\bar{\partial}u$ is conjugate-analytic, and therefore has a power series expansion of the form

$$w_\alpha(z)^{-1}\bar{\partial}u(z) = \sum_{k=0}^{\infty} a_k \bar{z}^k, \quad z \in \mathbb{D},$$

where $\limsup_{k\to\infty}|a_k|^{1/k} \le 1$. Consider now the function $f$ given by

$$f(z) = u(z) - \sum_{k=1}^{\infty} c_{-k}\tilde{e}_{\alpha,-k}(z), \quad z \in \mathbb{D}, \tag{1.6}$$

where the functions $\tilde{e}_{\alpha,-k}$ are as in Lemma 1.1 and $c_{-k} = a_{k-1}$ for $k \ge 1$. A differentiation using Lemma 1.1 gives

$$\bar{\partial}f(z) = \sum_{k=0}^{\infty} a_k w_\alpha(z)\bar{z}^k - \sum_{k=1}^{\infty} a_{k-1} w_\alpha(z)\bar{z}^{k-1} = 0, \quad z \in \mathbb{D},$$

showing that the function $f$ is analytic in $\mathbb{D}$. Solving for $u$ in (1.6) we obtain

$$u(z)=\sum_{k=1}^{\infty}c_{-k}\tilde{e}_{\alpha,-k}(z)+f(z)=\sum_{k=1}^{\infty}c_{-k}\tilde{e}_{\alpha,-k}(z)+\sum_{k=0}^{\infty}c_kz^k,\quad z\in\mathbb{D}.$$

This completes the proof of the theorem. □

Remark 1.3. Recall that the space $C^\infty(\mathbb{D})$ is naturally equipped with the topology given by all the semi-norms of the form

$$P_N(u)=\max\{|\partial^n\bar{\partial}^m u(z)|:z\in K_N,\ n+m\le N\},\quad u\in C^\infty(\mathbb{D}),\tag{1.7}$$

where $n,m\ge 0$ and the $K_i$ form a nested sequence of compact sets growing to $\mathbb{D}$. In view of (1.3) and the estimates (1.4) for the functions $p_{\alpha,k}$ it is then clear that the power series expansion (1.5) converges in $C^\infty(\mathbb{D})$.

Remark 1.4. We mention also that the power series expansion in Theorem 1.2 can be viewed as a special instance of a homogeneous expansion with respect to the group of rotations of the disc. Indeed, let the group action be given by

$$R_{e^{i\theta}}u(z)=u(e^{i\theta}z),\quad z\in\mathbb{D},$$

for $u\in C^\infty(\mathbb{D})$ and $e^{i\theta}\in\mathbb{T}$. It is well-known that every function $u\in C^\infty(\mathbb{D})$ admits a unique expansion of the form

$$u(z)=\sum_{k=-\infty}^{\infty}u_k(z),\quad z\in\mathbb{D},$$

convergent in $C^\infty(\mathbb{D})$, where $u_k$ is homogeneous of degree $k$ with respect to rotations in the sense that $R_{e^{i\theta}}u_k(z)=e^{ik\theta}u_k(z)$ for all $e^{i\theta}\in\mathbb{T}$. Moreover, $u_k$ is given by

$$u_k(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(e^{i\theta}z)e^{-ik\theta}d\theta,\quad z\in\mathbb{D},$$

and $P_N(u_k)\le C_m/(1+k^2)^m$ for $m=1,2,\ldots$, where $C_m=C_{m,N}$ does not depend on $k$, and the semi-norms $P_N$ are given by (1.7). We leave it to the reader to confirm that this holds for the functions in the expansion (1.5).

We shall throughout this paper let $\phi_k$ denote the exponential monomials on $\mathbb{T}$, that is,

$$\phi_k(e^{i\theta}) = e^{ik\theta}, \quad e^{i\theta} \in \mathbb{T}, \tag{1.8}$$

for $k \in \mathbb{Z}$. The following proposition exemplifies the fact that the restriction $\alpha > -1$ in the parameter range is essential for a satisfactory existence theory for the $\alpha$-harmonic Dirichlet problem (0.4).

PROPOSITION 1.5. *Let $\alpha \in \mathbb{R}$. Assume that there exists an $\alpha$-harmonic function $u$ in $\mathbb{D}$ such that $\lim_{r\to 1} u_r = \phi_k$ in $\mathcal{D}'(\mathbb{T})$, where $k < 0$ is a negative integer and $\phi_k$ is given by* (1.8). *Then $\alpha > -1$.*

PROOF. Consider the power series expansion (1.5) from Theorem 1.2. By assumption we have that $\langle u_r, \phi_{-k}\rangle \to 1$ as $r \to 1$, where $\langle \cdot, \cdot \rangle$ indicates distributional pairing. A calculation shows that

$$\langle u_r, \phi_{-k}\rangle = c_{-k}\tilde{e}_{\alpha,k}(r)$$

for $0 \le r < 1$, where the function $\tilde{e}_{\alpha,k}$ is as in Lemma 1.1. The result now follows from the fact that $\tilde{e}_{\alpha,k}(r) \to \infty$ as $r \to 1$ if $\alpha \le -1$ by the monotone convergence theorem. □

From now on we will only treat the case $\alpha > -1$. For integers $k < 0$ we introduce the functions

$$e_{\alpha,k}(z) = \frac{\Gamma(|k|+\alpha+1)}{\Gamma(|k|)\Gamma(\alpha+1)}\left(\int_0^1 t^{|k|-1}(1-t|z|^2)^{\alpha}dt\right)\bar{z}^{|k|}, \quad z \in \mathbb{D}, \tag{1.9}$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the Gamma function. For integers $k \ge 0$ we set

$$e_{\alpha,k}(z) = z^k. \tag{1.10}$$

Note that with the functions $e_{\alpha,k}$, $k \in \mathbb{Z}$, the power series expansion of an $\alpha$-harmonic function $u$ takes the form

$$u(z) = \sum_{k=-\infty}^{\infty} c_k e_{\alpha,k}(z), \quad z \in \mathbb{D}, \tag{1.11}$$

by Theorem 1.2. Note also that the term $c_k e_{\alpha,k}$ in (1.11) is the $k$-th homogeneous part of $u$ in the terminology of Remark 1.4.

Factors of normalization are sometimes conveniently described using the Beta integral

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\quad x,y>0$$

(see [**5**, Theorem 1.1.4]). Using this formula it is easy to see that the factor of normalization in (1.9) ensures that $e_{\alpha,k}(e^{i\theta})=\phi_k(e^{i\theta})$ for $e^{i\theta}\in\mathbb{T}$. Recall in this context also the generalized binomial coefficients given by

$$\binom{x}{y}=\frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}=\frac{1}{(x+1)B(x-y+1,y+1)},\tag{1.12}$$

as well as the standard power series expansion

$$\frac{1}{(1-z)^\beta}=\sum_{n=0}^\infty\binom{n+\beta-1}{n}z^n=\sum_{n=0}^\infty\frac{\Gamma(n+\beta)}{n!\Gamma(\beta)}z^n,\quad z\in\mathbb{D},\tag{1.13}$$

valid when $\operatorname{Re}\beta>0$.

We next turn to the calculation of the functions $e_{\alpha,k}$. For this we need a few lemmas.

Lemma 1.6. *Let $\alpha>-1$ and let $k\geq 1$ be a positive integer. Denote by $u$ the function*

$$u(z)=\left(\sum_{n=0}^\infty\binom{n+k-1}{n}(1-|z|^2)^n-(1-|z|^2)^{\alpha+1}\sum_{n=0}^\infty\binom{n+\alpha+k}{n+\alpha+1}(1-|z|^2)^n\right)\bar z^k$$

*for $z\in\mathbb{D}\setminus\{0\}$. Then*

$$\bar\partial u(z)=\frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)(k-1)!}(1-|z|^2)^\alpha\bar z^{k-1},\quad z\in\mathbb{D}\setminus\{0\}.$$

Proof. By (1.13) we have

$$\left(\sum_{n=0}^\infty\binom{n+k-1}{n}(1-|z|^2)^n\right)\bar z^k=\frac{\bar z^k}{|z|^{2k}}=\frac{1}{z^k}$$

for $z\in\mathbb{D}\setminus\{0\}$. Introduce the auxiliary function

$$f_\alpha(t)=t^{\alpha+1}\sum_{n=0}^\infty\binom{n+\alpha+k}{n+\alpha+1}t^n,\quad 0<t<1.$$

With this notation, $u$ takes the form

$$u(z) = \frac{1}{z^k} - f_\alpha(1-|z|^2)\bar{z}^k, \quad z \in \mathbb{D} \setminus \{0\}. \tag{1.14}$$

Differentiating this identity we obtain

$$\bar{\partial} u(z) = \big((1-(1-|z|^2))f'_\alpha(1-|z|^2) - kf_\alpha(1-|z|^2)\big)\bar{z}^{k-1}, \quad z \in \mathbb{D} \setminus \{0\}. \tag{1.15}$$

A straightforward calculation using (1.12) now shows that

$$(1-t)f'_\alpha(t) - kf_\alpha(t) = \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)(k-1)!} t^\alpha$$

for $0 < t < 1$. By (1.15) this yields the conclusion of the lemma. □

The following lemma relates $e_{\alpha,-k}$ to the function $u$ in Lemma 1.6.

Lemma 1.7. *Let the function $u$ be as in Lemma* 1.6*. Then*

$$u(z) = e_{\alpha,-k}(z) + c/z^k, \quad z \in \mathbb{D} \setminus \{0\},$$

*for some complex number $c$.*

Proof. Let $f = u - e_{\alpha,-k}$ in $\mathbb{D} \setminus \{0\}$. In view of (1.9) we have $\bar{\partial} f = 0$ in $\mathbb{D} \setminus \{0\}$ by Lemmas 1.1 and 1.6. Hence $f$ is analytic in the punctured disc $\mathbb{D} \setminus \{0\}$ by Weil's lemma. Note also that $f$ is homogeneous of degree $-k$ with respect to rotations in the sense that $f(e^{i\theta}z) = e^{-ik\theta}f(z)$ for $e^{i\theta} \in \mathbb{T}$ and $z \in \mathbb{D} \setminus \{0\}$. Using this homogeneity property we conclude that the Laurent series expansion of $f$ in $\mathbb{D} \setminus \{0\}$ has the form $c/z^k$ for some complex number $c$. This completes the proof. □

We next investigate the limit behavior of $u(z)$ as $z \to 0$.

Lemma 1.8. *Let $u$ be as in Lemma* 1.6*. Then* $\lim_{z\to 0} z^k u(z) = 0$.

Proof. Consider the function $f_\alpha$ from the proof of Lemma 1.6. Note that the generalized binomial coefficient given by (1.12) can also be written as

$$\binom{n+\alpha+k}{n+\alpha+1} = \frac{1}{(k-1)!}\prod_{j=1}^{k-1}(n+\alpha+1+j).$$

Hence it is a polynomial in $n$ of degree $k-1$ with leading coefficient $1/(k-1)!$. It is also increasing in $\alpha$ for $\alpha > -1$. This leads to the estimates

$$\binom{n+k-1}{n} \leq \binom{n+\alpha+k}{n+\alpha+1} \leq \binom{n+k-1}{n} + c\binom{n+k-2}{n}$$

for $n = 0, 1, 2, \ldots$, where $c = c_{\alpha,k}$ is a positive constant. For the function $f_\alpha$ this gives the estimates

$$\frac{1}{(1-t)^k} \leq \frac{f_\alpha(t)}{t^{\alpha+1}} \leq \frac{1}{(1-t)^k} + \frac{c}{(1-t)^{k-1}}$$

for $0 < t < 1$. Passing to the function $u$ using (1.14) we have the asymptotics

$$\begin{aligned} u(z) &= \frac{1}{z^k} - f_\alpha(1-|z|^2)\bar{z}^k = \frac{1}{z^k} - (1-|z|^2)^{\alpha+1}\biggl(\frac{1}{|z|^{2k}} + \mathcal{O}(|z|^{-2k+2})\biggr)\bar{z}^k \\ &= (1-(1-|z|^2)^{\alpha+1})\frac{1}{z^k} + \mathcal{O}(|z|^{-k+2}) = \mathcal{O}(|z|^{-k+2}) \end{aligned}$$

as $z \to 0$. This gives the conclusion of the lemma. □

We can now finish the calculation of $e_{\alpha,k}$.

Theorem 1.9. *Let $\alpha > -1$ and let $k \geq 1$ be a positive integer. Then*

$$\begin{aligned} e_{\alpha,-k}(z) = \biggl(&\sum_{n=0}^{\infty}\binom{n+k-1}{n}(1-|z|^2)^n \\ &- (1-|z|^2)^{\alpha+1}\sum_{n=0}^{\infty}\binom{n+\alpha+k}{n+\alpha+1}(1-|z|^2)^n\biggr)\bar{z}^k \end{aligned} \tag{1.16}$$

*for $z \in \mathbb{D} \setminus \{0\}$.*

Proof. Let $u$ be as in Lemma 1.6. By Lemma 1.7 we have that

$$u(z) = e_{\alpha,-k}(z) + c/z^k, \quad z \in \mathbb{D} \setminus \{0\},$$

for some complex number $c$. In view of Lemma 1.8 it follows that $c = 0$. □

When $\alpha$ is a non-negative integer we have cancellation in (1.16).

COROLLARY 1.10. *Let $\alpha$ be a non-negative integer and let $k \geq 1$. Then*

$$e_{\alpha,-k}(z) = \Big( \sum_{n=0}^{\alpha} \binom{n+k-1}{n} (1-|z|^2)^n \Big) \bar{z}^k, \quad z \in \mathbb{D}.$$

PROOF. The result follows by Theorem 1.9. □

Let $\alpha > -1$ and let $k \geq 1$ be a positive integer. Recall formula (1.9). A change of variables shows that

$$e_{\alpha,-k}(z) = I_{|z|^2}(k, \alpha+1) \frac{1}{z^k}, \quad z \in \mathbb{D} \setminus \{0\},$$

where

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad 0 \leq x \leq 1,$$

is the incomplete Beta function ratio appearing in statistics. Using known properties of incomplete Beta functions we can give an alternative derivation of Theorem 1.9 (see [**2**, formulas (26.5.2) and (26.5.4)]). We omit the details. Further information on the Beta distribution can be found in [**16**].

Theorem 1.9 has the following consequence concerning boundary regularity of $\alpha$-harmonic functions.

THEOREM 1.11. *Assume that $\alpha > -1$ is not an integer and let $u$ be an $\alpha$-harmonic function in $\mathbb{D}$. If $u \in C^m(\bar{\mathbb{D}})$ and $m > \alpha + 1$, then $u$ is analytic in $\mathbb{D}$.*

PROOF. Let $k < 0$ be a negative integer. In view of Remark 1.4 and equation (1.11) the $k$-th homogeneous part $u_k$ of $u$ has the form $u_k = c_k e_{\alpha,k}$ for some complex number $c_k$. Also $u_k \in C^m(\bar{\mathbb{D}})$ since $u \in C^m(\bar{\mathbb{D}})$. By Theorem 1.9 the function $e_{\alpha,k}$ is not in $C^m(\bar{\mathbb{D}})$ since $m > \alpha+1$ and $\alpha$ is not an integer. We conclude that $c_k = 0$ for $k < 0$. Therefore $u$ is analytic. □

We mention that the lack of regularity alluded to in Theorem 1.11 sits in the radial component. Tangentially the kernel $P_\alpha$ has good regularizing properties (see Sections 5 and 6 below).

## 2. The $\alpha$-harmonic Poisson kernel.

In this section we introduce the *$\alpha$-harmonic Poisson kernel* and study some of its basic properties such as boundedness of $L^1$-means.

Definition 2.1. Let $\alpha > -1$. Define the *$\alpha$-harmonic Poisson kernel* by

$$P_\alpha(z) = \sum_{k=-\infty}^{\infty} e_{\alpha,k}(z), \quad z \in \mathbb{D}.$$

Note that the function $P_\alpha$ is $\alpha$-harmonic by Theorem 1.2, and that the sum defining $P_\alpha$ is convergent in $C^\infty(\mathbb{D})$ by Remark 1.3.

We have the following auxiliary integral formula for $P_\alpha$.

Proposition 2.2. *Let $\alpha > -1$. Then the $\alpha$-harmonic Poisson kernel $P_\alpha$ is given by the formula*

$$P_\alpha(z) = \frac{1}{1-z} + (\alpha+1)\bar{z} \int_0^1 \frac{(1-t|z|^2)^\alpha}{(1-t\bar{z})^{\alpha+2}} dt, \quad z \in \mathbb{D}.$$

Proof. Recall the power series expansion (1.13). By (1.9) we have that

$$\begin{aligned} P_\alpha(z) &= \frac{1}{1-z} + (\alpha+1)\bar{z} \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)} \int_0^1 t^k \bar{z}^k (1-t|z|^2)^\alpha dt \\ &= \frac{1}{1-z} + (\alpha+1)\bar{z} \int_0^1 \frac{(1-t|z|^2)^\alpha}{(1-t\bar{z})^{\alpha+2}} dt, \quad z \in \mathbb{D}. \end{aligned}$$

This completes the proof of the proposition. □

The following estimation will prove useful.

Lemma 2.3. *Let $\beta > 0$. Then*

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{(1-r^2)^\beta}{|1-re^{i\theta}|^{\beta+1}} d\theta \leq \frac{\Gamma(\beta)}{\Gamma((\beta+1)/2)^2}$$

*for $0 \leq r < 1$, where $\Gamma$ is the Gamma function.*

Proof. By Parseval's formula and (1.13) it follows that

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\theta}{|1-re^{i\theta}|^{\beta+1}} = \sum_{k=0}^{\infty} \left( \frac{\Gamma(k+(\beta+1)/2)}{k!\Gamma((\beta+1)/2)} \right)^2 r^{2k}$$

for $0 \leq r < 1$. By log-convexity of the Gamma function we have that $\Gamma(k+(\beta+1)/2)^2 \leq k!\Gamma(k+\beta)$ for $k \geq 0$ (see [**6**, Theorem 2.1]). We now conclude that

$$\frac{1}{2\pi}\int_{\mathbb{T}}\frac{d\theta}{|1-re^{i\theta}|^{\beta+1}}\leq\sum_{k=0}^{\infty}\frac{\Gamma(k+\beta)}{k!\Gamma((\beta+1)/2)^2}r^{2k}=\frac{\Gamma(\beta)}{\Gamma((\beta+1)/2)^2}\cdot\frac{1}{(1-r^2)^\beta}$$

for $0\leq r<1$, where the last equality follows by (1.13). □

REMARK 2.4. Note that the log-convexity estimate in the proof of Lemma 2.3 is sharp in the limit in the sense that

$$\lim_{k\to\infty}\frac{\Gamma(k+(\beta+1)/2)^2}{k!\Gamma(k+\beta)}=1 \tag{2.1}$$

which follows by a calculation using Stirling's formula

$$\Gamma(z)=\sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right)$$

as $\operatorname{Re} z\to\infty$ (see [**5**, Section 1.4]).

Note also that for $\beta=1$ we have equality in the estimate in Lemma 2.3 for all $0\leq r<1$. We next calculate $P_\alpha$.

THEOREM 2.5. *Let $\alpha>-1$. Then the Poisson kernel $P_\alpha$ is given by the formula*

$$P_\alpha(z)=\frac{(1-|z|^2)^{\alpha+1}}{(1-z)(1-\bar{z})^{\alpha+1}},\quad z\in\mathbb{D}.$$

PROOF. Consider the function

$$u(z)=\frac{(1-|z|^2)^{\alpha+1}}{(1-z)(1-\bar{z})^{\alpha+1}},\quad z\in\mathbb{D}.$$

In order to prove that $u=P_\alpha$, we first show that $u$ is $\alpha$-harmonic. By straightforward differentiation we have

$$\bar{\partial}u(z)=(\alpha+1)\frac{(1-|z|^2)^{\alpha}}{(1-\bar{z})^{\alpha+2}},\quad z\in\mathbb{D}.$$

By this formula it is clear that the function $w_\alpha^{-1}\bar{\partial}u$ is conjugate-analytic. Therefore $u$ is $\alpha$-harmonic.

We next show that

$$\frac{1}{2\pi}\int_{\mathbb{T}} u(re^{i\theta})d\theta = 1$$

for $0 \leq r < 1$. Recall the power series expansion (1.13). An application of Parseval's formula now yields

$$\frac{1}{2\pi}\int_{\mathbb{T}} u(re^{i\theta})d\theta = (1-r^2)^{\alpha+1}\sum_{k=0}^{\infty}\binom{k+\alpha}{k}r^{2k} = 1$$

for $0 \leq r < 1$, where the last equality again follows by (1.13).

By Lemma 2.3 we have the $L^1$-bound

$$\frac{1}{2\pi}\int_{\mathbb{T}} |u(re^{i\theta})|d\theta \leq \Gamma(\alpha+1)/\Gamma(\alpha/2+1)^2$$

for $0 \leq r < 1$, where $\Gamma$ is the Gamma function. By Helly's theorem there exists a sequence $r_j \to 1$ and a measure $\mu$ such that $u_{r_j} \to \mu$ in the weak$^*$ topology of measures, where the function $u_r$ is given by (0.5) for $0 \leq r < 1$. Next consider such a weak$^*$ limit point $u_{r_j} \to \mu$. Since $u_r \to 0$ uniformly away from the singleton set $\{1\}$ we find that $\mu = c\delta_1$ for some constant $c$ by a standard distribution theoretical argument (see Hörmander [**21**, Theorem 2.3.4]). The result of the previous paragraph then implies that $c = 1$. By this we conclude that $u_r \to \delta_1$ as $r \to 1$ in the weak$^*$ topology of measures. In particular, $u_r \to \delta_1$ as $r \to 1$ in $\mathcal{D}'(\mathbb{T})$.

We now show that $u = P_\alpha$. Recall the power series expansion (1.11) provided by Theorem 1.2. Note in particular that

$$u_r(e^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k e_{\alpha,k}(r)e^{ik\theta}, \quad e^{i\theta} \in \mathbb{T},$$

for $0 \leq r < 1$ in view of the homogeneity of the $e_{\alpha,k}$'s. A calculation of Fourier coefficients gives that $\hat{u}_r(k) = c_k e_{\alpha,k}(r)$ for $k \in \mathbb{Z}$ and $0 \leq r < 1$. Letting $r \to 1$ we see that $c_k = 1$ for all $k \in \mathbb{Z}$ since $u_r \to \delta_1$ in $\mathcal{D}'(\mathbb{T})$. By (1.11) we conclude that $u = P_\alpha$. This completes the proof. $\square$

The following structure formula is of some interest.

PROPOSITION 2.6. *Let $\alpha > -1$. Then*

$$P_\alpha(z) = \frac{1}{1-z} + \frac{\bar{z}}{1-\bar{z}} f_\alpha\left(\frac{1-|z|^2}{1-\bar{z}}\right), \quad z \in \mathbb{D},$$

*where* $f_\alpha(w) = (1 - w^{\alpha+1})/(1 - w)$ *using the principal branch of the logarithm.*

PROOF. Use Theorem 2.5 and decompose the function $P_\alpha$ as

$$P_\alpha(z) = \frac{1}{1-z} + \frac{1}{1-z}\left(\left(\frac{1-|z|^2}{1-\bar{z}}\right)^{\alpha+1} - 1\right), \quad z \in \mathbb{D}.$$

Note that

$$\frac{1-|z|^2}{1-\bar{z}} - 1 = \frac{\bar{z}(1-z)}{1-\bar{z}} \tag{2.2}$$

which follows by straightforward calculation. By this observation we get

$$\begin{aligned} P_\alpha(z) &= \frac{1}{1-z} + \frac{1}{1-z}\left(\frac{1-|z|^2}{1-\bar{z}} - 1\right) f_\alpha\left(\frac{1-|z|^2}{1-\bar{z}}\right) \\ &= \frac{1}{1-z} + \frac{\bar{z}}{1-\bar{z}} f_\alpha\left(\frac{1-|z|^2}{1-\bar{z}}\right), \end{aligned}$$

where the last equality follows by cancellation using (2.2). □

We remark that for $\alpha = 0$ the result of Proposition 2.6 simplifies to the well-known formula

$$\frac{1-|z|^2}{|1-z|^2} = \frac{1}{1-z} + \frac{\bar{z}}{1-\bar{z}}$$

for the classical Poisson kernel.

We next show that the estimate of Lemma 2.3 is sharp in the limit as $r \to 1$.

PROPOSITION 2.7. *Let* $\beta > 0$. *Then*

$$\lim_{r\to 1} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{(1-r^2)^\beta}{|1-re^{i\theta}|^{\beta+1}} d\theta = \frac{\Gamma(\beta)}{\Gamma((\beta+1)/2)^2},$$

*where* $\Gamma$ *is the Gamma function.*

PROOF. By Lemma 2.3 it suffices to show that

$$\liminf_{r\to 1} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{(1-r^2)^\beta}{|1-re^{i\theta}|^{\beta+1}} d\theta \geq \frac{\Gamma(\beta)}{\Gamma((\beta+1)/2)^2}. \tag{2.3}$$

Moreover, as in the proof of Lemma 2.3 we have that

$$\frac{1}{2\pi}\int_{\mathbb{T}}\frac{(1-r^2)^\beta}{|1-re^{i\theta}|^{\beta+1}}d\theta=(1-r^2)^\beta\sum_{k=0}^{\infty}\left(\frac{\Gamma(k+(\beta+1)/2)}{k!\Gamma((\beta+1)/2)}\right)^2 r^{2k} \tag{2.4}$$

for $0\le r<1$ by Parseval's formula and (1.13).

Let $0<c<1$. By (2.1) there exists $k_c\ge 0$ such that $\Gamma(k+(\beta+1)/2)^2\ge ck!\Gamma(k+\beta)$ for all $k\ge k_c$. Now by (2.4) we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}}\frac{(1-r^2)^\beta}{|1-re^{i\theta}|^{\beta+1}}d\theta &\ge c(1-r^2)^\beta\frac{\Gamma(\beta)}{\Gamma((\beta+1)/2)^2}\sum_{k=k_c}^{\infty}\frac{\Gamma(k+\beta)}{k!\Gamma(\beta)}r^{2k}\\ &=c(1-r^2)^\beta\frac{\Gamma(\beta)}{\Gamma((\beta+1)/2)^2}\left(\frac{1}{(1-r^2)^\beta}-\sum_{k=0}^{k_c-1}\frac{\Gamma(k+\beta)}{k!\Gamma(\beta)}r^{2k}\right)\end{aligned}$$

for $0\le r<1$, where the last equality follows by (1.13). Passing to the limit as $r\to 1$ we get

$$\liminf_{r\to 1}\frac{1}{2\pi}\int_{\mathbb{T}}\frac{(1-r^2)^\beta}{|1-re^{i\theta}|^{\beta+1}}d\theta\ge\frac{c\,\Gamma(\beta)}{\Gamma((\beta+1)/2)^2}.$$

Letting $c\to 1$, the inequality (2.3) follows. □

We now return to the function $P_\alpha$.

Corollary 2.8. *Let $\alpha>-1$. For $0\le r<1$ we have*

$$\frac{1}{2\pi}\int_{\mathbb{T}}|P_\alpha(re^{i\theta})|d\theta\le\frac{\Gamma(\alpha+1)}{\Gamma((\alpha+2)/2)^2}\quad\textit{and}$$
$$\lim_{r\to 1}\frac{1}{2\pi}\int_{\mathbb{T}}|P_\alpha(re^{i\theta})|d\theta=\frac{\Gamma(\alpha+1)}{\Gamma((\alpha+2)/2)^2},$$

*where $\Gamma$ is the Gamma function.*

Proof. If we set $z=re^{i\theta}$, then the two statements immediately follow from Theorem 2.5 together with Lemma 2.3 and Proposition 2.7, respectively. □

Let us comment on the size of the $L^1$-means bound of $P_\alpha$ from Corollary 2.8.

Proposition 2.9. *Consider the function*

$$M(\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma((\alpha+2)/2)^2}, \quad \alpha > -1,$$

*where* $\Gamma$ *is the Gamma function. Then the function* $M$ *is strictly decreasing on* $(-1,0)$*, strictly increasing on* $(0,\infty)$ *and* $M(0)=1$*. Furthermore,*

$$\lim_{\alpha\to-1}(\alpha+1)M(\alpha) = \frac{1}{\pi} \quad \textit{and} \quad \lim_{\alpha\to\infty}\frac{\sqrt{\alpha}}{2^\alpha}M(\alpha) = \sqrt{\frac{2}{\pi}}.$$

PROOF. It is evident that $M(0)=1$. Let us check the monotonicity properties of the function $M$. Recall the formula

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{z+k}\right)$$

for the logarithmic derivative of the Gamma function (see Artin [**6**, formula (2.10)]); here $\gamma$ is Euler's constant. A straightforward calculation using this formula gives that

$$\frac{d}{d\alpha}\log M(\alpha) = \alpha\sum_{k=0}^{\infty}\frac{1}{(\alpha+1+k)(\alpha+2+2k)}$$

for $\alpha > -1$. Analyzing the sign of the logarithmic derivative of $M$, the asserted monotonicity properties of $M$ are evident.

Let us turn to the asymptotic behavior of $M$. Using the functional equation for the Gamma function we have that

$$\lim_{\alpha\to-1}(\alpha+1)M(\alpha) = \lim_{\alpha\to-1}\Gamma(\alpha+2)/\Gamma(\alpha/2+1)^2 = \Gamma(1)/\Gamma(1/2)^2 = 1/\pi$$

since $\Gamma(1/2) = \sqrt{\pi}$.

Consider next the asymptotics of $M(\alpha)$ as $\alpha\to\infty$. Recall the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+1/2)$$

for the Gamma function (see [**5**, Section 1.5]). A straightforward calculation using this formula gives

$$M(\alpha) = \frac{2^\alpha}{\sqrt{\pi}} \frac{\Gamma((\alpha+1)/2)}{\Gamma((\alpha+2)/2)}$$

for $\alpha > -1$. An application of Stirling's formula now gives the asymptotics of $M(\alpha)$ as $\alpha \to \infty$ (see Remark 2.4). □

## 3. Distributions and homogeneous Banach spaces.

The purpose of this section is to recall some basic facts from distribution theory and homogeneous Banach spaces. A standard reference for homogeneous Banach spaces is Katznelson [**23**]; see also Shapiro [**31**, Chapter 9]. A standard reference for distribution theory is Hörmander [**21**]; see also [**23**].

A distribution $u \in \mathcal{D}'^m(\mathbb{T})$ on $\mathbb{T}$ of order less than or equal to $m \geq 0$ is a linear form $u$ on $C^\infty(\mathbb{T})$ such that

$$|\langle u, \varphi \rangle| \leq C\|\varphi\|_{C^m}, \quad \varphi \in C^\infty(\mathbb{T}),$$

where $\|\cdot\|_{C^m}$ is the norm of $m$ times continuously differentiable functions on $\mathbb{T}$. A function $f \in L^1(\mathbb{T})$ is identified with the distribution of order 0 given by

$$\langle f, \varphi \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(e^{i\theta}) f(e^{i\theta}) d\theta, \quad \varphi \in C^\infty(\mathbb{T}).$$

Note that the space $\mathcal{D}'^m(\mathbb{T})$ is naturally identified with the dual of $C^m(\mathbb{T})$. We identify $\mathcal{D}'^0(\mathbb{T})$ with the space $M(\mathbb{T})$ of complex measures on $\mathbb{T}$ equipped with the norm of total variation of measures which is natural in view of a classical result going back to F. Riesz (see [**30**, Chapters 2 and 6]).

A distribution $u \in \mathcal{D}'(\mathbb{T})$ on $\mathbb{T}$ is an element $u$ in $\mathcal{D}'^m(\mathbb{T})$ for some $m \geq 0$, that is, $\mathcal{D}'(\mathbb{T}) = \bigcup_{m \geq 0} \mathcal{D}'^m(\mathbb{T})$. The space of distributions $\mathcal{D}'(\mathbb{T})$ is topologized by means of the semi-norms

$$\mathcal{D}'(\mathbb{T}) \ni u \mapsto |\langle u, \varphi \rangle|$$

for $\varphi \in C^\infty(\mathbb{T})$. Note that $u_j \to u$ in $\mathcal{D}'(\mathbb{T})$ means that $\langle u_j, \varphi \rangle \to \langle u, \varphi \rangle$ for every $\varphi \in C^\infty(\mathbb{T})$.

By a Banach space of distributions on $\mathbb{T}$ we mean a Banach space $X$ which is continuously embedded in $\mathcal{D}'(\mathbb{T})$. Note that this continuity requirement means that if $u_j \to u$ in $X$ then $u_j \to u$ in $\mathcal{D}'(\mathbb{T})$. The following lemma is convenient to have available.

Lemma 3.1. *Let $X$ be a Banach space of distributions on $\mathbb{T}$. Then there* *exists an integer* $m \geq 0$ *such that* $X \subset \mathcal{D}'^m(\mathbb{T})$ *and*

$$\|f\|_{\mathcal{D}'^m} \leq C\|f\|_X, \quad f \in X,$$

*where* $\|\cdot\|_{\mathcal{D}'^m}$ *is the norm on the dual of* $C^m(\mathbb{T})$.

PROOF. Set

$$X_{m,N} = \{f \in X : \ f \in \mathcal{D}'^m(\mathbb{T}) \text{ and } \|f\|_{\mathcal{D}'^m} \leq N\}$$

for integers $m \geq 0$ and $N \geq 1$. Since every $f \in X$ is a distribution of finite order, it is clear that $X = \bigcup_{m,N} X_{m,N}$. It is straightforward to check that every set $X_{m,N}$ is closed in $X$. An application of Baire's category theorem yields the existence of numbers $m$ and $N$ such that the set $X_{m,N}$ has an interior point $f_0$, that is, there is an $\varepsilon > 0$ such that $f \in X$ and $\|f - f_0\|_X < \varepsilon$ implies $f \in X_{m,N}$.

The proof is completed by a symmetrization argument. For $f \in X$ with $\|f\|_X < 1$, we have $2\varepsilon f = (f_0 + \varepsilon f) - (f_0 - \varepsilon f)$ and an estimation gives

$$2\varepsilon\|f\|_{\mathcal{D}'^m} \leq \|f_0 + \varepsilon f\|_{\mathcal{D}'^m} + \|f_0 - \varepsilon f\|_{\mathcal{D}'^m} \leq 2N,$$

showing that the conclusion of the lemma holds with constant $C = N/\varepsilon$. □

Let us recall some more constructions from distribution theory. The Fourier coefficients of a distribution $u \in \mathcal{D}'(\mathbb{T})$ are defined by

$$\hat{u}(k) = \langle u, \phi_{-k}\rangle, \quad k \in \mathbb{Z},$$

where the $\phi_k$'s are given by (1.8). It is well-known that the Fourier series $\sum \hat{u}(k)\phi_k$ converges to $u$ in $\mathcal{D}'(\mathbb{T})$ in the usual sense, thus

$$\langle u, \varphi\rangle = \sum_{k=-\infty}^{\infty} \hat{u}(k)\hat{\varphi}(-k), \quad \varphi \in C^\infty(\mathbb{T}).$$

A trigonometric series $\sum c_k\phi_k$ is the Fourier series for some $u \in \mathcal{D}'(\mathbb{T})$ if and only if the sequence $\{c_k\}_{k\in\mathbb{Z}}$ of coefficients is of polynomial growth, that is, $|c_k| \leq C(1 + |k|)^N$, $k \in \mathbb{Z}$, for some positive constants $C$ and $N$.

The convolution of two distributions $u$ and $v$ in $\mathcal{D}'(\mathbb{T})$ is the distribution $u * v$ in $\mathcal{D}'(\mathbb{T})$ uniquely determined by the property that

$$(u * v)\hat{\ }(k) = \hat{u}(k)\hat{v}(k), \quad k \in \mathbb{Z}. \tag{3.1}$$

The translation of $u \in \mathcal{D}'(\mathbb{T})$ by $e^{i\theta} \in \mathbb{T}$ is the distribution $T_{e^{i\theta}}u = u_{e^{i\theta}}$ uniquely determined by the property that $(u_{e^{i\theta}})\hat{}(k) = e^{-ik\theta}\hat{u}(k)$ for $k \in \mathbb{Z}$. Note that

$$\langle u_{e^{i\theta}}, \varphi\rangle = \langle u, \varphi_{e^{-i\theta}}\rangle, \quad \varphi \in C^\infty(\mathbb{T}),$$

where $\varphi_{e^{i\theta}}(e^{i\tau}) = \varphi(e^{i(\tau-\theta)})$ for $e^{i\tau} \in \mathbb{T}$. Note also that $u_{e^{i\theta}} = u * \delta_{e^{i\theta}}$, where $\delta_{e^{i\theta}}$ is the unit Dirac measure at $e^{i\theta} \in \mathbb{T}$.

Recall from the introduction the definition of a homogeneous Banach space. Such spaces enjoy the following approximation property.

PROPOSITION 3.2. *Let $B$ be a homogeneous Banach space on $\mathbb{T}$. If $\mu \in M(\mathbb{T})$ is a complex measure and $f \in B$, then $\mu * f \in B$ and $\|\mu * f\|_B \le \|\mu\|\|f\|_B$, where $\|\mu\|$ is the norm of total variation of $\mu$. Furthermore, if $\mu_n \to \mu$ in the weak* topology of measures and $f \in B$, then $\mu_n * f \to \mu * f$ in $B$.*

The result of Proposition 3.2 is well-known (see Katznelson [**23**, Chapter I]). In particular, we have that $\sigma_n(f) = K_n * f \to f$ in $B$ if $f \in B$, where

$$K_n(e^{i\theta}) = \frac{1}{n+1}\left(\frac{\sin((n+1)\theta/2)}{\sin(\theta/2)}\right)^2 = \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ik\theta}, \quad e^{i\theta} \in \mathbb{T}, \tag{3.2}$$

for $n = 1, 2, \dots$ are the Fejér kernels.

For a Banach space $X$ of distributions on $\mathbb{T}$ we let

$$\operatorname{Spec} X = \{k \in \mathbb{Z} : \hat{f}(k) \neq 0 \text{ for some } f \in X\} \tag{3.3}$$

denote its Fourier spectrum. Note that if $B$ is a homogeneous Banach space then

$$\operatorname{Spec} B = \{k \in \mathbb{Z} : \ \phi_k \in B\}$$

by Proposition 3.2. Indeed, if $f \in B$, then $\hat{f}(k)\phi_k = f * \phi_k$ belongs to $B$ which proves the assertion. Note also that a trigonometric polynomial $p$ belongs to $B$ if and only if $\operatorname{supp}\hat{p} \subset \operatorname{Spec} B$. Here $\operatorname{supp}\hat{f} = \{k \in \mathbb{Z} : \ \hat{f}(k) \neq 0\}$ is the support of the Fourier transform $\hat{f}$.

Let $B$ be a homogeneous Banach space on $\mathbb{T}$ and consider its dual $B^*$ consisting of all continuous linear functionals on $B$ normed in the usual way. We set

$$\operatorname{Spec} B^* = \{-k : \ k \in \operatorname{Spec} B\}.$$

The Fourier coefficients of a continuous linear functional $\ell \in B^*$ are defined by

$$\hat{\ell}(k) = \begin{cases} \ell(\phi_{-k}) & \text{if } k \in \operatorname{Spec} B^*, \\ 0 & \text{if } k \in \mathbb{Z} \setminus \operatorname{Spec} B^*. \end{cases}$$

In view of Proposition 3.2, the action of $\ell$ is recovered from its Fourier coefficients by

$$\ell(f) = (f, \ell) = \lim_{n\to\infty} \sum_{k=-n}^{n} \Big(1 - \frac{|k|}{n+1}\Big) \hat{f}(k)\hat{\ell}(-k), \quad f \in B, \tag{3.4}$$

using Cesàro summation, where the symbol $(\cdot,\cdot)$ indicates dual pairing. We can consider a trigonometric polynomial $p$ with $\operatorname{supp}\hat{p} \subset \operatorname{Spec} B^*$ as an element in $B^*$ by defining

$$p(f) = \langle f, p\rangle = \sum_{k\in \operatorname{Spec} B} \hat{f}(k)\hat{p}(-k), \quad f \in B.$$

By (3.4) it is clear that the space of trigonometric polynomials $p$ with $\operatorname{supp}\hat{p} \subset \operatorname{Spec} B^*$ forms a weak$^*$ dense subspace of $B^*$. We denote by $(B^*)_c$ the norm closure in $B^*$ of the trigonometric polynomials $p$ with $\operatorname{supp}\hat{p} \subset \operatorname{Spec} B^*$. In view of the above considerations it is natural to associate to an element $\ell \in (B^*)_c$ its Fourier series

$$\ell \sim \sum_{k\in \operatorname{Spec} B^*} \hat{\ell}(k)\phi_k. \tag{3.5}$$

We point out that the formal series in (3.5) is in general not the Fourier series of a distribution, see Proposition 4.13 below.

The group of translation operators $T : \mathbb{T} \ni e^{i\theta} \mapsto T_{e^{i\theta}} \in \mathcal{L}(B)$ induces translation operators on $B^*$ by

$$(T_{e^{i\theta}}\ell)(f) = (T_{e^{-i\theta}}f, \ell), \quad f \in B,$$

for $\ell \in B^*$ and $e^{i\theta} \in \mathbb{T}$ using transposed action. Note that translation invariance of $B$ gives translation invariance of $B^*$ in the sense that $\|T_{e^{i\theta}}\ell\|_{B^*} = \|\ell\|_{B^*}$ for $\ell \in B^*$ and $e^{i\theta} \in \mathbb{T}$. Note also that this group action $T : \mathbb{T} \ni e^{i\theta} \mapsto T_{e^{i\theta}} \in \mathcal{L}(B^*)$ has the continuity property that for every $\ell \in B^*$ the map $\mathbb{T} \ni e^{i\theta} \mapsto T_{e^{i\theta}}\ell$ is continuous in the weak$^*$ topology on $B^*$ inherited from $B$. We mention also that the space $(B^*)_c$ admits the alternative description

$$(B^*)_c = \Big\{ \ell \in B^* : \lim_{e^{i\theta}\to 1} \|T_{e^{i\theta}}\ell - \ell\|_{B^*} = 0 \Big\}$$

using the group of translations (see Katznelson [**23**, Chapter I] for details).

## 4. The relative completion.

Let $X$ be a Banach space of distributions on $\mathbb{T}$. We say that the space $X$ is translation invariant if it has the property that $T_{e^{i\theta}} f \in X$ and $\|T_{e^{i\theta}} f\|_X = \|f\|_X$ if $f \in X$ and $e^{i\theta} \in \mathbb{T}$, where the translation operators $T : \mathbb{T} \ni e^{i\theta} \mapsto T_{e^{i\theta}}$ are defined in the usual way in the sense of distribution theory (see Section 3). We say that the space $X$ is relatively complete if it has the property, henceforth referred to as the relative completeness property, that

(i) $f_n \in X$, $\|f_n\|_X \leq C$ for $n \geq 1$ and $f_n \to f$ in $\mathcal{D}'(\mathbb{T})$ implies $f \in X$ and $\|f\|_X \leq C$.

By a relatively complete translation invariant Banach space of distributions on $\mathbb{T}$ we mean a Banach space of distributions on $\mathbb{T}$ which is both translation invariant and relatively complete in the sense explained above.

REMARK 4.1. Note that if $X$ is a Banach space of distributions on $\mathbb{T}$, then $X$ is relatively complete if and only if

(i)′ $f_n \in X$, $f_n \to f$ in $\mathcal{D}'(\mathbb{T})$ and $\liminf_{n\to\infty} \|f_n\|_X < \infty$ implies $f \in X$ and $\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X$.

We leave the straightforward proof of this claim to the reader.

We first show that a homogeneous Banach space can always be isometrically embedded in a relatively complete translation invariant Banach space of distributions on $\mathbb{T}$. As a matter of convenience we use the Cesàro summability method: $\sigma_n(f) = K_n * f$, where $K_n$ is the Fejér kernel given by (3.2).

THEOREM 4.2. *Let $B$ be a homogeneous Banach space on $\mathbb{T}$. Set*

$$X = \{f \in \mathcal{D}'(\mathbb{T}) : \ \sigma_n(f) \in B \textit{ for } n \geq 1 \textit{ and } \sup_{n\geq 1} \|\sigma_n(f)\|_B < \infty\},$$

*and equip the space $X$ with the norm*

$$\|f\|_X = \sup_{n\geq 1} \|\sigma_n(f)\|_B, \quad f \in X.$$

*Then the space $X$ is a relatively complete translation invariant Banach space of distributions on $\mathbb{T}$ such that $B$ embeds isometrically in $X$ and the Fourier spectrums of $B$ and $X$ coincide,* $\operatorname{Spec} B = \operatorname{Spec} X$.

Proof. It is easy to see that the map $X \ni f \mapsto \{\sigma_n(f)\}$ embeds $X$ isometrically as a closed subspace of $\ell^\infty(B)$. It is also straightforward to check that if $B$ embeds in $\mathcal{D}'^m(\mathbb{T})$, then $X$ also embeds in $\mathcal{D}'^m(\mathbb{T})$. This shows that $X$ is a Banach space of distributions on $\mathbb{T}$. The translation invariance of $X$ is inherited from the translation invariance of $B$ by the choice of norm. An application of Proposition 3.2 shows that the space $B$ embeds isometrically in $X$. It is straightforward to check that $\operatorname{Spec} B = \operatorname{Spec} X$.

We proceed to check the relative completeness property of $X$. For this purpose let $\{f_k\}$ be a sequence in $X$ such that $\|f_k\|_X \le C$ for $k \ge 1$ and $f_k \to f$ in $\mathcal{D}'(\mathbb{T})$. Now $\sigma_n(f_k) \in B$ for all $n, k \ge 1$ and it is easy to see that $\sigma_n(f_k) \to \sigma_n(f)$ in $B$ as $k \to \infty$ since only finitely many nonzero Fourier coefficients are involved. This gives that $\sigma_n(f) \in B$ for $n \ge 1$ and $\|\sigma_n(f)\|_B = \lim_k \|\sigma_n(f_k)\|_B \le C$. Thus $f \in X$ and $\|f\|_X \le C$. □

We proceed to discuss uniqueness properties of the space $X$ constructed in Theorem 4.2. First we need a lemma.

Lemma 4.3. *Let $\mu \in M(\mathbb{T})$ be a complex measure. Then there exists a sequence $\{\mu_n\}_{n=1}^\infty$ of discrete measures such that $\mu_n \to \mu$ in the weak$^*$ topology of measures and $\|\mu_n\| \le \|\mu\|$ for $n \ge 1$, where $\|\cdot\|$ is the norm of total variation.*

Proof. Let $n \ge 1$ be a positive integer and divide the circle $\mathbb{T}$ into $n$ disjoint half-open arcs

$$I_{n,k} = \{e^{i\theta} \in \mathbb{T} : \; 2\pi k/n \le \theta < 2\pi(k+1)/n\}$$

for $0 \le k \le n-1$. Consider next the discrete measures

$$\mu_n = \sum_{k=0}^{n-1} \mu(I_{n,k}) \delta_{e^{i2\pi k/n}}$$

for $n = 1, 2, \dots$, where $\delta_{e^{i\tau}}$ is a unit Dirac mass at $e^{i\tau} \in \mathbb{T}$. It is straightforward to check using uniform continuity that $\int_{\mathbb{T}} \varphi d\mu_n \to \int_{\mathbb{T}} \varphi d\mu$ for $\varphi \in C(\mathbb{T})$. The total variation estimate $\|\mu_n\| \le \|\mu\|$ for $n \ge 1$ is evident by construction. □

We next consider convolution with a general measure.

Theorem 4.4. *Let $X$ be a relatively complete translation invariant Banach space of distributions on $\mathbb{T}$. If $\mu \in M(\mathbb{T})$ is a complex measure and $f \in X$, then the convolution $\mu * f$ belongs to $X$ and the norm inequality $\|\mu * f\|_X \leq \|\mu\|\|f\|_X$ holds.*

Proof. Assume first that $\mu \in M(\mathbb{T})$ is discrete, $\mu = \sum_{k=1}^n a_k \delta_{e^{i\theta_k}}$. Then the convolution

$$\mu * f = \sum_{k=1}^n a_k T_{e^{i\theta_k}} f$$

belongs to $X$ and

$$\|\mu * f\|_X \leq \sum_{k=1}^n |a_k| \|T_{e^{i\theta_k}} f\|_X = \left( \sum_{k=1}^n |a_k| \right) \|f\|_X = \|\mu\|\|f\|_X$$

by the triangle inequality using translation invariance.

Consider next the case when $\mu \in M(\mathbb{T})$ is a general complex measure. By Lemma 4.3 there exists a sequence $\{\mu_n\}_{n=1}^\infty$ of discrete measures such that $\mu_n \to \mu$ in the weak* topology of measures and $\|\mu_n\| \leq \|\mu\|$ for $n \geq 1$. By the first part of the proof we have that $\mu_n * f \in X$ and $\|\mu_n * f\|_X \leq \|\mu\|\|f\|_X$. Since $\mu_n * f \to \mu * f$ in $\mathcal{D}'(\mathbb{T})$ and $X$ is relatively complete, we conclude that $\mu * f \in X$ and $\|\mu * f\|_X \leq \|\mu\|\|f\|_X$. □

Note that if $X$ is as in Theorem 4.4, then

$$\operatorname{Spec} X = \{k \in \mathbb{Z} : \phi_k \in X\}.$$

Indeed, if $f \in X$, then $\hat{f}(k)\phi_k = f * \phi_k$ belongs to $X$ which proves the assertion. Note also that a trigonometric polynomial $p$ belongs to $X$ if and only if $\operatorname{supp} \hat{p} \subset \operatorname{Spec} X$.

The following lemma will be used to calculate norms in the proof of Theorem 4.6 below.

Lemma 4.5. *Let $X$ be as in Theorem* 4.4. *Then*

$$\lim_{n\to\infty} \|\sigma_n(f)\|_X = \|f\|_X, \quad f \in X.$$

Proof. By Remark 4.1 we have that $\|f\|_X \leq \liminf_{n\to\infty} \|\sigma_n(f)\|_X$. Since $\sigma_n(f) = K_n * f$, we have by Theorem 4.4 that $\|\sigma_n(f)\|_X \leq \|K_n\|_{L^1} \|f\|_X = \|f\|_X$. This gives that $\limsup_{n\to\infty} \|\sigma_n(f)\|_X \leq \|f\|_X$. □

We can now prove the following uniqueness property of the space constructed in Theorem 4.2.

THEOREM 4.6. *Let $X_1$ and $X_2$ be relatively complete translation invariant Banach spaces of distributions on $\mathbb{T}$ such that* $\operatorname{Spec} X_1 = \operatorname{Spec} X_2$. *Assume that*

$$\|p\|_{X_1} = \|p\|_{X_2}$$

*for every trigonometric polynomial $p$ with* $\operatorname{supp} \hat{p} \subset \operatorname{Spec} X_1 = \operatorname{Spec} X_2$. *Then $X_1 = X_2$ with equality of norms.*

PROOF. By symmetry, we only have to show that $X_1$ is isometrically contained in $X_2$. To this end, let $f \in X_1$. By the spectral assumption, the Cesàro means $\sigma_n(f) = K_n * f$ belong to $X_j$ for $j = 1, 2$ and

$$\|\sigma_n(f)\|_{X_1} = \|\sigma_n(f)\|_{X_2}, \quad n = 1, 2, 3 \dots. \tag{4.1}$$

An application of Lemma 4.5 yields $\|f\|_{X_1} = \lim_{n\to\infty} \|\sigma_n(f)\|_{X_1}$, which implies that $\{\sigma_n(f)\}$ is a bounded sequence in $X_2$ in view of (4.1). Since $\sigma_n(f) \to f$ in $\mathcal{D}'(\mathbb{T})$, it follows by relative completeness of $X_2$ that $f \in X_2$. Thus $X_1 \subset X_2$ as sets. Moreover, applying Lemma 4.5 also to $f \in X_2$ yields $\|f\|_{X_1} = \|f\|_{X_2}$ in view of (4.1). This completes the proof. $\square$

We next define the relative completion mentioned in the introduction.

DEFINITION 4.7. Let $B$ be a homogeneous Banach space on $\mathbb{T}$. The *relative completion* $\tilde{B}$ of $B$ is the relatively complete translation invariant Banach space of distributions on $\mathbb{T}$ containing $B$ isometrically such that $\operatorname{Spec} \tilde{B} = \operatorname{Spec} B$.

By Theorems 4.2 and 4.6 this notion of relative completion exists and is uniquely determined by its defining properties. Note also that with this terminology a homogeneous Banach space $B$ is relatively complete if and only if it is equal to its own relative completion, that is, $B = \tilde{B}$.

Some easy examples of relative completions are: $C(\mathbb{T})^\sim = L^\infty(\mathbb{T})$, $L^1(\mathbb{T})^\sim = M(\mathbb{T})$, $L^p(\mathbb{T})^\sim = L^p(\mathbb{T})$ for $1 < p < \infty$, and $\operatorname{lip}_\alpha(\mathbb{T})^\sim = \operatorname{Lip}_\alpha(\mathbb{T})$ for $0 < \alpha \le 1$ (Lipschitz/Hölder spaces) as follows by the Ascoli-Arzelà theorem.

As a consequence of the isometric embedding of $B$ into $\tilde{B}$, the relative completion as defined originally in Gagliardo [**11**] and further studied in Braun and Feichtinger [**8**] coincides with our notion when $B$ is a homogeneous Banach space.

Let $X$ be a translation invariant Banach space of distributions on $\mathbb{T}$ and consider the subspace

$$X_c = \Big\{ f \in X : \lim_{e^{i\theta} \to 1} \|T_{e^{i\theta}} f - f\|_X = 0 \Big\}.$$

It is straightforward to check that the subspace $X_c$ of $X$ is a homogeneous Banach space on $\mathbb{T}$ (see Katznelson [**23**, Chapter I] for details).

COROLLARY 4.8. *If $X$ is a relatively complete translation invariant Banach space of distributions on $\mathbb{T}$, then $X$ is the relative completion of $X_c$, that is, $X = (X_c)\tilde{}$. Also, if $B$ is a homogeneous Banach space on $\mathbb{T}$, then $B = (\tilde{B})_c$.*

PROOF. For the first part note that $\operatorname{Spec} X = \operatorname{Spec} X_c$ and apply Theorem 4.6. Since $\operatorname{Spec} B = \operatorname{Spec} \tilde{B}$, the second assertion follows by denseness of trigonometric polynomials in homogeneous Banach spaces. □

Operators commuting with translations are usually referred to as Fourier multipliers or convolution operators. The following proposition is included for the sake of completeness.

PROPOSITION 4.9. *Let $B$ be a homogeneous Banach space on $\mathbb{T}$. A bounded operator $M \in \mathcal{L}(L^1(\mathbb{T}), B)$ commutes with translations if and only if it has the form $M = M_f$ for some $f \in \tilde{B}$, where*

$$M_f g = f * g, \quad g \in L^1(\mathbb{T}).$$

*The operator $M_f$ has norm $\|M_f\| = \|f\|_{\tilde{B}}$ and is compact if and only if $f \in B$.*

PROOF. Let $f \in \tilde{B}$ and let $p$ be a trigonometric polynomial. Note that $f * p \in B$ since $f * p$ is a trigonometric polynomial and $\operatorname{supp}(f * p)\hat{} \subset \operatorname{supp} \hat{f} \subset \operatorname{Spec} B$. Also, by Theorem 4.4 we have the inequality $\|f*p\|_B \le \|f\|_{\tilde{B}} \|p\|_{L^1}$ since $B$ is isometrically contained in $\tilde{B}$. By an approximation argument we conclude that the operator $M_f$ is bounded from $L^1(\mathbb{T})$ into $B$ and $\|M_f\| \le \|f\|_{\tilde{B}}$ if $f \in \tilde{B}$. It is evident that the operator $M_f$ commutes with translations since it is a convolution operator.

Assume next that $M \in \mathcal{L}(L^1(\mathbb{T}), B)$ is a bounded operator that commutes with translations. Note first that $T_{e^{i\theta}} \phi_k = e^{-ik\theta} \phi_k$ for $e^{i\theta} \in \mathbb{T}$. Applying the operator $M$ we have that $T_{e^{i\theta}} M \phi_k = M T_{e^{i\theta}} \phi_k = e^{-ik\theta} M \phi_k$ for $e^{i\theta} \in \mathbb{T}$. A straightforward calculation gives that $M\phi_k = m_k \phi_k$ for some $m_k \in \mathbb{C}$. Note also that $|m_k| \le \|M\|$ for $k \in \mathbb{Z}$. By a standard approximation argument we conclude that the operator $M$ has the form

$$Mg = f * g, \quad g \in L^1(\mathbb{T}),$$

where $f = \sum m_k \phi_k$ in $\mathcal{D}'(\mathbb{T})$. Choosing $g = K_n$ as the Fejér kernel, we have that $\sigma_n(f) \in B$ and $\|\sigma_n(f)\|_B \le \|M\|$ for $n \ge 1$. Relative completeness of $\tilde{B}$ now gives that $f \in \tilde{B}$ and $\|f\|_{\tilde{B}} \le \|M\|$, so that $M = M_f$ for some $f \in \tilde{B}$. As a byproduct we have that $\|f\|_{\tilde{B}} \le \|M_f\|$ if $f \in \tilde{B}$.

Note that $M_f$ is compact if $f$ is a trigonometric polynomial in $B$. Compactness of $M_f$ for $f \in B$ follows by polynomial approximation in $B$. Conversely, if $M_f$ is compact, then, arguing as in the previous paragraph, we see that the set $\{\sigma_n(f) : n \ge 1\}$ is relatively compact in $B$, which implies that $f \in B$. □

We shall next identify the relative completion as a certain canonical Banach space dual. Let $B$ be a homogeneous Banach space and recall from Section 3 the discussion about its dual $B^*$. An element $f$ in the relative completion $\tilde{B}$ naturally induces a continuous linear functional on the space $(B^*)_c$ by setting

$$(p, f) = \langle f, p \rangle = \sum_{k \in \operatorname{Spec} B} \hat{f}(k)\hat{p}(-k) \tag{4.2}$$

for each trigonometric polynomial $p \in B^*$, where the symbol $(\cdot, \cdot)$ indicates dual pairing. Indeed,

$$|\langle \sigma_n(f), p \rangle| \le \|\sigma_n(f)\|_B \|p\|_{B^*} \le \|f\|_{\tilde{B}} \|p\|_{B^*}$$

so passing to the limit gives the norm bound $|\langle f, p \rangle| \le \|f\|_{\tilde{B}} \|p\|_{B^*}$ for $f \in \tilde{B}$ and each trigonometric polynomial $p \in B^*$, showing that (4.2) extends uniquely to a continuous linear functional on $(B^*)_c$ of norm less than or equal to $\|f\|_{\tilde{B}}$.

THEOREM 4.10. *Let $B$ be a homogeneous Banach space on $\mathbb{T}$. Then the relative completion $\tilde{B}$ is naturally identified with the dual of the space $(B^*)_c$ by means of the pairing* (4.2). *Furthermore*

$$\|f\|_{\tilde{B}} = \sup_{\|p\|_{B^*} \le 1} |\langle f, p \rangle|, \quad f \in \tilde{B},$$

*where the supremum is taken over all trigonometric polynomials $p$ with* $\operatorname{supp} \hat{p} \subset \operatorname{Spec} B^*$.

PROOF. We first explain how elements in the dual $((B^*)_c)^*$ can be identified with distributions on $\mathbb{T}$. We associate to $\ell \in ((B^*)_c)^*$ the formal trigonometric series $S\ell = \sum_{k \in \operatorname{Spec} B} \hat{\ell}(k)\phi_k$, where

$$\hat{\ell}(k) = \begin{cases} (\phi_{-k}, \ell) & \text{if } k \in \operatorname{Spec} B, \\ 0 & \text{if } k \in \mathbb{Z} \setminus \operatorname{Spec} B. \end{cases}$$

Note that $|(\phi_{-k}, \ell)| \leq \|\ell\| \|\phi_{-k}\|_{B^*}$ and

$$\|\phi_{-k}\|_{B^*} = \sup_{\|f\|_B \leq 1} |\langle f, \phi_{-k} \rangle| \leq C\|\phi_{-k}\|_{C^m} \leq C'(1+|k|)^m$$

for $k \in \operatorname{Spec} B$ using the embedding of $B$ into $\mathcal{D}'^m(\mathbb{T})$ provided by Lemma 3.1. These estimations imply that the correspondence $\ell \mapsto S\ell$ embeds the space $((B^*)_c)^*$ continuously as a Banach space of distributions on $\mathbb{T}$. Note that $\langle S\ell, p \rangle = (p, \ell)$ if $p$ is a trigonometric polynomial with $\operatorname{supp} \hat{p} \subset \operatorname{Spec} B^*$.

It is clear by construction that $\operatorname{Spec}((B^*)_c)^* = \operatorname{Spec} B$. Since translations in duals are defined by transposed action, it follows by construction that the space $((B^*)_c)^*$ is translation invariant.

We next check relative completeness of $((B^*)_c)^*$. Let $\{\ell_n\}$ be a sequence in $((B^*)_c)^*$ such that $\|\ell_n\|_{((B^*)_c)^*} \leq C$ for all $n$ and $S\ell_n \to f$ in $\mathcal{D}'(\mathbb{T})$. Since $S\ell_n \to f$ in $\mathcal{D}'(\mathbb{T})$, we have that the limit of $(p, \ell_n)$ as $n \to \infty$ exists for every trigonometric polynomial $p$ in $(B^*)_c$. Since trigonometric polynomials in $B^*$ are dense in $(B^*)_c$ a standard argument gives the existence of $\ell \in ((B^*)_c)^*$ such that $\ell_n \to \ell$ in the weak* topolgy on $((B^*)_c)^*$ and $\|\ell\|_{((B^*)_c)^*} \leq C$. In particular, $S\ell = f$, so $((B^*)_c)^*$ is relatively complete.

We next show that $B$ is isometrically contained in $((B^*)_c)^*$. Recall by (4.2) that an element $f \in B$ acts on a trigonometric polynomial $p \in (B^*)_c$ by $(p, f) = \langle f, p \rangle$ using the distributional pairing. In the same paragraph preceding the theorem it was shown that $\|f\|_{((B^*)_c)^*} \leq \|f\|_B$ for $f \in B$. For the reverse inequality, fix $f \in B$. By the Hahn-Banach theorem there exists an $\ell \in B^*$ with $\|\ell\|_{B^*} = 1$ such that $\ell(f) = \|f\|_B$. Since $\sigma_n(f) \to f$ in $B$, we have $\ell(f) = \lim_n \ell(\sigma_n(f)) = \lim_n (f, \sigma_n'(\ell))$, where $\sigma_n'(\ell)$ is defined by transposed pairing. Note that $\sigma_n'(\ell) \in (B^*)_c$ and $\|\sigma_n'(\ell)\|_{B^*} \leq 1$. An estimation now gives $\|f\|_B \leq \|f\|_{((B^*)_c)^*}$ for $f \in B$. By an application of Theorem 4.6 we conclude that $((B^*)_c)^* = \tilde{B}$. $\square$

We remark that the proof of Theorem 4.10 gives an alternative existence proof of the relative completion $\tilde{B}$.

Corollary 4.11. *Let $X$ be a relatively complete translation invariant Banach space of distributions on $\mathbb{T}$. Then $X$ has a separable predual.*

Proof. The predual $((X_c)^*)_c$ of $(X_c)^{\sim}$ constructed using the recipe in Theorem 4.10 is clearly separable. Since $(X_c)^{\sim} = X$ by Corollary 4.8, this completes the proof. □

Corollary 4.12. *Let $B$ be a homogeneous Banach space on $\mathbb{T}$. If $B$ is relatively complete, then $B$ is the dual of $(B^*)_c$.*

Proof. In this case $B = \tilde{B}$ and the result is evident by Theorem 4.10. □

An interesting example of a relatively complete homogeneous Banach space is the Hardy space $H^1(\mathbb{T})$. The relative completeness property of $H^1(\mathbb{T})$ is a straightforward reformulation of a classical result by F. and M. Riesz often formulated as saying that if $\mu \in M(\mathbb{T})$ and $\hat{\mu}(n) = 0$ for $n < 0$ then $\mu$ is absolutely continuous (see Katznelson [**23**, Theorem III.3.13] or Zygmund [**34**, Theorem VII.8.2]). The predual (dual) of $H^1(\mathbb{T})$ with respect to Cauchy duality is the space VMOA($\mathbb{T}$) (BMOA($\mathbb{T}$)) which is the content of a well-known result by Sarason (Fefferman-Stein), see Garnett [**13**, Chapter VI]. We mention also that preduals of $Q_p$ spaces have recently been investigated by Aleman, Carlsson and Persson [**3**], [**4**].

We next discuss when the predual $(B^*)_c$ of $\tilde{B}$ is a space of distributions. For a set $\Lambda \subset \mathbb{Z}$ we write

$$C^\infty_\Lambda(\mathbb{T}) = \{\varphi \in C^\infty(\mathbb{T}) : \operatorname{supp} \hat{\varphi} \subset \Lambda\}$$

for the space of smooth functions spectral in $\Lambda$.

Proposition 4.13. *Let $B$ be a homogeneous Banach space on $\mathbb{T}$ and set $\Lambda = \operatorname{Spec} B$. Then the following assertions are equivalent*:

(1) *The inclusion $C^\infty_\Lambda(\mathbb{T}) \subset B$ holds.*
(2) *The dual $B^*$ embeds continuously in $\mathcal{D}'(\mathbb{T})$.*
(3) *The space $(B^*)_c$ embeds continuously in $\mathcal{D}'(\mathbb{T})$.*
(4) *The estimate*

$$\|\phi_k\|_B \le C(1 + |k|)^m, \quad k \in \Lambda,$$

*holds for some constants $C$ and $m \ge 0$.*

*Furthermore, under the above conditions, the dual $B^*$ of $B$ is naturally identified with the distributions $u \in \mathcal{D}'(\mathbb{T})$ such that*

$$|u(\varphi)| \le C' \|\varphi\|_B, \quad \varphi \in C^\infty_\Lambda(\mathbb{T}),$$

*and* $\operatorname{supp} \hat{u} \subset -\Lambda$*, where the best constant $C'$ is the norm $\|u\|_{B^*}$.*

SKETCH OF PROOF. Assume (1). By the closed graph theorem, the space $C^\infty_\Lambda(\mathbb{T})$ is continuously embedded in $B$, which gives an estimate $\|\varphi\|_B \le C\|\varphi\|_{C^m}$ for $\varphi \in C^\infty_\Lambda(\mathbb{T})$. Using this estimate it is straightforward to check that the Fourier coefficients of an element $\ell \in B^*$ satisfy a growth estimate

$$|\hat{\ell}(k)| \le C\|\ell\|_{B^*}(1+|k|)^m, \quad k \in \operatorname{Spec} B^*,$$

for some new constant $C$. This proves (2). It is evident that (2) implies (3).

Assume (3). By Lemma 3.1 there exists $m \ge 0$ and constant $C$ such that $\|\varphi\|_{\mathcal{D}'^m} \le C\|\varphi\|_{B^*}$ for $\varphi \in (B^*)_c$. We now use Theorem 4.10 and duality to verify (4).

Assume (4). For $\varphi \in C^\infty_\Lambda(\mathbb{T})$ we have $\sum|\hat{\varphi}(k)|\|\phi_k\|_B < \infty$, showing that $\varphi \in B$ by completeness of $B$. This proves (1). The last assertion is self-explanatory. $\square$

Let $1 \le p < \infty$ and let $\omega = \{\omega_k\}_{k=-\infty}^\infty$ be a doubly infinite positive weight sequence such that $1/\omega_k$ is of at most polynomial growth as $|k| \to \infty$. Denote by $A^p_\omega$ the space of all $f \in \mathcal{D}'(\mathbb{T})$ with finite norm

$$\|f\|^p_{p,\omega} = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^p\omega_k < \infty.$$

It is straightforward to check that the space $A^p_\omega$ is a homogeneous Banach space on $\mathbb{T}$ and also that $A^p_\omega$ is relatively complete, $(A^p_\omega)\tilde{} = A^p_\omega$. Note that $\|\phi_k\|_{p,\omega} = \omega_k^{1/p}$ for $k \in \mathbb{Z}$. If the weight $\omega_k$ grows more than any polynomial, then $B = A^p_\omega$ does not satisfy condition (4) in Proposition 4.13 showing that the canonical predual $((A^p_\omega)^*)_c$ of $A^p_\omega$ is not a space of distributions. Note however that $((A^p_\omega)^*)_c$ has a natural description as a sequence space. We omit the details.

## 5. An analysis of the $\alpha$-harmonic Dirichlet problem.

In this section we discuss the $\alpha$-harmonic Dirichlet problem. Let us first define the $\alpha$-harmonic Poisson integral.

DEFINITION 5.1. Let $\alpha > -1$ and $f \in \mathcal{D}'(\mathbb{T})$. The $\alpha$-harmonic Poisson integral of $f$ is the function

$$P_\alpha[f](z) = P_{\alpha,r} * f(e^{i\theta}), \quad z = re^{i\theta} \in \mathbb{D},$$

where $P_{\alpha,r}(e^{i\theta}) = P_\alpha(re^{i\theta})$ for $e^{i\theta} \in \mathbb{T}$ in accordance with (0.5).

We next calculate the power series expansion of the Poisson integral.

PROPOSITION 5.2. *Let $\alpha > -1$ and $f \in \mathcal{D}'(\mathbb{T})$. Then the function $P_\alpha[f]$ has the power series expansion*

$$P_\alpha[f](z) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e_{\alpha,k}(z), \quad z \in \mathbb{D},$$

*where $\hat{f}(k)$, $k \in \mathbb{Z}$, are the Fourier coefficients of $f$ and the functions $e_{\alpha,k}$ are given by* (1.9) *and* (1.10).

PROOF. In view of Definition 5.1 and the identity (3.1) for Fourier coefficients we have that

$$(P_{\alpha,r} * f)\hat{\ }(k) = e_{\alpha,k}(r)\hat{f}(k), \quad k \in \mathbb{Z},$$

where the Fourier coefficients of $P_{\alpha,r}$ are calculated using the power series expansion from Definition 2.1. This yields

$$P_\alpha[f](z) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e_{\alpha,k}(r) e^{ik\theta} = \sum_{k=-\infty}^{\infty} \hat{f}(k) e_{\alpha,k}(z), \quad z = re^{i\theta} \in \mathbb{D},$$

where the last equality follows by homogeneity of the $e_{\alpha,k}$'s. $\square$

Note that the power series in Proposition 5.2 converges in $C^\infty(\mathbb{D})$ since the Fourier coefficients $\hat{f}(k)$, $k \in \mathbb{Z}$, are of polynomial growth, so $P_\alpha[f]$ is $\alpha$-harmonic by Theorem 1.2.

THEOREM 5.3. *Let $\alpha > -1$. Let $u$ be an $\alpha$-harmonic function, and suppose that for some sequence of numbers $0 \le r_n < 1$ with $\lim_{n\to\infty} r_n = 1$ we have $u_{r_n} \to f$ in $\mathcal{D}'(\mathbb{T})$ as $n \to \infty$, where $u_{r_n}$ is given by* (0.5)*. Then*

$$u(z) = P_\alpha[f](z), \quad z \in \mathbb{D}.$$

PROOF. Recall the power series expansion (1.11) of $u$ provided by Theorem 1.2. Note in particular that

$$u_r(e^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k e_{\alpha,k}(r) e^{ik\theta}, \quad e^{i\theta} \in \mathbb{T},$$

for $0 \leq r < 1$ in view of the homogeneity of the $e_{\alpha,k}$'s. A calculation of Fourier coefficients gives that $\hat{u}_r(k) = c_k e_{\alpha,k}(r)$ for $k \in \mathbb{Z}$ and $0 \leq r < 1$. Now $u_{r_n} \to f$ in $\mathcal{D}'(\mathbb{T})$ so passing to the limit as $n \to \infty$ shows that $c_k = \hat{f}(k)$ for $k \in \mathbb{Z}$. In view of Proposition 5.2 we conclude that $u = P_\alpha[f]$. $\square$

Note that Theorem 5.3 yields uniqueness of solutions to the Dirichlet problem (0.4). We next calculate the boundary limit of the $\alpha$-harmonic Poisson integral.

Theorem 5.4. *Let $\alpha > -1$ and let $B$ be a homogeneous Banach space on $\mathbb{T}$. Let $f \in B$ and set $u = P_\alpha[f]$. Then $\lim_{r\to 1} u_r = f$ in $B$, where $u_r$ is given by* (0.5) *for* $0 \leq r < 1$.

Proof. It is straightforward to check using Proposition 5.2 that the limit assertion of the theorem holds true when $f = p$ is a trigonometric polynomial in $B$. By Corollary 2.8 we have that $\|P_{\alpha,r}\|_{L^1} \leq C$ for $0 \leq r < 1$, where $C = \Gamma(\alpha+1)/\Gamma((\alpha+2)/2)^2$. By denseness of trigonometric polynomials in $B$ we can find a trigonometric polynomial $p \in B$ such that $\|f - p\|_B < \varepsilon/(C+2)$. Since $P_{\alpha,r} * f - f = P_{\alpha,r} * (f - p) + (P_{\alpha,r} * p - p) + (p - f)$, we have that

$$\|P_{\alpha,r} * f - f\|_B \leq \|P_{\alpha,r}\|_{L^1}\|f - p\|_B + \|P_{\alpha,r} * p - p\|_B + \|p - f\|_B < \varepsilon$$

for $r$ close to 1. This completes the proof. $\square$

We end this section with a representation theorem for $\alpha$-harmonic functions.

Theorem 5.5. *Let $\alpha > -1$ and let $B$ be a homogeneous Banach space on $\mathbb{T}$. Then an $\alpha$-harmonic function $u$ in $\mathbb{D}$ is such that $u_r \in B$ for $0 \leq r < 1$ and*

$$\sup_{0\leq r<1} \|u_r\|_B < \infty \tag{5.1}$$

*if and only if it has the form of a Poisson integral $u = P_\alpha[f]$ of some $f \in \tilde{B}$, where $\tilde{B}$ is the relative completion of $B$.*

Proof. If $u = P_\alpha[f]$ for some $f \in \tilde{B}$, then (5.1) holds by Proposition 4.9 since the Poisson kernel $P_\alpha$ has bounded $L^1$-means by Corollary 2.8.

To prove the converse, recall the power series expansion (1.11) provided by Theorem 1.2. A straightforward calculation shows that $\hat{u}_r(k) = e_{\alpha,k}(r)c_k \to c_k$ as $r \to 1$ for every $k \in \mathbb{Z}$. By an application of Lemma 3.1 it follows that the family of functions $u_r$, $0 \leq r < 1$, is uniformly bounded in $\mathcal{D}'^m(\mathbb{T})$ for some $m \geq 0$. A standard approximation argument now gives that $\lim_{r\to1} u_r = f$ in the weak* topology of $\mathcal{D}'^m(\mathbb{T})$, where $f = \sum c_k\phi_k$ in $\mathcal{D}'(\mathbb{T})$. By the relative completeness property we conclude that $f \in \tilde{B}$. By Theorem 5.3 we have that $u = P_\alpha[f]$. $\square$

Remark 5.6. For $B = L^1(\mathbb{T})$ the result of Theorem 5.5 gives a well-known characterization of Poisson integrals of complex measures on $\mathbb{T}$. For $B = L^p(\mathbb{T})$, $1 < p < \infty$, the result of Theorem 5.5 gives a well-known characterization of Poisson integrals of functions from $L^p(\mathbb{T})$, $1 < p < \infty$. For $B = C(\mathbb{T})$ the result of Theorem 5.5 gives a well-known characterization of Poisson integrals of functions from $L^\infty(\mathbb{T})$. See for instance Rudin [**30**, Chapter 11], Stein [**33**, Section VII.1], Zygmund [**34**, Section IV.6] or Axler et al [**7**, Theorem 6.12].

We remark that there is an analogous theory for the formal adjoint $\Delta_\alpha^* = \bar\partial w_\alpha^{-1}\partial$ of $\Delta_\alpha$. We refrain from developing this direction further.

## 6. Fatou theorems.

In this section we shall discuss pointwise boundary limits almost everywhere of $\alpha$-harmonic functions in non-tangential approach regions of the form

$$\Gamma_\beta(e^{i\theta_0}) = \{re^{i\theta} \in \mathbb{D} : |\theta - \theta_0| < \beta(1-r),\ r \geq 0,\ \theta \in \mathbb{R}\},$$

where $e^{i\theta_0} \in \mathbb{T}$ and $\beta > 0$ is a positive constant. Approach regions of this type are often called Stolz angles and the terminology Fatou theorem refers to a classical result by Fatou (see Zygmund [**34**, Theorem III.7.9]). A standard reference is Stein [**33**]; see also [**7**, Chapter 6] or [**30**, Chapter 11]. We mention also Carlsson [**9**] and an interesting paper by Nagel and Stein [**26**] on related maximal estimates.

Let $\alpha > -1$, and consider the function

$$K_\alpha(z) = \frac{(1-|z|^2)^{\alpha+1}}{|1-z|^{\alpha+2}}, \quad z \in \mathbb{D}. \tag{6.1}$$

Recall that $K_\alpha(z) = |P_\alpha(z)|$ by Theorem 2.5. It is evident that for $0 \leq r < 1$ fixed, the quantity $K_\alpha(re^{i\theta})$ is even in $\theta \in (-\pi, \pi)$ and decreasing in $\theta \in (0, \pi)$.

Lemma 6.1. *Let $\alpha > -1$, and consider the function $K_\alpha$ given by* (6.1). *Then*

$$K_\alpha(z) \leq (1+\beta)^{\alpha+2} K_\alpha(re^{i\theta_0})$$

*for $z = re^{i\theta} \in \Gamma_\beta(e^{i\theta_0})$ and $\beta > 0$.*

Proof. By the triangle inequality we have that

$$|1 - re^{i\theta_0}| \leq |1 - re^{i\theta}| + |re^{i\theta} - re^{i\theta_0}| \leq |1 - re^{i\theta}| + |\theta - \theta_0|,$$

where the last inequality follows by a geometric consideration. Using the inequality $|\theta - \theta_0| < \beta(1-r)$ we obtain

$$|1 - re^{i\theta_0}| \leq |1 - re^{i\theta}| + \beta(1-r) \leq (1+\beta)|1 - re^{i\theta}|,$$

which leads to the estimation

$$K_\alpha(re^{i\theta}) = \frac{(1-r^2)^{\alpha+1}}{|1-re^{i\theta}|^{\alpha+2}} \leq (1+\beta)^{\alpha+2}\frac{(1-r^2)^{\alpha+1}}{|1-re^{i\theta_0}|^{\alpha+2}} = (1+\beta)^{\alpha+2}K_\alpha(re^{i\theta_0})$$

for $re^{i\theta} \in \Gamma_\beta(e^{i\theta_0})$. $\square$

For $e^{i\theta} \in \mathbb{T}$ and $0 < h \leq \pi$ we denote by $A(e^{i\theta}, h)$ the open arc

$$A(e^{i\theta}, h) = \{e^{i\tau} \in \mathbb{T} : |\tau - \theta| < h\}.$$

Recall that the maximal function $M\mu$ of a complex measure $\mu \in M(\mathbb{T})$ is defined by

$$M\mu(e^{i\theta}) = \sup_{0<h\leq\pi} \frac{1}{2h}|\mu|(A(e^{i\theta}, h)),$$

where $|\mu|$ is the total variation of $\mu$. The symmetric derivative of a measure $\mu \in M(\mathbb{T})$ at the point $e^{i\theta} \in \mathbb{T}$ is defined by

$$D\mu(e^{i\theta}) = \lim_{h\to 0+} \frac{\pi}{h}\mu(A(e^{i\theta}, h))$$

whenever the limit exists. It is known that $D\mu(e^{i\theta}) = 0$ for a.e. $e^{i\theta} \in \mathbb{T}$ if $\mu \in M(\mathbb{T})$ is singular with respect to Lebesgue measure on $\mathbb{T}$ (see [**30**, Theorem 7.13]).

Theorem 6.2. *Let $\alpha > -1$. Let $\mu \in M(\mathbb{T})$ and set $u = P_\alpha[\mu]$. Assume that $e^{i\theta_0} \in \mathbb{T}$ is such that $D|\mu|(e^{i\theta_0}) = 0$. Then*

$$\lim_{\Gamma_\beta(e^{i\theta_0})\ni z\to e^{i\theta_0}} u(z) = 0$$

*for every $\beta > 0$.*

Proof. Let $\varepsilon > 0$ be given. Since $D|\mu|(e^{i\theta_0}) = 0$, there exists $\delta > 0$ such that

$$\frac{1}{2h}|\mu|(A(e^{i\theta_0},h))<\varepsilon$$

for $0<h\leq\delta$. Let $\mu_1$ be the restriction of $|\mu|$ to the arc $A(e^{i\theta_0},\delta)$ and set $\mu_2=|\mu|-\mu_1$. Observe that $M\mu_1(e^{i\theta_0})\leq\varepsilon$ by construction. By the triangle inequality we have that

$$|u(z)|\leq\int_{\mathbb{T}}K_\alpha(ze^{-i\tau})d\mu_1(e^{i\tau})+\int_{\mathbb{T}}K_\alpha(ze^{-i\tau})d\mu_2(e^{i\tau}). \tag{6.2}$$

Since the support of $\mu_2$ is at a positive distance away from the point $e^{i\theta_0}$ we have that

$$\int_{\mathbb{T}}K_\alpha(ze^{-i\tau})d\mu_2(e^{i\tau})\to 0$$

as $\mathbb{D}\ni z\to e^{i\theta_0}$ by uniform convergence. We next estimate the leftmost integral on the right-hand side in (6.2). By Lemma 6.1 we have

$$\begin{aligned}\int_{\mathbb{T}}K_\alpha(ze^{-i\tau})d\mu_1(e^{i\tau})&\leq(1+\beta)^{\alpha+2}\int_{\mathbb{T}}K_\alpha(re^{i(\theta_0-\tau)})d\mu_1(e^{i\tau})\\&=(1+\beta)^{\alpha+2}(K_{\alpha,r}*\mu_1)(e^{i\theta_0})\end{aligned}$$

for $z=re^{i\theta}\in\Gamma_\beta(e^{i\theta_0})$ using the notation (0.5). Now apply [**23**, Lemma III.2.4] to conclude that

$$\int_{\mathbb{T}}K_\alpha(ze^{-i\tau})d\mu_1(e^{i\tau})\leq(1+\beta)^{\alpha+2}\|K_{\alpha,r}\|_{L^1}M\mu_1(e^{i\theta_0})\leq C_{\alpha\beta}\varepsilon$$

for $z=re^{i\theta}\in\Gamma_\beta(e^{i\theta_0})$, where $C_{\alpha\beta}=(1+\beta)^{\alpha+2}\Gamma(\alpha+1)/\Gamma(\alpha/2+1)^2$. By (6.2) we have that

$$\limsup_{\Gamma_\beta(e^{i\theta_0})\ni z\to e^{i\theta_0}}|u(z)|\leq C_{\alpha\beta}\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, the conclusion of the theorem follows. $\square$

Recall that a point $e^{i\theta_0}\in\mathbb{T}$ is called a Lebesgue point for $f\in L^1(\mathbb{T})$ if

$$\lim_{h\to 0}\frac{1}{h}\int_{\theta_0-h}^{\theta_0+h}|f(e^{i\theta})-f(e^{i\theta_0})|d\theta=0.$$

It is well-known that almost every $e^{i\theta_0} \in \mathbb{T}$ is a Lebesgue point for $f \in L^1(\mathbb{T})$ (see [**33**, Section I.1]).

Corollary 6.3. *Let $\alpha > -1$. Let $f \in L^1(\mathbb{T})$ and set $u = P_\alpha[f]$. Assume that $e^{i\theta_0} \in \mathbb{T}$ is a Lebesgue point for $f$. Then*

$$\lim_{\Gamma_\beta(e^{i\theta_0}) \ni z \to e^{i\theta_0}} u(z) = f(e^{i\theta_0})$$

*for every $\beta > 0$.*

Proof. Set $d\mu(e^{i\theta}) = (f(e^{i\theta}) - f(e^{i\theta_0}))d\theta/(2\pi)$ and apply Theorem 6.2. □

The following corollary gives the non-tangential boundary behavior almost everywhere of an $\alpha$-harmonic function with bounded $L^1$-means.

Corollary 6.4. *Let $\alpha > -1$. Let $\mu \in M(\mathbb{T})$ be a complex measure and consider its Lebesgue decomposition*

$$d\mu(e^{i\theta}) = f(e^{i\theta})d\theta/(2\pi) + d\mu_s(e^{i\theta}), \quad e^{i\theta} \in \mathbb{T},$$

*where $f \in L^1(\mathbb{T})$ and the measure $\mu_s$ is singular with respect to Lebesgue measure on $\mathbb{T}$. Set $u = P_\alpha[\mu]$. Then for almost every $e^{i\theta} \in \mathbb{T}$ it holds that*

$$\lim_{\Gamma_\beta(e^{i\theta}) \ni z \to e^{i\theta}} u(z) = f(e^{i\theta})$$

*for every $\beta > 0$.*

Proof. It is well-known that every $\mu \in M(\mathbb{T})$ has a Lebesgue decomposition (see [**30**, Theorem 6.10]). Since $\mu_s$ is singular, so is its total variation $|\mu_s|$. The result follows by Theorem 6.2, Corollary 6.3 and comments made above. □

It is known that, even for analytic functions, the non-tangential convergence regions in Fatou's theorem can not be extended to larger tangential Jordan regions (see for instance Zygmund [**34**, Theorem VII.7.44]).

Note that the essential property of $P_\alpha$ used in this section is that it is pointwise majorized by the function $K_\alpha$. We mention that kernels of the form $K_\alpha$ appear naturally in the study of integral representations of solutions of Dirichlet problems for higher order Laplacians (see for instance [**1**], [**27**], [**28**], [**29**]).

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## References

[1] A. Abkar and H. Hedenmalm, A Riesz representation formula for super-biharmonic functions, Ann. Acad. Sci. Fenn. Math., **26** (2001), 305–324.

[2] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Applied Mathematics Series, **55**, 1964.

[3] A. Aleman, M. Carlsson and A.-M. Persson, Preduals of $Q_p$-spaces, Complex Var. Elliptic Equ., **52** (2007), 605–628.

[4] A. Aleman, M. Carlsson and A.-M. Persson, Preduals of $Q_p$-spaces. II. Carleson imbeddings and atomic decompositions, Complex Var. Elliptic Equ., **52** (2007), 629–653.

[5] G. E. Andrews, R. Askey and R. Roy, Special Functions, Enoyclopedia Math. Appl., **71**, Cambridge University Press, 1999.

[6] E. Artin, The Gamma Function, Holt, Rinehart and Winston, 1964.

[7] S. Axler, P. Bourdon and W. Ramey, Harmonic Function Theory, Second edition, Grad. Texts in Math., **137**, Springer-Verlag, 2001.

[8] W. Braun and H. G. Feichtinger, Banach spaces of distributions having two module structures, J. Funct. Anal., **51** (1983), 174–212.

[9] M. Carlsson, Fatou-type theorems for general approximate identities, Math. Scand., **102** (2008), 231–252.

[10] J. Dixmier, Sur un théorème de Banach, Duke Math. J., **15** (1948), 1057–1071.

[11] E. Gagliardo, A unified structure in various families of function spaces, In: Compactness and Closure Theorems, Proc. Internat. Sympos. Linear Spaces, Jerusalem, 1960, Jerusalem Academic Press, Jerusalem, Pergamon, Oxford, 1961, pp. 237–241.

[12] P. R. Garabedian, A partial differential equation arising in conformal mapping, Pacific J. Math., **1** (1951), 485–524.

[13] J. B. Garnett, Bounded Analytic Functions, Revised first edition, Grad. Texts in Math., **236**, Springer-Verlag, 2007.

[14] G. Godefroy, Existence and uniqueness of isometric preduals: a survey, In: Banach Space Theory, Iowa City, IA, 1987, (ed. B.-L. Lin), Contemp. Math., **85**, Amer. Math. Soc., Providence, RI, 1989, pp. 131–193.

[15] A. Grothendieck, Une caractérisation vectorielle-métrique des espaces $L^1$, Canad. J. Math., **7** (1955), 552–561.

[16] A. K. Gupta and S. Nadarajah, Handbook of Beta Distribution and Its Applications, Statist. Textbooks Monogr., **174**, Marcel Dekker, Inc., New York, 2004.

[17] H. Hedenmalm, B. Korenblum and K. Zhu, Theory of Bergman Spaces, Springer-Verlag, 2000.

[18] H. Hedenmalm and A. Olofsson, Hele-Shaw flow on weakly hyperbolic surfaces, Indiana Univ. Math. J., **54** (2005), 1161–1180.

[19] H. Hedenmalm and Y. Perdomo G., Mean value surfaces with prescribed curvature form, J. Math. Pures Appl., **83** (2004), 1075–1107.

[20] H. Hedenmalm and S. Shimorin, Hele-Shaw flow on hyperbolic surfaces, J. Math. Pures Appl. (9), **81** (2002), 187–222.

[21] L. Hörmander, The Analysis of Linear Partial Differential Operators. I, Springer-Verlag, 1990.
[22] S. Kaijser, A note on dual Banach spaces, Math. Scand., **41** (1977), 325–330.
[23] Y. Katznelson, An Introduction to Harmonic Analysis, Dover, 1976.
[24] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry. Vol. I, Wiley, 1963.
[25] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry. Vol. II, Wiley, 1969.
[26] A. Nagel and E. M. Stein, On certain maximal functions and approach regions, Adv. in Math., **54** (1984), 83–106.
[27] A. Olofsson, A representation formula for radially weighted biharmonic functions in the unit disc, Publ. Mat., **49** (2005), 393–415.
[28] A. Olofsson, Regularity in a singular biharmonic Dirichlet problem, Monatsh. Math., **148** (2006), 229–239.
[29] A. Olofsson, A computation of Poisson kernels for some standard weighted biharmonic operators in the unit disc, Complex Var. Elliptic Equ., **53** (2008), 545–564.
[30] W. Rudin, Real and Complex Analysis, McGraw-Hill Book Co., New York, 1987.
[31] H. S. Shapiro, Topics in Approximation Theory, Lecture Notes in Math., **187**, Springer-Verlag, 1971.
[32] S. Shimorin, On Beurling-type theorems in weighted $\ell^2$ and Bergman spaces, Proc. Amer. Math. Soc., **131** (2003), 1777–1787.
[33] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Math. Ser., **30**, Princeton University Press, 1970.
[34] A. Zygmund, Trigonometric Series: Vol. I, II, Cambridge University Press, 1968.

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