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Low frequency expansion in thermoelasticity with second sound in three dimensions

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Abstract. We consider the initial-boundary value problem in hyperbolic thermoelasticity with second sound in a three-dimensional exterior domain. The low frequency expansion of solutions to the corresponding stationary resolvent problem is given and the limit to the classical thermoelastic problem is investigated.

1. Introduction.

In this paper, we consider the low frequency expansion of the resolvent problem corresponding to linear thermoelasticity with second sound in a three-dimensional exterior domain. The modeling of the second sound effect turns the classical hyperbolic-parabolic thermoelastic system into a purely hyperbolic one with a damping term. This is done using Cattaneo's law instead of Fourier's law of heat conduction. Thus the physical paradox of infinite propagation of heat pulses is removed.

Let Ω be an exterior domain in $\boldsymbol{R}^3$ with $C^{1,1}$ boundary Γ. The linear hyperbolic thermoelastic system with second sound in Ω is formulated as follows:

$$\begin{aligned} &u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \beta\nabla\theta = 0 \\ &\theta_t + \gamma \mathrm{div}\, q + \delta \mathrm{div}\, u_t = 0 \\ &\tau_0 q_t + q + \kappa\nabla\theta = 0 \end{aligned} \tag{1.1}$$

in $\Omega\times(0,\infty)$ subject to the initial condition:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad q(x,0) = q_0(x) \quad \text{in } \Omega.$$

As boundary condition in this paper, we consider the Dirichlet condition:

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$$u = 0, \quad \theta = 0 \quad \text{on } \Gamma \times (0, \infty)$$

Here, μ, β, γ, δ and κ are positive constants while λ is a constant such that $2\mu + \lambda > 0$, $\mu > 0$, and u and q are three vectors of unknown functions while θ is a unknown scalar function. $\tau_0 > 0$ is the so-called relaxation parameter, while $\tau_0 = 0$ leads to the classical hyperbolic-parabolic thermoelastic equations in Ω:

$$\begin{aligned} &u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla \mathrm{div}\, u + \beta\nabla\theta = 0 \\ &\theta_t - \gamma\kappa\Delta\theta + \delta \mathrm{div}\, u_t = 0 \end{aligned} \tag{1.2}$$

in $\Omega \times (0, \infty)$ subject to the initial condition:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x) \quad \text{in } \Omega$$

and the boundary condition:

$$u = 0, \quad \theta = 0 \quad \text{on } \Gamma \times (0, \infty).$$

For a survey on results in classical thermoelasticity see [**7**], for a survey on hyperbolic heat conduction models see [**1**]. The model used here was introduced by Lord and Shulman [**9**]. Results on the well-posedness both for linear and nonlinear thermoelasticity with second sound in one or three dimensions, and on the time-asymptotic behavior for bounded domains or for the Cauchy problem are given in [**12**], [**13**], [**5**], [**4**], [**19**], [**14**] and also the references therein. The time-asymptotic behavior in exterior domains for the system with second sound has not yet been studied. For this purpose the low frequency expansion for the associated resolvent problem is of interest.

We are interested in the low frequency expansion of the corresponding resolvent problems to (1.1) and (1.2), which is especially important to investigate the decay property of solutions to (1.1) and (1.2) as time goes to infinity, cf. [**6**], [**15**], [**16**], [**17**], [**18**]. This is possible since the solutions can be represented via resolvents (essentially: Laplace transform). In fact, according to a general theory due to Vainberg [**16**], [**17**], [**18**], in order to obtain the local energy decay of solutions to the evolution equations of hyperbolic type in exterior domains, the following observations are crucial:

(1) Asymptotic expansion at the origin of the low frequency part of the generalized resolvent operator.
(2) Absence of the point spectrum in the lower half-plane including the real axis.
(3) Summability property of the the high frequency part of generalized resolvent operator.

Here, the generalized resolvent operator is obtained by shrinking the definition domain to the space of functions having compact support and expanding the range to $L_{q,\mathrm{loc}}$ functions.

Indeed, if we devide the solution formula into the high frequency part and low frequency part, then the properties (2) and (3) guarantee the exponential stability (the exponential decay property) of the high frequency part, and the property (1) implies the decay rate of the low frequency part. As a result, we can decide the rate of the local energy decay of solutions to the evolution equations of hyperbolic type by the low frequency expansion formula of the generalized resolvent operator. Since the equation (1.1) is of hyperbolic type, this strategy should work, that is combining the low frequency expansion obtained in this paper and the investigation of the summability property which will be the future work, we will have the rate of local energy decay of solutions to the initial boundary value problems (1.1), (1.2) and (1.3).

For wave equations, elasticity or Maxwell equations, a collection of references for results on low frequency asymptotics is given in the work of Pauly [**11**].

An essential difference between the classical thermoelastic equations and the thermoelastic equations with second sound appears in the high frequency part. In fact, in case of the Cauchy problem, according to the result due to Naito [**14**], although the decay rate of the high frequency part is the same, one part of the solution formulae in the second sound case has the form like $e^{-\tau_0^{-1}t}$ as $t \to \infty$, which means that a vanishing part appears in the expansion formula for the second sound case as $\tau_0 \to 0$. On the other hand, the low frequency part in the case of second sound converges to the corresponding low frequency part in the classical case, which will be seen in the last section of this paper.

For an expansion in classical thermoelasticity ($\tau_0 = 0$) in connection with scattering following an incident plane wave see [**2**].

Moreover, we will discuss some convergence property of the resolvent as τ_0 tends to zero. We remark that it has been observed ([**3**]) for other systems that the behavior for $\tau_0 > 0$ and that for $\tau_0 = 0$ might be quite different. Indeed, for bounded domains there may be exponential stability of the system if $\tau_0 = 0$, while for $\tau_0 > 0$ there is no exponential stability. Here we show that the systems are close to each other.

To state our results precisely, we consider the resolvent problem corresponding to (1.1) and (1.2), which is formulated as follows:

$$\begin{aligned}
&k^2u - \mu\Delta u - (\mu+\lambda)\nabla\mathrm{div}\, u + \beta\nabla\theta = f && \text{in } \Omega\\
&k\theta + \gamma\mathrm{div}\, q + \delta k\mathrm{div}\, u = g && \text{in } \Omega\\
&\tau_0 kq + q + \kappa\nabla\theta = h && \text{in } \Omega\\
&u = 0, \quad \theta = 0 && \text{on } \Gamma
\end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &k^2u - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \beta\nabla\theta = f && \text{in } \Omega \\ &k\theta - \gamma\kappa\Delta\theta + \delta k \mathrm{div}\, u = g && \text{in } \Omega \\ &u = 0, \quad \theta = 0 && \text{on } \Gamma \end{aligned} \tag{1.4}$$

respectively.

As main results we shall obtain the low frequency expansion (near $k = 0$) in Theorem 3.3 and Theorem 3.4, and the conclusion in Section 4 on the continuous dependence of the parameter τ_0.

The paper is organized as follows: In Section 2 we consider the spectral analysis for the Cauchy problem where Ω is all of $\boldsymbol{R}^3$, in Section 3 the case of an arbitrary exterior domain Ω is considered, and in Section 4 the conclusion on the dependence of the relaxation parameter τ_0 is presented.

2. Spectral analysis of the thermoelastic equations with second sound for $\Omega = \boldsymbol{R}^3$.

In this section, we consider the resolvent problem in $\boldsymbol{R}^3$:

$$\begin{aligned} &k^2u - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \beta\nabla\theta = f && \text{in } \boldsymbol{R}^3 \\ &k\theta + \gamma \mathrm{div}\, q + \delta k \mathrm{div}\, u = g && \text{in } \boldsymbol{R}^3 \\ &\tau_0 k q + q + \kappa\nabla\theta = h && \text{in } \boldsymbol{R}^3. \end{aligned} \tag{2.1}$$

From the third equation of (2.1) we have, for $k \neq -1/\tau_0$,

$$q = (\tau_0 k + 1)^{-1}(h - \kappa\nabla\theta) \tag{2.2}$$

and therefore, inserting this formula into the second equation in (2.1), we have

$$\begin{aligned} &k^2u - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \beta\nabla\theta = f && \text{in } \boldsymbol{R}^3 \\ &k\theta - \gamma\kappa(\tau_0 k + 1)^{-1}\Delta\theta + \delta k \mathrm{div}\, u = g - \gamma(\tau_0 k + 1)^{-1}\mathrm{div}\, h && \text{in } \boldsymbol{R}^3. \end{aligned} \tag{2.3}$$

Therefore, for the simplicity instead of (2.3) we consider the following equation:

$$\begin{aligned} &k^2u - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \beta\nabla\theta = f && \text{in } \boldsymbol{R}^3 \\ &k\theta - \gamma\kappa(\tau_0 k + 1)^{-1}\Delta\theta + \delta k \mathrm{div}\, u = g && \text{in } \boldsymbol{R}^3. \end{aligned} \tag{2.4}$$

To solve (2.4), we introduce the Helmholtz decomposition. In general, given $f = {}^t(f_1, f_2, f_3) \in L_p(\boldsymbol{R}^3)^1$, we set

$$
\begin{aligned}
g &= Pf = \mathscr{F}_\xi^{-1}\big[\hat{P}(\xi)\hat{f}(\xi)\big](x) \\
\pi &= Qf = \mathscr{F}_\xi^{-1}\Big[-\frac{i\xi\cdot\hat{f}(\xi)}{|\xi|^2}\Big](x)
\end{aligned}
\tag{2.5}
$$

where $\hat{f} = \mathscr{F}[f]$ and $\mathscr{F}^{-1}$ stand for the Fourier transform and its inversion formula, respectively; $\hat{P}(\xi)$ is the 3×3 matrix given by the formula:

$$
\hat{P}(\xi) = \left(\delta_{j\ell} - \frac{\xi_j\xi_\ell}{|\xi|^2}\right), \quad \delta_{j\ell} = \begin{cases} 1 & j = \ell, \\ 0 & j \neq \ell \end{cases}
$$

and $\cdot$ stands for the usual inner product in $\boldsymbol{R}^3$. Using these symbols, we have

$$
f = Pf + \nabla Qf. \tag{2.6}
$$

In particular, we know from the Fourier multiplier theorem that

$$
\operatorname{div} Pf = 0, \quad \|Pf\|_{L_p(\boldsymbol{R}^3)} \le C\|f\|_{L_p(\boldsymbol{R}^3)} \tag{2.7}
$$

provided that $1 < p < \infty$.

Applying P and Q in (2.4) and using the fact that $\operatorname{div} u = \Delta Qu$, we have

$$
\begin{aligned}
&k^2 Pu - \mu\Delta Pu = Pf && \text{in } \boldsymbol{R}^3 \\
&k^2 Qu - (2\mu+\lambda)\Delta Qu + \beta\theta = Qf && \text{in } \boldsymbol{R}^3 \\
&k\theta - \gamma\kappa(\tau_0 k + 1)^{-1}\Delta\theta + \delta k\Delta Qu = g && \text{in } \boldsymbol{R}^3.
\end{aligned}
\tag{2.8}
$$

We can solve the first equation of (2.8) easily. In fact, we have

$$
Pu = \mathscr{F}_\xi^{-1}\left[\frac{\hat{P}(\xi)\hat{f}(\xi)}{k^2 + \mu|\xi|^2}\right](x). \tag{2.9}
$$

On the other hand, to solve the 2nd and 3rd equations in (2.8), for notational simplicity we set

[1] ${}^tM$ denotes the transposed M

$$w = Qu, \quad F = Qf, \quad G = g. \tag{2.10}$$

And then, we have

$$\begin{aligned} &k^2 w - (2\mu + \lambda)\Delta w + \beta\theta = F \\ &k\theta - \gamma\kappa(\tau_0 k + 1)^{-1}\Delta\theta + \delta k \Delta w = G \end{aligned} \tag{2.11}$$

in $\boldsymbol{R}^3$. Note that w and θ are both scalar functions, so that (2.11) is a system of two partial differential equations. Applying the Fourier transform to (2.11), we have a system of two linear equations:

$$\begin{aligned} &\big(k^2 + (2\mu + \lambda)|\xi|^2\big)\hat{w}(\xi) + \beta\hat{\theta}(\xi) = \hat{F}(\xi) \\ &\big(k + \gamma\kappa(\tau_0 k + 1)^{-1}|\xi|^2\big)\hat{\theta}(\xi) - \delta k|\xi|^2\hat{w}(\xi) = \hat{G}(\xi). \end{aligned}$$

Setting

$$\hat{A}_k(\xi) = \begin{pmatrix} k^2 + (2\mu + \lambda)|\xi|^2 & \beta \\ -\delta k|\xi|^2 & k + \gamma\kappa(\tau_0 k + 1)^{-1}|\xi|^2 \end{pmatrix}$$

we finally arrive at the linear equation:

$$\hat{A}_k(\xi)\begin{bmatrix} \hat{w}(\xi) \\ \hat{\theta}(\xi) \end{bmatrix} = \begin{bmatrix} \hat{F}(\xi) \\ \hat{G}(\xi) \end{bmatrix}. \tag{2.12}$$

To obtain the low frequency expansion in $\boldsymbol{R}^3$, we start with the analysis of the inverse matrix of $\hat{A}_k(\xi)$. We have

$$\begin{aligned} \det\, \hat{A}_k(\xi) &= (2\mu + \lambda)\gamma\kappa(\tau_0 k + 1)^{-1}|\xi|^4 + \big((2\mu + \lambda + \delta\beta)k \\ &\quad + \gamma\kappa(\tau_0 k + 1)^{-1}k^2\big)|\xi|^2 + k^3 \\ &= (2\mu + \lambda)\gamma\kappa(\tau_0 k + 1)^{-1}\tilde{P}(t, k) \quad (t = |\xi|^2) \end{aligned} \tag{2.13}$$

where we have set

$$\begin{aligned} \tilde{P}(t, k) &= t^2 + ((2\mu + \lambda)\gamma\kappa)^{-1}(\tau_0 k + 1)\big[(2\mu + \lambda + \delta\beta)k + \gamma\kappa(\tau_0 k + 1)^{-1}k^2\big]t \\ &\quad + ((2\mu + \lambda)\gamma\kappa)^{-1}(\tau_0 k + 1)k^3 \\ &= t^2 + \left(\frac{2\mu + \lambda + \delta\beta}{(2\mu + \lambda)\gamma\kappa}(\tau_0 k + 1)k + \frac{k^2}{2\mu + \lambda}\right)t + \frac{(\tau_0 k + 1)k^3}{(2\mu + \lambda)\gamma\kappa}. \end{aligned}$$

We start with the following lemma.

LEMMA 2.1. *Let $\tilde{P}(t,k)$ be the polynomial defined as above. Then, there exist two functions $\mu_j(k,\tau_0)$ $(j=1,2)$ such that*

$$\tilde{P}(t,k) = (t+\mu_1(k,\tau_0))(t+\mu_2(k,\tau_0))$$

$$\mu_1(k,\tau_0) = -\frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}k + \sum_{j=1}^{\infty} s_j^1(\tau_0)k^{j+1}$$

$$\mu_2(k,\tau_0) = -\frac{1}{2\mu+\lambda+\delta\beta}k^2 + \sum_{j=1}^{\infty} s_j^2(\tau_0)k^{j+2}.$$

Here, $s_j^1(\tau_0)$ and $s_j^2(\tau_0)$ are polynomials in τ_0 and the expansion formulas converge absolutely when $|k| \leq k_0$ and $|\tau_0| \leq 1$ for some positive constant k_0 which is independent of τ_0.

PROOF. To obtain the formula for $\mu_1(k,\tau_0)$, we set $t = ks$, and then we have

$$s^2 + \frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}s + k\left\{\left(\frac{1}{2\mu+\lambda} + \frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}\tau_0\right)s + \frac{1}{(2\mu+\lambda)\gamma\kappa}\right\}$$
$$+ k^2\frac{\tau_0}{(2\mu+\lambda)\gamma\kappa} = 0. \tag{2.14}$$

If we set

$$s_1(k,\tau_0) = -\frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa} + \sum_{j=1}^{\infty} s_j^1(\tau_0)k^j \tag{2.15}$$

we have $s_j^1(\tau_0) = (j!)^{-1}s_1^{(j)}(0,\tau_0)$, and therefore differentiating (2.14) j times, setting $k=0$ in the resultant equation and writing $s_1^{(j)}(0,\tau_0) = s_1^{(j)}$ $(j \geq 1)$ for simplicity, we have

$$2s_1(0,\tau_0)s_1^{(j)} + 2\sum_{\ell=1}^{j-1}\binom{j-1}{\ell}s_1^{(j-\ell)}s_1^{(\ell)} + \frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}s_1^{(j)}$$
$$+ j\left(\frac{1}{2\mu+\lambda} + \frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}\tau_0\right)s_1^{(j-1)} + \frac{\delta_{1j}}{(2\mu+\lambda)\gamma\kappa} + \frac{2\delta_{2j}\tau_0}{(2\mu+\lambda)\gamma\kappa} = 0.$$

Since

$$2s_1(0,\tau_0)+\frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}=-\frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}$$

we have

$$\begin{aligned}s_1^{(j)}=\frac{(2\mu+\lambda)\gamma\kappa}{2\mu+\lambda+\delta\beta}\bigg\{&2\sum_{\ell=1}^{j-1}\binom{j-1}{\ell}s_1^{(\ell)}s_1^{(j-\ell)}+j\bigg(\frac{1}{2\mu+\lambda}+\frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}\tau_0\bigg)s_1^{(j-1)}\\&+\frac{\delta_{1j}}{(2\mu+\lambda)\gamma\kappa}+\frac{2\delta_{2j}\tau_0}{(2\mu+\lambda)\gamma\kappa}\bigg\}.\end{aligned}$$

From this formula we see that $s_1^{(j)}$ are polynomials in τ_0 for all $j\geq 1$, which implies the assertion for $\mu_1(k,\tau_0)=ks_1(k,\tau_0)$.

To obtain the formula for $\mu_2(k,\tau_0)$, we set $t=k^2s$, and then we have

$$\begin{aligned}&\frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}s+\frac{1}{(2\mu+\lambda)\gamma\kappa}\\&\quad+k\bigg\{s^2+\bigg(\frac{1}{2\mu+\lambda}+\frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}\tau_0\bigg)s+\frac{\tau_0}{(2\mu+\lambda)\gamma\kappa}\bigg\}=0.\end{aligned}\tag{2.16}$$

If we set

$$s_2(k,\tau_0)=-\frac{1}{2\mu+\lambda+\delta\beta}+\sum_{j=1}^{\infty}s_j^2(\tau_0)k^j$$

differentiating (2.16) j times and setting $k=0$ in the resultant equation, we have

$$\begin{aligned}&\frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}s_2^{(j)}(0,\tau_0)\\&+j\bigg(\frac{d}{dk}\bigg)^{j-1}\bigg\{s_2(k,\tau_0)^2+\bigg(\frac{1}{2\mu+\lambda}+\frac{2\mu+\lambda+\delta\beta}{(2\mu+\lambda)\gamma\kappa}\bigg)s_2(k,\tau_0)+\frac{\tau_0}{(2\mu+\lambda)\gamma\kappa}\bigg\}\bigg|_{k=0}\\&=0\end{aligned}$$

from which the assertion for $\mu_2(k,\tau)$ follows analogously. This completes the proof of the lemma. □

Now, we shall give a solution formula. Since

$$\hat{A}_k(\xi)^{-1} = \frac{1}{\det \hat{A}_k(\xi)} \begin{pmatrix} k + \gamma\kappa(\tau_0 k + 1)^{-1}|\xi|^2 & -\beta \\ \delta k|\xi|^2 & k^2 + (2\mu + \lambda)|\xi|^2 \end{pmatrix}$$

$$\det \hat{A}_k(\xi) = (2\mu + \lambda)\gamma\kappa(\tau_0 k + 1)^{-1}\tilde{P}(|\xi|^2, k)$$

we have

$$\begin{aligned} \hat{w}(\xi) &= \frac{1}{(2\mu + \lambda)\gamma\kappa\tilde{P}(|\xi|^2, k)} \left[\{(\tau_0 k + 1)k + \gamma\kappa|\xi|^2\}\hat{F}(\xi) - \beta(\tau_0 k + 1)\hat{G}(\xi)\right] \\ \hat{\theta}(\xi) &= \frac{\tau_0 k + 1}{(2\mu + \lambda)\gamma\kappa\tilde{P}(|\xi|^2, k)} \left\{\delta k|\xi|^2\hat{F}(\xi) + (k^2 + (2\mu + \lambda)|\xi|^2)\hat{G}(\xi)\right\}. \end{aligned} \tag{2.17}$$

From (2.6) and (2.10), we have $u(x) = Pu(x) + \nabla Qu(x) = Pu(x) + \nabla w(x)$, and then using (2.9) and (2.17) we have

$$\begin{aligned} \hat{u}(\xi) = \frac{\hat{P}(\xi)\hat{f}(\xi)}{k^2 + \mu|\xi|^2} &+ \frac{i\xi}{(2\mu + \lambda)\gamma\kappa\tilde{P}(|\xi|^2, k)} \\ &\times \left[\{(\tau_0 k + 1)k + \gamma\kappa|\xi|^2\}\hat{F}(\xi) - \beta(\tau_0 k + 1)\hat{G}(\xi)\right]. \end{aligned} \tag{2.18}$$

Recalling (2.5) and (2.10) and denoting the j-th component of $\hat{P}(\xi)\hat{f}(\xi)$ by $\hat{P}(\xi)\hat{f}(\xi)\mid_j$, we have

$$\hat{P}(\xi)\hat{f}(\xi)\mid_j = \hat{f}_j(\xi) - \sum_{\ell=1}^{3} \frac{\xi_i\xi_\ell}{|\xi|^2}\hat{f}_\ell(\xi), \quad \hat{F}(\xi) = \widehat{Qf}(\xi) = -i\sum_{\ell=1}^{3} \frac{\xi_\ell\hat{f}_\ell(\xi)}{|\xi|^2}, \quad \hat{G}(\xi) = \hat{g}(\xi).$$

Finally we arrive at the following formulas:

$$\begin{aligned} \hat{u}_j(\xi) &= \frac{\hat{f}_j(\xi)}{k^2 + \mu|\xi|^2} - \sum_{\ell=1}^{3} \frac{\xi_j\xi_\ell}{(k^2 + \mu|\xi|^2)|\xi|^2}\hat{f}_\ell(\xi) + \sum_{\ell=1}^{3} \frac{(\tau_0 k + 1)k\xi_j\xi_\ell}{(2\mu + \lambda)\gamma\kappa\tilde{P}(|\xi|^2, k)|\xi|^2}\hat{f}_\ell(\xi) \\ &\quad + \sum_{\ell=1}^{3} \frac{\xi_j\xi_\ell}{(2\mu + \lambda)\tilde{P}(|\xi|^2, k)}\hat{f}_\ell(\xi) - \frac{i\beta(\tau_0 k + 1)\xi_j}{(2\mu + \lambda)\gamma\kappa\tilde{P}(|\xi|^2, k)}\hat{g}(\xi) \\ \hat{\theta}(\xi) &= -i\sum_{\ell=1}^{3} \frac{(\tau_0 k + 1)\delta k\xi_\ell}{(2\mu + \lambda)\gamma\kappa\tilde{P}(|\xi|^2, k)}\hat{f}_\ell(\xi) + \frac{(\tau_0 k + 1)k^2}{(2\mu + \lambda)\gamma\kappa\tilde{P}(|\xi|^2, k)}\hat{g}(\xi) \\ &\quad + \frac{(\tau_0 k + 1)|\xi|^2}{\gamma\kappa\tilde{P}(|\xi|^2, k)}\hat{g}(\xi). \end{aligned} \tag{2.19}$$

From (2.19) we get the following theorem.

Theorem 2.2. *Let* $1 < q < \infty$ *and* $0 < \tau_0 \leq 1$. *Then, for any small* $\epsilon > 0$ *there exist a constant* $\sigma_0 > 0$ *depending on* ϵ *and an operator* $S_k \in \mathrm{Anal}\,(U_{\sigma_0,\epsilon}, \mathscr{B}(L_q(\boldsymbol{R}^3)^3 \times L_q(\boldsymbol{R}^3), W_q^2(\boldsymbol{R}^3)^3 \times W_q^2(\boldsymbol{R}^3)))$ *such that for any* $(f, g) \in L_q(\boldsymbol{R}^3)^3 \times L_q(\boldsymbol{R}^3)$, $(u, \theta) = S_k(f, g)$ *solves equation* (2.4) *uniquely. Here, for two Banach spaces* X *and* Y, $\mathscr{B}(X, Y)$ *denotes the set of all bounded linear operators from* X *into* Y, $U_{\sigma_0,\epsilon}$ *denotes an open set in* $\boldsymbol{C}$ *defined by the formula*:

$$U_{\sigma_0,\epsilon} = \left\{ k \in \boldsymbol{C} \setminus \{0\} \mid |\arg k| \leq \left(\frac{\pi}{2}\right) - \epsilon,\ |k| \leq \sigma_0 \right\}$$

and $\mathrm{Anal}\,(U_{\sigma_0,\epsilon}, X)$ *denotes the set of all holomorphic functions defined on* $U_{\sigma_0,\epsilon}$ *with their values in* X.

Proof. In view of Lemma 2.1, we see that for any small $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ depending only on ϵ such that

$$\left|\mu|\xi|^2 + k^2\right| \geq c_\epsilon(\mu|\xi|^2 + |k|^2)$$

provided that $|\arg k| \leq (\pi/2) - \epsilon$. In view of Lemma 2.1, we also see that there exist positive numbers σ_0 and c_ϵ depending on ϵ such that

$$\left|\tilde{P}(|\xi|^2, k)\right| \geq c_\epsilon(|\xi|^2 + |k|)(|\xi|^2 + |k|^2)$$

provided that $|\arg k| \leq (\pi/2) - \epsilon$ and $|k| \leq \sigma_0$ whenever $0 < \tau_0 \leq 1$. Therefore, if we define an operator S_k by the formula:

$$S_k(f, g) = \left(\mathscr{F}_\xi^{-1}[\hat{u}_1], \mathscr{F}_\xi^{-1}[\hat{u}_2], \mathscr{F}_\xi^{-1}[\hat{u}_3], \mathscr{F}_\xi^{-1}[\hat{\theta}]\right)$$

where $\hat{u}_j(\xi)$ $(j = 1, 2, 3)$ and $\hat{\theta}(\xi)$ are functions given in (2.19), then applying the Fourier multiplier theorem, we see that S_k is a holomorphic function with respect to $k \in \{k \in \boldsymbol{C} \mid \mathrm{Re}\, k > 0 \text{ and } |k| < \sigma_0\}$ with values in $\mathscr{B}(L_q(\boldsymbol{R}^3)^3 \times L_q(\boldsymbol{R}^3), W_q^2(\boldsymbol{R}^3)^3 \times W_q^2(\boldsymbol{R}^3))$ and $(u, \theta) = S_k(f, g)$ solves (2.4) for $(f, g) \in L_q(\boldsymbol{R}^3)^3 \times L_q(\boldsymbol{R}^3)$. □

Now, we shall discuss some expansion formula of S_k in a neighborhood of the origin: $k = 0$ of a complex plane, which can be done by shrinking the definition domain of S_k and widening the range of S_k in a suitable sense (see Vainberg [**16**], [**17**], [**18**]). The main theorem will be stated at the end of this section. To give an expansion formula for S_k, we shall give several lemmas in what follows.

LEMMA 2.3. *Let* $\mathrm{Re}\sqrt{a}>0$. *Then, we have*

$$\mathscr{F}_\xi^{-1}\left[\frac{1}{|\xi|^2+a}\right](x)=\frac{e^{-\sqrt{a}|x|}}{4\pi|x|},\quad \mathscr{F}_\xi^{-1}\left[\frac{1}{|\xi|^2}\right](x)=\frac{1}{4\pi|x|} \tag{2.20}$$

$$\mathscr{F}_\xi^{-1}\left[\frac{1}{(|\xi|^2+a)|\xi|^2}\right](x)=\frac{1}{4\pi\sqrt{a}}-\frac{|x|}{8\pi}+\frac{\sqrt{a}|x|^2}{8\pi}\int_0^1(1-s)^2e^{-s\sqrt{a}|x|}\,ds. \tag{2.21}$$

PROOF. The formulas in (2.20) are well-known, so that we may omit its proof.

Since

$$\frac{1}{(|\xi|^2+a)|\xi|^2}=\frac{-1}{a}\left(\frac{1}{|\xi|^2+a}-\frac{1}{|\xi|^2}\right)$$

by (2.20) we have

$$\mathscr{F}_\xi^{-1}\left[\frac{1}{(|\xi|^2+a)|\xi|^2}\right](x)=-\frac{1}{4\pi|x|a}\big(e^{-\sqrt{a}|x|}-1\big).$$

Making an integration by parts two times, we have

$$\begin{aligned}
e^{-\sqrt{a}|x|}-1&=\int_0^1\frac{d}{ds}e^{-s\sqrt{a}|x|}\,ds=-\sqrt{a}|x|\int_0^1 e^{-s\sqrt{a}|x|}\,ds\\
&=-\sqrt{a}|x|\left\{\big[-(1-s)e^{-s\sqrt{a}|x|}\big]_0^1-\sqrt{a}|x|\int_0^1(1-s)e^{-s\sqrt{a}|x|}\,ds\right\}\\
&=-\sqrt{a}|x|+(\sqrt{a}|x|)^2\int_0^1(1-s)e^{-s\sqrt{a}|x|}\,ds\\
&=-\sqrt{a}|x|+\frac{1}{2}(\sqrt{a}|x|)^2-\frac{1}{2}(\sqrt{a}|x|)^3\int_0^1(1-s)^2e^{-s\sqrt{a}|x|}\,ds.
\end{aligned} \tag{2.22}$$

Therefore, we have

$$\begin{aligned}
&\mathscr{F}_\xi^{-1}\left[\frac{1}{(|\xi|^2+a)|\xi|^2}\right](x)\\
&=-\frac{1}{4\pi|x|a}\left[-\sqrt{a}|x|+\frac{1}{2}a|x|^2-\frac{1}{2}a^{\frac{3}{2}}|x|^3\int_0^1(1-s)^2e^{-s\sqrt{a}|x|}\,ds\right]\\
&=\frac{1}{4\pi\sqrt{a}}-\frac{|x|}{8\pi}+\frac{\sqrt{a}|x|^2}{8\pi}\int_0^1(1-s)^2e^{-s\sqrt{a}|x|}\,ds.
\end{aligned}$$

This shows (2.21), which completes the proof of the lemma. □

Lemma 2.4. *When* $\operatorname{Re} k > 0$, *we have the following formulas*:

$$\mathscr{F}_\xi^{-1}\left[\frac{1}{k^2+\mu|\xi|^2}\right](x) = \frac{1}{4\pi\mu|x|} - \frac{k}{4\pi\mu^{\frac{3}{2}}}\int_0^1 e^{-s\frac{k}{\sqrt{\mu}}|x|}\,ds \tag{2.23}$$

$$\mathscr{F}_\xi^{-1}\left[\frac{\xi_j\xi_\ell}{(k^2+\mu|\xi|^2)|\xi|^2}\right](x) = \frac{1}{8\pi\mu}\left(\frac{\delta_{j\ell}}{|x|} - \frac{x_jx_\ell}{|x|^3}\right) + kG_{j\ell}(k,|x|) \tag{2.24}$$

where we have set

$$G_{j\ell}(k,|x|) = \frac{-1}{8\pi\mu^{\frac{3}{2}}}\int_0^1 (1-s)^2\left\{2\delta_{j\ell} - \frac{k}{\sqrt{\mu}}s\left(3\frac{x_jx_\ell}{|x|} + \delta_{j\ell}|x|\right)\right.$$
$$\left. + s^2\left(\frac{k}{\sqrt{\mu}}\right)^2 x_jx_\ell\right\}e^{-s\frac{k}{\sqrt{\mu}}|x|}\,ds. \tag{2.25}$$

Proof. Since $k^2+\mu|\xi|^2 = \mu(|\xi|^2+(k/\sqrt{\mu})^2)$, by (2.20) with $a = k/\sqrt{\mu}$ we have

$$\mathscr{F}_\xi^{-1}\left[\frac{1}{k^2+\mu|\xi|^2}\right](x) = \frac{e^{-\frac{k}{\sqrt{\mu}}|x|}}{4\pi|x|\mu}.$$

By (2.22) we have

$$e^{-\frac{k}{\sqrt{\mu}}|x|} = 1 - \frac{k}{\sqrt{\mu}}|x|\int_0^1 e^{-s\frac{k}{\sqrt{\mu}}|x|}\,ds.$$

Summing up, we have proved (2.23).

To prove (2.24), using (2.21) we observe that

$$\begin{aligned}
&\mathscr{F}_\xi^{-1}\left[\frac{\xi_j\xi_\ell}{(k^2+\mu|\xi|^2)|\xi|^2}\right](x)\\
&= -\frac{1}{\mu}\frac{\partial^2}{\partial x_j\partial x_\ell}\mathscr{F}_\xi^{-1}\left[\frac{1}{\left(|\xi|^2+\left(\frac{k}{\sqrt{\mu}}\right)^2\right)|\xi|^2}\right](x)\\
&= \frac{1}{8\pi\mu}\frac{\partial^2}{\partial x_j\partial x_\ell}|x| - \frac{k}{8\pi\mu^{\frac{3}{2}}}\frac{\partial^2}{\partial x_j\partial x_\ell}\left[|x|^2\int_0^1(1-s)^2e^{-s\frac{k}{\sqrt{\mu}}|x|}\,ds\right].
\end{aligned}$$

To proceed we observe that

$$\frac{\partial}{\partial x_j}|x| = \frac{x_j}{|x|}, \quad \frac{\partial^2}{\partial x_j \partial x_\ell}|x| = \frac{\delta_{j\ell}}{|x|} - \frac{x_j x_\ell}{|x|^3}, \quad \frac{\partial}{\partial x_j}|x|^2 = 2x_j,$$

$$\frac{\partial^2}{\partial x_j \partial x_\ell}|x|^2 = 2\delta_{j\ell}, \quad \frac{\partial}{\partial x_j} e^{-s\frac{k}{\sqrt{\mu}}|x|} = -s\frac{k}{\sqrt{\mu}}\frac{x_j}{|x|} e^{-s\frac{k}{\sqrt{\mu}}|x|},$$

$$\frac{\partial^2}{\partial x_j \partial x_\ell} e^{-s\frac{k}{\sqrt{\mu}}|x|} = -s\frac{k}{\sqrt{\mu}}\left(\frac{\delta_{j\ell}}{|x|} - \frac{x_j x_\ell}{|x|^3}\right) e^{-s\frac{k}{\sqrt{\mu}}|x|} + s^2\left(\frac{k}{\sqrt{\mu}}\right)^2 \frac{x_j x_\ell}{|x|^2} e^{-s\frac{k}{\sqrt{\mu}}|x|}.$$

In particular, we have

$$\begin{aligned}
&\frac{\partial^2}{\partial x_j \partial x_\ell}\left[|x|^2 e^{-s\frac{k}{\sqrt{\mu}}|x|}\right] \\
&= \left\{2\delta_{j\ell} - 4\frac{x_j x_\ell}{|x|} s\frac{k}{\sqrt{\mu}} - s\frac{k}{\sqrt{\mu}}\left(\delta_{j\ell}|x| - \frac{x_j x_\ell}{|x|}\right) + s^2\left(\frac{k}{\sqrt{\mu}}\right)^2 x_j x_\ell\right\} e^{-s\frac{k}{\sqrt{\mu}}|x|} \\
&= \left\{2\delta_{j\ell} - \frac{k}{\sqrt{\mu}} s\left(3\frac{x_j x_\ell}{|x|} + \delta_{j\ell}|x|\right) + s^2\left(\frac{k}{\sqrt{\mu}}\right)^2 x_j x_\ell\right\} e^{-s\frac{k}{\sqrt{\mu}}|x|}.
\end{aligned}$$

Therefore, defining $G_{j\ell}(k, |x|)$ by the formula (2.25), we have (2.24), which completes the proof of the lemma. □

In the following two lemmas, we treat the other terms.

LEMMA 2.5. *Let $\tilde{P}(t,k)$ and $\mu_j = \mu_j(k,\tau_0)$ be the same functions as in Lemma 2.1. Set*

$$H_\ell(x) = |x|^2 \int_0^1 (1-s)^2 e^{-s\mu_\ell(k,\tau_0)^{\frac{1}{2}}|x|}\, ds, \quad \ell = 1, 2. \tag{2.26}$$

Then, we have the following formulas:

$$\mathscr{F}_\xi^{-1}\left[\frac{1}{\tilde{P}(|\xi|^2,k)}\right](x) = \frac{1}{4\pi\left(\mu_1^{\frac{1}{2}} + \mu_2^{\frac{1}{2}}\right)} - \frac{|x|}{8\pi} + \frac{\mu_1^{\frac{3}{2}} H_1(x) - \mu_2^{\frac{3}{2}} H_2(x)}{8\pi(\mu_1 - \mu_2)} \tag{2.27}$$

$$\mathscr{F}_\xi^{-1}\left[\frac{i\xi_j}{\tilde{P}(|\xi|^2,k)}\right](x) = -\frac{x_j}{8\pi|x|} + \frac{\mu_1^{\frac{3}{2}}}{8\pi(\mu_1-\mu_2)}\frac{\partial H_1}{\partial x_j}(x) - \frac{\mu_2^{\frac{3}{2}}}{8\pi(\mu_1-\mu_2)}\frac{\partial H_2}{\partial x_j}(x) \tag{2.28}$$

$$\mathscr{F}_\xi^{-1}\left[\frac{-\xi_j\xi_\ell}{\tilde{P}(|\xi|^2,k)}\right](x) = -\frac{\delta_{j\ell}}{8\pi|x|} + \frac{x_jx_\ell}{8\pi|x|^3} + \frac{\mu_1^{\frac{3}{2}}}{8\pi(\mu_1-\mu_2)}\frac{\partial^2 H_1}{\partial x_j\partial x_\ell}(x) - \frac{\mu_2^{\frac{3}{2}}}{8\pi(\mu_1-\mu_2)}\frac{\partial^2 H_2}{\partial x_j\partial x_\ell}(x) \tag{2.29}$$

$$\mathscr{F}_\xi^{-1}\left[\frac{|\xi|^2}{\tilde{P}(|\xi|^2,k)}\right](x) = \frac{1}{4\pi|x|} - \frac{\mu_1^{\frac{1}{2}}}{4\pi}\int_0^1 e^{-s\mu_1^{\frac{1}{2}}|x|}\,ds - \frac{\mu_2}{4\pi(\mu_1^{\frac{1}{2}}+\mu_2^{\frac{1}{2}})} + \frac{\mu_2}{8\pi}|x| - \frac{\mu_1^{\frac{3}{2}}\mu_2}{8\pi(\mu_1-\mu_2)}H_1(x) + \frac{\mu_2^{\frac{5}{2}}}{8\pi(\mu_1-\mu_2)}H_2(x). \tag{2.30}$$

Lemma 2.6. *Let $\tilde{P}(t,k)$ and $\mu_j = \mu_j(k,\tau_0)$ be the same functions as in Lemma* 2.1. *Set*

$$H_\ell^m(x) = \int_0^1 (1-s)^2 s^m e^{-s\mu_\ell(k,\tau_0)^{\frac{1}{2}}|x|}\,ds, \quad \ell=1,2,\ m=0,1,2. \tag{2.31}$$

Then, we have

$$\begin{aligned}
&\mathscr{F}_\xi^{-1}\left[\frac{-\xi_j\xi_\ell}{\tilde{P}(|\xi|^2,k)|\xi|^2}\right](x)\\
&= \frac{-1}{8\pi(\mu_1-\mu_2)}\Big[2\delta_{j\ell}\big(\sqrt{\mu_1}H_1^0(x)-\sqrt{\mu_2}H_2^0(x)\big)\\
&\quad - \left(3\frac{x_jx_\ell}{|x|}+\delta_{j\ell}|x|\right)\big(\mu_1H_1^1(x)-\mu_2H_2^1(x)\big) + x_jx_\ell\big(\mu_1^{\frac{3}{2}}H_1^2(x)-\mu_2^{\frac{3}{2}}H_2^2(x)\big)\Big].
\end{aligned} \tag{2.32}$$

A Proof of Lemma 2.5. To obtain (2.27), we write

$$\begin{aligned}
\frac{1}{\tilde{P}(|\xi|^2,k)} &= \frac{1}{(|\xi|^2+\mu_1)(|\xi|^2+\mu_2)} = \frac{1}{\mu_2-\mu_1}\left[\frac{1}{|\xi|^2+\mu_1} - \frac{1}{|\xi|^2+\mu_2}\right]\\
&= \frac{\mu_1}{\mu_1-\mu_2}\frac{1}{(|\xi|^2+\mu_1)|\xi|^2} - \frac{\mu_2}{\mu_1-\mu_2}\frac{1}{(|\xi|^2+\mu_2)|\xi|^2}.
\end{aligned}$$

By (2.21) we have (2.27) immediately. From (2.27), we have (2.28) and (2.29) by the following observation:

$$\begin{aligned}
\mathscr{F}_\xi^{-1}\left[\frac{i\xi_j}{\tilde{P}(|\xi|^2,k)}\right](x) &= \frac{\partial}{\partial x_j}\mathscr{F}_\xi^{-1}\left[\frac{1}{\tilde{P}(|\xi|^2,k)}\right](x) \\
&= -\frac{x_j}{8\pi|x|} + \frac{\mu_1^{\frac{3}{2}}}{8\pi(\mu_1-\mu_2)}\frac{\partial H_1}{\partial x_j}(x) - \frac{\mu_2^{\frac{3}{2}}}{8\pi(\mu_1-\mu_2)}\frac{\partial H_2}{\partial x_j}(x), \\
\mathscr{F}_\xi^{-1}\left[\frac{-\xi_j\xi_\ell}{\tilde{P}(|\xi|^2,k)}\right](x) &= \frac{\partial^2}{\partial x_j\partial x_\ell}\mathscr{F}_\xi^{-1}\left[\frac{1}{\tilde{P}(|\xi|^2,k)}\right](x) \\
&= \frac{\partial}{\partial x_\ell}\left[-\frac{x_j}{8\pi|x|} + \frac{\mu_1^{\frac{3}{2}}}{8\pi(\mu_1-\mu_2)}\frac{\partial H_1}{\partial x_j}(x) - \frac{\mu_2^{\frac{3}{2}}}{8\pi(\mu_1-\mu_2)}\frac{\partial H_2}{\partial x_j}(x)\right] \\
&= -\frac{\delta_{j\ell}}{8\pi|x|} + \frac{x_jx_\ell}{8\pi|x|^3} + \frac{\mu_1^{\frac{3}{2}}}{8\pi(\mu_1-\mu_2)}\frac{\partial^2 H_1}{\partial x_j\partial x_\ell}(x) \\
&\quad - \frac{\mu_2^{\frac{3}{2}}}{8\pi(\mu_1-\mu_2)}\frac{\partial^2 H_2}{\partial x_j\partial x_\ell}(x).
\end{aligned}$$

To show (2.30), we write

$$\begin{aligned}
\frac{|\xi|^2}{\tilde{P}(|\xi|^2,k)} &= \frac{|\xi|^2}{(|\xi|^2+\mu_1)(|\xi|^2+\mu_2)} = \frac{|\xi|^2+\mu_2-\mu_2}{(|\xi|^2+\mu_1)(|\xi|^2+\mu_2)} \\
&= \frac{1}{|\xi|^2+\mu_1} - \frac{\mu_2}{\tilde{P}(|\xi|^2,k)}.
\end{aligned}$$

Combining (2.20) and (2.27) and writing

$$e^{-\mu_1^{\frac{1}{2}}|x|} = 1 - \mu_1^{\frac{1}{2}}|x|\int_0^1 e^{-s\mu_1^{\frac{1}{2}}|x|}\,ds$$

we have (2.30). This completes the proof of Lemma 2.5. □

A Proof of Lemma 2.6. To show (2.32), we write

$$\frac{1}{\tilde{P}(|\xi|^2,k)} = \frac{-1}{\mu_1-\mu_2}\left[\frac{1}{|\xi|^2+\mu_1} - \frac{1}{|\xi|^2+\mu_2}\right]$$

and then by (2.21) we have

$$
\begin{aligned}
&\mathscr{F}_\xi^{-1}\left[\frac{-\xi_j\xi_\ell}{\tilde{P}(|\xi|^2,k)|\xi|^2}\right](x) \\
&\quad = \frac{-1}{8\pi(\mu_1-\mu_2)}\frac{\partial^2}{\partial x_j\partial x_\ell}\left\{\int_0^1(1-s)^2|x|^2\big(\sqrt{\mu_1}e^{-s\sqrt{\mu_1}|x|}-\sqrt{\mu_2}e^{-s\sqrt{\mu_2}|x|}\big)\,ds\right\}.
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\frac{\partial}{\partial x_j}|x|^2 = 2x_j, \quad \frac{\partial^2}{\partial x_j\partial x_\ell}|x|^2 = 2\delta_{j\ell}, \\
&\frac{\partial}{\partial x_j}\big(\sqrt{\mu_1}e^{-s\sqrt{\mu_1}|x|}-\sqrt{\mu_2}e^{-s\sqrt{\mu_2}|x|}\big) \\
&\quad = -s\frac{x_j}{|x|}\big(\mu_1e^{-s\sqrt{\mu_1}|x|}-\mu_2e^{-s\sqrt{\mu_2}|x|}\big) \\
&\frac{\partial^2}{\partial x_j\partial x_\ell}\big(\sqrt{\mu_1}e^{-s\sqrt{\mu_1}|x|}-\sqrt{\mu_2}e^{-s\sqrt{\mu_2}|x|}\big) \\
&\quad = -s\left(\frac{\delta_{j\ell}}{|x|}-\frac{x_jx_\ell}{|x|^3}\right)\big(\mu_1e^{-s\sqrt{\mu_1}|x|}-\mu_2e^{-s\sqrt{\mu_2}|x|}\big) \\
&\qquad + s^2\frac{x_jx_\ell}{|x|^2}\big(\mu_1^{\frac{3}{2}}e^{-s\sqrt{\mu_1}|x|}-\mu^{\frac{3}{2}}e^{-s\sqrt{\mu_2}|x|}\big).
\end{aligned}
$$

By Leibniz's formula, we have

$$
\begin{aligned}
&\mathscr{F}_\xi^{-1}\left[\frac{-\xi_j\xi_\ell}{\tilde{P}(|\xi|^2,k)|\xi|^2}\right](x) \\
&\quad = \frac{-1}{8\pi(\mu_1-\mu_2)}\int_0^1(1-s)^2\bigg[2\delta_{j\ell}\big(\sqrt{\mu_1}e^{-s\sqrt{\mu_1}|x|}-\sqrt{\mu_2}e^{-s\sqrt{\mu_2}|x|}\big) \\
&\qquad\qquad + 4x_j(-s)\frac{x_\ell}{|x|}\big(\mu_1e^{-s\sqrt{\mu_1}|x|}-\mu_2e^{-s\sqrt{\mu_2}|x|}\big) \\
&\qquad\qquad + |x|^2(-s)\left(\frac{\delta_{j\ell}}{|x|}-\frac{x_jx_\ell}{|x|^3}\right)\big(\mu_1e^{-s\sqrt{\mu_1}|x|}-\mu_2e^{-s\sqrt{\mu_2}|x|}\big) \\
&\qquad\qquad + |x|^2s^2\frac{x_jx_\ell}{|x|^2}\big(\mu_1^{\frac{3}{2}}e^{-s\sqrt{\mu_1}|x|}-\mu_2^{\frac{3}{2}}e^{-s\sqrt{\mu_2}|x|}\big)\bigg]\,ds \\
&\quad = \frac{-1}{8\pi(\mu_1-\mu_2)}\bigg[2\delta_{j\ell}\big(\sqrt{\mu_1}H_1^0(x)-\sqrt{\mu_2}H_2^0(x)\big) \\
&\qquad - \left(\frac{3x_jx_\ell}{|x|}+\delta_{j\ell}|x|\right)\big(\mu_1H_1^1(x)-\mu_2H_2^1(x)\big)+x_jx_\ell\big(\mu_1^{\frac{3}{2}}H_1^2(x)-\mu_2^{\frac{3}{2}}H_2^2(x)\big)\bigg].
\end{aligned}
$$

This completes the proof of the lemma. □

Applying Lemmas 2.4, 2.5 and 2.6 to (2.19), we have

$$
\begin{aligned}
u_j(x) = {} & \left[\frac{1}{4\pi\mu|x|} - \frac{k}{4\pi\mu^{\frac{3}{2}}}\int_0^1 e^{-s(k/\sqrt{\mu})|x|}\,ds\right] * f_j \\
& - \sum_{\ell=1}^{3}\left[\frac{1}{8\pi\mu}\left(\frac{\delta_{j\ell}}{|x|} - \frac{x_j x_\ell}{|x|^3}\right) + kG_{j\ell}(k,|x|)\right] * f_\ell \\
& + \frac{1}{8\pi(\mu_1-\mu_2)}\sum_{\ell=1}^{3}\frac{(\tau_0 k+1)k}{(2\mu+\lambda)\gamma\kappa}\biggl[2\delta_{j\ell}\sqrt{\mu_1}H_1^0(x) - \sqrt{\mu_2}H_2^2(x) \\
& \qquad - \left(3\frac{x_j x_\ell}{|x|} + \delta_{j\ell}|x|\right)\bigl(\mu_1 H_1^1(x) - \mu_2 H_2^1(x)\bigr) \\
& \qquad + x_j x_\ell\bigl(\mu_1^{\frac{3}{2}} H_1^2(x) - \mu_2^{\frac{3}{2}} H_2^2(x)\bigr)\biggr] * f_\ell \\
& + \sum_{\ell=1}^{3}\frac{1}{8\pi(2\mu+\lambda)}\biggl[\left(\frac{\delta_{j\ell}}{|x|} - \frac{x_j x_\ell}{|x|^3}\right) \\
& \qquad - \frac{1}{\mu_1-\mu_2}\left(\mu_1^{\frac{3}{2}}\frac{\partial^2 H_1}{\partial x_j\partial x_\ell}(x) - \mu_2^{\frac{3}{2}}\frac{\partial^2 H_2}{\partial x_j\partial x_\ell}(x)\right)\biggr] * f_\ell \\
& + \frac{\beta(\tau_0 k+1)}{8\pi(2\mu+\lambda)\gamma\kappa}\left[\frac{x_j}{|x|} - \frac{1}{\mu_1-\mu_2}\left(\mu_1^{\frac{3}{2}}\frac{\partial H_1}{\partial x_j}(x) - \mu_2^{\frac{3}{2}}\frac{\partial H_2}{\partial x_j}(x)\right)\right] * g, \\
\theta(x) = {} & \sum_{\ell=1}^{3}\frac{(\tau_0 k+1)\delta k}{8\pi(2\mu+\lambda)\gamma\kappa}\left[\frac{x_\ell}{|x|} - \frac{1}{\mu_1-\mu_2}\left(\mu_1^{3/2}\frac{\partial H_1}{\partial x_\ell}(x) - \mu_2^{\frac{3}{2}}\frac{\partial H_2}{\partial x_\ell}(x)\right)\right] * f_\ell \\
& + \frac{(\tau_0 k+1)k^2}{(8\pi(2\mu+\lambda)\gamma\kappa}\left[\frac{2}{\mu_1^{\frac{1}{2}}+\mu_2^{\frac{1}{2}}} - |x| + \frac{1}{\mu_1-\mu_2}\bigl(\mu_1^{\frac{3}{2}}H_1(x) - \mu_2^{\frac{3}{2}}H_2(x)\bigr)\right] * g \\
& + \frac{\tau_0 k+1}{8\pi\gamma\kappa}\biggl[\frac{2}{|x|} - 2\sqrt{\mu_1}\int_0^1 e^{-s\sqrt{\mu_1}|x|}\,ds - \frac{2\mu_2}{\mu_1^{\frac{1}{2}}+\mu_2^{\frac{1}{2}}} + \mu_2|x| \\
& \qquad - \frac{1}{\mu_1-\mu_2}\bigl(\mu_1^{\frac{3}{2}}\mu_2 H_1(x) - \mu_2^{\frac{5}{2}}H_2(x)\bigr)\biggr] * g.
\end{aligned}
$$

Here and hereafter, $*$ stands for the usual convolution operator, namely

$$f * g(x) = \int_{\boldsymbol{R}^3} f(x-y)g(y)\,dy = \int_{\boldsymbol{R}^3} f(y)g(x-y)\,dy.$$

Let $f \in L_{1,\mathrm{loc}}(\boldsymbol{R}^3)$ and $g \in L_q(\boldsymbol{R}^3)$. Assume that $1 < q < \infty$ and that $g(x) = 0$ for $|x| \geq R$. Then, by Hölder's inequality we have

$$
\begin{aligned}
|f * g(x)| &\leq \int_{|y|\leq R} |f(x-y)||g(y)|\,dy \\
&\leq \left\{ \int_{|y|\leq R} |f(x-y)|\,dy \right\}^{\frac{1}{q'}} \left\{ \int_{|y|\leq R} |f(x-y)||g(y)|^q\,dy \right\}^{\frac{1}{q}}
\end{aligned}
$$

where $q' = q/(q-1)$. Then, for any $L > 0$ we have

$$
\begin{aligned}
&\int_{|x|\leq L} |(f*g)(x)|^q\,dx \\
&\quad \leq \int_{|x|\leq L} \left[\left\{ \int_{|y|\leq R} |f(x-y)|\,dy \right\}^{\frac{q}{q'}} \int_{|y|\leq R} |f(x-y)||g(y)|^q\,dy \right] dx \\
&\quad \leq \left\{ \int_{|x|\leq R+L} |f(x)|\,dx \right\}^{\frac{q}{q'}} \int_{|y|\leq R} \left(\int_{|x|\leq L} |f(x-y)|\,dx \right) |g(y)|^q\,dy \\
&\quad \leq \left\{ \int_{|x|\leq L+R} |f(x)|\,dx \right\}^{1+(\frac{q}{q'})} \left\{ \int_{\boldsymbol{R}^3} |g(y)|^q\,dy \right\}^{\frac{1}{q}}
\end{aligned}
$$

which implies that

$$
\|f*g\|_{L_q(B_L)} \leq \|f\|_{L_q(B_{L+R})} \|g\|_{L_q(\boldsymbol{R}^3)}.
$$

Moreover, by Lemma 2.1 we can write

$$
\sqrt{\mu_1} = k^{\frac{1}{2}} g_{11}(k,\tau_0) + k^{\frac{3}{2}} g_{12}(k,\tau_0), \quad \sqrt{\mu_2} = k g_{21}(k,\tau_0) + k^2 g_{22}(k,\tau_0)
$$

with some holomorphic functions $g_{j\ell}(k,\tau_0)$ which are defined on $U_\sigma := \{k \in \boldsymbol{C} \mid |k| \leq \sigma\}$. Here, σ is a rather small positive number which is chosen independently of τ_0 whenever $0 < \tau_0 \leq 1$. From this observation, $u_j(x)$ and $\theta(x)$ depend on $k \in U_\sigma$ analytically as $W^2_{q,\mathrm{loc}}(\boldsymbol{R}^3)$ function provided that $(f,g) \in L_q(\boldsymbol{R}^3)^3 \times L_q(\boldsymbol{R}^3)$ and (f,g) vanishes for $|x| \geq R$. Moreover, we have

$$
\begin{aligned}
u_j(x) &= \frac{1}{4\pi\mu|x|} * f_j - \sum_{\ell=1}^{3} \frac{1}{8\pi\mu}\left(\frac{\delta_{j\ell}}{|x|} - \frac{x_j x_\ell}{|x|^3}\right) * f_\ell \\
&\quad + \sum_{\ell=1}^{3} \frac{1}{8\pi(2\mu+\lambda)}\left(\frac{\delta_{j\ell}}{|x|} - \frac{x_j x_\ell}{|x|^3}\right) * f_\ell + \frac{\beta}{8\pi(2\mu+\lambda)\gamma\kappa}\frac{x_j}{|x|} * g + O(|k|^{\frac{1}{2}}), \\
\theta(x) &= \frac{1}{4\pi\gamma\kappa|x|} * g + O(|k|^{\frac{1}{2}}).
\end{aligned}
$$

Summing up, we have proved the following theorem.

THEOREM 2.7. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, $0 < \tau_0 \le 1$ and $R > 0$. Let σ_0 and S_k be the same number and solution operator as in Theorem 2.2, respectively. Set*

$$
\begin{aligned}
&\mathscr{L}_{q,R}(\boldsymbol{R}^3) = \{(f,g) \in L_q(\boldsymbol{R}^3)^3 \times L_q(\boldsymbol{R}^3) \mid (f,g) \text{ vanishes for } |x| > R\} \\
&\mathscr{W}_{q,\mathrm{loc}}(\boldsymbol{R}^3) = W^2_{q,\mathrm{loc}}(\boldsymbol{R}^3)^3 \times W^2_{q,\mathrm{loc}}(\boldsymbol{R}^3).
\end{aligned}
$$

Then, there exist a σ $(0 < \sigma \le \sigma_0)$ and $G_j(k) \in \mathrm{Anal}\,(U_\sigma, \mathscr{B}(\mathscr{L}_{q,R}(\boldsymbol{R}^3), \mathscr{W}_{q,\mathrm{loc}}(\boldsymbol{R}^3)))$ $(j = 0, 1)$ such that when $(f,g) \in \mathscr{L}_{q,R}(\boldsymbol{R}^3)$, $G_k(f,g) = (k^{1/2}G_0(k) + G_1(k))(f,g)$ solves equation (2.4) uniquely for $k \in U_\sigma$ and $G_k(f,g) = S_k(f,g)$ for $k \in U_{\sigma,\epsilon}$.

Moreover, if we set $(u_0, \theta_0) = G_1(0)(f,g)$, then $(u_0, \theta_0) \in \mathscr{W}_{q,\mathrm{loc}}(\boldsymbol{R}^3)$ and (u_0, θ_0) solves the equation

$$
\begin{aligned}
&-\mu\Delta u_0 - (\mu+\lambda)\nabla \mathrm{div}\, u_0 + \beta\nabla\theta_0 = f \quad \text{in } \boldsymbol{R}^3 \\
&-\kappa\gamma\Delta\theta_0 = g \quad \text{in } \boldsymbol{R}^3
\end{aligned}
\tag{2.33}
$$

and

$$
\begin{aligned}
u_{0,j}(x) &= \frac{1}{4\pi\mu|x|} * f_j - \sum_{\ell=1}^{3} \frac{1}{8\pi\mu}\left(\frac{\delta_{j\ell}}{|x|} - \frac{x_j x_\ell}{|x|^3}\right) * f_\ell \\
&\quad + \sum_{\ell=1}^{3} \frac{1}{8\pi(2\mu+\lambda)}\left(\frac{\delta_{j\ell}}{|x|} - \frac{x_j x_\ell}{|x|^3}\right) * f_\ell + \frac{\beta}{8\pi(2\mu+\lambda)\gamma\kappa}\frac{x_j}{|x|} * g, \qquad (2.34) \\
\theta_0(x) &= \frac{1}{4\pi\gamma\kappa|x|} * g.
\end{aligned}
$$

We remark that we can derive the solution formula (2.34) to the equation

(2.33) directly, using

$$\Delta^2\left(-\frac{|x|}{8\pi}\right) = \delta(x) \quad \text{in } \boldsymbol{R}^3 \tag{2.35}$$

applying P and Q to (2.33), then applying the Fourier transform, solving in Fourier space, and finally transforming back.

3. Spectral analysis of the thermoelastic equations with second sound in $\Omega \subset \boldsymbol{R}^3$.

In this section, we consider the resolvent problem:

$$\begin{aligned} &k^2 u - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \beta\nabla\theta = f && \text{in } \Omega \\ &k\theta + \gamma \mathrm{div}\, q + \delta k \mathrm{div}\, u = g && \text{in } \Omega \\ &\tau_0 k q + q + \kappa\nabla\theta = h && \text{in } \Omega \end{aligned} \tag{3.1}$$

subject to the boundary condition:

$$u = \theta = 0 \quad \text{on } \Gamma \tag{3.2}$$

where Γ denotes the boundary of an exterior domain Ω of $C^{1,1}$ class. Since $q = (1+\tau_0 k)^{-1}(h - \kappa\nabla\theta)$, inserting this formula into the second equation of (3.1) we have

$$\begin{aligned} &k^2 u - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \beta\nabla\theta = f && \text{in } \Omega \\ &k\theta - \gamma\kappa(\tau_0 k+1)^{-1}\Delta\theta + \delta k \mathrm{div}\, u = g - \gamma(\tau_0 k+1)^{-1}\mathrm{div}\, h && \text{in } \Omega \end{aligned}$$

subject to the boundary condition (3.2). Therefore, for the simplicity we consider the following boundary value problem below:

$$\begin{aligned} &k^2 u - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \beta\nabla\theta = f && \text{in } \Omega \\ &k\theta - \gamma\kappa(\tau_0 k+1)^{-1}\Delta\theta + \delta k \mathrm{div}\, u = g && \text{in } \Omega \\ &u = \theta = 0 && \text{on } \Gamma. \end{aligned} \tag{3.3}$$

We shall discuss the low frequency expansion of solutions to (3.3) in this section, which corresponds to Theorem 2.7 in Section 2. For this purpose, we shall construct a parametrix of (3.3). Let $R > 0$ be a fixed large number such that $\boldsymbol{R}^3 \setminus \Omega \subset B_R = \{x \in \boldsymbol{R}^3 \mid |x| < R\}$. Set

$$\begin{aligned}
&\mathscr{L}_{q,R}(\Omega) = \big\{(f, g) \in L_q(\Omega)^3 \times L_q(\Omega) \mid (f, g) \text{ vanishes for } |x| > R\big\}\\
&\mathscr{W}^2_{q,\mathrm{loc}}(\Omega) = W^2_{q,\mathrm{loc}}(\Omega)^3 \times W^2_{q,\mathrm{loc}}(\Omega).
\end{aligned}$$

Let σ, S_k, $G_0(k)$ and $G_1(k)$ be the same constant and operators as in Theorem 2.7 and set

$$G_k = k^{1/2} G_0(k) + G_1(k). \tag{3.4}$$

We always assume that $0 < \tau_0 \leq 1$ throughout this section. By Theorem 2.7, we know that given $(f, g) \in \mathscr{L}_{q,R}(\boldsymbol{R}^3)$, $G_k(f, g)$ solves equation (2.4) for $k \in U_\sigma$ and that $G_k(f, g) = S_k(f, g)$ for $k \in U_{\sigma,\epsilon}$. In particular, $G_k(f, g) \in W^2_q(\boldsymbol{R}^3)^4$ whenever $k \in U_{\sigma,\epsilon}$, because it follows from Theorem 2.2 that $S_k(f, g) \in W^2_q(\boldsymbol{R}^3)^4$. We also know that

$$G_0(k), \quad G_1(k) \in \mathrm{Anal}\big(U_\sigma, \mathscr{B}(\mathscr{L}_{q,R}(\boldsymbol{R}^3), \mathscr{W}^2_{q,\mathrm{loc}}(\boldsymbol{R}^3))\big).$$

As an auxiliary problem, we consider the boundary value problem:

$$\begin{aligned}
&-\mu\Delta U - (\mu + \lambda)\nabla \mathrm{div}\, U + \beta\nabla\Theta = f && \text{in } \Omega_{R+5}\\
&-\kappa\gamma\Delta\Theta = g && \text{in } \Omega_{R+5}\\
&U = \Theta = 0 && \text{on } \partial\Omega_{R+5}
\end{aligned} \tag{3.5}$$

where $\Omega_{R+5} = \Omega \cap B_{R+5}$ and $\partial\Omega_{R+5}$ denotes the boundary of Ω_{R+5} which is given by the formula: $\partial\Omega_{R+5} = S_{R+5} \cup \Gamma$ with $S_{R+5} = \{x \in \boldsymbol{R}^3 \mid |x| = R + 5\}$. It is well-known (cf. [**8**], [**7**]) that equation (3.5) admits a unique solution $(U, \Theta) \in W^2_q(\Omega_{R+5})^3 \times W^2_q(\Omega_{R+5})$ for any $(f, g) \in L_q(\Omega_{R+5})^3 \times L_q(\Omega_{R+5})$. We define a linear operator $T : L_q(\Omega_{R+5})^3 \times L_q(\Omega_{R+5}) \to W^2_q(\Omega_{R+5})^3 \times W^2_q(\Omega_{R+5})$ by the formula: $T(f, g) = (U, \Theta)$. Let $\varphi = \varphi(x)$ be a function in $C^\infty_0(\boldsymbol{R}^3)$ such that $\varphi(x) = 1$ for $|x| \leq R+2$ and $\varphi(x) = 0$ for $|x| \geq R+3$. Given a function f defined on Ω, f_0 denotes the zero extension of f to the whole space and $\mathscr{R}f$ the restriction of f to Ω_{R+5}. Now, let us define the operator A_k by the formula:

$$A_k(f, g) = (1 - \varphi)G_k(f_0, g_0) + \varphi T(\mathscr{R}f, \mathscr{R}g)$$

for $(f, g) \in \mathscr{L}_{q,R}(\Omega)$ and we write $A_k(f, g) = (A^1_k(f, g), A^2_k(f, g)) = (u_k, \theta_k)$. Since

$G_k(f_0, g_0)$ and $T(\mathscr{R}f, \mathscr{R}g)$ satisfy equations (2.4) and (3.5), replacing (f, g) by (f_0, g_0) and $(\mathscr{R}f, \mathscr{R}g)$, respectively, we have

$$\begin{aligned} &k^2 u_k - \mu\Delta u_k - (\mu+\lambda)\nabla \operatorname{div} u_k + \beta\nabla\theta_k = f + B_k^1(f,g) && \text{in } \Omega \\ &k\theta_k - \gamma\kappa(\tau_0 k+1)^{-1}\Delta\theta_k + \delta k \operatorname{div} u_k = g + B_k^2(f,g) && \text{in } \Omega \\ &u_k = \theta_k = 0 && \text{on } \Gamma \end{aligned} \tag{3.6}$$

where we have set

$$\begin{aligned} B_k^1(f,g) &= \varphi k^2 U + \mu\{2(\nabla u_k - \nabla U)(\nabla\varphi) + (\Delta\varphi)(u_k - U)\} \\ &\quad + (\mu+\lambda)\big\{\nabla[(\nabla\varphi)\cdot(u_k - U)] + (\nabla\varphi)(\operatorname{div} u_k - \operatorname{div} U)\big\} \\ &\quad + \beta(\nabla\varphi)(\theta_k - \Theta) \\ B_k^2(f,g) &= \varphi k\Theta + \gamma\kappa(\tau_0 k+1)^{-1}\big[(\nabla\varphi)\cdot(\nabla\theta_k - \nabla\Theta) + (\Delta\varphi)(\theta_k - \Theta)\big] \\ &\quad - \delta k(\nabla\varphi)\cdot(u_k - U). \end{aligned}$$

We see that $B_k^j(f,g)$ $(j = 1, 2)$ are compact operators on $\mathscr{L}_{q,R}(\Omega)$, because they belong to $W_q^1(\Omega)^4$ and vanish for $|x| > R+3$. Set $B_k(f,g) = (B_k^1(f,g), B_k^2(f,g))$ and

$$\mathscr{P}_k(u,\theta) = \big(-\mu\Delta u - (\mu+\lambda)\nabla\operatorname{div} u + \beta\nabla\theta,\ -\gamma\kappa(\tau_0 k+1)^{-1}\Delta\theta + \delta k \operatorname{div} u\big)$$

for the sake of notational simplicity. By Theorem 2.7 and (3.6) we see that

$$\begin{aligned} &(k^2 u_k, k\theta_k) + \mathscr{P}_k A_k(f,g) = (I + B_k)(f,g) && \text{in } \Omega, \\ &A_k(f,g) = (0,0) && \text{on } \Gamma \end{aligned} \tag{3.7}$$

and

$$\lim_{k\to 0} \|B_k(f,g) - B_0(f,g)\|_{L_q(\Omega)} = 0 \tag{3.8}$$

where I denotes the identity operator on $(\mathscr{L}_{q,R}(\Omega))^4$. If we show the existence of the inverse operator $(I+B_k)^{-1}$ of $I+B_k$ on $(\mathscr{L}_{q,R}(\Omega))^4$, then $A_k(I+B_k)^{-1}$ is the solution operator of (3.3). In view of (3.8), to prove the existence of $(I+B_k)^{-1}$ it suffices to show the existence of $(I+B_0)^{-1}$. Therefore, the main task of this section is to prove the following lemma.

LEMMA 3.1. *Let $1 < q < \infty$. Then, $(I + B_0)^{-1}$ exists as a bounded linear operator on $\mathscr{L}_{q,R}(\Omega)$.*

PROOF. Since B_0 is a compact operator on $(\mathscr{L}_{q,R}(\Omega))^4$, to prove the lemma it suffices to show the injectivity of $I + B_0$. Let (f, g) be in $\mathscr{L}_{q,R}(\Omega)$ such that $(I + B_0)(f, g) = 0$. By (3.7) with $k = 0$ we see that

$$\mathscr{P}_0 A_0(f, g) = (0, 0) \quad \text{in } \Omega, \quad A_0(f, g) = (0, 0) \quad \text{on } \Gamma. \tag{3.9}$$

Set $(u, \theta) = A_0(f, g)$, and then we can write (3.9) componentwise as follows:

$$\begin{aligned} &-\mu\Delta u - (\mu + \lambda)\nabla \operatorname{div} u + \beta\nabla\theta = 0 && \text{in } \Omega \\ &\gamma\kappa\Delta\theta = 0 && \text{in } \Omega \\ &u = \theta = 0 && \text{on } \Gamma. \end{aligned} \tag{3.10}$$

Moreover, by (2.34) in Theorem 2.7 we have

$$\begin{aligned} u_j(x) = {} & \frac{1}{4\pi\mu|x|} * f_{0,j} - \sum_{\ell=1}^{3} \frac{1}{8\pi\mu}\left(\frac{\delta_{j\ell}}{|x|} - \frac{x_j x_\ell}{|x|^3}\right) * f_{0,\ell} \\ & + \sum_{\ell=1}^{3} \frac{1}{8\pi(2\mu + \lambda)}\left(\frac{\delta_{j\ell}}{|x|} - \frac{x_j x_\ell}{|x|^3}\right) * f_{0,\ell} + \frac{\beta}{8\pi(2\mu + \lambda)\gamma\kappa}\frac{x_j}{|x|} * g_0 \end{aligned} \tag{3.11}$$

$$\theta(x) = \frac{1}{4\pi\gamma\kappa|x|} * g_0 \tag{3.12}$$

for $|x| \geq R + 3$, because $A_0(f, g) = G_1(0)(f_0, g_0)$ for $|x| \geq R + 3$. Here and hereafter, we write $f_0 = {}^t(f_{0,1}, f_{0,2}, f_{0,3})$. To complete the proof of the lemma, we shall use the following well-known facts (cf. e.g. [**8**]).

THEOREM 3.2. *Let $1 < q < \infty$.* (1) *Let $\theta \in W^2_{q,\mathrm{loc}}(\Omega)$ satisfy the homogeneous equation*:

$$\Delta\theta = 0 \quad \textit{in } \Omega, \quad \theta = 0 \quad \textit{on } \Gamma$$

and the radiation condition:

$$\theta(x) = O(|x|^{-1}), \quad \nabla\theta(x) = O(|x|^{-2}) \quad \textit{as } |x| \to \infty \tag{3.13}$$

then θ must vanish identically.

(2) *Let* $u \in W^2_{q,\mathrm{loc}}(\Omega)^3$ *satisfy the homogeneous equation*:

$$-\mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u = 0 \quad \textit{in } \Omega, \qquad u = 0 \quad \textit{on } \Gamma \tag{3.14}$$

and the radiation condition:

$$u(x) = O(|x|^{-1}), \qquad \nabla u(x) = O(|x|^{-2}) \quad \textit{as } |x| \to \infty \tag{3.15}$$

then u *must vanish identically.*

Since (f_0, g_0) vanishes for $|x| > R+3$, it follows from (3.12) that θ satisfies the radiation condition (3.13), so that by Theorem 3.2 we see that $\theta = 0$. If we insert this into the first equation of (3.10), then we see that u satisfies (3.14). Therefore, our task is to show that u also satisfies (3.15) to conclude that $u = 0$. From (3.12), we have

$$\begin{aligned} 0 &= \frac{1}{|x|} * g_0 = \int_{\boldsymbol{R}^3} \frac{g_0(y)}{|x-y|}\, dy \\ &= \int_{\boldsymbol{R}^3} \left(\frac{1}{|x-y|} - \frac{1}{|x|}\right) g_0(y)\, dy + \frac{1}{|x|}\int_{\boldsymbol{R}^3} g_0(y)\, dy \quad \text{for } |x| > R+3. \end{aligned} \tag{3.16}$$

If we write

$$\frac{1}{|x-y|} - \frac{1}{|x|} = \int_0^1 \frac{\partial}{\partial s}\frac{1}{|x-sy|}\, ds = \int_0^1 \frac{\sum_{j=1}^3 (x_j - sy_j)y_j}{|x-sy|^3}\, ds$$

using the fact that $g_0(y) = 0$ for $|y| > R+3$, we have

$$\left| \int_{\boldsymbol{R}^3} \left(\frac{1}{|x-y|} - \frac{1}{|x|}\right) g_0(y)\, dy \right| \le C_R |x|^{-2} \quad \text{for } |x| > R+4$$

which combined with (3.16) implies that

$$\int_{\boldsymbol{R}^3} g_0(y)\, dy = 0. \tag{3.17}$$

Therefore, if we write

$$\frac{\beta}{8\pi(2\mu+\lambda)\gamma\kappa}\frac{x_j}{|x|} * g_0 = \frac{\beta}{8\pi(2\mu+\lambda)\gamma\kappa}\int_{\boldsymbol{R}^3}\left(\frac{x_j - y_j}{|x-y|} - \frac{x_j}{|x|}\right) g_0(y)\, dy$$

in the formula (3.11), we see that

$$u(x) = O(|x|^{-1}), \quad \nabla u(x) = O(|x|^{-2}) \quad \text{as } |x| \to \infty$$

which combined with the assertion (2) of Theorem 3.2 implies that $u(x)$ also vanishes identically. Now, we have $A_0(f, g) = 0$, from which it follows that

$$(1 - \varphi)G_0(f_0, g_0) + \varphi T(\mathscr{R}f, \mathscr{R}g) = 0 \quad \text{in } \Omega. \tag{3.18}$$

If we write $G_0(f_0, g_0) = (u_0, \theta_0)$ and $T(\mathscr{R}f, \mathscr{R}g) = (U, \Theta)$, then (3.18) reads as follows:

$$(1 - \varphi)u_0 + \varphi U = 0, \quad (1 - \varphi)\theta_0 + \varphi\Theta = 0 \quad \text{in } \Omega. \tag{3.19}$$

Since $\varphi(x) = 1$ for $|x| \le R + 2$ and $\varphi(x) = 0$ for $|x| \ge R + 3$, from (3.19) we have

$$u_0 = 0, \quad \theta_0(x) = 0 \quad \text{for } |x| \ge R + 3 \tag{3.20}$$

$$U = 0, \quad \Theta(x) = 0 \quad \text{for } |x| \le R + 2. \tag{3.21}$$

Note that $(u_0, \theta_0) \in W^2_{q,\mathrm{loc}}(\boldsymbol{R}^3)^4$ and $(U, \Theta) \in W^2_q(\Omega_{R+5})^4$ satisfy the equations:

$$\begin{cases} -\mu\Delta u_0 - (\mu + \lambda)\nabla \mathrm{div}\, u_0 + \beta\nabla\theta_0 = f_0 & \text{in } \boldsymbol{R}^3 \\ -\kappa\gamma\Delta\theta_0 = g_0 & \text{in } \boldsymbol{R}^3 \end{cases} \tag{3.22}$$

$$\begin{cases} -\mu\Delta U - (\mu + \lambda)\nabla \mathrm{div}\, U + \beta\nabla\Theta = \mathscr{R}f & \text{in } \Omega_{R+5} \\ -\kappa\gamma\Delta\Theta = \mathscr{R}g & \text{in } \Omega_{R+5} \\ U = \Theta = 0 & \text{on } \partial\Omega_{R+5} \end{cases} \tag{3.23}$$

respectively. If we set $(U_0, \Theta_0)(x) = (U, \Theta)(x)$ for $x \in \Omega_{R+5}$ and $(U_0, \Theta_0)(x) = (0, 0)$ for $x \in \boldsymbol{R}^3 \setminus \Omega$, then by (3.21) and (3.23) we have $(U_0, \Theta_0) \in W^2_q(B_{R+5})^4$ and

$$\begin{aligned} &-\mu\Delta U_0 - (\mu + \lambda)\nabla \mathrm{div}\, U_0 + \beta\nabla\Theta_0 = f_0 && \text{in } B_{R+5} \\ &-\kappa\gamma\Delta\Theta_0 = g_0 && \text{in } B_{R+5} \\ &U_0 = \Theta_0 = 0 && \text{on } S_{R+5}. \end{aligned} \tag{3.24}$$

From (3.20) and (3.22) it follows that the restriction of (u_0, θ_0) to B_{R+5} also satisfies (3.24), which combined with the uniqueness of solutions to (3.24) implies that $(u_0, \theta_0) = (U_0, \Theta_0)$ in B_{R+5}, that is $(u_0, \theta_0) = (U, \Theta)$ in Ω_{R+5}. Plugging this into (3.19), we have

$$0 = u_0 + \varphi(U - u_0) = u_0, \quad 0 = \theta_0 + \varphi(\Theta - \theta_0) = \theta_0 \quad \text{in } \Omega$$

which implies that $(f, g) = 0$ immediately. This completes the proof of the lemma. □

Combining Lemma 3.1 and (3.8), we see that there exists a small σ' $(0 < \sigma' \le \sigma)$ such that

$$\begin{aligned}(I + \mathscr{B}_k)^{-1} &= \big(I - (I + \mathscr{B}_0)^{-1}(\mathscr{B}_0 - \mathscr{B}_k)\big)^{-1}(I + \mathscr{B}_0)^{-1} \\ &= \left\{\sum_{j=0}^{\infty} \big((I + \mathscr{B}_0)^{-1}(\mathscr{B}_0 - \mathscr{B}_k)\big)^j\right\}(I + \mathscr{B}_0)^{-1}\end{aligned}$$

when $k \in \boldsymbol{C}$ and $|k| < \sigma'$. Moreover, $A_k(I + \mathscr{B}_k)^{-1}$ is a solution operator to (3.3) and the analytical property of $A_k(I + \mathscr{B}_k)^{-1}$ inherits from that of G_k mentioned in Theorem 2.7. Therefore, setting $H_k = A_k(I + \mathscr{B}_k)^{-1}$, we have the following theorem.

THEOREM 3.3. *Let $1 < q < \infty$ and $0 < \tau_0 \le 1$. Let R be a large fixed number such that $\boldsymbol{R}^3 \setminus \Omega \subset B_R$. Then, there exists a small number σ' $(0 < \sigma' \le \sigma)$ and an operator $H_k \in \mathscr{B}(\mathscr{L}_{q,R}, \mathscr{W}^2_{q,\mathrm{loc}}(\Omega))$ for each $k \in U_{\sigma'} = \{k \in \boldsymbol{C} \mid |k| \le \sigma'\}$ such that $H_k(f, g)$ satisfies equation (3.3) uniquely for any $(f, g) \in \mathscr{L}_{q,R}(\Omega)$ and $k \in U_{\sigma'}$ and H_k has the expansion formula*:

$$H_k = k^{1/2} H_0(k) + H_1(k) \quad \textit{for } k \in U_{\sigma'}$$

where $H^0(k)$, $H^1(k) \in \mathrm{Anal}(U_{\sigma'}, \mathscr{B}(\mathscr{L}_{q,R}, \mathscr{W}^2_{q,\mathrm{loc}}(\Omega)))$.

Since $S_k = G_k$ for $k \in U_{\sigma,\epsilon}$, we see that $A_k(I + \mathscr{B}_k)^{-1}(f, g) \in W^2_q(\Omega)^4$ provided that $(f, g) \in \mathscr{L}_{q,R}$ and $k \in U_{\sigma,\epsilon}$. And therefore, combining the whole space solution with $A_k(I + \mathscr{B}_k)^{-1}(f, g)$ by cut-off technique we have the following theorem.

THEOREM 3.4. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $0 < \tau_0 \le 1$. Let $\sigma' > 0$ be the same constant as in Theorem 3.3. Then, there exists an operator $T_k \in \mathrm{Anal}\,(U_{\sigma',\epsilon}, \mathscr{B}(L_q(\Omega)^4, W^2_q(\Omega)^4)$ such that $T_k(f, g)$ satisfies equation (3.3) for any $(f, g) \in L_q(\Omega)^4$ and $k \in U_{\sigma',\epsilon}$.*

PROOF. Let $k \in U_{\sigma',\epsilon}$. Let $\varphi \in C_0^\infty(\boldsymbol{R}^3)$ be a cut-off function such that $\varphi(x) = 1$ for $|x| \leq R+2$ and $\varphi(x) = 1$ for $|x| \geq R+3$. For any $(f,g) \in L_q(\boldsymbol{R}^3)^4$, we set $(v,\chi) = (1-\varphi)S_k(f_0,g_0)$, where (f_0,g_0) denotes the zero extension of (f,g) to the whole space. Obviously, $(v,\chi) \in W_q^4(\Omega)$ and satisfies the equation:

$$(k^2 v, k\chi) + \mathscr{P}_k(v,\chi) = (f,g) + (F,G) \quad \text{in } \Omega, \quad \mathscr{P}_k(u,\theta) = (0,0) \quad \text{on } \Gamma$$

for some $(F,G) \in \mathscr{L}_{q,R}(\Omega)$. If we set $(w,\omega) = A_k(I+\mathscr{B}_k)^{-1}(F,G)$, then as noted after Theorem 3.3, $(w,\omega) \in W_q^2(\Omega)^4$. Therefore, $(u,\theta) = (v,\chi) - (w,\omega) \in W_q^2(\Omega)^4$ and (u,θ) solves equation (3.3). In the above argument, obviously the dependence of (u,θ) on $k \in U_{\sigma',\epsilon}$ is holomorphic, which completes the proof of the theorem. □

4. The limit $\tau_0 \to 0$.

Employing the same argument, we can show the theorems corresponding to Theorems 3.3 and 3.4 in the classical thermoelastic case (cf. (1.4)). Moreover, we have, using Lemma 2.1,

THEOREM 4.1. *The solution operators H_k constructed in Theorem 3.3 and T_k in Theorem 3.4 depend on $\tau_0 \in (0,1]$ continuously. The limit of H_k and T_k as $\tau_0 \to 0$ are the corresponding operators of the classical thermoelastic equations, where the limit is given in the operator norm of $\mathscr{B}(\mathscr{L}_{q,R}(\Omega), \mathscr{W}^2_{q,\mathrm{loc}}(\Omega))$ when $k \in U_{\sigma'}$ and $\mathscr{B}(L_q(\Omega)^4, W_q^2(\Omega)^4)$ when $\operatorname{Re} k > 0$ and $|k| < \sigma'$, respectively.*

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